GLOBAL RATIO LIMIT THEOREMS FOR SOME NONLINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS. II

BY

STEPHEN GROSSBERG

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1. Introduction. A previous note [1] introduced some systems of nonlinear functional-differential equations of the form

$$\dot{X}(t) = AX(t) + B(X_t)X(t-\tau) + C(t) \qquad t \geq 0, \tag{1}$$

where $X=(x_1, \cdots, x_n)$ is nonnegative, $B(X_t)$ is a matrix of nonlinear functionals of $X(w)$ evaluated at all past times $w \in [-\tau, t]$, and $C=(C_1, \cdots, C_n)$ is a nonnegative and continuous input function. Some global ratio limit theorems were then stated for one of these systems. Here two other cases are considered. In particular, we study the dependence of the stability properties of (1) on the time lag τ.

Our systems are defined as follows. Given any positive integer n; any real numbers α, u, $\beta > 0$, and $\tau \geq 0$; and any $n \times n$ semistochastic matrix $P = \|p_{ij}\|$ (i.e., $p_{ij} \geq 0$ and $\sum_{k=1}^{n} p_{ik} = 0$ or 1), let

$$\dot{x}_i(t) = -\alpha x_i(t) + \beta \sum_{k=1}^{n} x_k(t-\tau) y_{ki}(t) + C_i(t), \tag{2}$$

$$y_{jk}(t) = p_{jk} z_{jk}(t) \left[\sum_{m=1}^{n} p_{jm} z_{jm}(t) \right]^{-1}, \tag{3}$$

and

$$\dot{z}_{jk}(t) = [-u z_{jk}(t) + \beta x_j(t-\tau) x_k(t)] \theta(p_{jk}), \tag{4}$$

for all $i, j, k = 1, 2, \cdots, n$, where

$$\theta(p) = 1 \quad \text{if } p > 0,$$
$$= 0 \quad \text{if } p \leq 0.$$

The initial data in $[-\tau, 0]$ is always chosen continuous, nonnegative, and with $z_{jk}(0) > 0$ iff $p_{jk} > 0$.

In Grossberg [1], we announced some results for the case

$$P = \begin{pmatrix} 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ & & 0 & & \end{pmatrix}.$$

Here we state some results for the cases $P = E_n/n$ and $P = 1/(n-1) \cdot (E_n - I_n)$, where E_n is the $n \times n$ matrix with 1's everywhere and I_n is the $n \times n$ identity matrix. In both cases, we choose all C_j to be identically zero.

2. Complete graphs with loops. Suppose $P = E_n/n$. In terms of the graph theoretical interpretation previously given [1], this means that every vertex v_i is connected to every vertex v_j with an equal weight $p_{ij} = 1/n$. The graph of this system is therefore *complete*, and since $i = j$ is permissible it is a complete graph *with loops*.

For fixed $\tau \geq 0$, let $S(\tau)$ be the largest real part of the zeros of $R_\tau(S) = S + \alpha - \beta e^{-\tau S}$, and let $\sigma(\tau) = u + 2S(\tau)$. The sign of $\sigma(\tau)$ influences the limiting behavior of the ratios $y_{jk}(t)$ and $X_i(t) = x_i(t)[\sum_{k=1}^{n} x_k(t)]^{-1}$ as $t \to \infty$, and in particular the behavior of $y_i(t) = \min\{y_{ki}(t): k = 1, 2, \cdots, n\}$ and $Y_i(t) = \max\{y_{ki}(t): k = 1, 2, \cdots, n\}$.

THEOREM 1. *For any fixed $n \geq 2$ and $\tau \geq 0$ with $\sigma(\tau) > 0$, let (2)–(4) have arbitrary nonnegative and continuous initial data. Then the limits $Q_i = \lim_{t\to\infty} X_i(t)$ and $P_{jk} = \lim_{t\to\infty} y_{jk}(t)$ exist and satisfy the equations*

$$Q_i = P_{ji}, \quad i, j = 1, 2, \cdots, n. \tag{5}$$

Moreover $Q_i \in [m_i, M_i]$ where $m_i = \min(X_i(0), y_i(0))$ and $M_i = \max(X_i(0), Y_i(0))$. The functions $X_i - y_i$, $X_i - Y_i$, $\dot{y}_i$, and $\dot{Y}_i$ change sign at most once and not at all if $y_i(0) \leq X_i(0) \leq Y_i(0)$.

