Targeted Testing of Dynamic Stochastic General Equilibrium Models*

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Abstract

We develop targeted specification tests for Dynamic Stochastic General Equilibrium (DSGE) models, which can separately examine a model's steady state properties, its overall dynamic properties, and its properties in selected frequency bands, such as business cycle frequencies. These tests can be applied to a subset of variables alongside the full model to pinpoint the sources of misspecification. We address the issues of indeterminacy and weak identification. Our empirical results indicate that a small-scale DSGE model is rejected based on the full spectrum test over the period from 1960 to 2007, revealing issues related to inflation dynamics and comovements between variables over business cycle frequencies. The same model is not rejected when a regime change is allowed in 1979. The Smets-Wouters model is not rejected over the same period. Furthermore, a medium-scale model incorporating news shocks is rejected based on business cycle frequencies, and issues related to hours worked are detected. These proposed methods are also applicable to Gaussian (factor augmented) Vector Autoregressions.

Keywords: DSGE, misspecification, frequency domain methods, weak identification.

JEL classification: C3, C52, E1.

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1 Introduction

Dynamic Stochastic General Equilibrium (DSGE) models aim to provide a comprehensive framework for analyzing business cycles, understanding monetary and fiscal policies, and making forecasts. Despite significant improvements over the past two decades, important misspecifications may still persist in various parts of these models. For instance, Schorfheide (2013) documented that DSGE models tend to underperform in capturing low-frequency fluctuations, which could lead to erroneous conclusions about the drivers of business cycles. As new versions of DSGE models continue to be proposed, it is crucial for practitioners to have a robust set of statistical methods to diagnose the compatibility of DSGE models with the data, as well as to identify key areas for further improvement.

This paper introduces a family of specification tests for diagnosing misspecifications in DSGE models. We call these tests "targeted tests" because they can separately examine a model's steady state properties, overall dynamic properties, and properties in selected frequency bands, such as business cycle frequencies. The ability to focus on specific aspects means that the tests do not require the DSGE model to match the dynamic behavior of the data in every fine detail. The tests can be applied to a subset of variables in addition to the full model, e.g., testing inflation and GDP dynamics without involving interest rates. This feature is helpful for pinpointing the variables most affected by misspecification, suggesting directions for model improvement.

The tests are constructed in the frequency domain based on weighted integrated periodograms. Bartlett (1955) and Grenander and Rosenblatt (1957) considered a periodogram-based approach to specification testing for univariate models. We generalize their idea to a multivariate setting to examine the specification of DSGE models. We first introduce a Kolmogorov-Smirnov test for checking the dynamic specification over the full spectrum. The test can be motivated from a likelihood perspective, as it is related to the divergence between frequency domain Gaussian likelihood and its expected value under the correct model specification. Subsequently, the test is generalized in several directions to: 1) allow weights over frequencies, 2) test the model's properties over a selected frequency band such as the business cycle frequencies; 3) test the model's steady state properties; and 4) test any of the above for a subset of the observables rather than all the variables. We tabulate the relevant 10%, 5%, and 1% critical values. The tests are consistent against global alternatives.

The tests can be computed at any given structural parameter value. Subsequently, we propose a two-step procedure to address parameter uncertainty. In the first step, we obtain a set of plausible parameter values via an inference method. In the second step, we check the compatibility of these parameter values with the data using the proposed tests, controlling the overall significance level with a Bonferroni correction. If all parameters are rejected, we reject the model. Otherwise, the model is not rejected, and the remaining parameter values offer a chance to re-evaluate the model's implications. In the second step, we consider selected frequency bands and subsets of variables to further understand the source of misspecification.

We obtain the set of plausible parameter values in two ways. The first is by inverting a statistic robust to weak identification. For this purpose, we use the score test of Qu (2014), although the tests of Guerron-Quintana et al. (2013) and Andrews and Mikusheva (2015) may also be useful. The second is by using draws from a proper Bayesian posterior distribution. Although the latter do not have a frequentist interpretation in this context, they enable us to examine whether the findings regarding misspecification are sensitive to particular parameter values. Moreover, considering these parameter values fosters a dialogue with the Bayesian DSGE literature since posterior distributions are commonly used to formulate policy recommendations. In particular, we may gain insights into whether these policy recommendations can withstand model testing.

Our framework encompasses both determinacy and indeterminacy in order to accommodate a wide range of empirical applications. Lubik and Schorfheide (2004) have found that indeterminacy is a feature of US monetary policy practices during 1960-1979. Other related studies examining monetary or fiscal policies include, among others, Leeper (1991), Clarida, et al. (2000), Benhabib, et al. (2001), Boivin and Giannoni (2006), Benati and Surico (2009), Mavroeidis (2010), Cochrane (2011, 2014), and Leeper et al. (2017). In our empirical application, we use the specification tests to compare a model's fit under different policy regimes within the same sample period and to evaluate its fit across different subsample periods after introducing a change in policy regimes. A related analysis, undertaken from a Bayesian perspective, is found in Lubik and Schorfheide (2004).

We evaluate the size and power properties of the tests using the small-scale model of Lubik and Schorfheide (2004) as the data generating process. We consider both determinacy and indeterminacy. The results indicate that the tests have excellent size properties even in small samples. We also examine the tests' power in diverse scenarios, including evaluating the complete model, subsets of variables, and specific frequency ranges.

We then move on to the empirical applications, where we examine three DSGE models: the small-scale model by Lubik and Schorfheide (2004), and two medium-scale models - the Smets and Wouters (2007) model and the news shocks model of Schmitt-Grohé and Uribe (2012). The Lubik and Schorfheide (2004) model contrasts determinacy and indeterminacy within a small-scale framework. The Smets and Wouters (2007) model extends the standard New Keynesian model by incorporating additional frictions and real rigidities and allows us to examine how model specification improves compared to the baseline small-scale model. Additionally, the Schmitt-Grohé and Uribe (2012) model provides an opportunity to evaluate whether the proposed information structure generates dynamics that fit the data adequately, as well as how the proposed structure compares to the standard structure assumed in the small- and medium-scale models.

Our results indicate that the small-scale DSGE model is rejected at the 10% significance level for both determinacy and indeterminacy specifications for the period 1960-2007 based on the full spectrum test, and it is nearly rejected based on the business cycle frequencies only (with only 0.08% draws surviving the test). Further analysis reveals misspecifications in most segments of the model, particularly in the inflation dynamics and comovements between variables. Specifically, the imaginary parts of the cross-spectra, which measure the lead and lag relationships between variables, deviate the most from the data. These conclusions are reinforced by using draws from posterior distributions. By splitting the full sample at 1979:II, we find that the model is no longer rejected if indeterminacy is used for the first subsample and determinacy for the second. This supports Lubik and Schorfheide's (2004) conclusion that U.S. monetary policy post-1982 is consistent with determinacy, while the pre-Volcker period exhibits greater uncertainty. We also analyze changes in data dynamics, especially the cross-spectrum, that bring the data closer to alignment with the model.

For the Smets and Wouters (2007) model, we find that it is not rejected at either full spectrum or business cycle frequencies at the 10% significance level using the full sample 1960:I-2007:IV. This contrasts with the case of the small-scale model examined earlier. Using draws from the posterior distribution produces qualitatively similar results. Regardless of which set of draws is used, over 60% and 80% of the draws are rejected based on the weighted spectrum and business cycle frequencies, respectively, indicating significant room for model improvement. We found qualitatively similar results on Smets and Wouters' original sample 1965:I-2004:IV.

Moving on to the news shocks model, our results show that it is rejected by the business cycle frequency test at the 10% significance level using the original Schmitt-Grohé and Uribe (2012) sample, while the weighted full spectrum test produces a near-rejection. Further examination reveals that the main source of incompatibility between the model and the data is the per capita labor hours and its comovements with all the other observables.

Our analysis builds on the literature of diagnosing DSGE models. In this literature, a common practice is to compare models based on marginal likelihoods or forecasting accuracy. A model is preferred over another if its marginal likelihood value or forecasting accuracy is significantly higher. This approach is informative if the purpose is to rank models; however, it does not show if the preferred model is consistent with the data. An alternative approach is to use a structural vector autoregression (SVAR) as a benchmark and compare the impulse responses of the DSGE model with those of the SVAR; see, e.g., Christiano et al. (2005). However, the identification conditions for impulse responses may not be compatible between the DSGE and SVAR models, as discussed in Del Negro et al. (2007). Recent studies have investigated alternative methods to overcome these limitations. Del Negro et al. (2007) developed a hybrid DSGE-VAR model to evaluate DSGE models. Del Negro and Schorfheide (2009) applied this framework to assess the DSGE model's policy predictions. Inoue et al. (2020) introduced stochastic specification errors into DSGE models and evaluated the improvement in model fit through forecasting error decompositions and marginal likelihood comparisons. Our methods differ from the above in several aspects: the tests do not involve a parametric reference model, they examine subsets of variables and selected frequency bands to pinpoint misspecification, and they address the issue of weak identification. To our knowledge, our methods are among the first to test DSGE models with valid frequentist coverage properties.

Our analysis also draws insights from the literature that evaluates rational expectations models from a frequency domain perspective. Key studies in this area include Watson (1993), King and Watson (1996), and Diebold et al. (1998). Watson (1993) recommended the use of model and data spectra plots as diagnostic tools. Diebold et al. (1998) underscored the importance of examining model fit across frequencies and highlighted the benefits of adopting a graphical approach. We aim to extend this line of inquiry to the current generation of DSGE models, using new tests while accounting for weak identification.

The rest of the paper is structured as follows. In Section 2, we explain how to compute a DSGE model's spectrum allowing for indeterminacy, and highlighting the issue of weak identification. Section 3 introduces the specification tests, provides a likelihood perspective, and characterizes their asymptotic properties under the null and alternative hypotheses. This section also describes a model-dependent prewhitening filter that improves the tests' finite sample properties under the null hypothesis. In this section, the parameter values are assumed to be known, and the issue of parameter uncertainty is addressed in Section 4. Section 5 provides calibrated simulations for finite sample properties. Section 6 presents three empirical applications, and Section 7 provides concluding remarks. The Appendix includes proofs of the results, details on the empirical applications, and additional tables and figures that complement the main analysis.

2 The spectrum of a DSGE model

In this section, we describe the spectrum of a log linearized DSGE model to provide a basis for our analysis. Consider a DSGE model, log linearized around its steady state (Sims, 2002):

$$\Gamma_0 S_t = \Gamma_1 S_{t-1} + \Psi \varepsilon_t + \Pi \eta_t, \tag{1}$$

where S_t is a vector that includes endogenous variables, conditional expectations, and variables from exogenous shock processes if they are serially correlated. The vector ε_t contains serially uncorrelated structural shocks and η_t contains expectation errors. The elements of Γ_0, Γ_1, Ψ and Π are functions of structural parameters of the model. Depending on the values of Γ_0 and Γ_1, the system can have none, a unique, or multiple stable solutions (indeterminacy). Under indeterminacy, the structural parameters alone do not uniquely determine the dynamics of the model. The above formulation is sufficiently flexible and it allows for medium-scale model such as Smets and Wouters (2007) and Schmitt-Grohé and Uribe (2012).

Lubik and Schorfheide (2003) show that the full set of solutions to the above model is representable as

$$S_t = \Phi_1 S_{t-1} + \Phi_\varepsilon \varepsilon_t + \Phi_\epsilon \epsilon_t, \tag{2}$$

or equivalently,

$$S_t = (1 - \Phi_1 L)^{-1} [\Phi_\varepsilon \ \Phi_\epsilon] \begin{bmatrix} \varepsilon_t \\ \epsilon_t \end{bmatrix},$$

where L is the lag operator. In (2), Φ_1, Φ_ε and Φ_ϵ depend only on Γ_0, Γ_1, Ψ and Π, therefore, are functions of the structural parameters only. The term ϵ_t contains the sunspot shocks, which arises only under indeterminacy. The DSGE model alone imposes few restrictions on ϵ_t, i.e., it needs to be a martingale difference, so that $E_t \epsilon_{t+1} = 0$, but it can be arbitrarily contemporaneously correlated with the fundamental shocks ε_t. Intuitively, the properties of ϵ_t depend on how agents form their expectations, which is not fully revealed by the DSGE model under indeterminacy. To capture these features, Qu and Tkachenko (2017) adopted the following parameterization which expresses ϵ_t as an orthogonal projection onto ε_t and a residual term $\widetilde{\epsilon}_t$:

$$\epsilon_t = M \varepsilon_t + \widetilde{\epsilon}_t,$$

where M is a matrix of constants and $\widetilde{\epsilon}_t$ is uncorrelated with ε_t, with $Var(\widetilde{\epsilon}_t) = \Sigma_\epsilon$. Let θ^D be a p-by-1 vector consisting of all the structural parameters in (1). Let θ^U be a q-by-1 vector consisting of the sunspot parameters $\theta^U = (\text{vec}(\Sigma_\epsilon)', \text{vec}(M)')'$. We define an augmented parameter vector as follows:

$$\theta = \begin{bmatrix} \theta^D \\ \theta^U \end{bmatrix}.$$

This augmented parameter vector uniquely determines the dynamics of the model under both determinacy and indeterminacy.

In practice, the estimation is typically based on a subset of S_t or some linear transformations involving its current and lagged values. To be consistent with this practice, we use a matrix $A(L)$ of finite-order lag polynomials to specify the observables and define

$$Y_t(\theta) = A(L) S_t = H(L;\theta)(\varepsilon_t' \ \epsilon_t')',$$

with

$$H(L;\theta) = A(L)(1 - \Phi_1 L)^{-1} [\Phi_\varepsilon \ \Phi_\epsilon]. \tag{3}$$

Using this notation, the spectral density of $Y_t(\theta)$ is given by

$$f_\theta(\omega) = \frac{1}{2\pi} H(\exp(-i\omega);\theta) \Sigma(\theta) H(\exp(-i\omega);\theta)^*, \tag{4}$$

where * denotes the conjugate transpose and

$$\Sigma(\theta) = \begin{pmatrix} I & 0 \\ M & I \end{pmatrix} \begin{pmatrix} \Sigma_\varepsilon & 0 \\ 0 & \Sigma_\epsilon \end{pmatrix} \begin{pmatrix} I & 0 \\ M & I \end{pmatrix}'.$$

In practice, estimation and model diagnosis are always based on comparing the properties of $Y_t(\theta)$ implied by the model with that of the data.

Turning to the data, we let $\{Y_t\}$ denote realizations from a stochastic process, which will be used for model diagnosis. The process $\{Y_t\}$ is usually assumed to be stationary after model-dependent detrending operations. (For example, its elements might contain GDP growth and inflation.) However, its population mean, related to the model's steady state, is typically nonzero. To capture this, we let $\mu(\theta_0)$ denote the mean of $\{Y_t\}$ implied by the model and write

$$Y_t = \mu(\theta_0) + Y_t(\theta_0).$$

The above provides a system of equations connecting the data (i.e., the left hand side) to the model (i.e., the right hand side). The model is fully correctly specified if these equations hold at some θ_0 for all t. The model's steady-state properties are correctly specified if the mean of Y_t equals $\mu(\theta_0)$. The model's overall dynamic properties are correctly specified if the spectral density of Y_t aligns with that of $Y_t(\theta_0)$. Moreover, the model is correctly specified over a frequency band if the two spectral densities agree with each other over that band. These connections enable us to develop targeted tests for various aspects of the DSGE model in a unified framework.

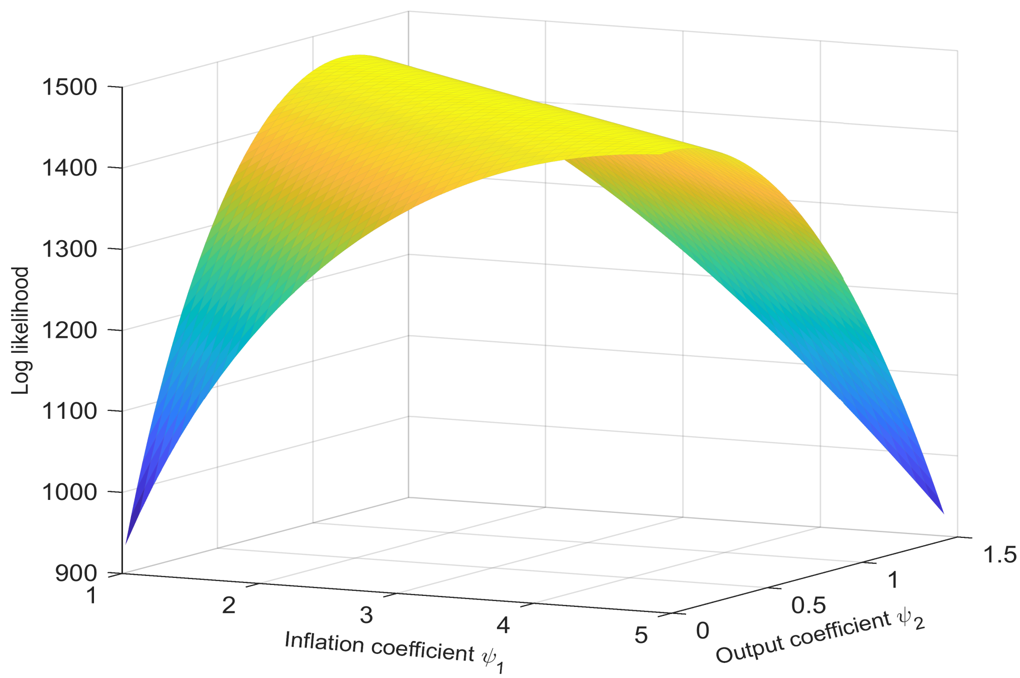
Before proceeding further, we present two examples to illustrate the interplay between parameter identification, equilibrium indeterminacy, and model specification analysis.

Example 1 *Suppose $E_t x_{t+1} = \alpha x_t$, where x_t is a scalar random variable (e.g., inflation), and α is a structural parameter. Assume $\lim_{k\to\infty} E_t x_{t+k}$ is finite. The solution to this model depends on whether $|\alpha| > 1$ or $|\alpha| \leq 1$. If $|\alpha| > 1$, solving the model forward yields $x_t = \lim_{k\to\infty}(1/\alpha)^k E_t x_{t+k} = 0$. In this regime, the model has a unique equilibrium and α is not identified; that is, it is impossible to uniquely determine α's value, even with infinite sample size. If $|\alpha| \leq 1$, solving the model forward is no longer informative. Instead, we can introduce a sunspot shock, $\epsilon_{t+1} = x_{t+1} - E_t x_{t+1}$, and solve the model backward to obtain*

$$x_{t+1} = \alpha x_t + \epsilon_{t+1}, \text{ with } E_t \epsilon_{t+1} = 0.$$

In this regime, the model exhibits indeterminacy (multiple equilibria) because x_t displays stochastic fluctuations in the absence of any shocks to fundamentals. Furthermore, if ϵ_t is stationary with a positive variance, then α is globally identified because x_t follows an AR(1) process. In summary, this simple example highlights three properties that, in fact, apply to general DSGE models: a) parameters can fail to be identified in DSGE models; b) identification properties can differ *across regimes; and c) a model's dynamic properties can differ across regimes, and, as a result, alternative regimes can produce different fits to empirical data. Recent studies have examined these issues; see, e.g., Canova and Sala (2009), Qu and Tkachenko (2017), and Lubik and Schorfheide (2004) on these three issues, respectively.*

Figure 1: Log likelihood surface with respect to Taylor rule coefficients.



Example 2 *An and Schorfheide (2007) used the following model to showcase the Bayesian analysis of DSGE models, where y_t, π_t, and r_t are log deviations of output, inflation, and interest rate from their steady states:*

$$
\begin{aligned}
y_t &= E_t y_{t+1} + g_t - E_t g_{t+1} - \frac{1}{\tau}(r_t - E_t \pi_{t+1} - E_t z_{t+1}), \\
\pi_t &= \beta E_t \pi_{t+1} + \kappa (y_t - g_t), \\
r_t &= \rho_r r_{t-1} + (1-\rho_r)\psi_1 \pi_t + (1-\rho_r)\psi_2 (y_t - g_t) + \varepsilon_{rt},
\end{aligned}
$$

where $g_t = \rho_g g_{t-1} + \varepsilon_{gt}, z_t = \rho_z z_{t-1} + \varepsilon_{zt}, \varepsilon_{rt} \overset{iid}{\sim} N(0, \sigma_r^2), \varepsilon_{gt} \overset{iid}{\sim} N(0, \sigma_g^2),\ \varepsilon_{zt} \overset{iid}{\sim} N(0, \sigma_z^2)$, and the three shocks are mutually independent. This model exhibits both determinate and indeterminate regimes as in the previous example. For the determinate regime, it has been shown that the four parameters in the Taylor equation (ρ_r, ψ_1, ψ_2, and σ_r) are not separately identified, meaning that their values can be moved along a curve without altering the model's dynamic properties (Qu and Tkachenko, 2012). To illustrate the effect of this property on inference, we plot the log *likelihood surface with respect to ψ_1 and ψ_2 while fixing all other parameters at their posterior mean values reported in An and Schorfheide (2007) for a simulated sample of size 1000. Note that, if the model were well identified, we would expect the log likelihood to resemble an elliptical dome, uniquely peaked at the maximum likelihood estimate (MLE) and displaying curvature in all directions. Instead, here the surface displays a ridge, which becomes nearly flat when moving the two parameters in opposite directions. Furthermore, if all four Taylor rule parameters (ρ_r, ψ_1, ψ_2, and σ_r) are allowed to change, then the likelihood surface will be completely flat along one direction, which means that the MLE is inconsistent and the information matrix is singular. Consequently, we expect that a specification test will likely have a nonstandard distribution (e.g., non Chi-square) if it requires the consistent estimation of the model's parameters.*

The above two examples demonstrate the importance of considering indeterminacy and weak identification when conducting specification analysis for DSGE models.

3 Frequency domain specification tests

In this section, we present the frequency domain tests and examine their asymptotic properties under null and alternative hypotheses. The value of θ_0 is assumed known and fixed. We address the issue of parameter uncertainty in the next section.

3.1 Proposed tests

We propose a family of tests based on integrated periodograms. The use of integrated periodograms for misspecification testing in the univariate case dates back to Grenander and Rosenblatt (1957) and Bartlett (1955). We extend this idea to the multivariate setting to test the specification of a DSGE model.

Suppose we have a sample of T observations: $\{Y_1, Y_2, ..., Y_T\}$, for which the Fourier frequencies are given by $\omega_j = 2\pi j/T$ ($j = 0, 1, ..., T-1$). The Fourier transform and periodogram are defined, respectively, as follows:

$$w_T(\omega_j) = (2\pi T)^{-1/2} \sum_{t=1}^{T} (Y_t - \mu(\theta_0)) \exp(-i\omega_j t),$$

and

$$I_T(\omega_j) = w_T(\omega_j)\, w_T(\omega_j)^*.$$

The Fourier transform projects the process $Y_t - \mu(\theta_0)$ onto the frequency domain, preserving all the information contained in the original series. Since the Fourier transform of a constant is equal to zero for any nonzero frequency, we can omit $\mu(\theta_0)$ and equivalently express $w_T(\omega_j)$ as $(2\pi T)^{-1/2} \sum_{t=1}^{T} Y_t \exp(-i\omega_j t)$ for $j = 1, ..., T-1$.

We first consider the testing of the model's dynamic properties (i.e., variance and serial correlation properties), which corresponds to the following null and alternative hypotheses:

$$\begin{aligned} H_0 &: \text{The spectral density of } Y_t \text{ equals } f_{\theta_0}(\omega) \text{ over } \omega \in [-\pi, \pi], \\ H_1 &: \text{The spectral density of } Y_t \text{ differs from } f_{\theta_0}(\omega) \text{ for some } \omega \in [-\pi, \pi]. \end{aligned}$$

Recall that if the model's dynamic properties are correctly specified, then $I_T(\omega_j) - f_{\theta_0}(\omega_j)$ are approximately uncorrelated across frequencies with a zero mean. Based on this property, we propose the following Kolmogorov-Smirnov test for checking the model's dynamic specification:

$$\mathcal{H}_{dT}(\theta_0) = \sup_{r\in[0,1]} \left\| (T/2)^{-1/2} \sum_{j=1}^{[Tr/2]} \mathrm{vec}\left\{ f_{\theta_0}(\omega_j)^{-1/2} \left(I_T(\omega_j) - f_{\theta_0}(\omega_j) \right) f_{\theta_0}(\omega_j)^{-1/2} \right\} \right\|_\infty .$$

The main part of the test is $I_T(\omega_j) - f_{\theta_0}(\omega_j)$, where $f_{\theta_0}(\omega_j)$ is the spectral density function implied by the model, $I_T(\omega_j)$ is the periodogram computed from the data, and the division by $f_{\theta_0}(\omega_j)$ acts as a normalization to ensure that the test is asymptotically pivotal. The norm $\|.\|_\infty$ represents the supremum norm, which is used to search for the strongest evidence over frequencies against the null hypothesis, i.e., for a generic vector $z = (z_1, ..., z_k) \in \mathbb{C}^k$, $\|z\|_\infty = \max(|z_1|, ..., |z_k|)$. The zero frequency is excluded, therefore, the test is invariant to the model's steady state properties. The test is straightforward to compute. No simulation is needed.

An immediate generalization of $\mathcal{H}_{dT}(\theta_0)$ is to assign weights to different frequencies. This feature is useful because DSGE models are not intended to capture high frequency fluctuations in the data. Let $W(\omega_j)$ be a smooth scalar-valued function or an indicator function to select the target frequencies. We propose the weighted test statistic

$$\mathcal{H}_{dT}^{W}(\theta_0) = \sup_{r\in[0,1]} \left\| (T/2)^{-1/2} \sum_{j=1}^{[Tr/2]} W(\omega_j)\, \mathrm{vec}\left\{ f_{\theta_0}(\omega_j)^{-1/2} \left(I_T(\omega_j) - f_{\theta_0}(\omega_j) \right) f_{\theta_0}(\omega_j)^{-1/2} \right\} \right\|_\infty . \tag{5}$$

We consider two specifications for $W(\omega_j)$. In the first case, $W(\omega)$ equals one for business cycle frequencies ($\pi/16 \le \omega \le \pi/3$ for quarterly data), and zero otherwise. In the second case, $W(\omega)$ is a linear decreasing function that assigns lower weights to high frequencies ($W(\omega) = 1 - \omega/\pi$). Note that the variance of the partial sum in (5) grows linearly with r, and setting $W(\omega) = 1 - \omega/\pi$ counterbalances this tendency by putting more emphasis on business cycle and lower frequencies. We provide critical values for these two cases and provide code for simulating critical values for other choices.

Now we turn to the model's steady state properties. In this case, the null and alternative hypotheses are:

$$\begin{aligned} H_0 &: \text{The mean of } Y_t \text{ equals } \mu(\theta_0), \\ H_1 &: \text{The mean of } Y_t \text{ differs from } \mu(\theta_0). \end{aligned}$$

Under the null hypothesis, the Fourier transform at the zero frequency, $w_T(0) = (2\pi T)^{-1/2} \sum_{t=1}^{T} (Y_t - \mu(\theta_0))$, is asymptotically normally distributed with a zero mean. The transformed values at nonzero Fourier frequencies are irrelevant for testing $\mu(\theta_0)$ because they are invariant to a location shift. We obtain a test for the model's steady state properties as follows:

$$\mathcal{H}_{sT}(\theta_0) = \sup_{r\in[0,1]} \left\|(2\pi T f_{\theta_0}(0))^{-1/2} \textstyle\sum_{j=1}^{[Tr]} (Y_t - \mu(\theta_0))\right\|_\infty .$$

This statistic has been used previously to test for structural changes in the mean of an otherwise stationary time series. The test has power even if $E(Y_t) = \mu(\theta_0)$ is violated for only a portion of the sample.

Finally, we can combine the aforementioned statistics to obtain a joint test for the model's static and dynamic specifications:

$$\mathcal{H}_T(\theta_0) = \max\left(\mathcal{H}_{sT}(\theta_0), \mathcal{H}_{dT}(\theta_0)\right).$$

Alternatively, using weights, the joint test can be expressed as:

$$\mathcal{H}_T^W(\theta_0) = \max\left(\mathcal{H}_{sT}(\theta_0), \mathcal{H}_{dT}^W(\theta_0)\right).$$

The tests presented above are for the full model. In practice, there might be an interest in closely examining a subset of variables. For instance, King and Watson (1996) compared three rational expectations models based on their ability to capture the comovement of a real (GDP) and a nominal (interest rate) variable. This kind of analysis becomes particularly valuable when the full model is rejected, as it can help pinpoint the source of the rejection. Doing this in our framework is straightforward. Let A be a variable selection matrix. For example, if the model comprises three variables and we aim to examine the first two, then set $A = [1,0,0;0,1,0]$. Similarly, to focus on the difference between the first and second variable, set $A = [1,-1,0]$. To analyze the relationship between their growth rates, set $A = [1-L,0,0;0,1-L,0]$, where L represents the lag operator. After defining A, we formulate the tests by replacing Y_t and $f_{\theta_0}(\omega)$ with AY_t and $Af_{\theta_0}(\omega)A'$, respectively. The subsequent analysis remains the same.

3.1.1 A likelihood perspective

The tests can be derived from a likelihood perspective. Recall that the Whittle likelihood is a frequency domain approximation to the time domain Gaussian likelihood. For a DSGE model, the Whittle likelihood (using dynamic properties only and omitting an additive constant) has the expression

$$L_T(\theta) = -\frac{1}{2T} \textstyle\sum_{j=1}^{T-1} \left[\log\det\left(f_\theta(\omega_j)\right) + \operatorname{tr}\left(f_\theta^{-1}(\omega_j) I_T(\omega_j)\right)\right].$$

Suppose the true spectral density generating the data is given by $f_0(\cdot)$, which may differ from the DSGE model-implied spectral density $f_\theta(\cdot)$. The Whittle log likelihood computed using the true spectral density is

$$L_{0,T} = -\frac{1}{2T} \textstyle\sum_{j=1}^{T-1} \left[\log\det\left(f_0(\omega_j)\right) + \operatorname{tr}\left(f_0^{-1}(\omega_j) I_T(\omega_j)\right)\right],$$

whose expected value is

$$E(L_{0,T}) = -\frac{1}{2T}\sum_{j=1}^{T-1}\left[\log\det(f_0(\omega_j)) + n_Y)\right] + o\left(T^{-1/2}\right),$$

where n_Y is dimension of Y_t, and the $o\left(T^{-1/2}\right)$ term arises because $E(I_T(\omega_j)) - f_0(\omega_j) = o(T^{-1/2})$. The divergence $T^{1/2}(L_T(\theta) - E(L_{0,T}))$ is therefore equal to

$$\frac{1}{2T^{1/2}}\sum_{j=1}^{T-1}\operatorname{tr}\left\{f_\theta(\omega_j)^{-1/2}\left(f_\theta(\omega_j) - I_T(\omega_j)\right)f_\theta(\omega_j)^{-1/2}\right\}$$
$$+\frac{1}{2T^{1/2}}\sum_{j=1}^{T-1}\left[\log\det(f_0(\omega_j)) - \log\det(f_\theta(\omega_j))\right] + o(1).$$

The second term is independent of the data, and the first term yields the $\mathcal{H}_{dT}(\theta_0)$ test upon replacing the trace operator with the vec operator and taking the supremum. Therefore, the $\mathcal{H}_{dT}(\theta_0)$ statistic is based on the divergence between the log likelihood and its expected value under the true data generating process (DGP), with the sup operator acting as a searching mechanism for the strongest evidence of model misspecification. This likelihood interpretation also provides insight into why the test is consistent against global alternatives, a result that will be proved later in this section.