3. Stability properties are graded in the time lag τ. Theorem 1 has an unusual consequence when $\alpha > \beta$. This case is characterized by the property that $\lim_{t\to\infty} x_i(t) = 0$ for all $i = 1, 2, \cdots, n$ and all $\tau \geq 0$. Heuristically this is the case for which the effect of all perturbations C_i over a finite time interval eventually die out.

PROPOSITION 1. *If $\alpha > \beta$, then $\sigma(\tau)$ is monotone increasing in $\tau \geq 0$, and $\sigma(0) = \sigma \equiv u + 2(\beta - \alpha)$.*

Thus if $\alpha > \beta$, then $\sigma(\tau_0) > 0$ implies that Theorem 1 holds for all $\tau \geq \tau_0$. We therefore say that the stability properties when $P = E_n/n$ are *graded* in $\tau \geq 0$. In particular, if $u > 2(\alpha - \beta) > 0$, then Theorem 1 holds for all $\tau \geq 0$ and all $n \geq 2$.

4. Dependence of limiting equations on the time lag τ. The condition $\sigma(\tau) > 0$ is not superfluous to guaranteeing the limiting equations (5), as we now illustrate in the case $\tau = 0$ for simplicity.

PROPOSITION 2. *If $\sigma < 0$, then*

$$| y_{jk}(t) - y_{jk}(0) | \leqq 2 \log \left(1 + \frac{\beta x^2(0)}{4 | \sigma | z^{(j)}(0)}\right)$$

for all $t \geqq 0$, *where* $z^{(j)} = \sum_{m=1}^{n} z_{jm}$ *and* $x = \sum_{m=1}^{n} x_k$.

In particular if $|\sigma|$ is chosen so large that

$$| y_{jk}(0) - y_{ik}(0) | > 2 \log \left(1 + \frac{\beta x^2(0)}{4 | \sigma | z^{(j)}(0)}\right)\left(1 + \frac{\beta x^2(0)}{4 | \sigma | z^{(i)}(0)}\right),$$

then the equations $P_{jk} = Q_k$ and $P_{ik} = Q_k$ cannot be simultaneously fulfilled.

5. **Complete graphs without loops.** In the complete graph with loops, any probability distribution Q_i can arise as a limit when $t \to \infty$ if $\sigma(\tau) > 0$. Simply let $m_i = M_i$ be the desired distribution. When the loops are removed from the complete graph, this is no longer true in general. This latter case is characterized by the matrix $P = (E_n - I_n)/(n-1)$ since then $p_{ii} = 0$, $i = 1, 2, \cdots, n$. We illustrate this fact in the complete 3-graph without loops.

THEOREM 2. *Let* $P = \frac{1}{2}(E_3 - I_3)$ *and* $\tau = 0$. *Then for any positive initial data satisfying* $z_{ij}(0) = z_{ji}(0)$, $i, j = 1, 2, 3$, *the following conclusions hold.*

(A) (*Limiting behavior.*) *All the ratios* X_i *and* y_{jk} *have limits* $Q_i = \lim_{t\to\infty} X_i(t)$ *and* $P_{jk} = \lim_{t\to\infty} y_{jk}(t)$ *which satisfy the equations*

$$\frac{1}{2} \geqq Q_i = Q_j P_{ji} + Q_k P_{ki}, \quad \{i, j, k\} = \{1, 2, 3\}. \tag{6}$$

In particular

$$\lim_{t\to\infty} x_i(t) e^{(\alpha-\beta)t} = Q_i \sum_{k=1}^{3} x_k(0).$$

(B) (*Uniqueness.*) *If moreover the coefficients satisfy the inequality* $\sigma = u + 2(\beta - \alpha) > 0$, *then* $Q_i = 1/3$, *and* $P_{jk} = p_{jk} = \frac{1}{2}(1 - \delta_{jk})$, $i, j, k = 1, 2, 3$.

That is, the dynamical "limiting transition probabilities" P_{jk} always equal the geometrical "path weights" p_{jk}, in sharp contrast to the complete graph with loops. $\sigma > 0$ can be guaranteed if $\alpha > \beta$, or $\alpha < \beta$, or $\alpha = \beta$, for appropriate choices of u (e.g., if $u > 2|\alpha - \beta|$). Since $x = \sum_{k=1}^{3} x_k$ obeys the equation $\dot{x} = (\beta - \alpha)x$, $\lim_{t\to\infty} x_i(t) = 0$ if $\alpha > \beta$, $\lim_{t\to\infty} x_i(t) = \infty$ if $\alpha < \beta$, and $\sum_{k=1}^{3} x_k(t) = \text{constant}$ if $\alpha = \beta$. In all cases $\lim_{t\to\infty} X_i(t) = 1/3$. The absolute size of the outputs $x_i(t)$ is thus a bad index of the stability of the ratios $X_i(t)$ as $t \to \infty$.