3.1.2 Applications beyond DSGE models

The above tests are applicable to other linear models such as Gaussian SVARs and Factor Augmented VARs. We highlight these connections below.

Structural Vector Autoregressions. A typical SVAR model (Sims, 1980) is

$$\Phi_0 Y_t = \mu + \sum_{j=1}^{p}\Phi_j Y_{t-j} + \varepsilon_t,$$

where Y_t is an n-dimensional column vector, $\Phi_0, \Phi_1, ..., \Phi_p$ are coefficient matrices, and ε_t is an n-by-1 vector of serially uncorrelated structural disturbances with $Var(\varepsilon_t) = \Sigma$. We define $\theta = (\Phi_0, \Phi_1, ..., \Phi_p, \Sigma)$ and $\Pi(L;\theta) = \Phi_0 - \sum_{j=1}^{p}\Phi_j L^j$. The spectral density of Y_t is given by

$$f_\theta(\omega) = \frac{1}{2\pi}\left[\Pi(\exp(-i\omega);\theta)^{-1}\right]\Sigma\left[\Pi(\exp(-i\omega);\theta)^{-1}\right]^*.$$

For any given parameter value θ, the proposed tests can be constructed using Y_t and $f_\theta(\omega)$, or AY_t and $Af_\theta(\omega)A'$ for a subset of variables, where A is a selection matrix as defined previously.

Factor Augmented VAR. A typical model (see, e.g., Stock and Watson (2005)) is

$$\begin{aligned} Y_t &= \lambda(L)f_t + D(L)Y_{t-1} + v_t, \\ f_t &= \Gamma(L)f_{t-1} + \zeta_t, \end{aligned}$$

where Y_t is an n-by-1 vector of observables, f_t comprises the latent factors, ζ_t is a serially uncorrelated structural disturbance with $Var(\zeta_t) = I$, $Var(v_t) = \Sigma$ and $E(\zeta_t v_s') = 0$ for all t and s. $\lambda(L)$, $D(L)$ and $\Gamma(L)$ are matrix lag polynomials with $D(L)$ typically being diagonal.

The parameter vector θ consists of the elements in $\lambda(L)$, $D(L), \Gamma(L)$ and Σ. Under stationarity, Y_t has the following moving average representation:

$$Y_t = H_1(L;\theta)\zeta_t + H_2(L;\theta)v_t,$$

where $H_1(L;\theta) = [I - D(L)L]^{-1}\lambda(L)[I - \Gamma(L)L]^{-1}$ and $H_2(L;\theta) = [I - D(L)L]^{-1}$. The spectral density of Y_t is thus given by

$$f_\theta(\omega) = \frac{1}{2\pi}H_1(\exp(-i\omega);\theta)H_1(\exp(-i\omega);\theta)^* + \frac{1}{2\pi}H_2(\exp(-i\omega);\theta)\Sigma H_2(\exp(-i\omega);\theta)^*.$$

Stock and Watson (2005) discussed several identification strategies for this model, including (1) contemporaneous timing restrictions on the zero-order term in $H_1(L;\theta)$, (2) long-run restrictions on $H_1(1;\theta)$, (3) factor loading restrictions on $\lambda(L)$ and (4) Uhlig's (2005) sign restrictions on the coefficients of $H_1(L;\theta)$. Since these restrictions only affect $f_\theta(\omega)$, they are implementable in the frequency domain, and model diagnostics can be carried out under such restrictions. Importantly, our framework works directly with the structural parameter vector, so we can avoid making assumptions about the reduced form parameters. However, it is important to note that our framework assumes that the dimension of Y_t, n, is finite, which excludes the direct analysis of high-dimensional factor models. One potential generalization in that direction is to estimate the factors first and then treat them as part of the data, which can be implemented in a two-step procedure in the time domain (Stock and Watson, 2005). It would be interesting to explore these aspects in the frequency domain as well.

3.2 Asymptotic properties under null and alternative hypotheses

We need the following assumptions in order to study the tests' asymptotic properties under the null hypothesis of correct model specification.

Assumption 1. $\theta_0 \in \Theta \subset \mathbb{R}^q$ with Θ compact.

Assumption 2. The model solution can be written as

$$Y_t(\theta) = H(L;\theta)u_t(\theta) \quad \text{with} \quad H(L;\theta) = \textstyle\sum_{j=0}^{\infty} h_j(\theta)L^j,$$

where $u_t(\theta)$ is serially uncorrelated with a nonsingular covariance matrix denoted by $\Sigma(\theta)$.

Assumption 3. For all $\omega \in [-\pi, \pi]$ and $\theta \in \Theta$, there exist positive constants C_L and C_U such that: (i) the eigenvalues of $f_\theta(\omega)$ satisfy $C_L \leq \mathrm{eig}(f_\theta(\omega)) \leq C_U$; (ii) $f_\theta(\omega)$ belongs to the Lipschitz class of order β (Lip(β)) with $\beta > 1/2$; (iii) $||\partial \operatorname{vec} f_\theta(\omega)/\partial\theta'|| \leq C_U$; (iv) $\partial \operatorname{vec} f_\theta(\omega)/\partial\theta$ belongs to Lip(β) with $\beta > 1/2$; (v) $||\partial\mu(\theta)/\partial\theta'|| \leq C_U$.

Assumption 4. $\{Y_t\}_{t=1}^T$ are multivariate normal random vectors.

Assumption 1 is standard. Assumption 2 states that the model solution is a linear process as in (3). Assumptions 3(i)-(ii) are satisfied by stationary finite-order VARMA processes with finite variances, which are typically the forms of solutions in linearized DSGE models. Assumptions 3(iii)-(iv) hold if parameters enter smoothly into $f_\theta(\omega)$ and $\mu(\theta)$. Qu and Tkachenko (2017) used assumptions similar to 1-3 in their study of the identification properties of log linearized DSGE models. Assumption 4 imposes normality, a common specification in DSGE models. Without this assumption, it remains feasible to derive the tests' distributions but they will depend on the third and fourth cumulants of the shocks $u_t(\theta)$.

To present the limiting distributions under the null hypothesis, let

$$\widetilde{B}(r) = (B_1(r) + iB_2(r))/\sqrt{2},$$

where i is the imaginary unit and $B_1(r)$ and $B_2(r)$ are two independent Wiener processes.

Theorem 1 *Suppose $\{Y_t\}_{t=1}^T$ is a covariance stationary process with spectral density $f_{\theta_0}(\omega)$ over $\omega \in [-\pi, \pi]$ (or over the frequencies selected by $W(\cdot)$ if $\mathcal{H}_{dT}^W$ is computed). Under Assumptions 1-4, and assuming $E(Y_t) = \mu(\theta_0)$ when testing the steady state, we have*

1. $\mathcal{H}_{dT}(\theta_0) \Rightarrow \sup_{r\in[0,1]} \|G_d(r)\|_\infty$, *where $G_d(r)$ is an $n_Y(n_Y+1)/2$ vector of independent processes, with the first n_Y elements being independent Wiener processes and the last $n_Y(n_Y-1)/2$ elements being independent copies of $\widetilde{B}(r)$.*

2. $\mathcal{H}_{dT}^W(\theta_0) \Rightarrow \sup_{r\in[0,1]} \|\int_0^r W(s)dG_d(s)\|_\infty$, *where $W(s)$ is an indicator function or a bounded smooth function specified by the user.*

3. $\mathcal{H}_{sT}(\theta_0) \Rightarrow \sup_{r\in[0,1]} \|G_s(r)\|_\infty$, *where $G_s(r)$ is an n_Y vector of independent Wiener processes.*

4. $\mathcal{H}_T(\theta_0) \Rightarrow \max\left(\sup_{r\in[0,1]} \|G_d(r)\|_\infty, \sup_{r\in[0,1]} \|G_s(r)\|_\infty\right)$, *where the elements of $[G_d(r), G_s(r)]$ are mutually independent.*

The first two results do not involve the steady state. The elements of $G_d(r)$ can be divided into two parts: the first n_Y elements represent the diagonal elements in $I_T(.)$, and the remaining $n_Y(n_Y-1)/2$ elements represent the off-diagonal elements. The limiting distributions of $\mathcal{H}_{dT}(\theta_0)$, $\mathcal{H}_{sT}(\theta_0)$ and $\mathcal{H}_T(\theta_0)$ are pivotal, meaning that they depend only on the number of variables being tested (n_Y) and they can be easily simulated. Table 1 presents the 10%, 5%, and 1% critical values for n_Y ranging from 1 to 10. When a subset of model variables is tested, n_Y refers to the dimension of AY_t. Critical values for $\mathcal{H}_{dT}^W$ depend on $W(\omega)$, and they need to be tabulated on a case-by-case basis. In the table, we provide the values for two cases: a smooth function that underweights high frequencies and an indicator function that only selects business cycle frequencies. We provide computer code for simulating other scenarios that supports user-defined functions $W(\omega)$.

Table 1: Critical values of the specification tests

Test	Size	Number of observables tested (n_Y)									
		1	2	3	4	5	6	7	8	9	10
Full Spectrum	10	1.946	2.261	2.423	2.529	2.615	2.677	2.735	2.779	2.816	2.856
$\mathcal{H}_{dT}(\theta_0)$	5	2.231	2.510	2.649	2.748	2.832	2.885	2.930	2.976	3.010	3.045
	1	2.804	3.015	3.143	3.219	3.287	3.339	3.373	3.403	3.440	3.475
Full Spectrum	10	0.867	0.981	1.040	1.081	1.110	1.136	1.155	1.172	1.187	1.201
Weighted by	5	0.977	1.077	1.128	1.167	1.196	1.216	1.235	1.251	1.264	1.277
$1-\omega/\pi$	1	1.190	1.271	1.311	1.347	1.372	1.387	1.403	1.418	1.428	1.447
Business Cycle	10	1.004	1.166	1.253	1.310	1.351	1.384	1.415	1.436	1.459	1.478
Frequencies	5	1.151	1.295	1.370	1.424	1.460	1.491	1.519	1.540	1.558	1.579
	1	1.446	1.559	1.619	1.668	1.705	1.724	1.756	1.766	1.787	1.804
Steady State	10	1.944	2.219	2.357	2.466	2.540	2.605	2.659	2.703	2.744	2.771
$\mathcal{H}_{sT}(\theta_0)$	5	2.224	2.480	2.614	2.709	2.778	2.845	2.894	2.937	2.968	2.989
	1	2.794	3.017	3.117	3.199	3.275	3.310	3.370	3.403	3.433	3.456
Steady State and	10	2.218	2.485	2.624	2.722	2.799	2.858	2.907	2.953	2.983	3.012
Full Spectrum	5	2.478	2.729	2.851	2.944	3.014	3.060	3.114	3.153	3.186	3.214
$\mathcal{H}_T(\theta_0)$	1	3.012	3.217	3.326	3.395	3.460	3.502	3.552	3.591	3.609	3.640

Note. The critical values are obtained through simulation using a sample size of 1000 and 100,000 replications. n_Y denotes the number of variables being tested.

The tests have power against local alternatives of order $T^{-1/2}$. The next result shows that they are consistent against stationary global alternatives.

Theorem 2 *Suppose $\{Y_t\}$ is a covariance stationary process with $E(Y_t)=\mu_0$ and spectral density $f_0(\omega)$ that satisfy Assumptions 1-4. Let $\mu(\theta_0)$ and $f_{\theta_0}(\omega)$ be the mean and spectral density of $\{Y_t\}$ implied by the DSGE model, satisfying Assumption 3. Let $\delta>0$ be an arbitrary constant independent of T. Then:*

1. *$\mathcal{H}_{dT}(\theta_0)\to\infty$ if $\|f_0(\omega)-f_{\theta_0}(\omega)\|>\delta$ for some $\omega\in[0,\pi]$.*
2. *$\mathcal{H}^W_{dT}(\theta_0)\to\infty$ if $\|f_0(\omega)-f_{\theta_0}(\omega)\|>\delta$ for some ω with $W(\omega)=1$.*
3. *$\mathcal{H}_{sT}(\theta_0)\to\infty$ if $\|\mu_0-\mu(\theta_0)\|>\delta$.*
4. *$\mathcal{H}_T(\theta_0)\to\infty$ if $\|f_0(\omega)-f_{\theta_0}(\omega)\|>\delta$ for some $\omega\in[0,\pi]$ or $\|\mu_0-\mu(\theta_0)\|>\delta$.*

The power of $\mathcal{H}_{dT}$ approaches 1 when the model and data spectra differ significantly over a set of frequencies. Similarly, the power of $\mathcal{H}^W_{dT}$ approaches 1 when there is a significant difference in spectra within the frequencies selected by $W(\omega)$. The $\mathcal{H}_{sT}$ test is consistent for misspecification in the mean, while the $\mathcal{H}_T$ test combines information from both the mean and spectrum and is consistent in detecting misspecification when either of them differs significantly.

3.2.1 Improving finite sample properties with prewhitening

Periodograms are known to be downward biased relative to the spectral density near the zero frequency for persistent time series, which can affect our tests. The fully parametric nature of a DSGE model provides a simple solution to address this issue. By using the model to simulate a long time series at θ_0 (denoted as $Y_t(\theta_0)$, where $t = 1, ..., \bar{T}$), we can construct a filter to prewhiten the empirical data, flattening the spectral density near the zero frequency and significantly reducing the bias even in small samples. It is crucial to use the DSGE model, rather than the actual empirical data, to compute the filter, which ensures that the filtering procedure does not influence the test distributions by choosing a large $\bar{T}$.

The steps of the filtering operation are as follows: (i) Simulate a long time series using the DSGE model at θ_0, denoted by $Y_t(\theta_0)$ for $t = 1, ..., \bar{T}$. (ii) Estimate a VAR(1) model using the simulated data: $Y_t(\theta_0) = BY_{t-1}(\theta_0) + e_t$. (iii) Use the estimated value of B, denoted as $\hat{B}$, and the actual empirical data to compute the filtered data $(I - \hat{B}L)Y_t$ for $t = 1, ..., T$. (iv) Compute the tests using the filtered data by replacing Y_t and $f_{\theta_0}(\omega)$ with $(I - \hat{B}L)Y_t$ and $(I - \hat{B}\exp(-i\omega))f_{\theta_0}(\omega)(I - \hat{B}\exp(-i\omega))^*$, respectively. To test a subset of variables, we estimate a VAR(1) for $AY_t(\theta_0)$ and apply the filtering to AY_t. Other aspects of the procedure remain the same. The asymptotic null distributions of the tests remain unchanged if $\bar{T}$ is sufficiently large (i.e., $\bar{T}/T \to \infty$) because the estimation uncertainty of the filter is asymptotically negligible. The tests remain consistent against stationary global alternatives. We use this prewhitening procedure when presenting our results for both simulations and empirical applications.

4 Accounting for parameter uncertainty

In the previous section, we considered a prespecified θ_0. In this section, we propose a testing procedure that accounts for parameter uncertainty and addresses the issue of weak identification. The main idea is to implement the test in two steps with a Bonferroni correction. First, we obtain an identification-robust confidence set for θ at the α_S level. This is achieved by inverting (i.e., sampling) the identification-robust score test proposed by Qu (2014), although other statistics such as those proposed by Guerron-Quintana et al. (2013) and Andrews and Mikusheva (2015) may also be used. Next, we compute the specification test at the $\alpha_{\mathcal{H}} = (\alpha - \alpha_S)$ level using all sampled values in this confidence set. If the model is not rejected, the parameter values that pass the specification checks are obtained. These values are valuable as they offer a chance to re-evaluate the model's implications. We begin by describing Qu's (2014) test.

4.1 The score test of Qu (2014)

The test is based on the Whittle likelihood. For any θ_0, the score function of the Whittle likelihood is equal to

$$D_T(\theta_0) = \frac{1}{2\sqrt{T}} \sum_{j=0}^{T-1} W(\omega_j) \left(\frac{\partial \operatorname{vec} f_{\theta_0}(\omega_j)}{\partial \theta'} \right)^* \left(f_{\theta_0}^{-1}(\omega_j)' \otimes f_{\theta_0}^{-1}(\omega_j) \right) \operatorname{vec}\left(I_T(\omega_j) - f_{\theta_0}(\omega_j) \right)$$
$$+ \frac{1}{2\pi\sqrt{T}} \sum_{t=1}^{T} W(0) \frac{\partial \mu(\theta_0)'}{\partial \theta} f_{\theta_0}^{-1}(0)(Y_t - \mu(\theta_0)). \quad (6)$$

A consistent approximation to the information matrix under normality is given by

$$M_T(\theta_0) = \frac{1}{2T} \sum_{j=0}^{T-1} W(\omega_j) \left(\frac{\partial \operatorname{vec} f_{\theta_0}(\omega_j)}{\partial \theta'} \right)^* \left(f_{\theta_0}^{-1}(\omega_j)' \otimes f_{\theta_0}^{-1}(\omega_j) \right) \frac{\partial \operatorname{vec} f_{\theta_0}(\omega_j)}{\partial \theta'} \quad (7)$$
$$+ \frac{1}{2\pi} W(0) \frac{\partial \mu(\theta_0)'}{\partial \theta} f_{\theta_0}^{-1}(0) \frac{\partial \mu(\theta_0)}{\partial \theta'},$$

where $W(.)$ is the same weighting function as before, which is symmetric around $\omega_j = \pi$, to select the frequencies, and $W(0)$ is set to zero if only dynamic properties are used to compute the score test. Using this notation, the score test for testing the null hypothesis of $\theta = \theta_0$ can be expressed as

$$S_T(\theta_0) = D_T(\theta_0)' M_T^+(\theta_0) D_T(\theta_0), \quad (8)$$

where $M_T^+(\theta)$ denotes the Moore-Penrose generalized inverse of $M_T(\theta)$. The generalized inverse is needed because $M_T(\theta)$ is not full rank if some parameters are unidentified. Qu (2014) showed that $S_T(\theta_0)$ is related to a linear multivariate regression. The dependent variables are related to $\operatorname{vec}(I_T(\omega_j) - f_{\theta_0}(\omega_j))$, and the regressors to $\frac{\partial \operatorname{vec} f_{\theta_0}(\omega_j)}{\partial \theta'}$. The rank of the regressors matrix is always bounded from above by the dimension of the structural parameter vector, irrespective of the strength of identification. When some parameters are unidentified, some columns of the regressor matrix become linearly dependent. Consequently, the dependent variables are projected onto a lower-dimensional space, resulting in a smaller test value. Formally, for any $0 \leq c < \infty$,

$$\lim_{T \to \infty} \Pr\left(S_T(\theta_0) \leq c \right) \to \Pr\left(\chi^2_{q-q_1} \leq c \right) \leq \Pr\left(\chi^2_q \leq c \right),$$

where $q = \dim(\theta_0)$ and q_1 is the number of unidentified parameter directions (i.e., the number of zero eigenvalues of $M_T(\theta_0)$). If the number of unidentified parameters is unknown, using the critical value of a χ^2_q distribution results in conservative inference.

4.2 Implementation

We explain the implementation of the tests using $\mathcal{H}_{dT}$ as an example. Suppose the desired significance level is $\alpha\%$. We first define positive constants α_S and $\alpha_{\mathcal{H}}$ such that $\alpha = \alpha_S + \alpha_{\mathcal{H}}$.

Next, we invert (i.e., sample) the $S_T(\theta_0)$ test to obtain an $(1-\alpha_S)\%$ confidence set for the parameters, denoted by $C_\theta(1-\alpha_S)$. Then, we calculate

$$\inf_{\theta \in C_\theta(1-\alpha_S)} \mathcal{H}_{dT}(\theta),$$

and reject the null hypothesis if it exceeds the $\alpha_{\mathcal{H}}\%$ critical value of $\sup_{r\in[0,1]} \|G_d(r)\|_\infty$. This two-step procedure first identifies a set of plausible parameter values and then assesses whether the most favorable model among them is supported by the data. In our implementation, for a 10% test, we set $\alpha_{\mathcal{H}} = \alpha_S = 5$, although other combinations might yield better power properties depending on the model. We leave such a power analysis for future work.

To determine the confidence set $C_\theta(1-\alpha_S)$, a grid search is not feasible for even small-scale DSGE models. Instead, we employ a modified version of a Markov chain Monte Carlo (MCMC) algorithm as described in Qu (2014). This algorithm uses Metropolis steps to generate frequent draws from areas in the parameter space where the values of $S_T(\theta)$ are low, and infrequent draws where $S_T(\theta)$ are high, creating an adaptive grid that is dense in important regions and sparse in unimportant areas. As in Qu (2014), we adjust the algorithm to account for any potential ridges or local minima in the surface of $S_T(\theta)$. The algorithm uses different proposal distributions to generate new parameter values. Specifically, the new draw is written as $\theta^* = \theta^{(j)} + \varepsilon$, where the first distribution gives $\varepsilon \sim N(0, cM_T(\theta^{(j)}))$ with c being a tuning constant, and the second gives $\varepsilon = cV_T(\theta^{(j)})$ or $-cV_T(\theta^{(j)})$ with $V_T(\theta^{(j)})$ being the eigenvector corresponding to the smallest eigenvalue of $M_T(\theta^{(j)})$. These two distributions produce draws that move both across and along the ridges of $S_T(\theta)$. The tuning parameter c is also allowed to take on multiple values to avoid getting stuck in a small neighborhood around a local minimum. Finally, multiple Markov chains are run with different initial values, and the confidence set is obtained by merging the accepted values from all chains. This set can then be approximated using the values of θ for which $S_T(\theta)$ does not exceed the critical value of the Chi-square distribution.

The reason for implementing the testing in two steps is two-fold. The first is computational: it becomes challenging to find the minimum of $\mathcal{H}_{dT}$ when the number of parameters is high. The second arises from a modeling perspective: researchers often have interests in both inference and model testing. Our procedure provides both results: the score test provides confidence sets for the parameters, while the specification test subsequently checks whether any of them are compatible with the data. The parameters that survive the testing can then be further analyzed, for example, by plotting their impulse responses, to gain insights into the model implications that are not rejected by the data.

Note that the confidence sets in the first step can be computed in various ways: using both the mean and the spectrum, using the spectrum only, or focusing solely on business cycle frequencies. This flexibility is useful. For instance, it can help determine if a model would still be rejected based on the spectrum test when we avoid estimating its steady state properties. We examine this in the empirical application section.

In our empirical applications, we also apply the specification tests to Bayesian posterior distributions of DSGE models. In this case, we first obtain the posterior distribution under informative priors and then apply our test to the parameters within the credible region. This analysis provides additional insights into the results obtained from the frequentist two-step procedure. If both methods lead to a rejection or acceptance of the model, we have greater confidence in the conclusion. If the posterior distribution results in a rejection while the frequentist two-step procedure does not, we can further investigate the surviving parameter values to assess their economic interpretations.

5 Simulations of size and power properties

We evaluate the size and power properties of our proposed specification tests using the model from Lubik and Schorfheide (2004) with empirically calibrated parameter values. The model is

$$\begin{aligned} y_t &= E_t y_{t+1} - \tau(r_t - E_t \pi_{t+1}) + g_t, \\ \pi_t &= \beta E_t \pi_{t+1} + \kappa(y_t - z_t), \\ r_t &= \rho_r r_{t-1} + (1-\rho_r)\psi_1 \pi_t + (1-\rho_r)\psi_2(y_t - z_t) + \varepsilon_{rt}, \\ g_t &= \rho_g g_{t-1} + \varepsilon_{gt}, \quad z_t = \rho_z z_{t-1} + \varepsilon_{zt}, \end{aligned} \tag{9}$$

where y_t, π_t, and r_t are log deviations of output, inflation, and nominal interest rate from their steady states, respectively. The shocks ε_{rt}, ε_{gt}, and ε_{gt} are independently and identically distributed as $N(0, \sigma_r^2), N(0, \sigma_g^2)$, and $N(0, \sigma_z^2)$, respectively, and ε_{gt} and ε_{zt} are cross correlated with correlation coefficient ρ_{gz}. The observables are log levels of output, inflation, and interest rate (both annualized), which are represented as $Y_t = (0,\, \pi^*,\, \pi^* + r^*)' + (y_t,\, 4\pi_t,\, 4r_t)'$, where the output is detrended, and π^* and r^* are annualized steady state rates of inflation and real interest rate with $\beta = (1 + r^*/100)^{-1/4}$. Under determinacy, the parameters and their values (posterior mean estimates from Lubik and Schorfheide, 2004) are

$$\begin{aligned} \theta &= (\tau, \beta, \kappa, \psi_1, \psi_2, \rho_r, \rho_g, \rho_z, \sigma_r, \sigma_g, \sigma_z, \rho_{gz}, \pi^*)' \\ &= (0.54, 0.992, 0.58, 2.19, 0.30, 0.84, 0.83, 0.85, 0.18, 0.18, 0.64, 0.36, 3.43)'. \end{aligned}$$

Under indeterminacy, the sunspot parameters are added, so that

$$\begin{aligned} \theta &= (\tau, \beta, \kappa, \psi_1, \psi_2, \rho_r, \rho_g, \rho_z, \sigma_r, \sigma_g, \sigma_z, \rho_{gz}, M_{r\epsilon}, M_{g\epsilon}, M_{z\epsilon}, \sigma_\epsilon, \pi^*)' \\ &= (0.69, 0.997, 0.77, 0.77, 0.17, 0.60, 0.68, 0.82, 0.23, 0.27, 1.13, 0.14, -0.68, 1.74, -0.69, 0.20, 4.28). \end{aligned}$$

Lubik and Schorfheide (2004) transformed the model's solution to ensure that its impulse responses are continuous at the boundary between the determinacy and indeterminacy regions.

The transformation applied is $S_t = \Theta_1 S_{t-1} + \widetilde{\Theta}_\varepsilon \varepsilon_t + \Theta_\epsilon \epsilon_t$ with $\widetilde{\Phi}_\varepsilon = \Phi_\varepsilon + \Phi_\epsilon(\Phi_\epsilon' \Phi_\epsilon)^{-1}\Phi_\epsilon'(\Phi_\varepsilon^b - \Phi_\varepsilon)$, where Φ_ε^b is the counterpart of Φ_ε with ψ_1 replaced by $\widetilde{\psi}_1 = 1 - (\beta\psi_2/\kappa)(1/\beta - 1)$. We apply the same transformation in order to be consistent with their analysis. Finally, the sunspot shock ϵ_t and the sunspot parameter are specified in the same way as described previously.

5.1 Size properties

We examine the size properties of the tests under both determinacy and indeterminacy, considering tests of the full model as well as subsets of one or two observables. The tests are based on the business cycle frequencies only, the full spectrum, the full spectrum with the weighting function $W(\omega) = 1 - \omega/\pi$ and, finally, the mean and spectrum, with all test statistics computed with the prewhitening procedure described in Subsection 3.2.1. The sample sizes are chosen to reflect the typical values in practice when working with DSGE models. All simulations are conducted over 5000 replications.

Table 2: Rejection frequencies under the null hypothesis (testing all variables)

Level	T	BC frequencies	Full spectrum	Weighted spectrum	Mean and spectrum
			Determinacy		
10%	80	0.099	0.085	0.105	0.085
	160	0.098	0.083	0.100	0.088
	240	0.090	0.087	0.098	0.101
	320	0.096	0.085	0.090	0.096
5%	80	0.061	0.045	0.070	0.048
	160	0.063	0.044	0.065	0.046
	240	0.046	0.044	0.057	0.049
	320	0.052	0.042	0.053	0.049
			Indeterminacy		
10%	80	0.106	0.088	0.122	0.108
	160	0.102	0.097	0.111	0.102
	240	0.104	0.092	0.109	0.099
	320	0.104	0.091	0.095	0.098
5%	80	0.072	0.049	0.081	0.059
	160	0.058	0.047	0.070	0.050
	240	0.059	0.048	0.064	0.051
	320	0.060	0.044	0.057	0.047

Note. T: sample size; all tests computed with prewhitening.

Table 2 shows the results for testing all three observables. In the determinate case, at the 10% nominal level, the tests using business cycle frequencies and weighted full spectrum have size close to the nominal level for all sample sizes. The full spectrum and mean and spectrum tests tend to be slightly conservative for smaller sample sizes, but improve with larger sample sizes.

At the 5% nominal level, the mean and spectrum and full spectrum based tests have size close to nominal, while the business cycle frequencies and weighted full spectrum based tests have slight upward size distortions at 80 and 160 observations. This is improved for larger samples. In the indeterminate case, all tests generally perform well at the 10% nominal level. At the 5% nominal level, small upward size distortions are seen in the business cycle frequencies and weighted full spectrum based tests for smaller samples, while the size is well-controlled for the rest. Overall, the tests show good size control across various sample sizes and policy regimes of the DGP when all observables are tested. We report the results for testing pairs of variables and three variables individually in the Appendix as they are very similar to the full model case. See Tables A1 and A2 for these two cases, respectively.

In conclusion, our examination of empirical sizes for tests of the full model and subsets of observables suggests that the proposed tests exhibit good size control across various sample sizes and regimes of the DGP. It is important to emphasize that prewhitening the data before performing the tests is a critical step in maintaining accurate size control, as substantial distortions were observed when using unfiltered data. For example, without prewhitening, the rejection rates corresponding to the first row of Table 2 were 0.183, 0.285, 0.217, and 0.268, respectively. The fact that a DSGE model is fully parametric enables us to stabilize the size properties without introducing additional estimation uncertainty.