The manner in which these limits are approached can also be qualitatively studied.

(C) (*Oscillations.*) *For all indices* $\{i, j, k\} = \{1, 2, 3\}$, *the functions* $f_{ij} = x_i - x_j$, $g_{ijk} = z_{ij} - z_{ik}$, $h_{ijk} = x_i z_{jk} - x_k z_{ji}$, *and* $\dot{y}_{ij}$ *change sign at most once.* f_{ij} *and* g_{kij} *do not change sign at all if* $f_{ij}(0) g_{kij}(0) \geqq 0$, *while* $\dot{y}_{ij}$ *and* h_{jki} *do not change sign at all if* $f_{ij}(0) g_{kij}(0) \geqq 0$ *and* $h_{jki}(0) g_{kij}(0) \geqq 0$. *Moreover* $f_{ij}(0) g_{kij}(0) > 0$ *implies* $f_{ij}(t) g_{kij}(t) > 0$ *for all* $t \geqq 0$, *while* $f_{ij}(0) g_{kij}(0) > 0$ *and* $h_{jki}(0) g_{kij}(0) > 0$ *imply* $f_{ij}(t) g_{kij}(t) > 0$ *and* $h_{jki}(t) g_{kij}(t) > 0$ *for all* $t \geqq 0$.

For example, if $x_i(0) > x_j(0)$ and $z_{ki}(0) > z_{kj}(0)$, then $x_i(t) > x_j(t)$ and $z_{ki}(t) > z_{kj}(t)$ for all $t \geqq 0$. That is, a common ordering in corresponding vertices and edges "propagates through time" and therefore is a geometrical property of the graph. If moreover $x_j(0) z_{ki}(0) > x_i(0) z_{kj}(0)$, then $y_{ij}(t)$ approaches its limit monotonically but does not reach this limit in finite time.

Proposition 2 also holds in the complete 3-graph without loops. When $\sigma < 0$ and $|\sigma| \gg 0$, the ratios $y_{ik}(t)$ are approximately constant. Nonetheless the limiting equations (6) hold because the ratios $X_i(t)$ adjust themselves as much as is required to reach an "equilibrium" state as $t \to \infty$.

6. **The variational system.** In this section we linearize the complete graphs with and without loops. We compare these linearizations with their nonlinear counterparts and in the graph without loops treat the general case $P = (E_n - I_n)/(n-1)$. Although conditions under which ratio limits exist coincide, the limiting equations are not always the same.

(2)–(4) can be written in matrix form when all $C_j \equiv 0$ as

$$U(t) = f(U(t), U(t - \tau)), \tag{7}$$

with

$$U = (x_1, \cdots, x_n, z_{11}, z_{12}, \cdots, z_{n,n-1}, z_{nn}),$$

$$f = (f_1, \cdots, f_n, f_{11}, f_{12}, \cdots, f_{n,n-1}, f_{nn}),$$

$$f_i = -\alpha x_i + \beta \sum_{k=1}^{n} x_k(t - \tau) p_{ki} z_{ki} \left(\sum_{m=1}^{n} p_{km} z_{km}\right)^{-1}$$

and

$$f_{jk} = [-u z_{jk} + \beta x_j(t - \tau) x_k] \theta(p_{jk}).$$

A *positive* solution U of (7) is one for which $x_i(t) > 0$ and $z_{jk}(t) > 0$ iff $p_{jk} > 0$ for all $t \geqq 0$. A *positive uniform* solution U_0 of (7) is one for which $x_i(t)=\gamma(t)>0$ and $z_{jk}(t)=\delta(t)\theta(p_{jk})$, where $\delta(t)>0$, for all $t\geqq 0$. If $V=U-U_0$, then [2] using the notation $f=f(\xi,\eta)$,

$$V(t)=f_\xi(U_0(t),U_0(t-\tau))V(t)+f_\eta(U_0(t),U_0(t-\tau))V(t-\tau)+o(\|V\|).$$