5.2 Power properties

We next examine the empirical power properties of the proposed tests. To do so, we compute the size-adjusted power at the 10% level, against alternatives that perturb a random element of the parameter vector by a fixed percentage (we consider 20% and 40%). The tests are computed using prewhitening, as in the previous section.

Table 3 displays the results of testing all observables. The tests that use the full spectrum and the combination of the spectrum and mean have the highest rejection rates. The rejection frequencies for the mean and spectrum tend to be similar or lower than the full spectrum, likely due to the presence of an extra parameter in the steady state. In the determinate case, tests based on the business cycle frequencies have a power of 52-62% compared to the full spectrum, and tests based on the weighted full spectrum achieve 62-67% of the full spectrum's power when the parameter is perturbed by 20%. These ranges become 47%-75% and 56-84%, respectively, when the perturbation is 40%. In the indeterminate case, the rejection frequencies are often lower because of the additional four parameters in the model. Otherwise, the patterns are similar to the determinate case: the business cycle frequencies and the weighted full spectrum tests have 60-70% and 68-82% of the full spectrum's power, respectively, when the random parameter is perturbed by 20%. These ranges become 63-79% and 70-89% when the perturbation is 40%. The reason why the tests based on weighting functions appear to be more powerful (measured as a percentage of the full spectrum based test) in the indeterminate case is likely because the spectral densities of observables under indeterminacy have much more mass at lower and business cycle frequencies. Since these frequencies receive higher weight, the business cycle and weighted full spectrum based tests become relatively more informative compared to the determinate case.

Table 3: Rejection frequencies under the alternative hypothesis (10%, testing all variables)

T	BC frequencies	Full spectrum	Weighted spectrum	Mean and spectrum
		Determinacy		
		Perturb a random element of θ by 20%		
80	0.210	0.341	0.228	0.305
160	0.266	0.506	0.312	0.425
240	0.330	0.627	0.413	0.520
320	0.384	0.691	0.458	0.598
		Perturb a random element of θ by 40%		
80	0.290	0.618	0.345	0.544
160	0.459	0.790	0.536	0.700
240	0.566	0.833	0.666	0.752
320	0.653	0.872	0.729	0.780
		Indeterminacy		
		Perturb a random element of θ by 20%		
80	0.232	0.332	0.271	0.317
160	0.299	0.436	0.324	0.394
240	0.332	0.525	0.358	0.448
320	0.348	0.576	0.416	0.497
		Perturb a random element of θ by 40%		
80	0.320	0.510	0.359	0.480
160	0.403	0.644	0.481	0.579
240	0.502	0.688	0.571	0.635
320	0.577	0.731	0.648	0.671

Note. T: sample size; all tests computed with prewhitening.

Tables A3 and A4 in the Appendix contain the results for pairs of observables and individual observables, respectively. The tests are informative as the rejection frequencies show nontrivial power even at small sample sizes.

In summary, the results above show that the proposed tests have nontrivial power in empirically relevant sample sizes, that the tests using weighting functions, such as business cycle frequency indicator or a smoothing function de-emphasizing higher frequencies, can be informative, and that the tests still have power when considering subsets of observables. The power properties can depend on the structure of the DGP, e.g., business cycle based tests could be more informative if the model implies that the spectral densities of observables have a substantive mass in that band.

6 Empirical applications

In this section, we examine three DSGE models: a small-scale model of Lubik and Schorfheide (2004), previously considered in Section 5, and two medium-scale models: the Smets and Wouters (2007) model and the news shocks model of Schmitt-Grohé and Uribe (2012). The Lubik and Schorfheide (2004) model is a popular choice for contrasting determinacy and indeterminacy within a small-scale framework. The Smets and Wouters (2007) model is a benchmark medium-scale New Keynesian model in academia and central banks. This model extends the standard New Keynesian model by incorporating additional frictions and real rigidities, allowing us to examine how model specification improves compared to the baseline small-scale model. The Schmitt-Grohé and Uribe (2012) model provides an opportunity to evaluate whether the proposed information structure generates dynamics that fit the data adequately.

6.1 The small-scale model

We perform specification testing on the model described in Section 5 at the 10% level under both determinacy and indeterminacy. The data are linearly detrended US log GDP, and annualized inflation and interest rates for the period 1960:I-2007:IV. We do not use the Hodrick-Prescott filter to avoid potential filtering-induced discrepancies near the zero frequency. We consider this full sample period as a starting point, and to evaluate to what extent the results are driven by potential differences in monetary policy regimes over time, we also consider two subsamples: the pre-Volcker period (1960:I-1979:II) and the post-Volcker period (1979:III-2007:IV), which are associated with indeterminate and determinate policy regimes, respectively, see Clarida et al. (2000) and Lubik and Schorfheide (2004). The partitioning of the sample using 1979:II is the same as in Clarida et al. (2000).

We obtain results using the two-step procedure outlined in Subsection 4.2. First, a 95% confidence set is obtained by inverting the score test of Qu (2014) in (8), using information from both the mean and the spectrum (we also compute the 95% confidence set using information from the spectrum only and repeat the subsequent analysis). The modified Metropolis algorithm from Qu (2014) is used to generate 100 Markov chains from different initial values, each of which is run until 1000 draws are accepted and subsequently merged. Then, the specification tests based on the weighted full spectrum, business cycle frequencies, and mean and full spectrum are conducted at the 5% level for each parameter vector in the confidence set to examine the specification of the full model, each observable separately and their pairs. The test statistic, critical value, and the percentage of draws rejected are reported for each case. All specification test statistics are computed with prewhitening and the test statistics based on the full spectrum are computed as $\mathcal{H}_{dT}^{W}(\theta_0)$ in (5) with the weight function $W(\omega) = 1 - \omega/\pi$ that puts less weight on higher frequencies that DSGE models are not designed to capture. Note that the test statistic for the mean and spectrum case is $\mathcal{H}_T(\theta_0)$, i.e., it does not apply the weight function. Otherwise, the steady state test $\mathcal{H}_{sT}(\theta_0)$ would dominate due to the critical values for the static test being much higher, essentially reducing the test to focusing on the steady state only. Finally, to relate our results to Bayesian DSGE literature, we obtain 200,000 draws from the posterior distribution with the likelihood under informative priors, drop 5% of the draws corresponding to the lowest density regions, and apply our specification tests to these parameters.

Below, for each case, we first summarize the main findings and then provide numerical details.

Result 1 *When using the full sample 1960:I-2007:IV, the model is rejected at the 10% level based on the full spectrum analysis for both determinacy and indeterminacy specifications. Limiting the analysis to just business cycle frequencies, the model continues to be rejected for determinacy and is nearly rejected for the indeterminacy specification. Subsequent analysis reveals that misspecification impacts most segments of the model, and, in particular, inflation dynamics and its comovements with GDP are incompatible with the data at these parameter values over business cycle frequencies. Using the MCMC draws from posterior distributions reinforces these conclusions.*

Table 4 presents the 95% confidence intervals for the 1960:I-2007:IV sample based on the mean and full spectrum for the determinacy specification. The intervals are most informative

Table 4: 95% confidence intervals, 1960:I-2007:IV, determinacy

θ	Parameter	Bounds	CI
τ	intertemporal substitution elasticity	[0.10, 1.00]	[0.10, 0.99]
β	discount factor	[0.98, 0.999]	[0.984, 0.999]
κ	Phillips curve slope	[0.01, 2.00]	[0.01, 1.998]
ψ_1	inflation target	[1.01, 3.00]	[1.01, 2.56]
ψ_2	output target	[0.01, 5.00]	[0.01, 4.99]
ρ_r	interest rate smoothing	[0.10, 0.90]	[0.68, 0.90]
ρ_g	exogenous spending AR	[0.10, 0.98]	[0.88, 0.98]
ρ_z	technology shock AR	[0.10, 0.98]	[0.92, 0.98]
σ_r	monetary policy shock SD	[0.01, 3.00]	[0.20, 0.40]
σ_g	exogenous spending SD	[0.01, 3.00]	[0.03, 0.15]
σ_z	technology shock SD	[0.01, 3.00]	[0.64, 2.09]
ρ_{gz}	exogenous spending-technology CORR	[-0.90, -0.90]	[-0.45, 0.90]
π^*	steady state inflation	[2.00, 8.00]	[2.00, 7.95]

Note. Values are based on the mean and full spectrum. Column 2: parameter interpretation. Column 3: bounds for permissible parameter values. Column 4: confidence intervals, obtained by sampling the score test and applying projections.

about the autoregressive coefficients and standard deviations for the exogenous shock processes, with the standard deviation of the technological shock having a relatively wider interval. Thus, the confidence set includes models with widely varying behavioral and policy parameters, but relatively restricted exogenous shock behavior.

Examining the specification test results in Table 5, the first notable conclusion is that 100% of all parameter values in the confidence set are rejected when considering either the weighted full spectrum or only business cycle frequencies. Therefore, the null of correct model specification is rejected at the 10% significance level in both cases. In order to pinpoint the sources of this misspecification, further tests based on individual and pairs of observables can be considered. Based on the full spectrum, all parameter draws are rejected for inflation and the GDP/inflation pair and less than 1% of draws survive for the GDP/interest rate pair. The results at business cycle frequencies are qualitatively similar, with only 1% of draws surviving for inflation and less than 1% surviving for the two pairs.

Table 5: Specification test results, 1960:I-2007:IV, determinacy

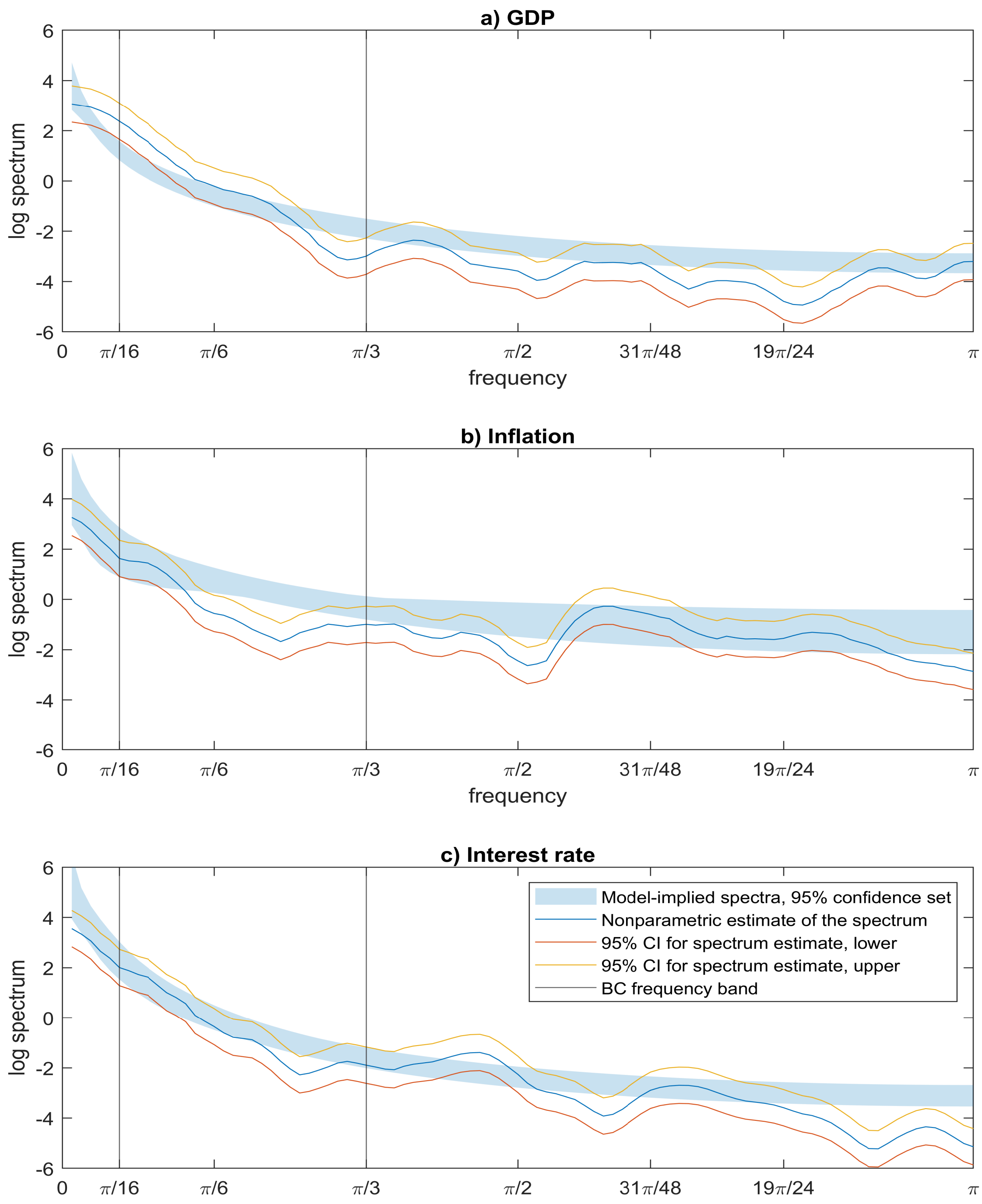
	Weighted spectrum			BC frequencies			Mean and spectrum		
	Test	CV	Rej.	Test	CV	Rej.	Test	CV	Rej.
Full model	1.239	1.128	100	1.779	1.370	100	2.614	2.851	99.97
GDP	0.619	0.977	93	0.452	1.151	82	1.715	2.478	30
Inflation	0.964	0.977	100	0.710	1.151	99	2.402	2.478	99.98
Interest rate	0.320	0.977	31	0.306	1.151	6	0.807	2.478	65
GDP-Inflation	1.140	1.077	100	1.086	1.295	99.95	2.286	2.729	99
GDP-Interest rate	0.953	1.077	99.91	1.090	1.295	99.69	1.894	2.729	98
Inflation-Interest rate	0.877	1.077	85	0.979	1.295	67	2.654	2.729	99.99

Note. The significance level is 10%. Test: the specification test value; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test.

To help visualize these results, Figure 2 contrasts nonparametrically estimated log spectral densities against the model-implied log spectra using the parameter values of the 95% confidence set. It is clear that there is a significant lack of overlap between the nonparametric 95% confidence interval and the model-implied spectral density for inflation within the business cycle frequency band. Specifically, the model overpredicts the distribution of the variation at these frequencies, which explains why both full spectrum and business cycle frequency-based specification tests reject the model when only inflation is considered. In comparison, the fit for GDP within the business cycle frequencies is slightly better, while the agreement for interest rates is much higher, consistent with the results in the table, where only 6% of the draws are rejected based on business cycle frequency tests.

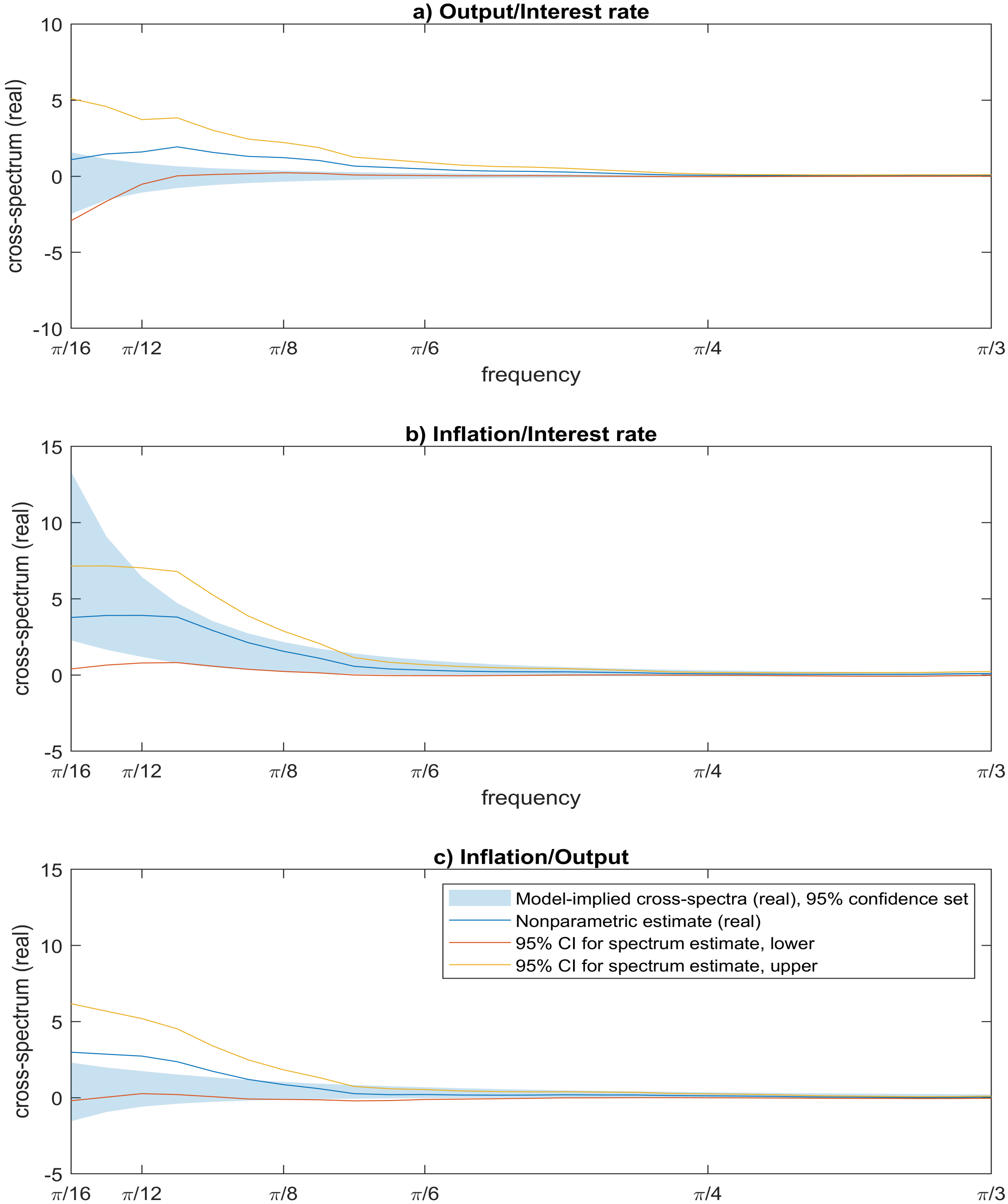
Figures 3 and 4 plot nonparametrically estimated real and imaginary parts of cross-spectra against their counterparts implied by the confidence set at business cycle frequencies. The real part of the cross-spectrum at frequency ω, also called the cospectrum, shows which portion of the covariance is due to cycles at that frequency. The imaginary part of the cross-spectrum, also

Figure 2: Log spectra under determinacy, 1960-2007.



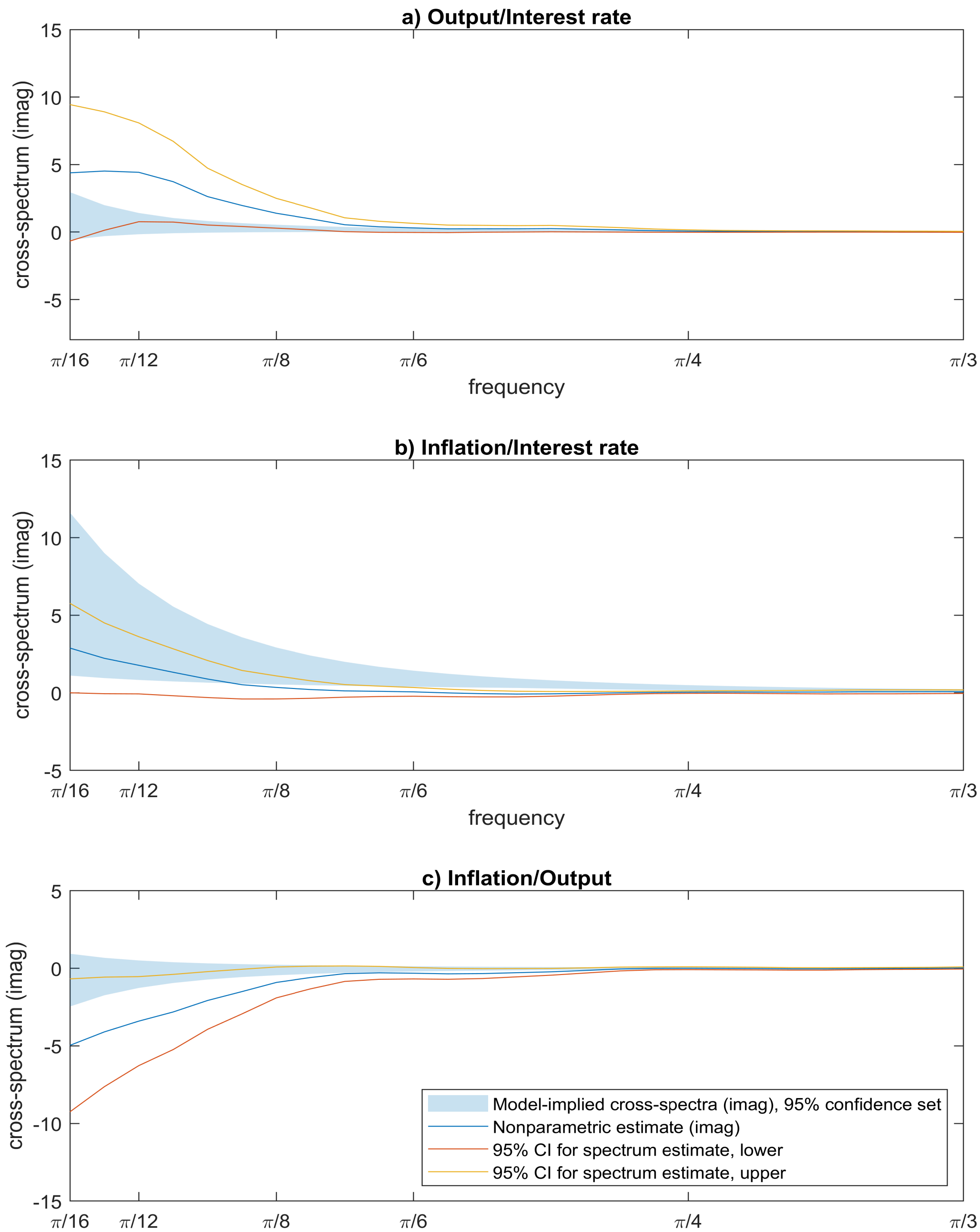
called the quadrature spectrum, indicates whether one series leads or lags the other series, as determined by its sign relative to the real part (i.e., a positive sign in both cases indicates that the first leads the second series). The figures show visible discrepancies between the inflation/output and output/interest rate pairs. For inflation/output, the data suggest that output leads inflation at business cycle frequencies (Figure 4(c)), whereas the model is ambiguous in this regard. For

Figure 3: Cross-spectra (real part) under determinacy, 1960-2007.



output/interest, the data suggest stronger out-of-phase comovement between these two variables than what is implied by the model (Figure 4(a)). Thus, the model struggles to capture dynamic correlations between two pairs of the three observables, suggesting limitations in its ability to explain the data. Finally, for the inflation/interest rate pair, the fit is better, with both the model and the data indicating that inflation leads interest rates. Nevertheless, the model still

Figure 4: Cross-spectra (imaginary part) under determinacy, 1960-2007.



tends to signal a stronger relationship than in the data.

Next, we examine the model specification for the full sample under indeterminacy. The relevant 95% confidence intervals are reported in Table 6. It can be seen that the intervals, again, are most informative about the parameters governing the dynamics of exogenous shocks. Compared to the determinacy case, all these intervals are wider. Different from the determinacy case, the interval for the slope of the Phillips curve is much tighter. Among the sunspot parameters, only the interval for the standard deviation of the sunspot shock is somewhat informative.

Table 6: 95% confidence intervals, 1960:1-2007:IV, indeterminacy

θ	Parameter	Bounds	CI
τ	intertemporal substitution elasticity	[0.10, 1.00]	[0.10, 0.9999]
β	discount factor	[0.98, 0.999]	[0.989, 0.999]
κ	Phillips curve slope	[0.01, 2.00]	[0.01, 0.458]
ψ_1	inflation target	[0.01, 0.99]	[0.01, 0.952]
ψ_2	output target	[0.01, 5.00]	[0.14, 4.999]
ρ_r	interest rate smoothing	[0.10, 0.90]	[0.10, 0.90]
ρ_g	exogenous spending AR	[0.10, 0.98]	[0.72, 0.97]
ρ_z	technology shock AR	[0.10, 0.98]	[0.88. 0.98]
σ_r	monetary policy shock SD	[0.01, 3.00]	[0.01, 0.39]
σ_g	exogenous spending SD	[0.01, 3.00]	[0.04, 0.68]
σ_z	technology shock SD	[0.01, 3.00]	[0.65, 2.69]
ρ_{gz}	exogenous spending-technology CORR	[-0.90, -0.90]	[-0.90, 0.83]
$M_{r\epsilon}$	sunspot-monetary coeff	[-3.00, 3.00]	[-2.956, 2.999]
$M_{g\epsilon}$	sunspot-exogenous spending coeff	[-3.00, 3.00]	[-3.00, 2.48]
$M_{z\epsilon}$	sunspot-technology coeff	[-3.00, 3.00]	[-0.54, 0.96]
σ_ϵ	sunspot shock SD	[0.01, 3.00]	[0.01, 0.84]
π^*	steady state inflation	[2.00, 8.00]	[2.00, 8.00]

Note. Values are based on the mean and full spectrum. Column 2: parameter interpretation. Column 3: bounds for permissible parameter values. Column 4: confidence intervals, obtained by sampling the score test and applying projections.

Proceeding to specification testing, results in Table 7 show that 100% and 99.92% of parameter values are rejected based on the weighted full spectrum and business cycle band, respectively – this conclusion is qualitatively similar to that under determinacy. It is apparent from Table 7 that comovements between variables are identified as a primary source of misspecification at both frequencies. Interest rate dynamics remain the model component that is most compatible with the data, particularly at business cycle frequencies, where 97% of draws cannot be rejected. Figures A1, A2, and A3 in the Appendix show plots of estimated versus model-implied log spectra and real and imaginary parts of estimated versus model-implied cross-spectra, respectively. The plots are qualitatively similar to those under determinacy. The main difference is that the model-implied confidence bands tend to be substantially wider, consistent with the wider confidence intervals in Table 6. These results show that the model fit has not fundamentally changed by switching from determinacy to indeterminacy for the entire sample period.

We also consider parameter values from a posterior distribution using priors from Lubik and Schorfheide (2004), and retain all posterior draws that are in the 95% highest density region. Specification test results under determinacy and indeterminacy can be found in Tables A5 and A6 in the Appendix, respectively. The overall conclusion remains unchanged: the full model is

Table 7: Specification test results, 1960:I-2007:IV, indeterminacy

	Weighted spectrum			BC frequencies			Mean and Spectrum		
	Test	CV	Rej.	Test	CV	Rej.	Test	CV	Rej.
Full model	1.137	1.128	100	1.277	1.370	99.92	2.504	2.851	98
GDP	0.471	0.977	97	0.302	1.151	16	1.359	2.478	41
Inflation	1.101	0.977	100	1.190	1.151	100	2.534	2.478	100
Interest rate	0.324	0.977	16	0.293	1.151	3	0.727	2.478	21
GDP-Inflation	1.214	1.077	100	1.300	1.295	100	2.556	2.729	99.67
GDP-Interest rate	0.662	1.077	95	0.799	1.295	86	1.521	2.729	35
Inflation-Interest rate	1.106	1.077	100	0.957	1.295	99.88	2.556	2.729	99.85

Note. The significance level is 10%. Test: the specification test value; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test.

rejected based on the weighted full spectrum in both cases, with comovements between variables identified as a primary source of misspecification. However, a difference from the previous results is that inflation dynamics are no longer rejected when considered individually under determinacy, with 38% and 44% of draws surviving based on full spectrum and business cycle frequencies, respectively. Under indeterminacy, inflation dynamics have 4% and 19% of draws surviving in the respective cases. This finding suggests that incorporating an informative prior can alter the fit of the model in some dimensions, but it does not make the model fully compatible with the data. Diagnostic plots for log spectra and cross-spectra can be found in Figures A4-A9 in the Appendix. As before, the imaginary parts of the output/interest and inflation/output pairs show the largest discrepancies, with their nonparametric counterparts remaining largely outside the 95% model-implied confidence sets. The model-implied confidence sets are smaller than their frequentist counterparts, reflecting the effect of the prior.

In summary, our analysis has shown that both determinacy and indeterminacy specifications are rejected based on the weighted full spectrum test when applied to the 1960-2007 period. It has also revealed that dynamic correlations, particularly the leads and lags relationships, are often at odds with data. Moreover, the analysis suggests that examining the imaginary part of the spectra is informative, particularly for understanding the leads and lags relationships.

Lubik and Schorfheide (2004) have convincingly demonstrated that a determinate regime fits better for the post-Volcker subsample, while an indeterminate regime fits the data better for the earlier subsample with heightened parameter uncertainty. Therefore, it is appropriate to examine the model specification that matches the two monetary policy regimes to their respective subsamples. Our findings for the 1979-2007 period are summarized below:

Result 2 *For the subsample 1979:III-2007:IV under determinacy, the model is no longer rejected when using weighted full spectrum or using business cycle frequencies. Further analysis revealed that the model-implied dynamic properties remain similar to the full sample determinacy* *estimates. However, in this part of the data, the comovements between variables are stronger between inflation and interest rate and weaker otherwise than those measured using the full sample, which results in a closer match between the model and the data. Using the MCMC draws from posterior distributions delivers qualitatively similar conclusions.*

Figure 5: Log spectra under determinacy, 1979-2007.

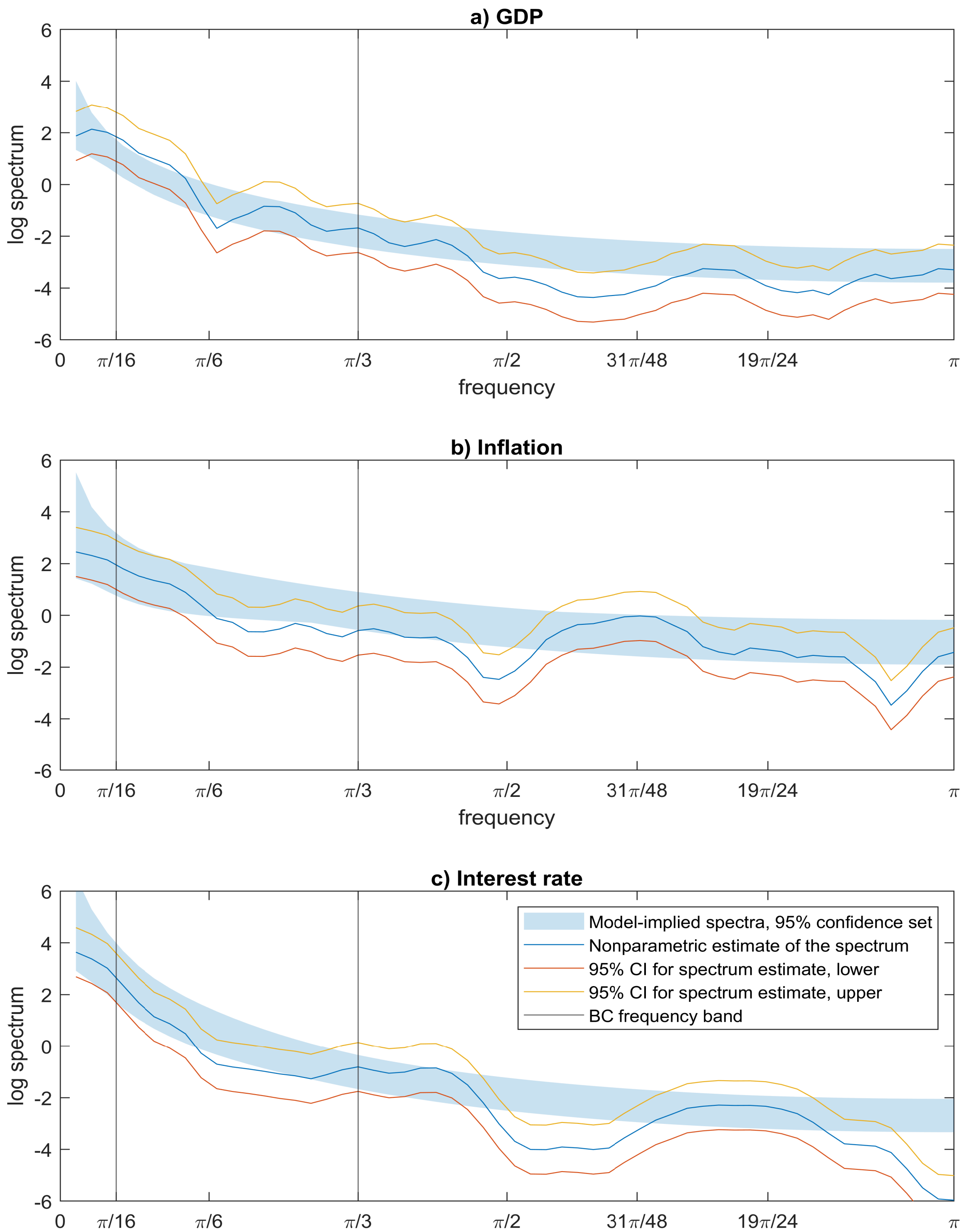
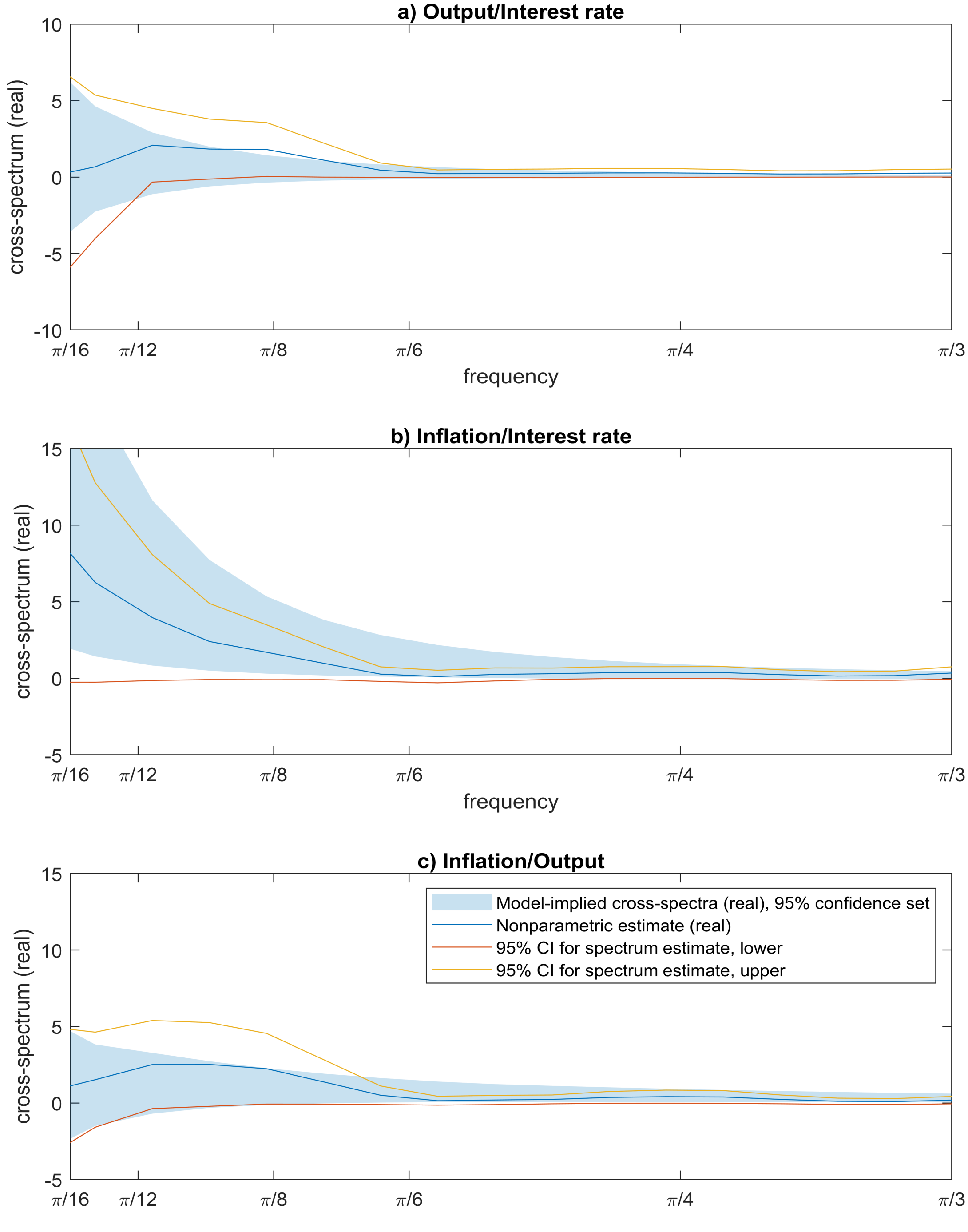


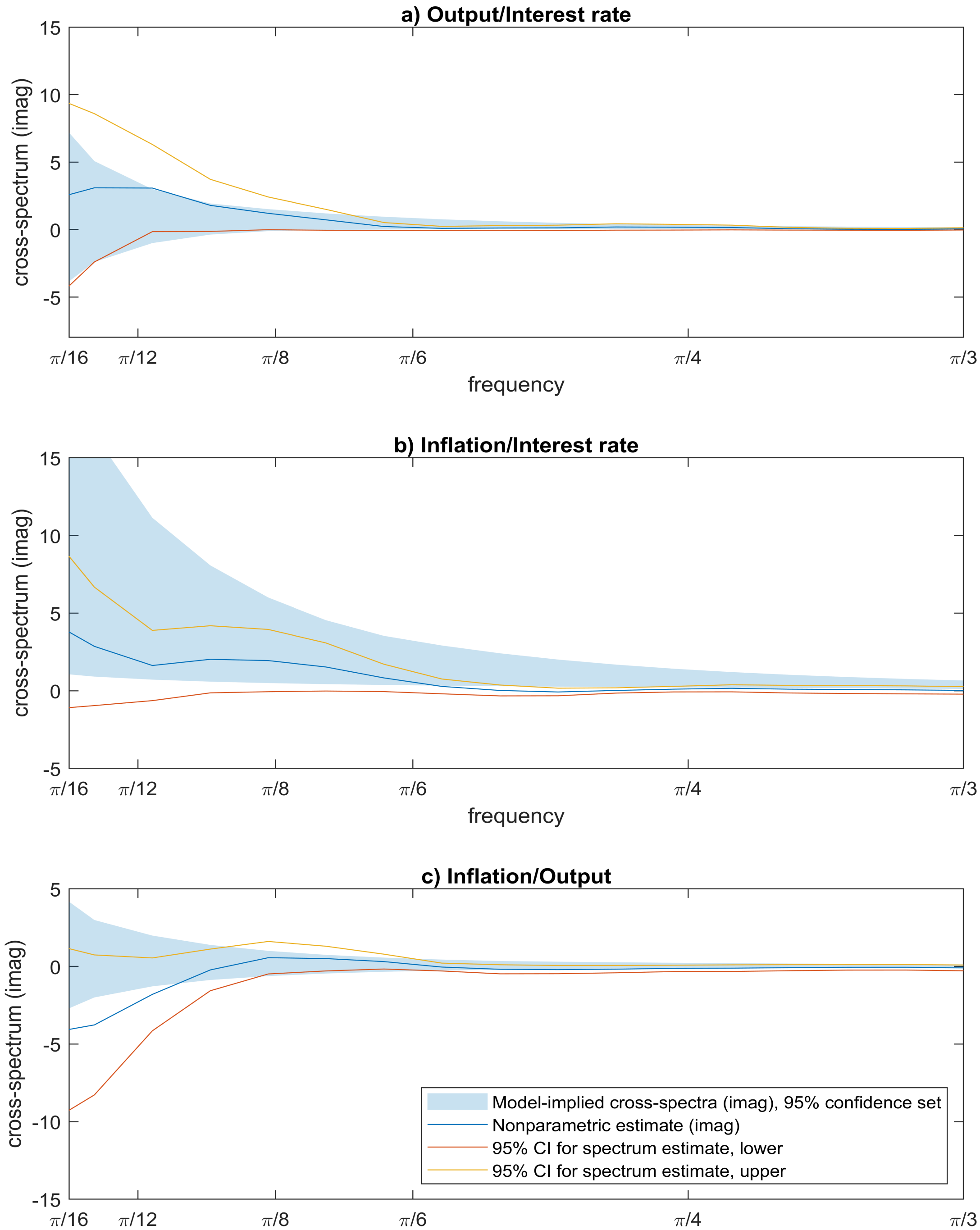
Figure 6: Cross-spectra (real part) under determinacy, 1979-2007.



The 95% confidence intervals for this case are collected in Appendix Table A7. Qualitatively, their pattern is similar to the full sample case under determinacy - the most informative intervals pertain to autoregressive coefficients and standard deviations of the shock processes. Most of those intervals are moderately wider, owing most likely to smaller sample size.

Turning to the specification tests in Table A8 in the Appendix, we observe that the tests

Figure 7: Cross-spectra (imaginary part) under determinacy, 1979-2007.



reject only 15% and 39% of draws for the weighted full spectrum and business cycle frequencies, respectively. Examining the results for individual variables and their comovements, it becomes clear that the model specification substantially improves, except for inflation dynamics and its comovement with GDP, which have the highest proportions of rejected draws based on the full spectrum (97% and 86%, respectively) that remain high at business cycle frequencies (92% and

76%, respectively).

Figures 5-7 show diagnostic spectral plots for this case. Figure 5 shows the model-implied spectra now visibly overlap more with the 95% confidence intervals of their nonparametrically estimated counterparts. Among them, the overlap in the inflation case is the smallest, which is consistent with the high rejection rates observed in Table A8. For the cross-spectrum, the data and model-implied cross-spectra of the inflation/interest rates pair agree well with each other because the contemporaneous comovement between these two variables (i.e., the real part of the cross-spectrum) is stronger in this part of the sample (Figure 6(b)). Additionally, the out-of-phase comovement between output and interest in the data is weaker in this part of the sample (Figure 7(a)), bringing the model closer to mirroring the data. Finally, for the inflation/output pair, the real part of the cross-spectrum is closer (Figure 6(c)), however, the lead-lag relationship in the data remains in tension with the model, in the sense that the nonparametric estimate partly lies outside of the model-implied confidence band (Figure 7(c)), consistent with the test rejecting most of the draws for this pair.

Table A9 contains the test results for this sample period using the Bayesian posterior draws. We obtain similar overall conclusions: the model is not rejected using either the full spectrum or business cycle frequencies; the comovement between inflation and interest rate is well-captured by the model, and inflation/output comovement constitutes a major tension between the model and data, with 78% of posterior draws rejected based on the weighted full spectrum. The spectra are displayed in Figures A10-A12. The marginal spectra results are comparable to their frequentist counterparts, while the model-implied intervals for the cross-spectra are significantly narrower, reflecting the effect of the prior. Next, we turn to the 1960-1979 period.

Result 3 *When examining the subsample from 1960:I to 1979:II under indeterminacy, the model is not rejected. An important contributing factor to this result is the wider model-implied confidence intervals, which can encompass richer patterns in the data. Meanwhile, using MCMC draws from posterior distributions results in a much higher percentage of rejected draws, specifically 99.99% when using the weighted full spectrum test for the full model. This may reflect the effect of the prior on a small sample size. Therefore, the use of frequentist confidence sets yields a more favorable evaluation of the model specification than using the Bayesian posterior distribution for this sample period.*

The 95% confidence intervals for this case are presented in Appendix Table A10. Compared to the full sample indeterminacy case, the pattern of the most informative intervals remains the same (slope of the Phillips curve, exogenous shock parameters and the sunspot shock standard deviation), however, the intervals become substantially longer than those in Table 6, most likely due to reduced sample size. Table A11 contains the specification test results, which show that the model is not rejected at the 10% level based on both the weighted full spectrum (55% of draws rejected) and the business cycle band (44% of draws rejected). Examining tests for specification of separate aspects of the model, it can be seen the agreement with data of the individual variables and their comovements is improved compared to the full sample case. Inflation dynamics and its comovement with GDP still have the highest proportions of rejected draws using the full spectrum (61% and 64%, respectively), however, these fall when only the business cycle band is used, to 11% and 49%, respectively. The plots in Figures A13-A15 further corroborate these findings. Table A12 and Figures A16-A18 provide results of using the Bayesian posterior distribution. The credible sets for the spectra plots are substantially narrower than the frequentist case, consistent with the high rejection rates of the tests.

We note that the data suggest that interest rates lead inflation over business cycle frequencies (Figure A18(b)), which is different from the lagging pattern observed during the 1979-2007 period (Figure A12(b)). The model tends to indicate the opposite relationship for this period, regardless of whether frequentist (Figure A15(b)) or Bayesian confidence sets (Figure A18(b)) are used. Still, this tension is insufficient to cause a rejection of the model.

So far in the analysis, we have used information from both the mean and spectrum to compute the parameter confidence sets. This is done to obtain sharp results; however, it might also introduce contamination. For instance, if the model's steady state is incomparable with the data, the estimation might skew other parameters, leading to a rejection of the model's dynamic properties. To examine this, we obtain the confidence sets using the spectrum only and recompute the rejection frequencies of the specification tests. The results for the full sample case are reported in Appendix Tables A13 and A14. For determinacy specification, the model is nearly rejected using the full spectrum weighted test (99.99% of draws rejected) and is rejected fully at business cycle frequencies. GDP and inflation are rejected individually as well as the GDP/inflation pair when using full spectrum. For the indeterminacy specification, the full spectrum rejects the model, while business cycle frequencies still have a few surviving draws (99.95% rejected). Inflation and GDP/inflation subsets are rejected for both frequency ranges. These results are broadly similar to those obtained previously, and the main difference is that we now obtain a near-rejection instead of a full rejection for the for determinacy case using the weighted full spectrum test. In conclusion, various aspects of the model remain at odds with the data even if we discard information from the steady state properties.

In summary, we have applied our specification tests to a small-scale DSGE model and provided a detailed analysis of the results combining numerical and graphical methods. One part of our findings reaffirms Lubik and Schorfheide's (2004) results obtained from a Bayesian perspective. Specifically, our tests support that the post-Volcker sample is consistent with a determinate monetary regime, while the earlier sample is consistent with an indeterminacy regime, albeit with heightened parameter uncertainty. Furthermore, our results show that the comovements between variables, particularly the lead and lag relationships indicated by the imaginary part of the cross-spectra, often constitute a significant source of tension between model and data. Comparing the model and data spectra, along with their confidence intervals, offers a useful way to gain insights into the model's specification. Finally, for this model, both frequentist and Bayesian confidence sets yield qualitatively similar conclusions. The most notable difference occurs for the pre-Volcker subsample, where the use of frequentist confidence sets leads to a more favorable evaluation of the model's specification properties compared to using the Bayesian posterior distribution.

6.2 The Smets and Wouters model

The next model we consider is the medium-scale model of Smets and Wouters (2007), which is described in the Appendix Section A.2. This model includes 36 free parameters and is estimated on seven observables: consumption growth, investment growth, output growth, labor hours, inflation, wage growth, and interest rate. Unlike the previous subsection, the variables including inflation and interest rate are not annualized here as in the Smets and Wouters' original analysis.

We perform specification testing at the 10% level using the same two-step procedure. Given the model's larger dimension, the confidence set is formed via merging output from 100 Markov chains each of which produces 5000 accepted draws. The specification testing proceeds in the same way as in the previous subsection. To relate to the Bayesian literature, the specification tests are also run on 0.5 million draws from the posterior distribution. In order to evaluate the contrast in specification for common observables such as inflation and interest rate, the application first considers US data for the same sample period of 1960:I-2007:IV as in the previous subsection. Subsequently, the tests are repeated on the subsample 1965:I-2004:IV, which corresponds to the original Smets and Wouters (2007) data.

Result 4 *The model is not rejected at either the full spectrum or business cycle frequencies at the 10% significance level on the full sample 1960:I-2007:IV. This contrasts with the case of the small-scale model examined earlier. Using draws from the posterior distribution produces qualitatively similar results. Regardless of which set of draws is used, over 60% and 80% of the draws are rejected based on the weighted spectrum and business cycle frequencies, respectively, indicating significant room for model improvement. Qualitatively similar results are found on the subsample 1965:I-2004:IV.*

Table 8 presents the results of specification tests using the 1960:I-2007:IV sample for the same variables as in the small-scale model. Results for the full set of variables are reported in the Appendix (Table A15). Panel (a) uses parameter values from the score test and (b) uses values from the 95% highest density region of the posterior distribution under Smets and Wouters' prior. In (a), the null of a correct specification cannot be rejected at the 10% level using either the weighted full spectrum or business cycle frequencies. Nonetheless, since the majority of the

Table 8: Specification test results for the SW model, 1960:I-2007:IV

	(a) Frequentist set			(b) Posterior distribution		
	Test	CV	Rej.	Test	CV	Rej.
Full model, weighted spectrum	0.849	1.235	78	0.878	1.235	67
Full model, BC frequencies	1.18	1.519	97	1.128	1.519	81
GDP growth	0.303	0.977	44	0.317	0.977	20
Inflation	0.393	0.977	56	0.398	0.977	58
Interest rate	0.332	0.977	19	0.372	0.977	30
GDP-Inflation	0.561	1.077	40	0.597	1.077	26
GDP-Interest rate	0.614	1.077	65	0.673	1.077	55
Inflation-Interest rate	0.509	1.077	64	0.552	1.077	81

Note. Each row represents a set of variables tested. The significance level is 10% for each case. Test: the specification test value, based on the weighted full spectrum unless indicated otherwise; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test. For (a) the parameter values are obtained by sampling the score test, and for (b) they are values from the 95% highest density region of the posterior distribution using SW's prior.

draws are rejected, additional testing for variable subsets is conducted to reveal the dimensions along which the model and the data are most at odds with each other. The results based on the frequentist confidence set reveal that the draws are most frequently rejected for consumption and its comovements with other variables. On the other hand, for the posterior distribution, the draws are frequently rejected for inflation dynamics and its comovements with other variables. Therefore, the prior can matter as in the small-scale model case.

Table 9: Specification test results for the SW model, 1965:I-2004:IV

	Frequentist set			Posterior distribution		
	Test	CV	Rej.	Test	CV	Rej.
Full model	0.867	1.235	68	0.901	1.235	89
Full model, BC frequencies	1.083	1.519	92	1.048	1.519	71
GDP growth	0.213	0.977	18	0.194	0.977	0.3
Inflation	0.319	0.977	22	0.371	0.977	42
Interest rate	0.310	0.977	20	0.331	0.977	47
GDP-Inflation	0.494	1.077	20	0.516	1.077	10
GDP-Interest rate	0.562	1.077	39	0.626	1.077	52
Inflation-Interest rate	0.454	1.077	32	0.628	1.077	88

Note. Each row represents a set of variables tested. The significance level is 10% for each case. Test: the specification test value, based on the weighted full spectrum unless indicated otherwise; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test. For (a) the parameter values are obtained by sampling the score test, and for (b) they are values from 95% highest density region of the posterior distribution using SW's prior.

Table 9 presents the specification test results for the sample period of 1965:I-2004:IV for the same subset of variables. Those for the entire set of variables are reported in Table A16. The results show that the correct specification null cannot be rejected at the 10% level for all the tests considered. The rejection frequencies overall are lower than in the full sample case. Using the Bayesian credible set yields similar conclusions.

The spectrum plots of individual variables for the two samples are reported in Figures A19 and A20 of the Appendix. The model-implied spectra (computed using parameter values in the frequentist confidence set) exhibit varying degrees of agreement with the nonparametric estimates. The pairwise comparisons (21 cases) are not included to save space, but they are available upon request.

Finally, one might be interested in re-examining the model's applications using the non-rejected parameter values. To illustrate this, we compute the impulse responses of the model using the surviving parameter values from the frequentist confidence set for the 1960:I-2007:IV sample; see Appendix Figure A21 for the responses of the output variable to the seven shocks. We report the median of these responses along with their 5th and 95th percentiles. The response computed using the posterior mean is also included, as this is what the Bayesian approach using Smets and Wouters' prior would have reported. An important message emerging from these plots is that the Bayesian posterior estimates always fall between the 5th and 95th percentiles of the frequentist values. Therefore, a researcher would not strongly disagree with the Bayesian conclusions regarding these responses after applying the model specification test. In a sense, these conclusions are robust.

6.3 A model with news shocks

We now turn to the model of Schmitt-Grohé and Uribe (2012). The model features anticipated shocks, the quantitative importance of which is actively investigated in the literature; see Milani and Treadwell (2012), Christiano et al. (2014), and Forni et al. (2017), among others. We outline the model in the Appendix Section A.3. There are seven exogenous shocks in the model, and all of them are assumed to have anticipated components. They are: 1) the stationary neutral productivity shock z_t, 2) the nonstationary neutral productivity shock X_t, 3) the stationary investment-specific productivity shock z_t^i, 4) the nonstationary investment-specific productivity shock A_t, 5) the government spending shock G_t, 6) the wage markup shock μ_t, 7) the preference shock ζ_t. The shocks X_t and A_t are made stationary using growth rates, with the respective variables being $\mu_t^x = X_t/X_{t-1}$ and $\mu_t^a = A_t/A_{t-1}$. G_t is detrended to form $g_t \equiv G_t/X_t^G$, where $X_t^G = (X_{t-1}^G)^{\rho_{xg}}(X_{t-1}A_{t-1}^{\alpha_K/(\alpha_K-1)})^{1-\rho_{xg}}$ is the trend in government spending. All seven processes x_t ($x = \{z, \mu^x, z^i, \mu^a, g, \mu, \zeta\}$) are assumed to follow $\ln(x_t/x) = \rho_x \ln(x_{t-1}/x) + \varepsilon_{x,t}$ with $\varepsilon_{x,t} = \varepsilon_{x,t}^0 + \varepsilon_{x,t-4}^4 + \varepsilon_{x,t-8}^8$, where x denote the steady state values of the variables and $\varepsilon_{x,t}^j \overset{iid}{\sim} N(0, \sigma_x^j)$. The total number of shocks is 21.

After log linearization, Schmitt-Grohé and Uribe (2012) estimate the model on seven demeaned observables: real GDP growth, real consumption growth, real investment growth, labor hours, real government spending growth, TFP growth, and relative price of investment growth;

Table 10: Specification test results for the SGU model, 1955:II-2006:IV

	Test	CV	Rej.
Full model, weighted spectrum	1.220	1.235	99.99
Full model, BC frequencies	1.643	1.519	100
GDP growth	0.254	0.977	15
Consumption growth	0.243	0.977	33
Investment growth	0.237	0.977	0.3
Labor hours growth	0.603	0.977	98
Government spending growth	0.334	0.977	26
TFP growth	0.297	0.977	51
Rel. price of investment growth	0.343	0.977	11
GDP-Consumption	0.662	1.077	50
GDP-Investment	0.768	1.077	75
GDP-Labor hours	1.212	1.077	100
GDP-Gov. spending	0.449	1.077	30
GDP-TFP	0.671	1.077	38
GDP-Rel. price of Investment	0.530	1.077	11
Consumption-Investment	0.547	1.077	20
Consumption-Labor hours	0.827	1.077	98
Consumption-Gov. spending	0.403	1.077	37
Consumption-TFP	0.510	1.077	19
Consumption-Rel. price of Investment	0.626	1.077	28
Investment-Labor hours	1.090	1.077	100
Investment-Gov. spending	0.423	1.077	17
Investment-TFP	0.766	1.077	50
Investment-Rel. price of Investment	0.382	1.077	6
Labor hours-Gov. spending	0.641	1.077	99
Labor hours-TFP	0.674	1.077	98
Labor hours-Rel. price of Investment	0.601	1.077	97
Gov. spending-TFP	0.610	1.077	48
Gov. spending-Rel. price of Investment	0.419	1.077	18
TFP-Rel. price of Investment	0.469	1.077	42

Note. Parameter values obtained by sampling the score test. Results based on the weighted full spectrum test unless indicated otherwise. Each row represents a set of variables being tested. Significance level: 10%. Test: the specification test value; CV: critical value; Rej.: percentage of parameter draws rejected.

we use the same set of variables and transformations. Consequently, the information from the steady state properties is not utilized in the analysis. The full vector of structural parameters is given by:

$$\begin{aligned}\phi \;=\; & [\theta, \gamma, \kappa, \delta_2, b, \rho_{xg}, \rho_{\mu^a}, \rho_{\mu^x}, \rho_{z^i}, \rho_z, \rho_\mu, \rho_g, \rho_\zeta, \sigma^0_{\mu^a}, \sigma^4_{\mu^a}, \sigma^8_{\mu^a}, \sigma^0_{\mu^x}, \sigma^4_{\mu^x}, \sigma^8_{\mu^x}, \sigma^0_{z^i}, \sigma^4_{z^i}, \sigma^8_{z^i}, \\ & \sigma^0_z, \sigma^4_z, \sigma^8_z, \sigma^0_\mu, \sigma^4_\mu, \sigma^8_\mu, \sigma^0_g, \sigma^4_g, \sigma^8_g, \sigma^0_\zeta, \sigma^4_\zeta, \sigma^8_\zeta, \sigma^{me}_{g^y}].\end{aligned}$$

Similar to the previous subsection, we use the two-step procedure to perform specification testing at the 10% level. The data sample used corresponds to that considered in Schmitt-Grohé and Uribe (2012) : 1955:II-2006:IV. The annotated parameter list and the corresponding robust 95% confidence intervals can be found in Table A17.

Result 5 *The model is rejected by the the business cycle frequencies test and nearly rejected by the weighted full spectrum test (99.99% of draws rejected) at the 10% significance level using the original Schmitt-Grohé and Uribe (2012) sample. This is in contrast with the case of the medium-scale model of Smets and Wouters (2007), where neither frequency range produced a rejection. Further examination reveals that the main source of incompatibility between the model and the data is the per capita hours worked and its comovements with all the other observables.*

Table 10 presents the specification test results. The business cycle frequencies results show that 100% of the draws are rejected when the full model is considered. Furthermore, only three draws are narrowly not rejected by the weighted full spectrum test. Testing individual observables reveals that only 2% of the draws survive for labor hours growth, while the other observables generate much lower rejection frequencies. Testing of pairs shows that all the comovements between labor hours and other observables are at odds with the data, with all draws rejected for their comovements with GDP and investment growth, while less than 2% of the draws survive for those with government spending growth, consumption growth, and TFP. These findings can be related to those of Schmitt-Grohé and Uribe (2012), whose Table III presented the model's predictions regarding standard deviations, correlations with output growth, and serial correlations of the seven observables. Their results indicated that the model's predicted second moments were overall similar to empirical second moments, but notable discrepancies were found in the serial correlation of the growth rate of hours and, to a lesser extent, in the correlation of hours and output. Our results show that these discrepancies are large enough from a statistical testing perspective to cause a rejection of the model over business cycle frequencies.

7 Conclusion

We have introduced new methods for assessing the specification of DSGE models. Our approach involves proposing a set of plausible parameter values and evaluating their compatibility with data. The methods can examine a model's steady state properties, overall dynamic properties, and behavior in selected frequency bands, such as business cycle frequencies, and they can focus on a subset of variables in addition to the full model. We have illustrated these methods through applications to both small- and medium-scale DSGE models. In future research, we hope to apply these methods to newly proposed DSGE models to assess their fit to the data and the robustness of their policy implications.

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Supplementary Appendix: Proofs, Model Details, and Additional Tables and Figures

A.1 Proofs of results in the paper

Proof of Theorem 1. Consider $\mathcal{H}_{dT}(\theta_0)$. We first prove finite dimensional convergence and then verify tightness.[1] The term inside the norm can be equivalently represented as

$$\Psi_T(r) = \left(\frac{T}{2}\right)^{-1/2} \sum_{j=1}^{[Tr/2]} \varphi_T\left(\omega_j\right)$$

with

$$\varphi_T\left(\omega\right) = \left(f_{\theta_0}^{-1/2}(\omega)' \otimes f_{\theta_0}^{-1/2}(\omega)\right) \operatorname{vec}\left(I_T\left(\omega\right) - f_{\theta_0}(\omega)\right).$$

For a fixed r, the asymptotic normality of $\Psi_T(r)$ follows directly from Lemma 2 in Qu (2014), by replacing $\phi_T(\omega)$ with $(f_{\theta_0}^{-1/2}(\omega_j)' \otimes f_{\theta_0}^{-1/2}(\omega_j))$. We now verify that the covariance matrix of $\Psi_T(r)$ has the desired structure. Note that $\varphi_T\left(\omega_j\right)$ are asymptotically independent in j, having zero mean and satisfying

$$E\left(\varphi_T\left(\omega_j\right) \varphi_T\left(\omega_j\right)^*\right) = \mathbb{I}_{n_Y^2} + O(T^{-1/2}),$$

where the last equality follows from

$$E\left(\operatorname{vec}\left\{I_T\left(\omega_j\right) - f_{\theta_0}(\omega_j)\right\} \operatorname{vec}\left\{I_T\left(\omega_j\right) - f_{\theta_0}(\omega_j)\right\}^*\right) = f_{\theta_0}(\omega_j)' \otimes f_{\theta_0}(\omega_j) + O(T^{-1/2}).$$

Therefore, for any fixed $r \in [0, 1]$,

$$E\left(\Psi_T(r)\Psi_T(r)^*\right) = r\mathbb{I}_{n_Y^2} + O(T^{-1/2}). \tag{A.1}$$

Further, because

$$f_{\theta_0}(\omega_j)^{-1/2}\left(I_T\left(\omega_j\right) - f_{\theta_0}(\omega_j)\right) f_{\theta_0}(\omega_j)^{-1/2} \tag{A.2}$$

is a Hermitian matrix, the elements of $\Psi_T(r)$ take particular forms. The element is real valued if it corresponds to a diagonal entry in (A.2) and is complex valued otherwise. For a closer look, we consider the special case with $n_Y = 2$. Then, $\Psi_T(r)$ takes the form $(a_{11}, a_{21} + ib_{21}, a_{21} - ib_{21}, a_{22})'$, where a_{11}, a_{21}, b_{21} and a_{22} are real numbers. Because of (A.1), we must have a_{11} and a_{22} converging to $N(0, 1)$ random variables and a_{21} and b_{21} converging to independent $N(0, 1/2)$ random variables. The case with a general n_Y follows similarly. Thus, we have established the finite dimensional convergence.

We now verify tightness, i.e., prove that for any $\varepsilon > 0$, there exists constants C and T_0, such that

$$P\left(\mathcal{H}_{dT}(\theta_0) > C\right) \leq \varepsilon \text{ for all } T > T_0.$$

[1] See P. 37 in Billingsley (1968) for the definition of tightness.

Applying Assumption 2, we have

$$I_T(\omega_j) - f_{\theta_0}(\omega_j) = H(\exp(-i\omega_j);\theta_0) I_\epsilon(\omega_j) H(\exp(-i\omega_j);\theta_0)^* - f_{\theta_0}(\omega_j) + R(\omega_j),$$

where $I_\epsilon(\omega_j)$ denotes the periodogram $\epsilon_t(\theta_0)$ at the frequency ω_j and $R(\omega_j)$ is a remainder term. Let $R_{kl}(\omega_j)$ denote the *(k,l)*-th element of $R(\omega_j)$, then Proposition 11.7.4 in Brockwell and Davis (1991, p. 445-446) implies

$$\max_{\omega_j \in [0,\pi]} E(|R_{kl}(\omega_j)|^2) = O(T^{-1}). \tag{A.3}$$

Applying the above decomposition, $\Psi_T(r)$ can be written as

$$\Psi_T(r) = \Psi_{T,1}(r) + \Psi_{T,2}(r)$$

with

$$\Psi_{T,1}(r) = \left(\frac{T}{2}\right)^{-1/2} \sum_{j=1}^{[Tr/2]} \left(f_{\theta_0}^{-1/2}(\omega_j)' \otimes f_{\theta_0}^{-1/2}(\omega_j)\right) \mathrm{vec}\left\{H(e^{-i\omega_j};\theta_0) I_\epsilon(\omega_j) H(e^{-i\omega_j};\theta_0)^* - f_{\theta_0}(\omega_j)\right\},$$

$$\Psi_{T,2}(r) = \left(\frac{T}{2}\right)^{-1/2} \sum_{j=1}^{[Tr/2]} \left(f_{\theta_0}^{-1/2}(\omega_j)' \otimes f_{\theta_0}^{-1/2}(\omega_j)\right) \mathrm{vec}\left(R(\omega_j)\right).$$

We now analyze the two terms separately. The summands of $\Psi_{T,1}(r)$ form a sequence of martingale differences. Applying a standard functional central limit theorem, we have

$$P\left(\sup_{r\in[0,1]} \|\Psi_{T,1}(r)\|_\infty > \frac{C}{2}\right) \le \varepsilon \text{ for some C and all } T > T_0.$$

For $\Psi_{T,2}(r)$, because of Assumption 3, there exists a finite constant D>0 such that

$$\left\|\mathrm{vec}\left(f_{\theta_0}^{-1/2}(\omega_j)' \otimes f_{\theta_0}^{-1/2}(\omega_j)\right)\right\|_\infty < D,$$

which implies

$$\|\Psi_{T,2}(r)\|_\infty \le \left(\frac{T}{2}\right)^{-1/2} D \sum_{k,l=1}^{n_Y} \sum_{j=1}^{[Tr/2]} |R_{kl}(\omega_j)|$$

because of the Cauchy-Schwarz inequality. Thus,

$$\begin{aligned} P\left(\sup_{r\in[0,1]} \|\Psi_{T,2}(r)\|_\infty > \frac{C}{2}\right) &\le P\left((T/2)^{-1/2} D \sum_{k,l=1}^{n_Y} \sum_{j=1}^{T/2} |R_{kl}(\omega_j)| > \frac{C}{2}\right) \\ &\le 16T^{-1}D^2 \frac{\sum_{k,l,u,v=1}^{n_Y} \sum_{j,h=1}^{T/2} E\left(|R_{kl}(\omega_j)|\,|R_{uv}(\omega_h)|\right)}{C^2}, \end{aligned}$$

where the first inequality is because $|R_{kl}(\omega_j)|$ are nonnegative and the second is due to the Chebyshev inequality. Applying (A.3), the numerator in the preceding display is of order O(T). Therefore, the whole term is of order O(1), which can be made small by choosing a large C. The above results imply the tightness.

The second result follows from the same argument. The third result follows from a standard functional central limit theorem. For the fourth result, the independence between $G_d(r)$ and $G_s(r)$ is implied by the Normality (Assumption 4). The proof is complete. ■

Proof of Theorem 2. Consider $\mathcal{H}_{dT}(\theta_0)$. Let $\mathcal{H}_{dT}(\theta_0; r)$ denote $\mathcal{H}_{dT}(\theta_0)$ before taking the supremum, i.e.,

$$\mathcal{H}_{dT}(\theta_0; r) = \left\| (T/2)^{-1/2} \sum_{j=1}^{[Tr/2]} \text{vec}\left\{ f_{\theta_0}(\omega_j)^{-1/2} \left(I_T(\omega_j) - f_{\theta_0}(\omega_j) \right) f_{\theta_0}(\omega_j)^{-1/2} \right\} \right\|_\infty .$$

Then, for any fixed $r \in [0, 1]$,

$$\begin{aligned}
& (T/2)^{-1/2} \mathcal{H}_{dT}(\theta_0; r) \\
= & \left\| \frac{2}{T} \sum_{j=1}^{[Tr/2]} \text{vec}\left\{ f_{\theta_0}(\omega_j)^{-1/2} \left(f_0(\omega_j) - f_{\theta_0}(\omega_j) \right) f_{\theta_0}(\omega_j)^{-1/2} \right\} \right\|_\infty + o_p(1) \\
= & \left\| \frac{1}{\pi} \int_0^{\pi r} \text{vec}\left(f_{\theta_0}(\omega)^{-1/2} \left(f_0(\omega) - f_{\theta_0}(\omega) \right) f_{\theta_0}(\omega)^{-1/2} \right) d\omega \right\|_\infty + o_p(1),
\end{aligned} \tag{A.4}$$

where the first equality is because of the law of large numbers and the second is due to the smoothness of the functions in ω.

Because $\|f_0(\omega) - f_{\theta_0}(\omega)\| > \delta$ for some ω, there exists a constant $C > 0$ such that

$$\left\| \text{vec}\left(f_{\theta_0}(\omega)^{-1/2} \left(f_0(\omega) - f_{\theta_0}(\omega) \right) f_{\theta_0}(\omega)^{-1/2} \right) \right\|_\infty > C\delta$$

holds for the same ω because of the positive definiteness of $f_{\theta_0}(\omega)^{-1/2}$. By the property of the supremum norm, one of the elements of $\text{vec}\left(f_{\theta_0}(\omega)^{-1/2} \left(f_0(\omega) - f_{\theta_0}(\omega) \right) f_{\theta_0}(\omega)^{-1/2} \right)$ must have a modulus greater than $C\delta$. Without loss of generality, assume it is the first element and denote it by $\zeta_{\theta_0}(\omega)$. Then, because of the continuity in ω, there is an interval with positive radius on which $\zeta_{\theta_0}(\omega) > C\delta/2$. Denote this interval by $[\omega_L, \omega_U]$.

Consider (A.4) with $r = \omega_U/\pi$,

$$\begin{aligned}
& (T/2)^{-1/2} \mathcal{H}_{dT}(\theta_0; \omega_U/\pi) \\
= & \left\| \frac{1}{\pi} \int_0^{\omega_U} \text{vec}\left(f_{\theta_0}(\omega)^{-1/2} \left(f_0(\omega) - f_{\theta_0}(\omega) \right) f_{\theta_0}(\omega)^{-1/2} \right) d\omega \right\|_\infty + o_p(1) \\
\geq & \left| \frac{1}{\pi} \int_0^{\omega_U} \zeta_{\theta_0}(\omega) d\omega \right| + o_p(1) \\
\geq & \frac{1}{\pi} \int_{\omega_L}^{\omega_U} \zeta_{\theta_0}(\omega) d\omega - \left| \frac{1}{\pi} \int_0^{\omega_L} \zeta_{\theta_0}(\omega) d\omega \right| + o_p(1) \\
\geq & \frac{C\delta}{2\pi} (\omega_U - \omega_L) - \left| \frac{1}{\pi} \int_0^{\omega_L} \zeta_{\theta_0}(\omega) d\omega \right| + o_p(1) \\
\geq & \frac{C\delta}{4\pi} (\omega_U - \omega_L) - \left| \frac{1}{\pi} \int_0^{\omega_L} \zeta_{\theta_0}(\omega) d\omega \right|,
\end{aligned} \tag{A.5}$$

where the first inequality uses the definition of $\zeta_{\theta_0}(\omega)$ and the property of the supremum norm, the second uses the triangle inequality, the third is because $\zeta_{\theta_0}(\omega)$ is greater than C$\delta/2$ over the interval and the last is because $\frac{C\delta}{4\pi}(\omega_U - \omega_L)$ is positive, thus dominating the $o_p(1)$ term. We now apply (A.5) to find a lower bound for $(T/2)^{-1/2}\mathcal{H}_{dT}(\theta_0)$. There are only two possibilities:

$$\begin{aligned}
\text{Case 1:} \quad & \left| \frac{1}{\pi} \int_0^{\omega_L} \zeta_{\theta_0}(\omega) d\omega \right| < \frac{C\delta}{8\pi} (\omega_U - \omega_L), \\
\text{Case 2:} \quad & \left| \frac{1}{\pi} \int_0^{\omega_L} \zeta_{\theta_0}(\omega) d\omega \right| \geq \frac{C\delta}{8\pi} (\omega_U - \omega_L).
\end{aligned}$$

In Case 1,

$$(T/2)^{-1/2}\mathcal{H}_{dT}(\theta_0) = (T/2)^{-1/2}\sup_{r\in[0,1]}\mathcal{H}_{dT}(\theta_0;r) \geq (T/2)^{-1/2}\mathcal{H}_{dT}(\theta_0;\omega_U/\pi) \geq \frac{C\delta}{8\pi}(\omega_U-\omega_L) > 0,$$

where the equality uses the definition of $\mathcal{H}_{dT}(\theta_0)$, the first inequality is because the supremum norm must be no less than any of its admissible values and the second inequality is because (A.5) and the definition of Case 1. In Case 2,

$$(T/2)^{-1/2}\mathcal{H}_{dT}(\theta_0) = (T/2)^{-1/2}\sup_{r\in[0,1]}\mathcal{H}_{dT}(\theta_0;r) \geq (T/2)^{-1/2}\mathcal{H}_{dT}(\theta_0;\omega_L/\pi) \geq \frac{C\delta}{8\pi}(\omega_U-\omega_L) > 0,$$

where the second inequality uses the definition of Case 2. Therefore, in both cases, $\mathcal{H}_{dT}(\theta_0) \to^p \infty$.

Now consider the order of $\mathcal{H}_{sT}(\theta_0)$ under global alternatives.

$$\begin{aligned}
T^{-1/2}\mathcal{H}_{sT}(\theta_0) &\geq \left\| f_{\theta_0}^{-1/2}(0)T^{-1}\sum_{j=1}^{T}(Y_t-\mu(\theta_0))\right\|_\infty \\
&\to^p \left\| f_{\theta_0}^{-1/2}(0)\left(\mu_0-\mu(\theta_0)\right)\right\|_\infty \\
&\geq \sqrt{n_Y^{-1}\left\| f_{\theta_0}^{-1/2}(0)\left(\mu_0-\mu(\theta_0)\right)\right\|^2} \\
&= \sqrt{n_Y^{-1}\left\| \left(\mu_0-\mu(\theta_0)\right)' f_{\theta_0}^{-1}(0)\left(\mu_0-\mu(\theta_0)\right)\right\|^2} \\
&> C,
\end{aligned}$$

where C is a positive constant and the last inequality follows because $f_{\theta_0}^{-1}(0)$ is positive definite. Therefore, $\mathcal{H}_{sT}(\theta_0) \to^p \infty$. The property of $\mathcal{H}_T(\theta_0)$ follows by combining the results for $\mathcal{H}_{sT}(\theta_0)$ and $\mathcal{H}_{dT}(\theta_0)$. ■

A.2 Smets and Wouters (2007) model equations

The vector of observable variables includes output growth (Δy_t), consumption growth (Δc_t), investment growth (Δi_t), wage growth (Δw_t), labor hours (l_t), inflation (π_t) and the interest rate (r_t). As in Smets and Wouters (2007), five parameters are fixed as follows: $\epsilon_p = \epsilon_w = 10, \delta = 0.025, g_y = 0.18, \phi_w = 1.50$. The analysis allows the remaining 36 structural parameters to vary. They are ordered as

$$\begin{aligned}
\theta^D = \; &(\rho_{ga}, \mu_w, \mu_p, \alpha, \psi, \varphi, \sigma_c, \lambda, \phi_p, \iota_w, \xi_w, \iota_p, \xi_p, \sigma_l, r_\pi, r_{\Delta y}, r_y, \rho, \rho_a, \rho_b, \rho_g, \\
&\rho_i, \rho_r, \rho_p, \rho_w, \sigma_a, \sigma_b, \sigma_g, \sigma_i, \sigma_r, \sigma_p, \sigma_w, \overline{\gamma}, 100(1/\beta-1), \overline{\pi}, \overline{l})'.
\end{aligned}$$

Below is the log linearized system consistent with Smets and Wouters' (2007) code.

The aggregate resource constraint: It satisfies

$$y_t = c_y c_t + i_y i_t + z_y z_t + \varepsilon_t^g.$$

Output (y_t) is composed of consumption (c_t), investment (i_t), capital utilization costs as a function of the capital utilization rate (z_t), and exogenous spending (ε_t^g). The latter follows an AR(1) model with an i.i.d. Normal error term (η_t^g), and is also affected by the productivity shock (η_t^a) as follows:

$$\varepsilon_t^g = \rho_g \varepsilon_{t-1}^g + \rho_{ga}\eta_t^a + \eta_t^g.$$

The coefficients c_y, i_y and z_y are functions of the steady state spending-output ratio (g_y), steady state output growth (γ), capital depreciation (δ), household discount factor (β), intertemporal elasticity of substitution (σ_c), fixed costs in production (ϕ_p), and share of capital in production (α): $i_y = (\gamma - 1 + \delta)k_y$, $c_y = 1 - g_y - i_y$, and $z_y = R_*^k k_y$. Here, k_y is the steady state capital-output ratio, and R_*^k is the steady state rental rate of capital: $k_y = \phi_p \left(L_*/k_*\right)^{\alpha-1} = \phi_p \left[((1-\alpha)/\alpha)\left(R_*^k/w_*\right)\right]^{\alpha-1}$ with $w_* = \left(\alpha^\alpha(1-\alpha)^{(1-\alpha)}/[\phi_p\left(R_*^k\right)^\alpha]\right)^{1/(1-\alpha)}$, and $R_*^k = \beta^{-1}\gamma^{\sigma_c} - (1-\delta)$.

Households: The consumption Euler equation is

$$c_t = c_1 c_{t-1} + (1-c_1)E_t c_{t+1} + c_2(l_t - E_t l_{t+1}) - c_3(r_t - E_t\pi_{t+1}) - \varepsilon_t^b, \tag{A.6}$$

where l_t is hours worked, r_t is the nominal interest rate, and π_t is inflation. The disturbance ε_t^b follows

$$\varepsilon_t^b = \rho_b \varepsilon_{t-1}^b + \eta_t^b.$$

The relationships of the coefficients in (A.6) to the habit persistence (λ), steady state labor market markup (ϕ_w), and other structural parameters highlighted above are

$$c_1 = \frac{\lambda/\gamma}{1+\lambda/\gamma}, c_2 = \frac{(\sigma_c - 1)\left(w_*^h L_*/c_*\right)}{\sigma_c\left(1+\lambda/\gamma\right)}, c_3 = \frac{1-\lambda/\gamma}{\left(1+\lambda/\gamma\right)\sigma_c},$$

where

$$w_*^h L_*/c_* = \frac{1}{\phi_w}\frac{1-\alpha}{\alpha}R_*^k k_y \frac{1}{c_y},$$

where R_*^k and k_y are defined as above, and $c_y = 1 - g_y - (\gamma - 1 + \delta)k_y$.

The dynamics of households' investment are given by

$$i_t = i_1 i_{t-1} + (1-i_1)E_t i_{t+1} + i_2 q_t + \varepsilon_t^i,$$

where ε_t^i is a disturbance to the investment-specific technology process, given by

$$\varepsilon_t^i = \rho_i \varepsilon_{t-1}^i + \eta_t^i.$$

The coefficients satisfy

$$i_1 = \frac{1}{1+\beta\gamma^{(1-\sigma_c)}},\ i_2 = \frac{1}{\left(1+\beta\gamma^{(1-\sigma_c)}\right)\gamma^2\varphi},$$

where φ is the steady state elasticity of the capital adjustment cost function. The corresponding arbitrage equation for the value of capital is given by

$$q_t = q_1 E_t q_{t+1} + (1-q_1)E_t r_{t+1}^k - (r_t - E_t\pi_{t+1}) - \frac{1}{c_3}\varepsilon_t^b\ , \tag{A.7}$$

with $q_1 = \beta\gamma^{-\sigma_c}(1-\delta) = (1-\delta)/(R_*^k + 1 - \delta)$.

Final and intermediate goods market: The aggregate production function is

$$y_t = \phi_p \left(\alpha k_t^s + (1-\alpha) l_t + \varepsilon_t^a \right),$$

where α captures the share of capital in production, and the parameter ϕ_p is one plus the fixed costs in production. Total factor productivity follows the AR(1) process

$$\varepsilon_t^a = \rho_a \varepsilon_{t-1}^a + \eta_t^a.$$

The current capital service usage (k_t^s) is a function of capital installed in the previous period (k_{t-1}) and the degree of capital utilization (z_t):

$$k_t^s = k_{t-1} + z_t.$$

Furthermore, the capital utilization is a positive fraction of the rental rate of capital (r_t^k):

$$z_t = z_1 r_t^k, \quad \text{where } z_1 = (1-\psi)/\psi,$$

and ψ is a positive function of the elasticity of the capital utilization adjustment cost function and normalized to be between zero and one. The accumulation of installed capital (k_t) satisfies

$$k_t = k_1 k_{t-1} + (1-k_1) i_t + k_2 \varepsilon_t^i,$$

where ε_t^i is the investment-specific technology process as defined before, and k_1 and k_2 satisfy

$$k_1 = \frac{1-\delta}{\gamma},\, k_2 = \left(1 - \frac{1-\delta}{\gamma}\right)\left(1 + \beta\gamma^{(1-\sigma_c)}\right)\gamma^2 \varphi.$$

The price markup satisfies

$$\mu_t^p = \alpha \left(k_t^s - l_t\right) + \varepsilon_t^a - w_t,$$

where w_t is the real wage. The New Keynesian Phillips curve is

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \varepsilon_t^p,$$

where ε_t^p is a disturbance to the price markup, following the ARMA(1,1) process given by

$$\varepsilon_t^p = \rho_p \varepsilon_{t-1}^p + \eta_t^p - \mu_p \eta_{t-1}^p.$$

The MA(1) term is intended to pick up some of the high frequency fluctuations in prices. The Phillips curve coefficients depend on price indexation (ι_p) and stickiness (ξ_p), the curvature of the goods market Kimball aggregator (ϵ_p), and other structural parameters:

$$\pi_1 = \frac{\iota_p}{1+\beta\gamma^{(1-\sigma_c)}\iota_p}, \pi_2 = \frac{\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}\iota_p}, \pi_3 = \frac{1}{1+\beta\gamma^{(1-\sigma_c)}\iota_p} \frac{\left(1-\beta\gamma^{(1-\sigma_c)}\xi_p\right)(1-\xi_p)}{\xi_p \left((\phi_p - 1)\epsilon_p + 1\right)}.$$

Finally, cost minimization by firms implies that the rental rate of capital satisfies

$$r_t^k = -\left(k_t^s - l_t\right) + w_t.$$

Labor market: The wage markup is

$$\mu_t^w = w_t - \left(\sigma_l l_t + \frac{1}{1-\lambda/\gamma}\left(c_t - (\lambda/\gamma)c_{t-1}\right)\right),$$

where σ_l is the elasticity of labor supply. Real wage w_t adjusts slowly according to

$$w_t = w_1 w_{t-1} + (1-w_1)\left(E_t w_{t+1} + E_t \pi_{t+1}\right) - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + \varepsilon_t^w,$$

where the coefficients are functions of wage indexation (ι_w) and stickiness (ξ_w) parameters, and the curvature of the labor market Kimball aggregator (ϵ_w):

$$\begin{aligned} w_1 &= \frac{1}{1+\beta\gamma^{(1-\sigma_c)}}, w_2 = \frac{1+\beta\gamma^{(1-\sigma_c)}\iota_w}{1+\beta\gamma^{(1-\sigma_c)}}, w_3 = \frac{\iota_w}{1+\beta\gamma^{(1-\sigma_c)}}, \\ w_4 &= \frac{1}{1+\beta\gamma^{(1-\sigma_c)}} \frac{\left(1-\beta\gamma^{(1-\sigma_c)}\xi_w\right)(1-\xi_w)}{\xi_w\left((\phi_w - 1)\,\epsilon_w\ + 1\right)}. \end{aligned}$$

The wage mark-up disturbance follows an ARMA(1,1) process:

$$\varepsilon_t^w = \rho_w \varepsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w.$$

Monetary policy: The empirical monetary policy reaction function is

$$r_t = \rho r_{t-1} + (1-\rho)\left(r_\pi \pi_t + r_y\left(y_t - y_t^*\right)\right) + r_{\Delta y}\left(\left(y_t - y_t^*\right) - \left(y_{t-1} - y_{t-1}^*\right)\right) + \varepsilon_t^r.$$

The monetary shock ε_t^r follows an AR(1) process:

$$\varepsilon_t^r = \rho_r \varepsilon_{t-1}^r + \eta_t^r.$$

The variable y_t^* stands for the time-varying optimal output level that is the result of a flexible price-wage economy. Since the equations for the flexible price-wage economy are essentially the same as above, but with the variables μ_t^p and μ_t^w set to zero, we omit the details.

A.3 Outline of the Schmitt-Grohé and Uribe (2012) model

The economy is populated with agents maximizing lifetime utility $E_0 \sum_{t=0}^{\infty} \beta^t \zeta_t U(V_t)$, where ζ_t is an exogenous preference shock, and $U(V_t) = (V_t^{1-\sigma} - 1)/(1-\sigma)$ with $V_t = C_t - bC_{t-1} - \psi h_t^\theta S_t$, where $S_t = (C_t - bC_{t-1})^\gamma S_{t-1}^{1-\gamma}$, so that consumer preferences are defined over V_t , which represents a bundle of consumption (C_t), labor (h_t) and an additional variable S_t. Jaimovich and Rebelo (2009) found that this form of preferences, together with other real rigidities, is key for generating aggregate comovement in response to news about fundamental shocks. Households own physical capital stock K_t, which evolves according to $K_t = (1-\delta(u_t))K_{t-1} + z_t^I I_t \left[1 - S\left(I_t/I_{t-1}\right)\right]$, where I_t is gross investment and u_t measures capacity utilization, so that the effective amount of capital supplied to firms is $u_t K_{t-1}$. The depreciation rate $\delta(u_t)$ satisfies $\delta(u_t) = \delta_0 + \delta_1(u_t - 1) + (\delta_2/2)(u_t - 1)^2$. The investment adjustment cost function $S(\cdot)$, due to Christiano et al. (2005), is given by $S(x) = (\kappa/2)(x - \mu^i)^2$, where μ^i is the steady state growth rate of investment. Finally, the stationary exogenous shock z_t^I affects the technology transforming investment goods into capital goods.

The production function is of Cobb-Douglas form:

$$Y_t = z_t(u_t K_{t-1})^{\alpha_k}(X_t h_t)^{\alpha_h}(X_t L)^{1-\alpha_k-\alpha_h}, \tag{A.8}$$

where Y_t is output, z_t is an exogenous productivity shock, X_t is a nonstationary labor-augmenting productivity shock, and L is a fixed factor of production. The capital and labor shares satisfy $\alpha_k, \alpha_h \in (0,1), \alpha_k + \alpha_h \leq 1$. The aggregate resource constraint is given by $Y_t = C_t + A_t I_t + G_t$, where G_t is government spending and A_t is a nonstationary shock to investment-specific technology.

The model features an imperfectly competitive labor market. The households supply labor to monopolistically competitive labor unions, which sell differentiated labor inputs to the final good producers. The elasticity of substitution between differentiated labor inputs is time-varying, with the wage markup denoted μ_t. In equilibrium, the wage rate paid by the union to its members is smaller than the wage rate firms pay to unions, and all unions charge the same wage rate.

A.4 Tables and figures

Table A1: Rejection frequencies under the null hypothesis (pairwise testing)

Level	T	BC frequencies			Full spectrum			Weighted spectrum			Mean and spectrum		
		Determinacy											
		(π,y)	(π,r)	(y,r)	(π,y)	(π,r)	(y,r)	(π,y)	(π,r)	(y,r)	(π,y)	(π,r)	(y,r)
10%	80	0.079	0.073	0.086	0.079	0.081	0.081	0.081	0.090	0.087	0.084	0.094	0.083
	160	0.085	0.079	0.092	0.087	0.082	0.090	0.082	0.089	0.088	0.081	0.091	0.083
	240	0.085	0.090	0.088	0.078	0.087	0.085	0.086	0.090	0.087	0.084	0.092	0.083
	320	0.092	0.094	0.082	0.093	0.092	0.085	0.090	0.095	0.093	0.089	0.094	0.093
5%	80	0.044	0.044	0.052	0.038	0.042	0.041	0.052	0.058	0.056	0.041	0.046	0.042
	160	0.047	0.041	0.055	0.046	0.042	0.044	0.046	0.053	0.051	0.041	0.049	0.042
	240	0.044	0.045	0.047	0.039	0.042	0.043	0.047	0.052	0.046	0.043	0.042	0.038
	320	0.044	0.050	0.044	0.048	0.044	0.041	0.045	0.050	0.047	0.049	0.048	0.042
		Indeterminacy											
10%	80	0.091	0.106	0.087	0.082	0.098	0.079	0.090	0.115	0.090	0.072	0.106	0.081
	160	0.086	0.098	0.089	0.089	0.093	0.090	0.085	0.096	0.088	0.083	0.115	0.079
	240	0.092	0.091	0.091	0.090	0.095	0.097	0.093	0.096	0.091	0.077	0.100	0.082
	320	0.097	0.098	0.086	0.092	0.095	0.094	0.078	0.097	0.091	0.086	0.102	0.086
5%	80	0.058	0.065	0.054	0.045	0.053	0.041	0.057	0.080	0.055	0.035	0.061	0.045
	160	0.047	0.056	0.051	0.043	0.052	0.047	0.052	0.057	0.051	0.043	0.061	0.040
	240	0.049	0.050	0.050	0.047	0.044	0.048	0.054	0.056	0.051	0.037	0.048	0.038
	320	0.049	0.055	0.045	0.044	0.045	0.046	0.041	0.051	0.051	0.041	0.053	0.045

Note. T: sample size; all tests are computed with prewhitening. π, y, and r denote inflation, output, and interest rate, respectively.

Table A2: Rejection frequencies under the null hypothesis (single variable testing)

Level	T	BC frequencies			Full spectrum			Weighted spectrum			Mean and spectrum		
							Determinacy						
		π	y	r	π	y	r	π	y	r	π	y	r
10%	80	0.068	0.066	0.068	0.078	0.080	0.084	0.066	0.067	0.072	0.096	0.074	0.066
	160	0.080	0.081	0.080	0.090	0.081	0.081	0.079	0.074	0.078	0.093	0.077	0.080
	240	0.083	0.081	0.092	0.096	0.091	0.094	0.081	0.087	0.082	0.086	0.085	0.085
	320	0.087	0.086	0.092	0.091	0.096	0.093	0.083	0.080	0.086	0.096	0.081	0.086
5%	80	0.035	0.039	0.038	0.042	0.038	0.039	0.037	0.038	0.044	0.048	0.039	0.034
	160	0.032	0.045	0.040	0.042	0.044	0.037	0.042	0.039	0.041	0.049	0.037	0.042
	240	0.045	0.040	0.050	0.047	0.044	0.046	0.042	0.050	0.043	0.046	0.042	0.047
	320	0.046	0.041	0.047	0.043	0.046	0.047	0.043	0.039	0.041	0.051	0.040	0.042
							Indeterminacy						
10%	80	0.074	0.066	0.074	0.091	0.080	0.088	0.079	0.069	0.065	0.075	0.083	0.067
	160	0.085	0.081	0.079	0.085	0.087	0.093	0.080	0.075	0.080	0.076	0.086	0.073
	240	0.082	0.086	0.091	0.094	0.092	0.098	0.091	0.080	0.086	0.081	0.089	0.088
	320	0.086	0.085	0.085	0.087	0.089	0.098	0.089	0.094	0.083	0.088	0.089	0.079
5%	80	0.042	0.035	0.043	0.044	0.038	0.045	0.047	0.037	0.039	0.037	0.046	0.036
	160	0.048	0.041	0.041	0.042	0.042	0.047	0.043	0.041	0.044	0.037	0.046	0.037
	240	0.040	0.042	0.049	0.051	0.044	0.050	0.044	0.041	0.047	0.039	0.041	0.046
	320	0.039	0.043	0.046	0.043	0.046	0.050	0.043	0.050	0.043	0.044	0.043	0.039

Note. T: sample size; all tests are computed with prewhitening. π, y, and r denote inflation, output, and interest rate, respectively.

Table A3: Rejection frequencies under the alternative hypothesis (pairwise testing; 10%)

T	BC frequencies			Full spectrum			Weighted spectrum			Mean and spectrum		
	(π, y)	(π, r)	(y, r)	(π, y)	(π, r)	(y, r)	(π, y)	(π, r)	(y, r)	(π, y)	(π, r)	(y, r)
	Determinacy											
	Perturb a random element of θ by 20%											
80	0.184	0.218	0.212	0.353	0.346	0.283	0.213	0.235	0.219	0.281	0.268	0.253
160	0.256	0.269	0.252	0.492	0.469	0.381	0.304	0.307	0.302	0.427	0.380	0.327
240	0.331	0.330	0.316	0.571	0.550	0.484	0.427	0.372	0.369	0.505	0.444	0.410
320	0.361	0.365	0.344	0.631	0.599	0.498	0.474	0.419	0.400	0.563	0.510	0.453
	Perturb a random element of θ by 40%											
80	0.255	0.283	0.276	0.600	0.553	0.456	0.319	0.316	0.318	0.512	0.466	0.396
160	0.445	0.411	0.409	0.737	0.679	0.576	0.531	0.465	0.473	0.658	0.596	0.528
240	0.555	0.494	0.481	0.800	0.724	0.626	0.646	0.561	0.504	0.704	0.644	0.582
320	0.617	0.555	0.500	0.834	0.759	0.658	0.700	0.605	0.549	0.742	0.675	0.592
	Indeterminacy											
	Perturb a random element of θ by 20%											
80	0.165	0.212	0.184	0.226	0.284	0.296	0.185	0.209	0.228	0.227	0.248	0.290
160	0.213	0.264	0.258	0.343	0.368	0.406	0.230	0.276	0.308	0.299	0.330	0.373
240	0.220	0.296	0.283	0.401	0.415	0.463	0.292	0.314	0.365	0.332	0.369	0.416
320	0.276	0.305	0.331	0.459	0.443	0.496	0.323	0.350	0.412	0.406	0.403	0.450
	Perturb a random element of θ by 40%											
80	0.267	0.287	0.299	0.439	0.421	0.479	0.284	0.332	0.333	0.416	0.380	0.443
160	0.361	0.371	0.400	0.572	0.536	0.580	0.416	0.409	0.478	0.511	0.472	0.534
240	0.433	0.435	0.494	0.630	0.608	0.641	0.518	0.452	0.567	0.581	0.524	0.589
320	0.514	0.460	0.539	0.685	0.629	0.685	0.578	0.528	0.602	0.639	0.577	0.609

Note. T: sample size; all tests are computed with prewhitening. π, y, and r denote inflation, output, and interest rate, respectively.

Table A4: Rejection frequencies under the alternative hypothesis (single variable testing; 10%)

T	BC frequencies			Full spectrum			Weighted spectrum			Mean and spectrum		
	π	y	r	π	y	r	π	y	r	π	y	r
Determinacy												
Perturb a random element of θ by 20%												
80	0.204	0.121	0.235	0.334	0.196	0.268	0.224	0.162	0.240	0.296	0.180	0.247
160	0.253	0.167	0.271	0.432	0.271	0.333	0.308	0.209	0.305	0.358	0.223	0.303
240	0.318	0.182	0.305	0.506	0.327	0.378	0.370	0.235	0.309	0.432	0.285	0.347
320	0.349	0.191	0.319	0.550	0.343	0.430	0.409	0.252	0.338	0.485	0.309	0.372
Perturb a random element of θ by 40%												
80	0.282	0.164	0.272	0.547	0.283	0.372	0.355	0.174	0.305	0.456	0.244	0.316
160	0.398	0.222	0.333	0.647	0.381	0.443	0.479	0.268	0.375	0.566	0.334	0.401
240	0.467	0.247	0.381	0.703	0.423	0.507	0.553	0.287	0.416	0.623	0.391	0.449
320	0.520	0.291	0.398	0.747	0.462	0.518	0.614	0.323	0.458	0.654	0.391	0.467
Indeterminacy												
Perturb a random element of θ by 20%												
80	0.135	0.136	0.166	0.178	0.181	0.259	0.155	0.158	0.207	0.188	0.174	0.244
160	0.163	0.165	0.219	0.239	0.241	0.340	0.199	0.203	0.273	0.227	0.221	0.297
240	0.197	0.167	0.254	0.302	0.272	0.368	0.231	0.227	0.310	0.271	0.252	0.337
320	0.219	0.217	0.295	0.336	0.309	0.417	0.266	0.265	0.355	0.300	0.265	0.358
Perturb a random element of θ by 40%												
80	0.225	0.192	0.270	0.337	0.274	0.379	0.238	0.237	0.322	0.321	0.245	0.360
160	0.289	0.244	0.350	0.450	0.369	0.443	0.347	0.305	0.398	0.400	0.324	0.425
240	0.337	0.281	0.398	0.488	0.403	0.526	0.380	0.338	0.432	0.447	0.364	0.467
320	0.369	0.303	0.416	0.534	0.440	0.546	0.429	0.394	0.445	0.484	0.403	0.515

Note. T: sample size; all tests are computed with prewhitening. π, y, and r denote inflation, output, and interest rate, respectively.

Table A5: Test results for the LS model using posterior draws, 1960:I-2007:IV, determinacy.

	Weighted spectrum			BC frequencies			Mean and spectrum		
	Test	CV	Rej.	Test	CV	Rej.	Test	CV	Rej.
Full model	1.214	1.128	100	1.629	1.370	100	2.306	2.851	99
GDP	0.875	0.977	99.98	0.442	1.151	50	2.068	2.478	73
Inflation	0.641	0.977	62	0.554	1.151	56	1.180	2.478	43
Interest rate	0.337	0.977	4	0.369	1.151	0.01	0.819	2.478	15
GDP-Inflation	0.963	1.077	99.91	0.927	1.295	75	1.988	2.729	58
GDP-Interest rate	0.902	1.077	99.92	0.957	1.295	98	2.132	2.729	79
Inflation-Interest rate	0.832	1.077	94	0.809	1.295	36	1.987	2.729	95

Note. The significance level is 10%. Test: the specification test value; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test.

Table A6: Test results for the LS model using posterior draws, 1960:I-2007:IV, indeterminacy

	Weighted spectrum			BC frequencies			Mean and spectrum		
	Test	CV	Rej.	Test	CV	Rej.	Test	CV	Rej.
Full model	1.694	1.128	100	2.073	1.370	100	3.448	2.851	100
GDP	0.359	0.977	90	0.280	1.151	83	0.875	2.478	18
Inflation	0.592	0.977	96	0.431	1.151	81	1.499	2.478	83
Interest rate	0.308	0.977	28	0.382	1.151	14	0.907	2.478	18
GDP-Inflation	0.909	1.077	99.61	0.953	1.295	95	1.854	2.729	79
GDP-Interest rate	1.439	1.077	100	1.675	1.295	100	3.051	2.729	100
Inflation-Interest rate	0.859	1.077	98	0.751	1.295	88	1.891	2.729	87

Note. The significance level is 10%. Test: the specification test value; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test.

Table A7: 95% confidence intervals for the LS model, 1979:III-2007:IV, determinacy.

θ	Parameter	Bounds	CI
τ	intertemporal substitution elasticity	[0.10, 1.00]	[0.10, 0.996]
β	discount factor	[0.98, 0.999]	[0.980, 0.999]
κ	Phillips curve slope	[0.01, 2.00]	[0.02, 1.999]
ψ_1	inflation target	[1.01, 3.00]	[1.010, 2.999]
ψ_2	output target	[0.01, 5.00]	[0.010, 4.999]
ρ_r	interest rate smoothing	[0.10, 0.90]	[0.62, 0.90]
ρ_g	exogenous spending AR	[0.10, 0.98]	[0.70, 0.98]
ρ_z	technology shock AR	[0.10, 0.98]	[0.85, 0.98]
σ_r	monetary policy shock SD	[0.01, 3.00]	[0.22, 0.51]
σ_g	exogenous spending SD	[0.01, 3.00]	[0.03, 0.38]
σ_z	technology shock SD	[0.01, 3.00]	[0.46, 1.47]
ρ_{gz}	exogenous spending-technology CORR	[-0.90, -0.90]	[-0.46, 0.90]
π^*	steady state inflation	[2.00, 8.00]	[2.00, 8.00]

Note. The results are obtained using the mean and the full spectrum.

Table A8: Specification test results, 1979:III-2007:IV, determinacy.

	Weighted spectrum			BC frequencies			Mean and spectrum		
	Test	CV	Rej.	Test	CV	Rej.	Test	CV	Rej.
Full model	0.682	1.128	15	0.769	1.370	39	2.519	2.851	99.93
GDP	0.468	0.977	7	0.306	1.151	0.03	2.180	2.478	99.99
Inflation	0.500	0.977	97	0.381	1.151	92	2.021	2.478	84
Interest rate	0.286	0.977	54	0.350	1.151	82	0.596	2.478	60
GDP-Inflation	0.597	1.077	86	0.424	1.295	76	2.356	2.729	99.82
GDP-Interest rate	0.581	1.077	45	0.639	1.295	72	2.612	2.729	99.99
Inflation-Interest rate	0.558	1.077	42	0.640	1.295	70	2.027	2.729	92

Note. The significance level is 10%. Test: the specification test value; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test.

Table A9: Test results for the LS model using posterior draws, 1979:III-2007:IV, determinacy.

	Weighted spectrum			BC frequencies			Mean and spectrum		
	Test	CV	Rej.	Test	CV	Rej.	Test	CV	Rej.
Full model	0.794	1.128	86	0.809	1.370	12	1.866	2.851	62
GDP	0.563	0.977	89	0.336	1.151	7	1.432	2.478	29
Inflation	0.401	0.977	16	0.337	1.151	10	0.729	2.478	32
Interest rate	0.279	0.977	2	0.339	1.151	1.5	0.556	2.478	2
GDP-Inflation	0.572	1.077	78	0.348	1.295	5	1.290	2.729	12
GDP-Interest rate	0.603	1.077	78	0.634	1.295	2	1.558	2.729	31
Inflation-Interest rate	0.537	1.077	26	0.354	1.295	2	1.515	2.729	47

Note. The significance level is 10%. Test: the specification test value; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test.

Table A10: 95% confidence intervals for the LS model, indeterminacy, 1960:I-1979:II.

θ	Parameter	Bounds	CI
τ	intertemporal substitution elasticity	[0.10, 1.00]	[0.10, 0.9999]
β	discount factor	[0.90, 0.999]	[0.984, 0.999]
κ	Phillips curve slope	[0.01, 2.00]	[0.01, 1.01]
ψ_1	inflation target	[0.01, 0.99]	[0.01, 0.989]
ψ_2	output target	[0.01, 5.00]	[0.01, 5.00]
ρ_r	interest rate smoothing	[0.10, 0.90]	[0.10, 0.90]
ρ_g	exogenous spending AR	[0.10, 0.98]	[0.26, 0.98]
ρ_z	technology shock AR	[0.10, 0.98]	[0.52, 0.98]
σ_r	monetary policy shock SD	[0.01, 3.00]	[0.01, 0.75]
σ_g	exogenous spending SD	[0.01, 3.00]	[0.03, 1.23]
σ_z	technology shock SD	[0.01, 3.00]	[0.42, 2.99]
ρ_{gz}	exogenous spending-technology CORR	[-0.90, -0.90]	[-0.90, 0.90]
$M_{r\epsilon}$	sunspot-monetary coeff	[-3.00, 3.00]	[-3.00, 3.00]
$M_{g\epsilon}$	sunspot-exogenous spending coeff	[-3.00, 3.00]	[-3.000, 2.999]
$M_{z\epsilon}$	sunspot-technology coeff	[-3.00, 3.00]	[-3.00, 1.66]
σ_ϵ	sunspot shock SD	[0.01, 3.00]	[0.01, 1.68]
π^*	steady state inflation	[2.00, 8.00]	[2.00, 8.00]

Note. The results are obtained using the mean and the full spectrum.

Table A11: Test results for the LS model, 1960:I-1979:II, indeterminacy.

	Weighted spectrum			BC frequencies			Mean and spectrum		
	Test	CV	Rej.	Test	CV	Rej.	Test	CV	Rej.
Full model	0.716	1.128	55	0.712	1.370	44	1.725	2.851	54
GDP	0.306	0.977	46	0.210	1.151	47	0.786	2.478	18
Inflation	0.440	0.977	61	0.315	1.151	11	1.063	2.478	94
Interest rate	0.315	0.977	17	0.190	1.151	2	1.071	2.478	93
GDP-Inflation	0.749	1.077	64	0.746	1.295	49	1.471	2.729	84
GDP-Interest rate	0.581	1.077	57	0.644	1.295	55	1.536	2.729	91
Inflation-Interest rate	0.650	1.077	9	0.682	1.295	1	1.458	2.729	66

Note. The significance level is 10%. Test: the specification test value; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test.

Table A12: Test results for the LS model using posterior draws, 1960:I-1979:II, indeterminacy.

	Weighted spectrum			BC frequencies			Mean and spectrum		
	Test	CV	Rej.	Test	CV	Rej.	Test	CV	Rej.
Full model	1.085	1.128	99.99	1.127	1.370	99	2.317	2.851	99
GDP	0.303	0.977	60	0.231	1.151	48	0.791	2.478	5
Inflation	0.239	0.977	4	0.255	1.151	4	0.420	2.478	3
Interest rate	0.277	0.977	61	0.193	1.151	73	0.971	2.478	31
GDP-Inflation	0.466	1.077	65	0.639	1.295	71	0.961	2.729	12
GDP-Interest rate	0.714	1.077	99.66	1.038	1.295	99.68	1.496	2.729	81
Inflation-Interest rate	0.566	1.077	86	0.748	1.295	91	1.209	2.729	63

Note. The significance level is 10%. Test: the specification test value; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test.

Table A13: Specification test results using spectrum only confidence set, 1960:I-2007:IV, determinacy

	Weighted spectrum			BC frequencies		
	Test	CV	Rej.	Test	CV	Rej.
Full model	1.126	1.128	99.99	1.742	1.370	100
GDP	0.988	0.977	100	0.953	1.151	99
Inflation	1.001	0.977	100	0.639	1.151	57
Interest rate	0.310	0.977	0.01	0.377	1.151	0
GDP-Inflation	1.186	1.077	100	1.130	1.295	96
GDP-Interest rate	0.875	1.077	98	1.141	1.295	99
Inflation-Interest rate	0.865	1.077	82	0.969	1.295	57

Note. The significance level is 10%. Test: the specification test value; CV: critical value; Rej.: percentage of draws rejected. The "Mean and spectrum" tests are not reported because only spectrum information is used to compute the confidence set.

Table A14: Specification test results using spectrum only confidence set, 1960:I-2007:IV, indeterminacy

	Weighted spectrum			BC frequencies		
	Test	CV	Rej.	Test	CV	Rej.
Full model	1.141	1.128	100	1.091	1.370	99.95
GDP	0.442	0.977	97	0.334	1.151	94
Inflation	1.093	0.977	100	1.230	1.151	100
Interest rate	0.309	0.977	10	0.292	1.151	2
GDP-Inflation	1.193	1.077	100	1.334	1.295	100
GDP-Interest rate	0.644	1.077	97	0.816	1.295	92
Inflation-Interest rate	1.088	1.077	100	0.909	1.295	99.80

Note. The significance level is 10%. Test: the specification test value; CV: critical value; Rej.: percentage of draws rejected. The "Mean and spectrum" tests are not reported because only spectrum information is used to compute the confidence set.

Table A15: Specification test results for the SW model, 1960:I-2007:IV

	(a) Frequentist set			(b) Posterior distribution		
	Test	CV	Rej.	Test	CV	Rej.
Full model, weighted spectrum	0.849	1.235	78	0.878	1.235	67
Full model, BC frequencies	1.18	1.519	97	1.128	1.519	81
GDP growth	0.303	0.977	44	0.317	0.977	20
Inflation	0.393	0.977	56	0.398	0.977	58
Interest rate	0.332	0.977	19	0.372	0.977	30
Consumption growth	0.226	0.977	82	0.229	0.977	18
Investment growth	0.276	0.977	75	0.242	0.977	2
Labor hours	0.169	0.977	1	0.175	0.977	6
Wage growth	0.261	0.977	45	0.264	0.977	10
Consumption-Labor hours	0.741	1.077	95	0.718	1.077	61
Consumption-GDP	0.683	1.077	94	0.732	1.077	67
Consumption-Inflation	0.581	1.077	86	0.581	1.077	38
Consumption-Investment	0.622	1.077	80	0.552	1.077	9
Consumption-Wages	0.393	1.077	61	0.342	1.077	4
Consumption-Interest rate	0.652	1.077	89	0.613	1.077	65
Investment-GDP	0.671	1.077	92	0.573	1.077	59
Investment-Labor hours	0.401	1.077	49	0.325	1.077	2
Investment-Inflation	0.491	1.077	62	0.506	1.077	26
Investment-Wages	0.542	1.077	54	0.375	1.077	0.3
Investment-Interest rate	0.681	1.077	75	0.627	1.077	42
GDP-Labor hours	0.419	1.077	33	0.489	1.077	39
GDP-Inflation	0.561	1.077	40	0.597	1.077	26
GDP-Wages	0.452	1.077	13	0.452	1.077	1
GDP-Interest rate	0.614	1.077	65	0.673	1.077	55
Labor hours-Inflation	0.460	1.077	41	0.469	1.077	65
Labor hours-Wages	0.494	1.077	19	0.548	1.077	26
Labor hours-Interest rate	0.515	1.077	7	0.496	1.077	7
Inflation-Wages	0.456	1.077	19	0.447	1.077	10
Inflation-Interest rate	0.509	1.077	64	0.552	1.077	81
Wages-Interest rate	0.424	1.077	41	0.458	1.077	63

Note. Each row represents a set of variables tested. The significance level is 10% for each case. Test: the specification test value, based on the weighted full spectrum unless indicated otherwise; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test. For (a) the parameter values are obtained by sampling the score test, and for (b) they are values from the 95% highest density region of the posterior distribution using SW's prior.

Table A16: Specification test results for the SW model, 1965:I-2004:IV

	(a) Frequentist set			(b) Posterior distribution		
	Test	CV	Rej.	Test	CV	Rej.
Full model, weighted spectrum	0.867	1.235	68	0.901	1.235	89
Full model, BC frequencies	1.083	1.519	92	1.048	1.519	71
GDP growth	0.214	0.977	18	0.194	0.977	0.3
Inflation	0.319	0.977	22	0.371	0.977	42
Interest rate	0.310	0.977	20	0.331	0.977	47
Consumption growth	0.198	0.977	60	0.181	0.977	1
Investment growth	0.308	0.977	82	0.304	0.977	7
Labor hours	0.217	0.977	8	0.240	0.977	26
Wage growth	0.266	0.977	4	0.270	0.977	0.3
Consumption-Labor hours	0.669	1.077	85	0.584	1.077	28
Consumption-GDP	0.635	1.077	69	0.729	1.077	26
Consumption-Inflation	0.579	1.077	78	0.610	1.077	22
Consumption-Investment	0.578	1.077	62	0.538	1.077	0.6
Consumption-Wages	0.307	1.077	25	0.281	1.077	0.8
Consumption-Interest rate	0.552	1.077	81	0.605	1.077	80
Investment-GDP	0.524	1.077	77	0.431	1.077	3
Investment-Labor hours	0.455	1.077	63	0.394	1.077	12
Investment-Inflation	0.550	1.077	64	0.514	1.077	16
Investment-Wages	0.411	1.077	60	0.365	1.077	6
Investment-Interest rate	0.737	1.077	82	0.707	1.077	62
GDP-Labor hours	0.501	1.077	58	0.456	1.077	47
GDP-Inflation	0.494	1.077	20	0.516	1.077	10
GDP-Wages	0.358	1.077	7	0.351	1.077	9
GDP-Interest rate	0.562	1.077	39	0.626	1.077	52
Labor hours-Inflation	0.443	1.077	25	0.526	1.077	42
Labor hours-Wages	0.381	1.077	10	0.456	1.077	41
Labor hours-Interest rate	0.482	1.077	7	0.491	1.077	51
Inflation-Wages	0.427	1.077	6	0.470	1.077	8
Inflation-Interest rate	0.455	1.077	32	0.628	1.077	88
Wages-Interest rate	0.355	1.077	31	0.415	1.077	69

Note. Each row represents a set of variables tested. The significance level is 10% for each case. Test: the specification test value, based on the weighted full spectrum unless indicated otherwise; CV: critical value; Rej.: percentage of parameter draws rejected by the specification test. For (a) the parameter values are obtained by sampling the score test, and for (b) they are values from the 95% highest density region of the posterior distribution using SW's prior.

Table A17: 95% confidence intervals, SGU model, 1955:II-2006:IV

	Parameter	Bounds	CI
θ	Frisch elasticity of labor supply (when $\gamma = b = 0$)	[1.00, 6.00]	[1.36, 6.00]
γ	Governs wealth elasticity of labor supply	[1E-06, 0.999]	[4E-04, 0.76]
κ	Investment adjustment cost parameter	[1.00, 30.00]	[5.58, 29.97]
$\delta_2\delta_1$	Ratio of depreciation parameters	[0.01, 3.00]	[0.01, 2.92]
b	Habit parameter	[1E-05, 0.98]	[0.74, 0.95]
ρ_{xg}	AR coeff. of government spending trend	[0.00, 0.98]	[0.27, 0.98]
ρ_{μ^a}	AR coeff. of nonstationary investment-specific prod. shock	[0.00, 0.70]	[0.28, 0.70]
ρ_{μ^x}	AR coeff. of nonstationary neutral productivity shock	[0.00, 0.70]	[5E-05, 0.70]
ρ_{z^i}	AR coeff. of stationary investment shock	[0.00, 0.98]	[9E-04, 0.98]
ρ_z	AR coeff. of stationary neutral productivity shock	[0.00, 0.99]	[0.02, 0.99]
ρ_μ	AR coeff. of wage markup shock	[0.00, 0.99]	[0.89, 0.99]
ρ_g	AR coeff. of gov. spending shock	[0.00, 0.99]	[0.89, 0.99]
ρ_ζ	AR coeff. of the preference shock	[0.00, 0.70]	[1E-05, 0.70]
$\sigma^0_{\mu^a}$	Std. dev. of unanticipated shock in μ^a_t	[0.01, 1.00]	[0.01, 0.42]
$\sigma^4_{\mu^a}$	Std. dev. of 4-period anticipated shock in μ^a_t	[0.01, 1.00]	[0.01, 0.41]
$\sigma^8_{\mu^a}$	Std. dev. of 8-period anticipated shock in μ^a_t	[0.01, 1.00]	[0.01, 0.41]
$\sigma^0_{\mu^x}$	Std. dev. of unanticipated shock in μ^x_t	[0.01, 1.00]	[0.01, 0.97]
$\sigma^4_{\mu^x}$	Std. dev. of 4-period anticipated shock in μ^x_t	[0.01, 1.00]	[0.01, 0.62]
$\sigma^8_{\mu^x}$	Std. dev. of 8-period anticipated shock in μ^x_t	[0.01, 1.00]	[0.01, 0.65]
$\sigma^0_{z^i}$	Std. dev. of unanticipated shock in z^i_t	[0.01, 40.00]	[4.36, 40.00]
$\sigma^4_{z^i}$	Std. dev. of 4-period anticipated shock in z^i_t	[0.01, 20.00]	[0.01, 19.94]
$\sigma^8_{z^i}$	Std. dev. of 8-period anticipated shock in z^i_t	[0.01, 20.00]	[0.01, 19.98]
σ^0_z	Std. dev. of unanticipated shock in z_t	[0.01, 2.00]	[0.01, 0.93]
σ^4_z	Std. dev. of 4-period anticipated shock in z_t	[0.01, 2.00]	[0.01, 0.67]
σ^8_z	Std. dev. of 8-period anticipated shock in z_t	[0.01, 2.00]	[0.01, 0.69]
σ^0_μ	Std. dev. of unanticipated shock in μ_t	[0.01, 10.00]	[0.01, 6.75]
σ^4_μ	Std. dev. of 4-period anticipated shock in μ_t	[0.01, 10.00]	[0.04, 9.24]
σ^8_μ	Std. dev. of 8-period anticipated shock in μ_t	[0.01, 10.00]	[0.01, 6.07]
σ^0_g	Std. dev. of unanticipated shock in g_t	[0.01, 2.00]	[0.01, 1.28]
σ^4_g	Std. dev. of 4-period anticipated shock in g_t	[0.01, 2.00]	[0.01, 1.34]
σ^8_g	Std. dev. of 8-period anticipated shock in g_t	[0.01, 2.00]	[0.01, 1.28]
σ^0_ζ	Std. dev. of unanticipated shock in ζ_t	[0.01, 10.00]	[0.01, 7.63]
σ^4_ζ	Std. dev. of 4-period anticipated shock in ζ_t	[0.01, 10.00]	[0.01, 9.01]
σ^8_ζ	Std. dev. of 8-period anticipated shock in ζ_t	[0.01, 10.00]	[0.01, 10.00]
$\sigma^{me}_{g^y}$	Std. dev. of measurement error in output growth	[0.01, 1.00]	[0.51, 0.73]

Note. Values are based on the mean and full spectrum. Column 2: parameter interpretation. Column 3: bounds for permissible parameter values. Column 4: confidence intervals, obtained by sampling the score test and applying projections.

Figure A1: Log spectra of the LS model under indeterminacy, 1960-2007.

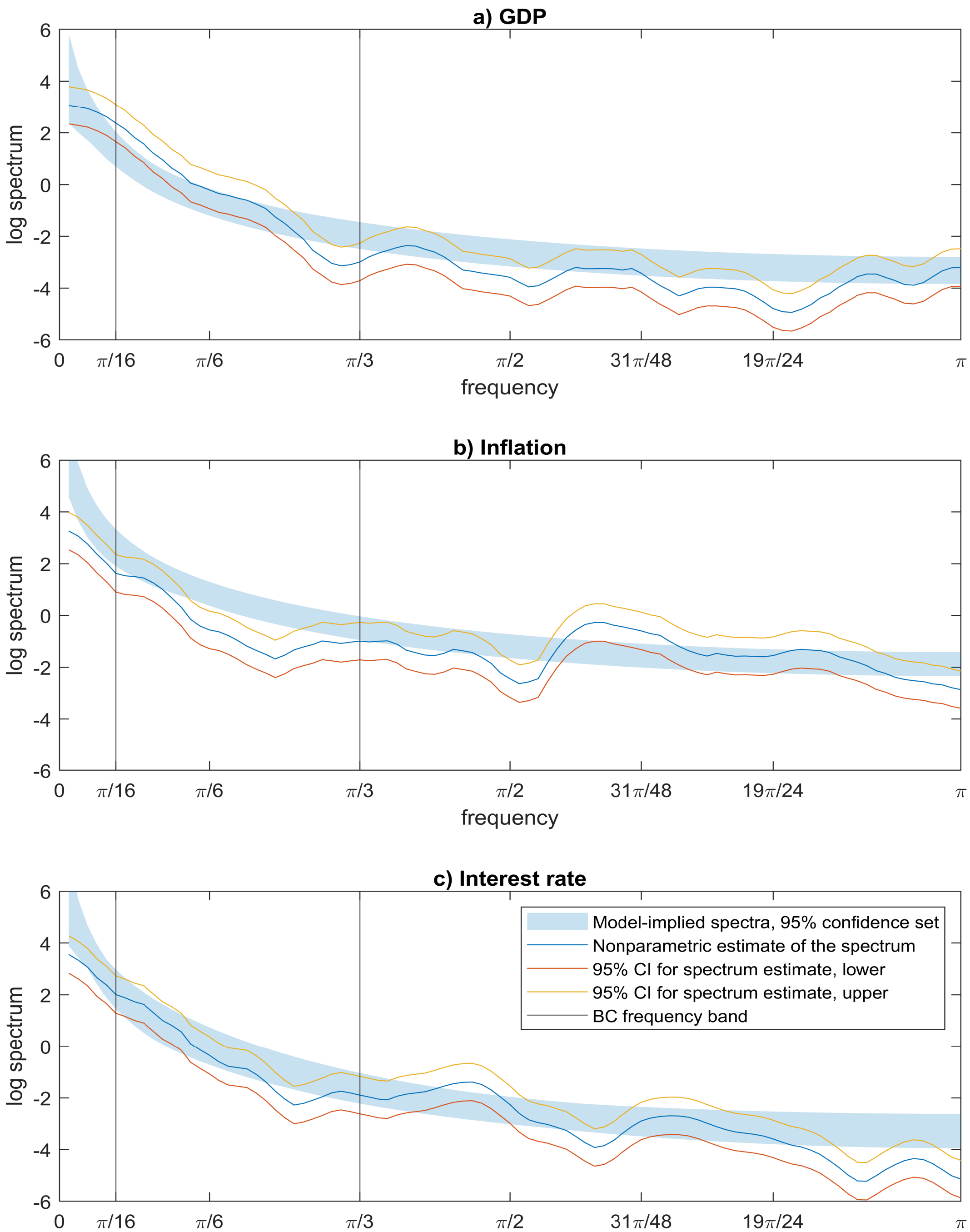


Figure A2: Cross-spectra (real part) under indeterminacy, 1960-2007.

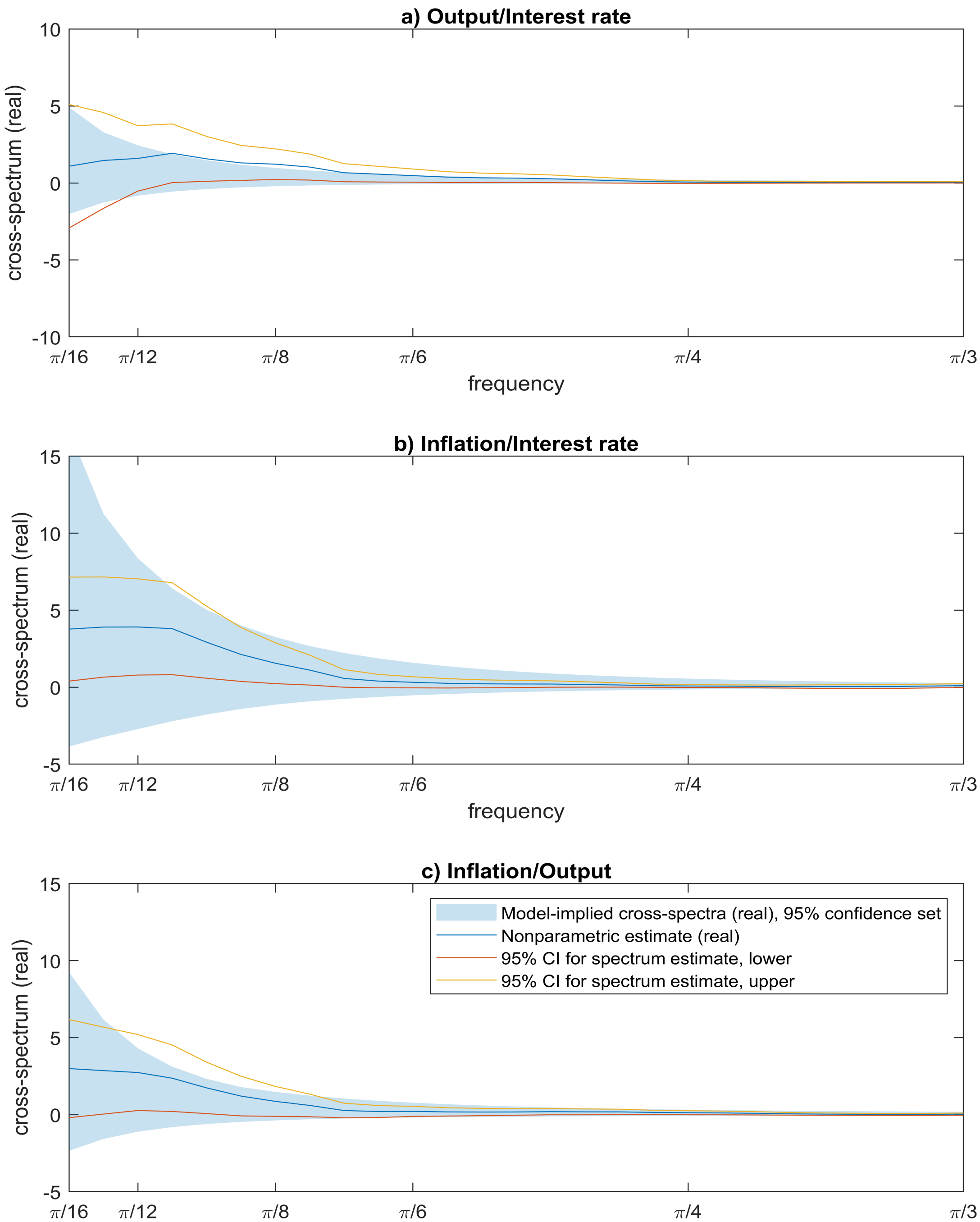


Figure A3: Cross-spectra (imaginary part) under indeterminacy, 1960-2007.

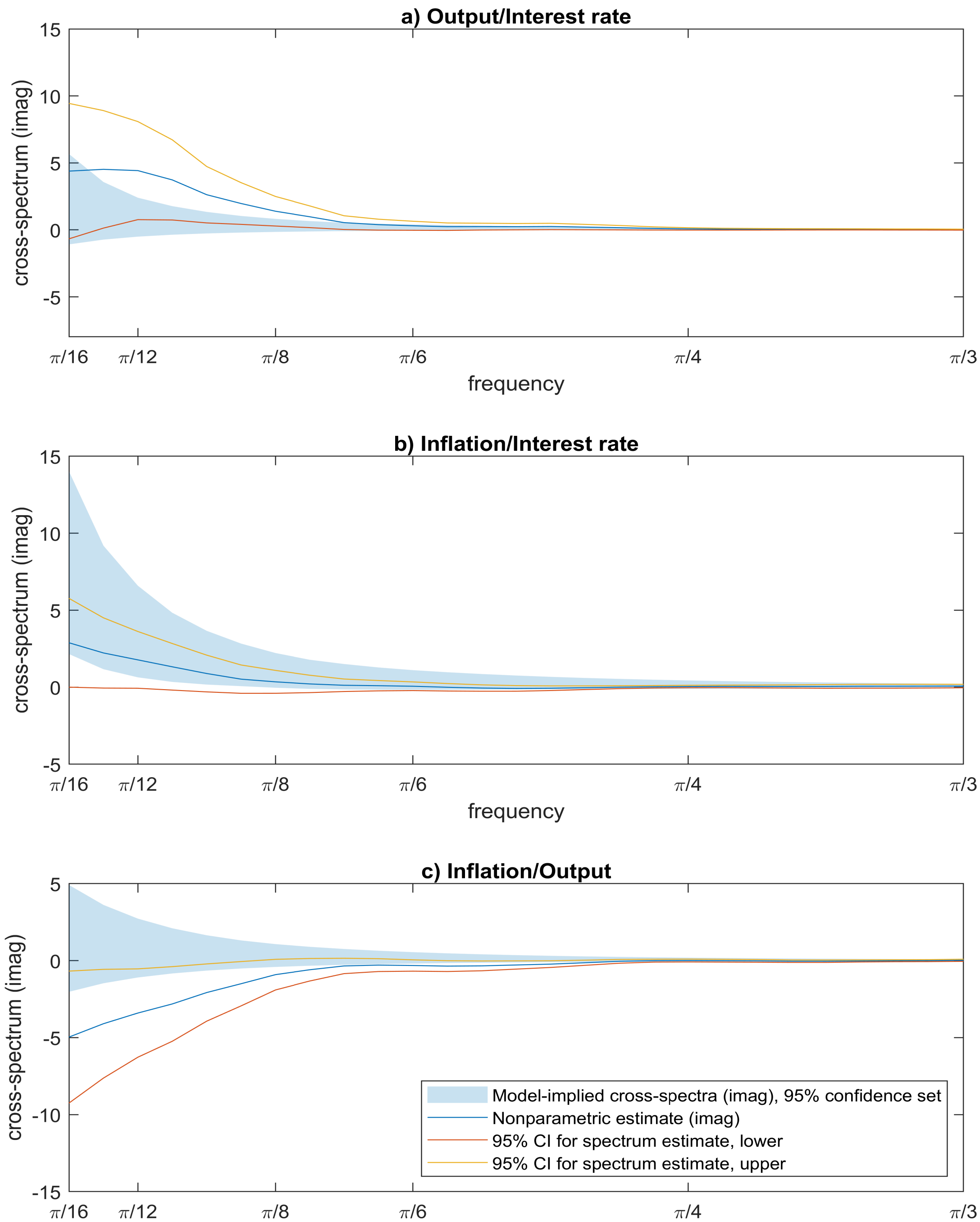


Figure A4: Log spectra using the posterior distribution under determinacy, 1960-2007.

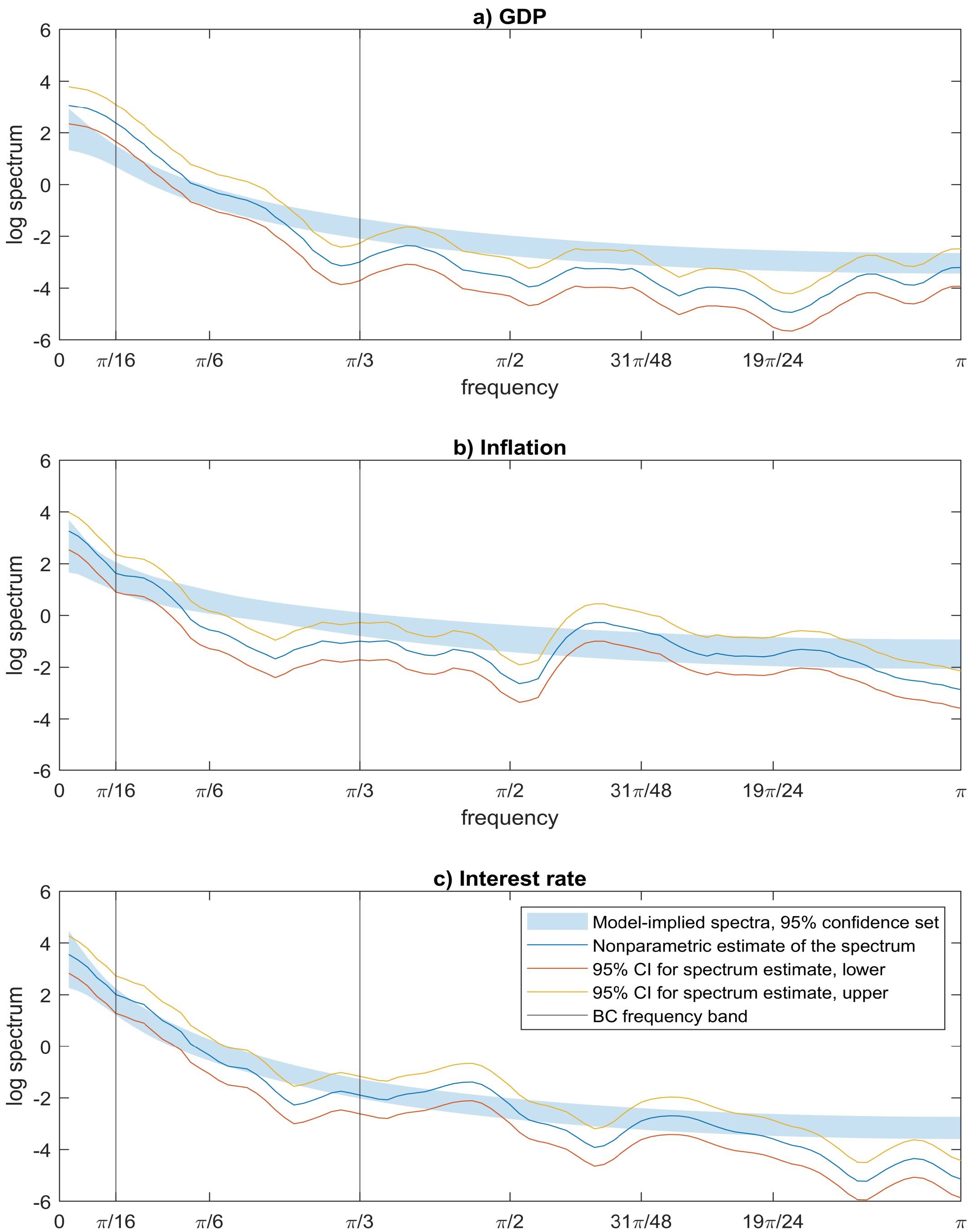


Figure A5: Cross-spectra (real part) using the posterior under determinacy, 1960-2007.

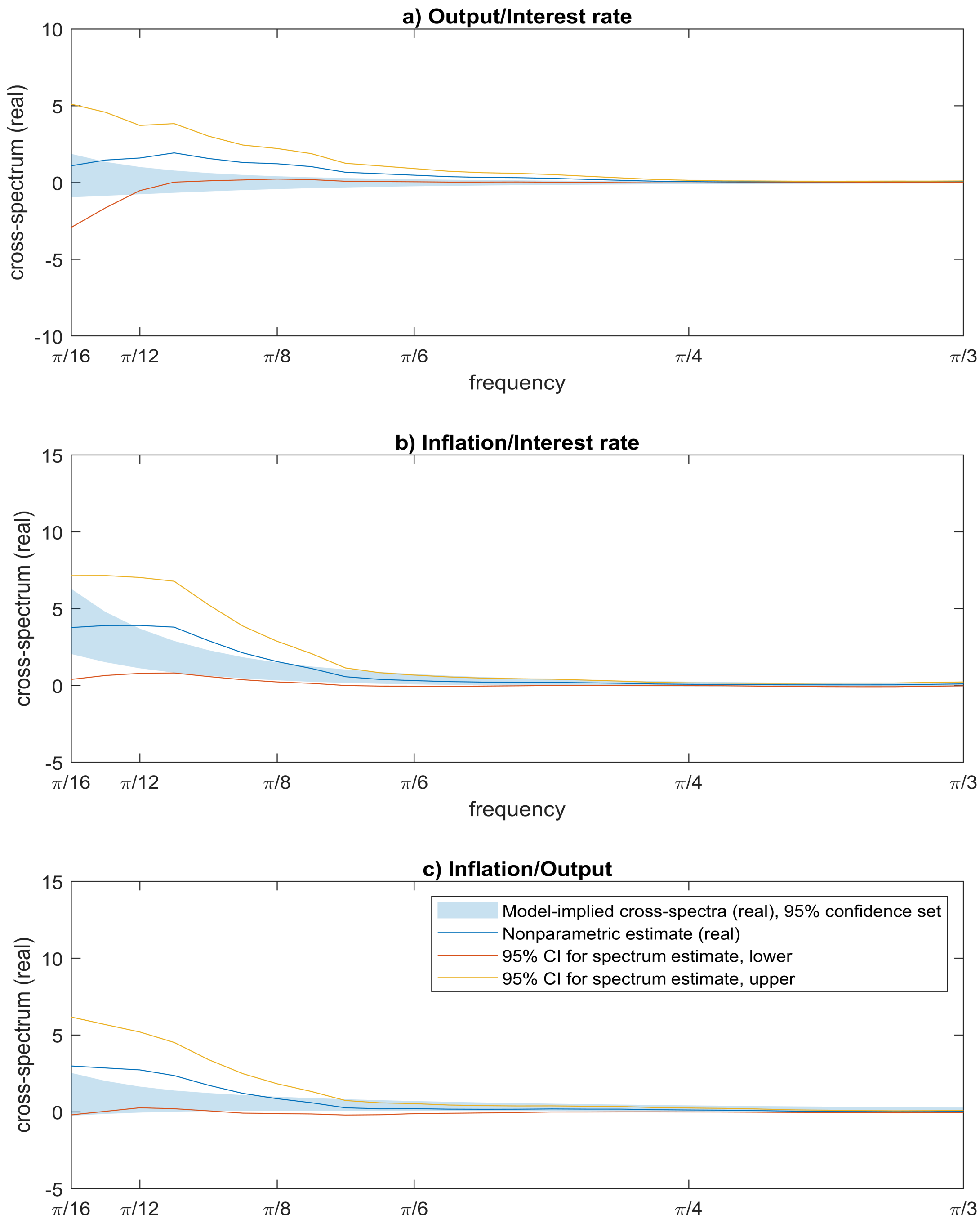


Figure A6: Cross-spectra (imaginary part) using the posterior under determinacy, 1960-2007.

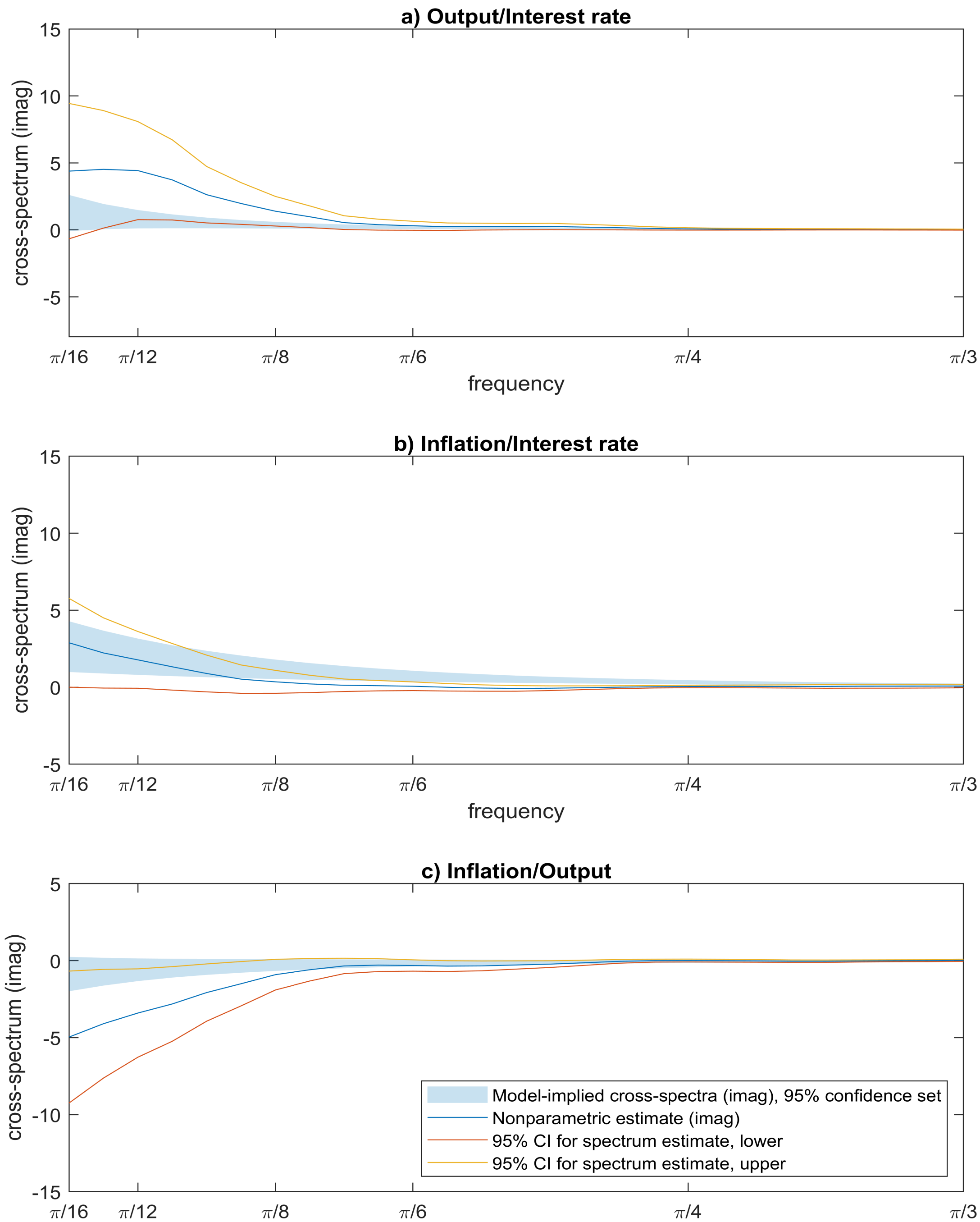


Figure A7: Log spectra using the posterior distribution under indeterminacy, 1960-2007.

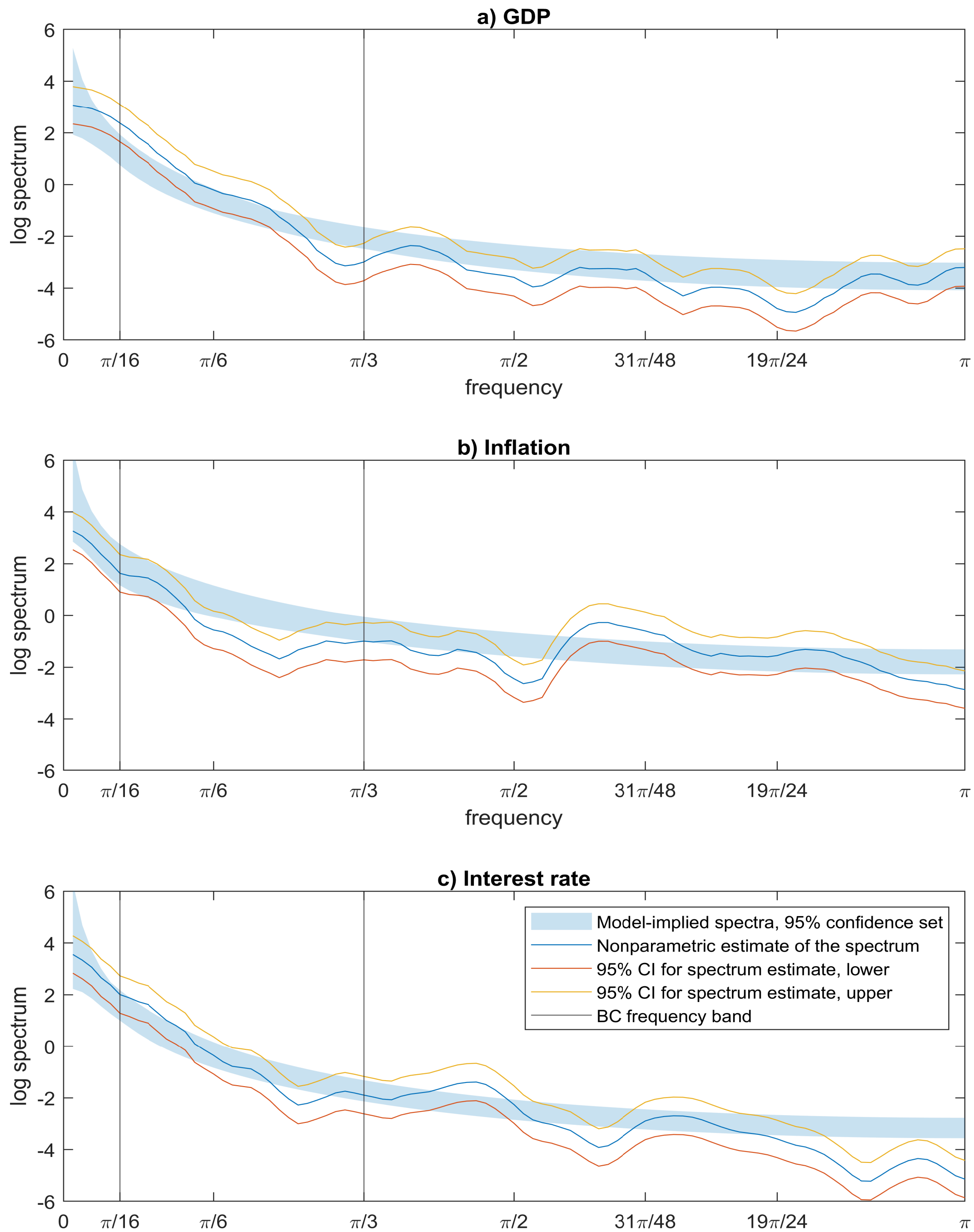


Figure A8: Cross-spectra (real part) using the posterior under indeterminacy, 1960-2007.

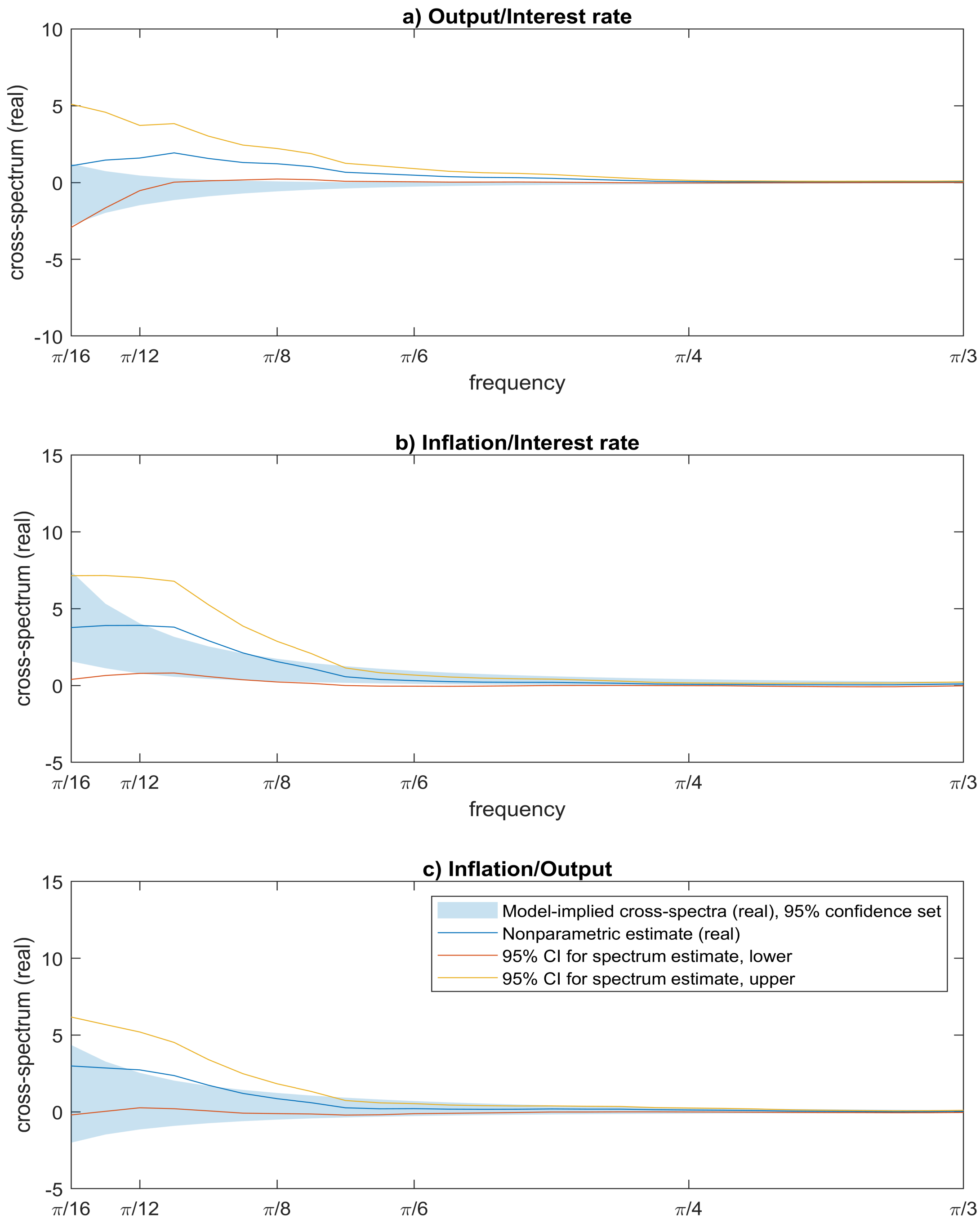


Figure A9: Cross-spectra (imaginary part) using the posterior under indeterminacy, 1960-2007.

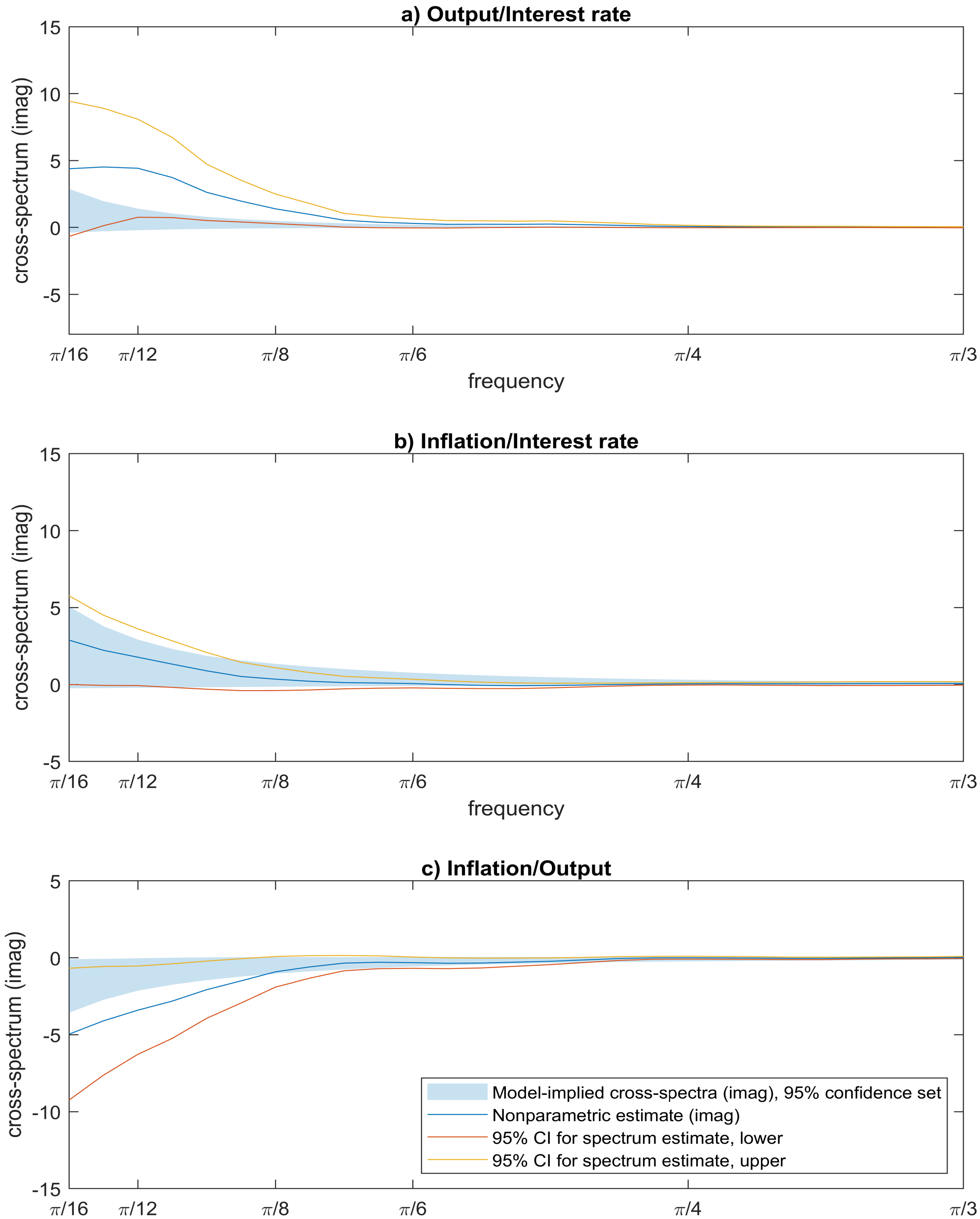


Figure A10: Log spectra using the posterior under determinacy, 1979-2007.

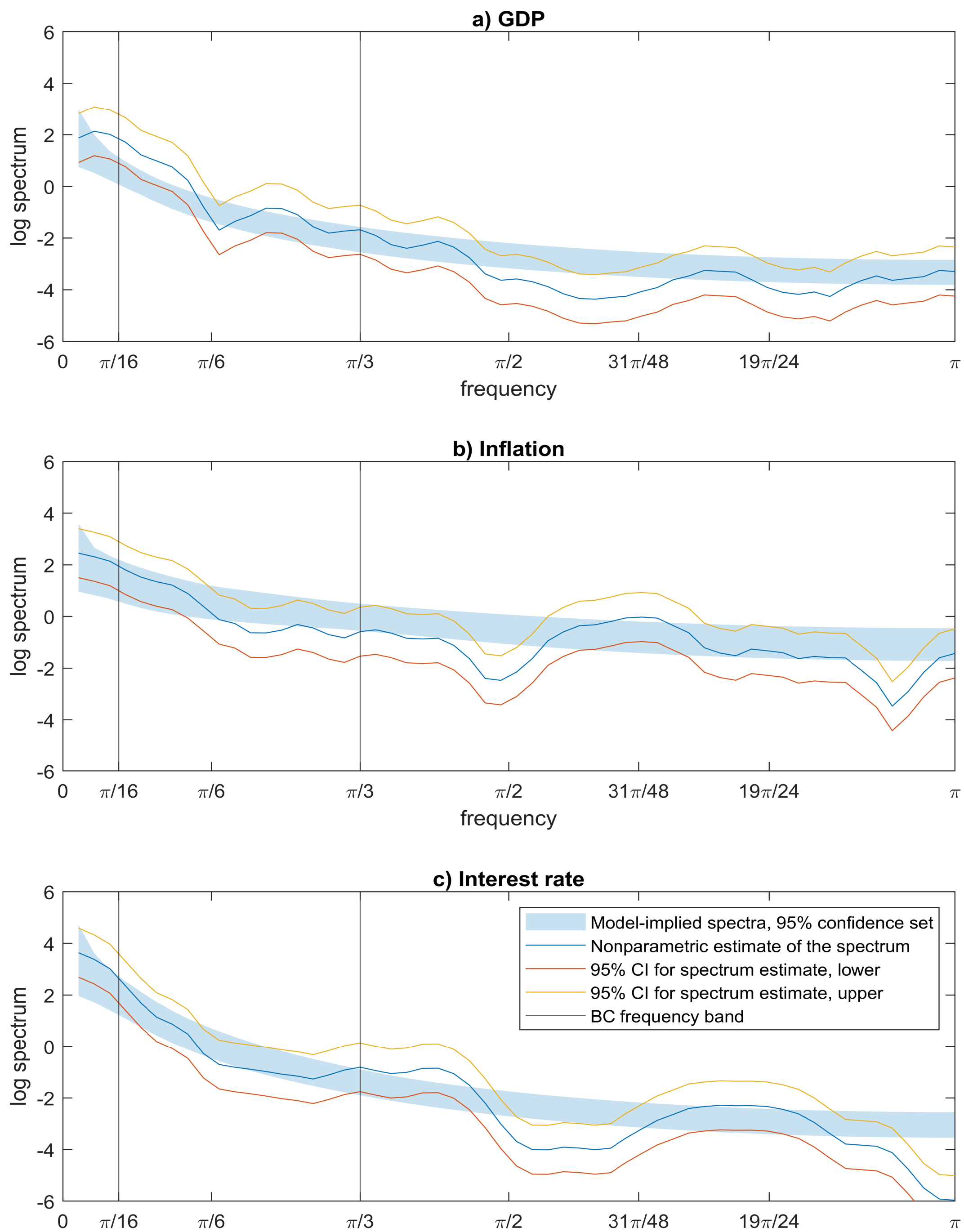


Figure A11: Cross-spectra (real part) using the posterior under determinacy, 1979-2007.

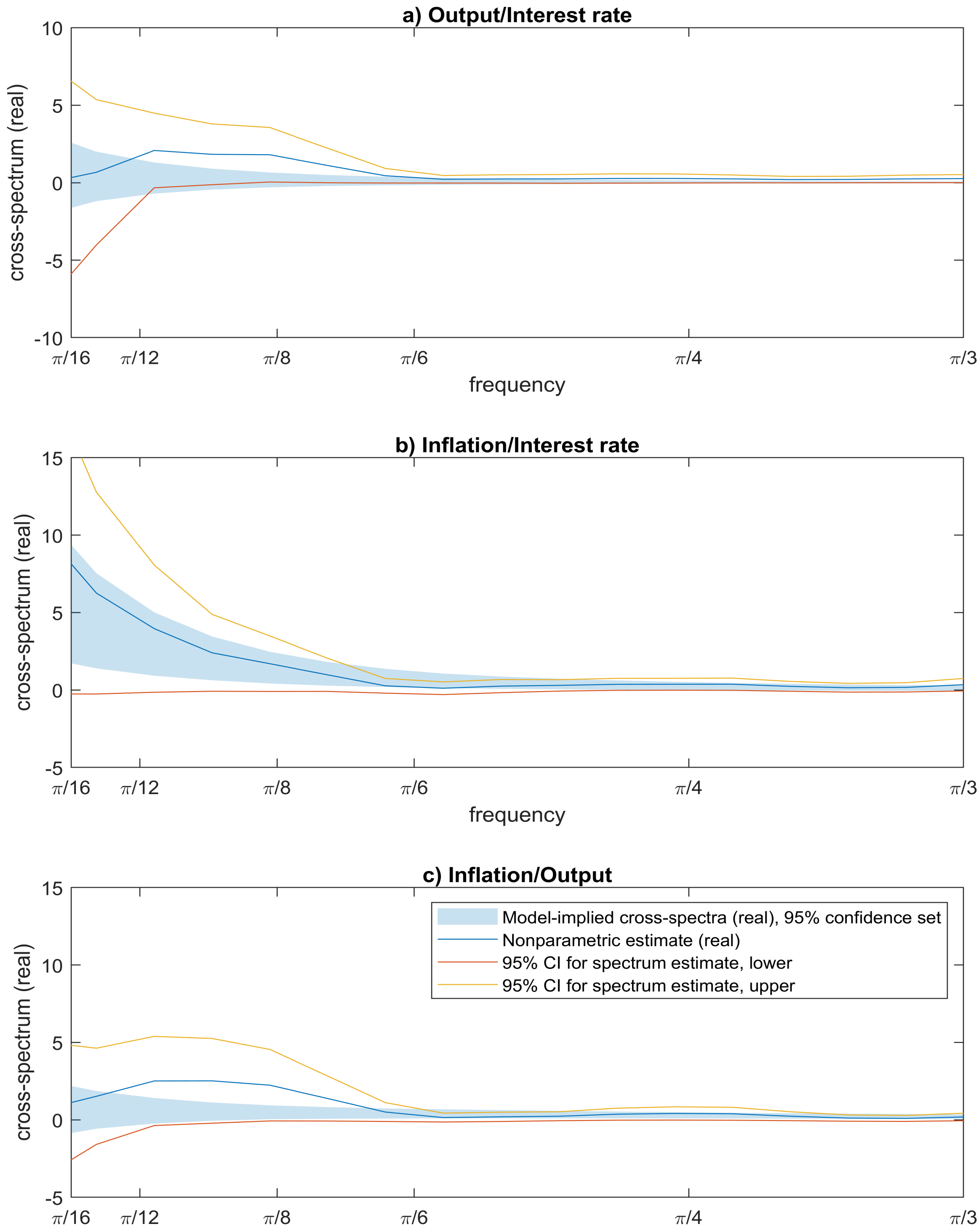


Figure A12: Cross-spectra (imaginary part) using the posterior under determinacy, 1979-2007.

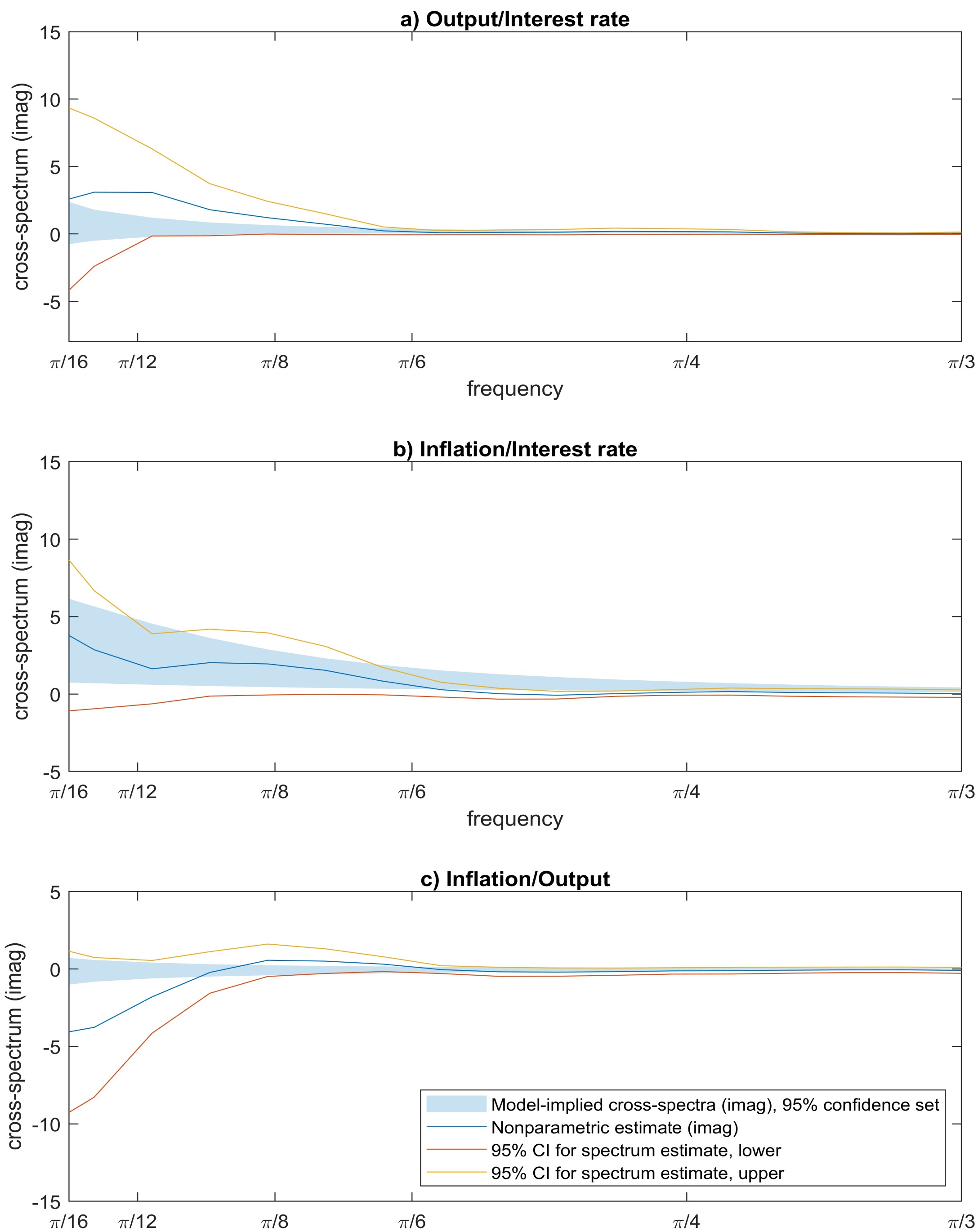


Figure A13: Log spectra under indeterminacy, 1960-1979.

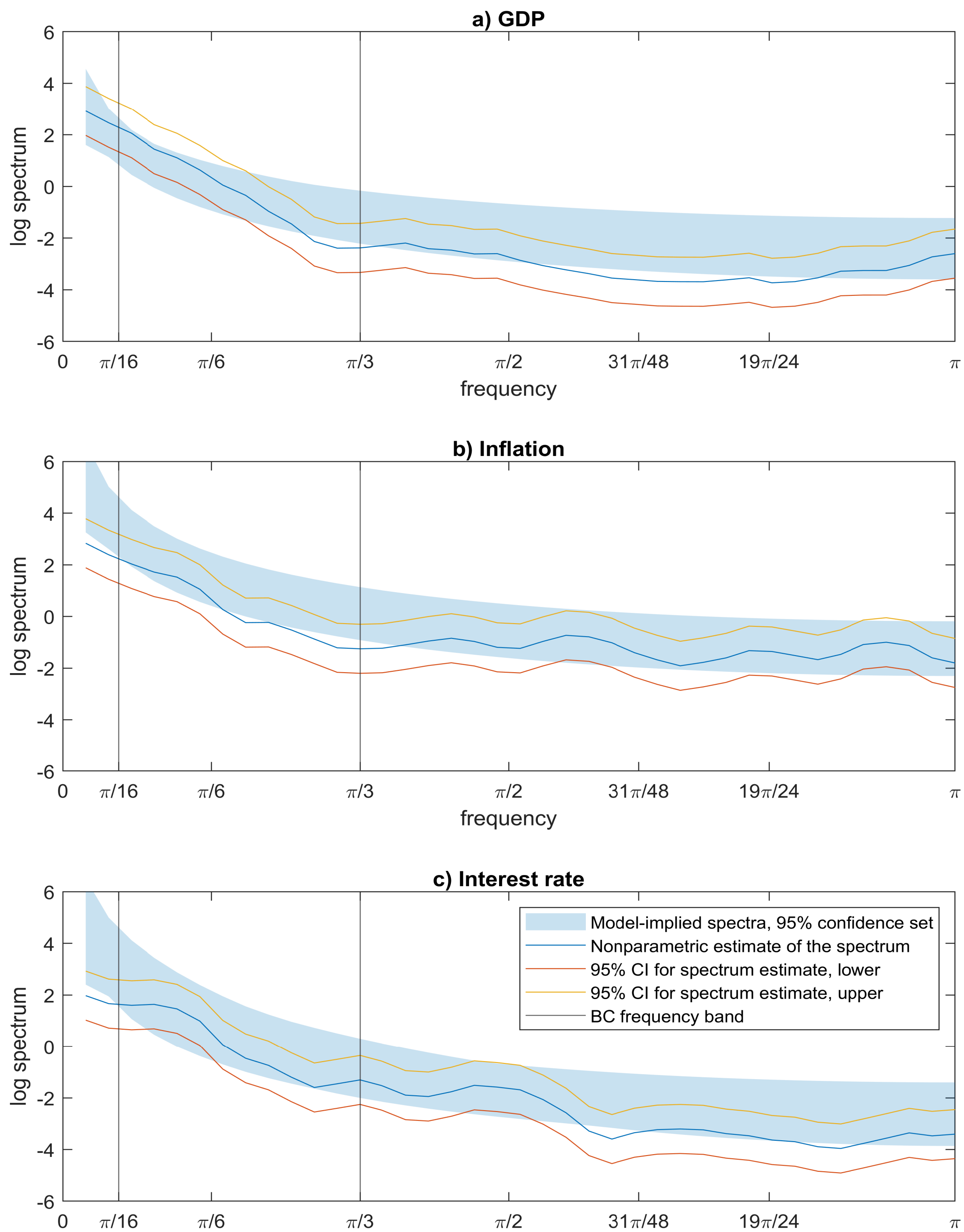


Figure A14: Cross-spectra (real part) under indeterminacy, 1960-1979.

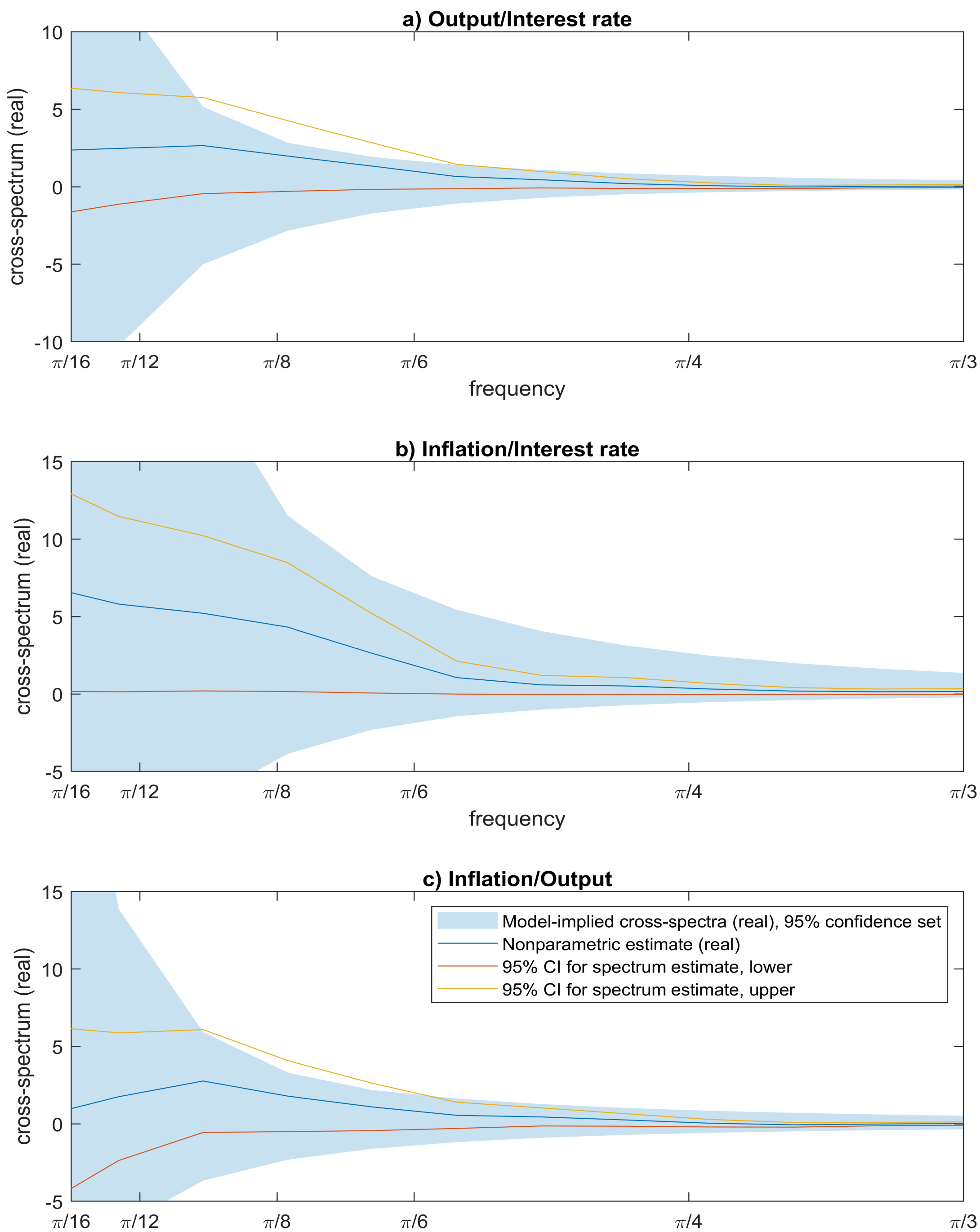


Figure A15: Cross-spectra (imaginary part) under indeterminacy, 1960-1979.

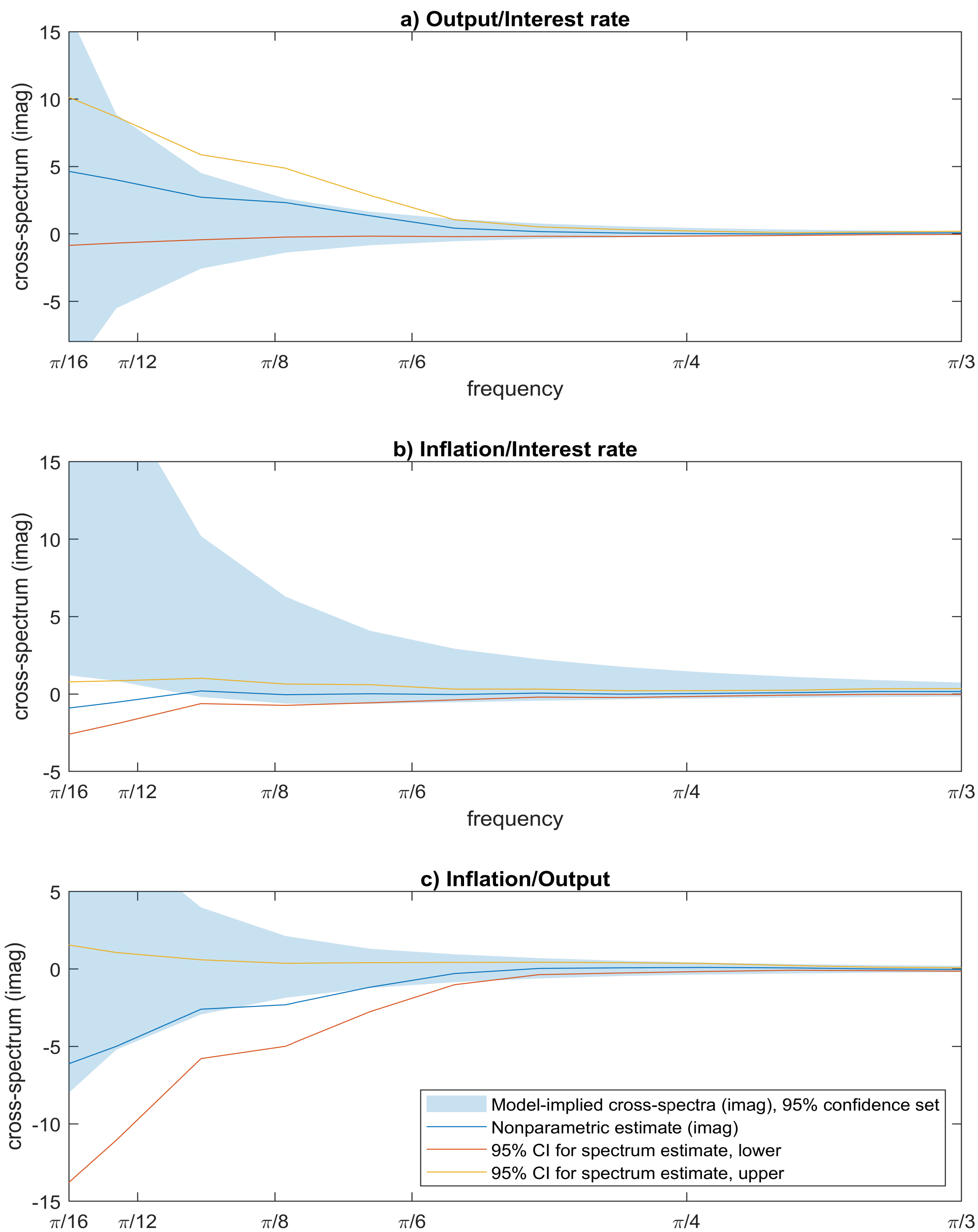


Figure A16: Log spectra using the posterior under indeterminacy, 1960-1979.

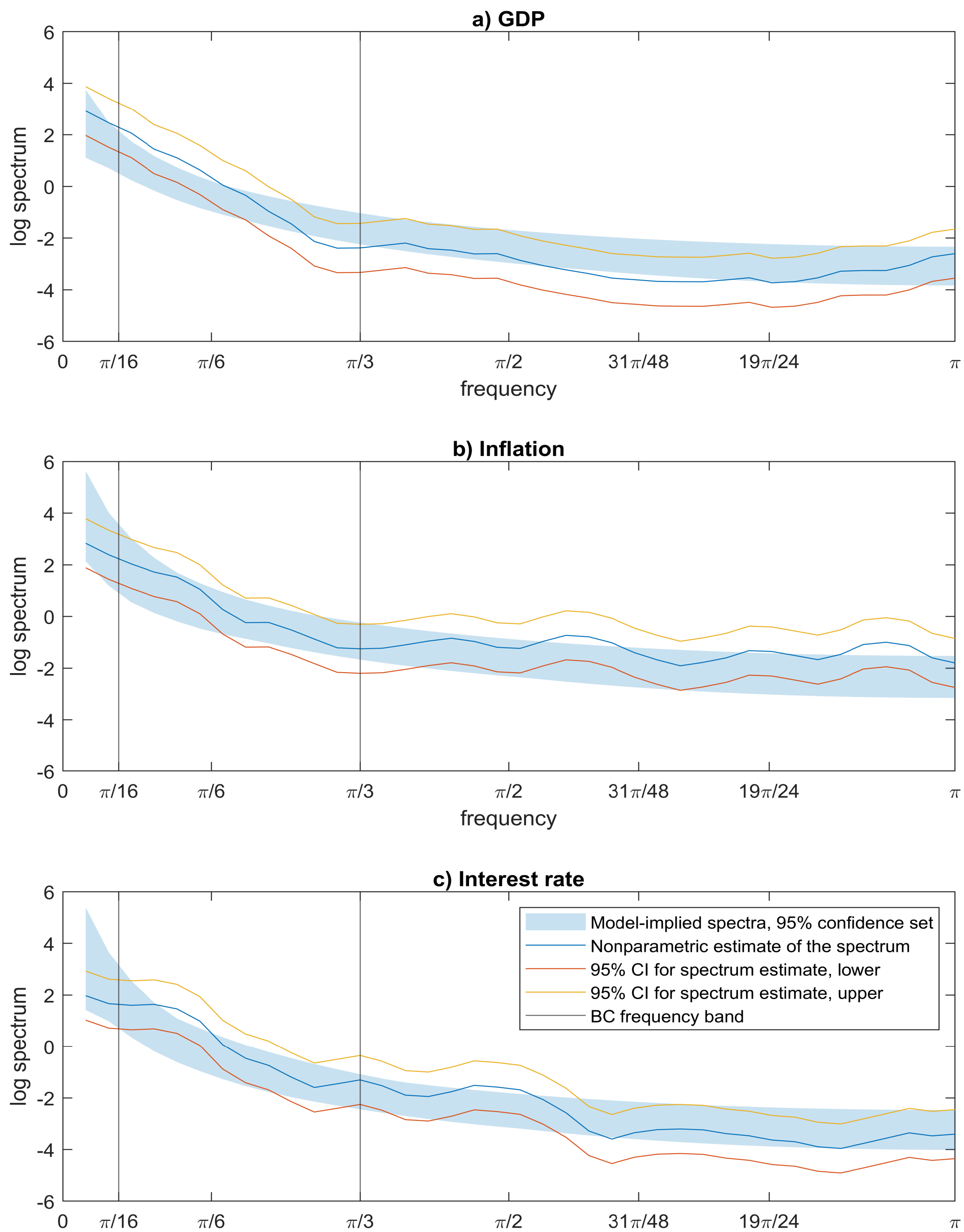


Figure A17: Cross-spectra (real part) using the posterior under indeterminacy, 1960-1979.

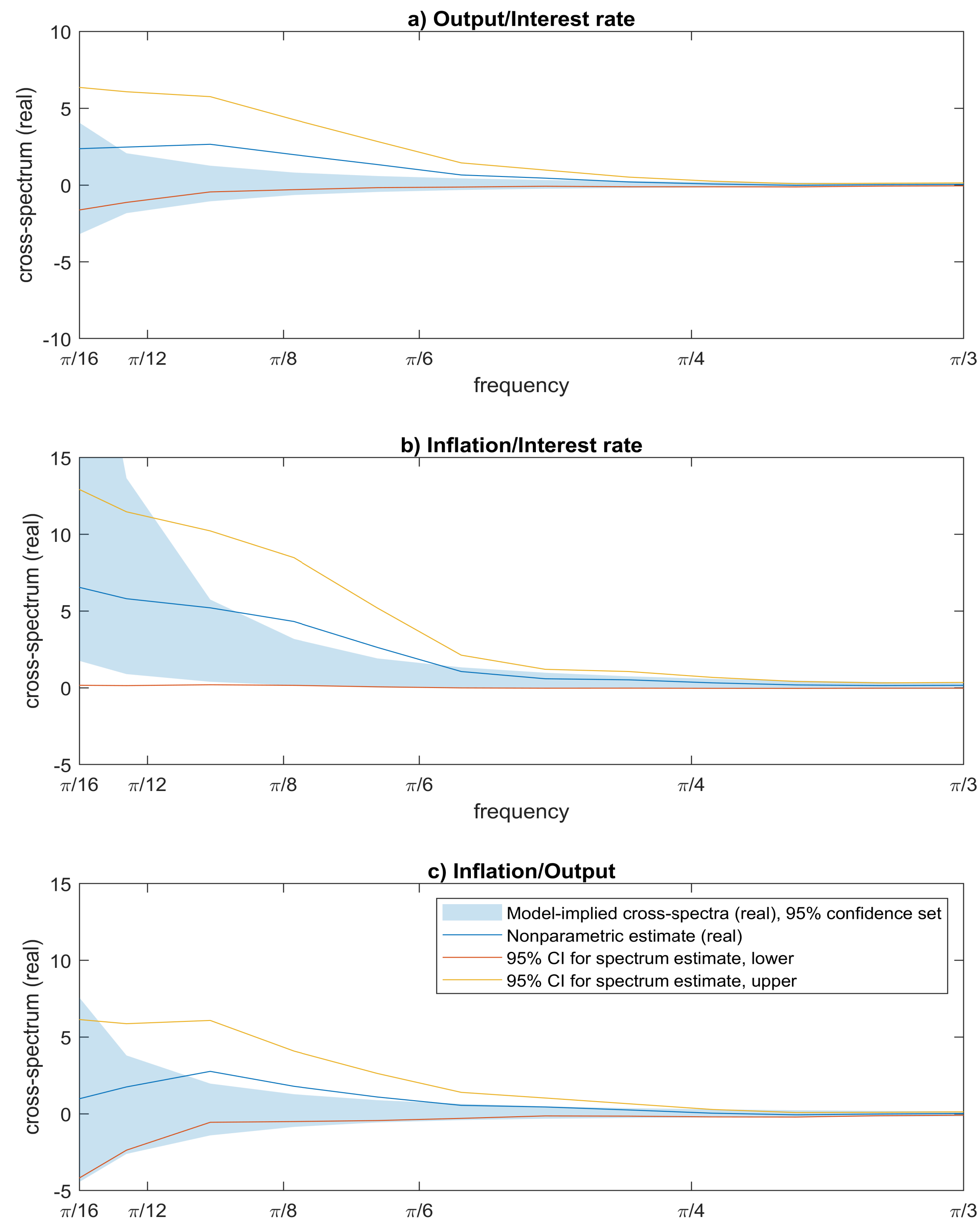


Figure A18: Cross-spectra (imaginary part) using the posterior under indeterminacy, 1960-1979.