This system is studied because

$$X_i-\frac{1}{n}=(1-n)\left(V_i\left[\sum_{k=1}^{n}V_k\right]^{-1}-\frac{1}{n}\right),\text{ where } V_i=x_i-\gamma,$$

whenever γ and $\sum_{k=1}^{n}x_k$ are given the same initial data. Ignoring the terms $o(\|V\|)$, which are $O(e^{(\beta-\alpha)t})$ and therefore exponentially small when $\alpha>\beta$, we find

$$\text{(8)}\qquad W(t)=f_\xi(U_0(t),U_0(t-\tau))W(t)+f_\eta(U_0(t),U_0(t-\tau))W(t-\tau),$$

which is the *variational system* of (7). We write W in component form as $W=(h_1,\cdots,h_n,h_{11},\cdots,h_{nn})$, and for every $f\in C^0[0,\tau]$ we define

$$K_\tau(f)=f(\tau)+\beta\int_0^\tau f(\xi)e^{-\xi s(\tau)}d\xi.$$

The linearized analog of Theorem 1 is then the following

THEOREM 3. *Let $P=E_n/n$ where $n\geqq 2$ and $\tau\geqq 0$ is chosen arbitrarily with $\sigma(\tau)>0$. Let U_0 be a fixed but arbitrary positive uniform solution of (7). For any solution of (8) whose initial data satisfies $K_\tau(\sum_{i=1}^{n}h_i)\neq 0$, the limits*

$$\tilde{Q}_i=\lim_{t\to\infty}h_i(t)\left[\sum_{m=1}^{n}h_m(t)\right]^{-1}$$

and

$$\tilde{P}_{jk}=\lim_{t\to\infty}h_{jk}(t)\left[\sum_{m=1}^{n}h_{jm}(t)\right]^{-1}$$

exist and satisfy the equations

$$\text{(9)}\qquad \tilde{P}_{jk}=(\tilde{Q}_j+\tilde{Q}_k)/(1+n\tilde{Q}_j).$$

Thus linearizing (7) as in (8) changes the distribution of its ratios as $t\to\infty$. The two conditions $Q_k=P_{jk}$ and (9) are compatible when $\tilde{Q}_j>0$ iff $\tilde{Q}_k=\tilde{P}_{jk}=1/n$.

The linearized analog of Theorem 2 is now given in terms of the functions $k(\tau)=\beta e^{-\tau s(\tau)}$ and $\sigma(\tau)$.

THEOREM 4. *Let $P=(E_n-I_n)/(n-1)$, where $n\geqq 3$ and $\tau\geqq 0$ are chosen to satisfy $\sigma(\tau)>0$ and $k(\tau)+\sigma(\tau)>k(\tau)(1+\tau\sigma(\tau))/(n-1)$. Let U_0 be any positive uniform solution of (7). Then there exist positive constants ω_1 and ω_2 such that for any solution of (8) whose initial data satisfies $K_\tau(\sum_{i=1}^{n}h_i)\neq 0$,*

$$h_i(t)\left[\sum_{m=1}^{n}h_m(t)\right]-\frac{1}{n}=O(e^{-\omega_1 t})$$

and

$$h_{jk}(t)\left[\sum_{m\neq j}h_{jm}(t)\right]^{-1}-\frac{1}{n-1}=O(e^{-\omega_2 t}).$$

COROLLARY 1. *For every $\tau\geq 0$ such that $\sigma(\tau)>0$, there exists an $n=n(\tau)$ such that Theorem 4 holds for n and τ.*

COROLLARY 2. *Theorem 4 is true for all systems with $n\geqq n_0$ and $\tau=\tau_0$ if it is true for $n=n_0$ and $\tau=\tau_0$. If $\tau=0$ and $\sigma>0$, Theorem 4 is true for all $n\geqq 3$.*

That is, stability is graded in n.

COROLLARY 3. *If Theorem 4 holds for $n=n_0$ and $\tau=\tau_0$, then it holds for $n=n_0$ and all τ in a neighborhood of τ_0.*

COROLLARY 4. *If $\alpha>\beta$ (i.e., $\lim_{t\to\infty}x_i(t)=0$ for all $\tau\geqq 0$, there exists a positive function $\mu(n)$ of $n\geqq 3$, which is monotone increasing in n with $\lim_{n\to\infty}\mu(n)=\infty$, such that Theorem 4 holds for all $n\geqq 3$ and $\tau\in[0,\mu(n))$.*

6. **Summary.** Global ratio limit theorems are stated for some nonlinear functional-differential equations and their linearizations. The possible limits depend on the matrix P characterizing the geometry of the system. A system is found whose stability becomes easier to guarantee as its time lag increases, and several of whose ratios oscillate no more than once and are monotonic as $t\to\infty$ no matter how large the time lag is.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY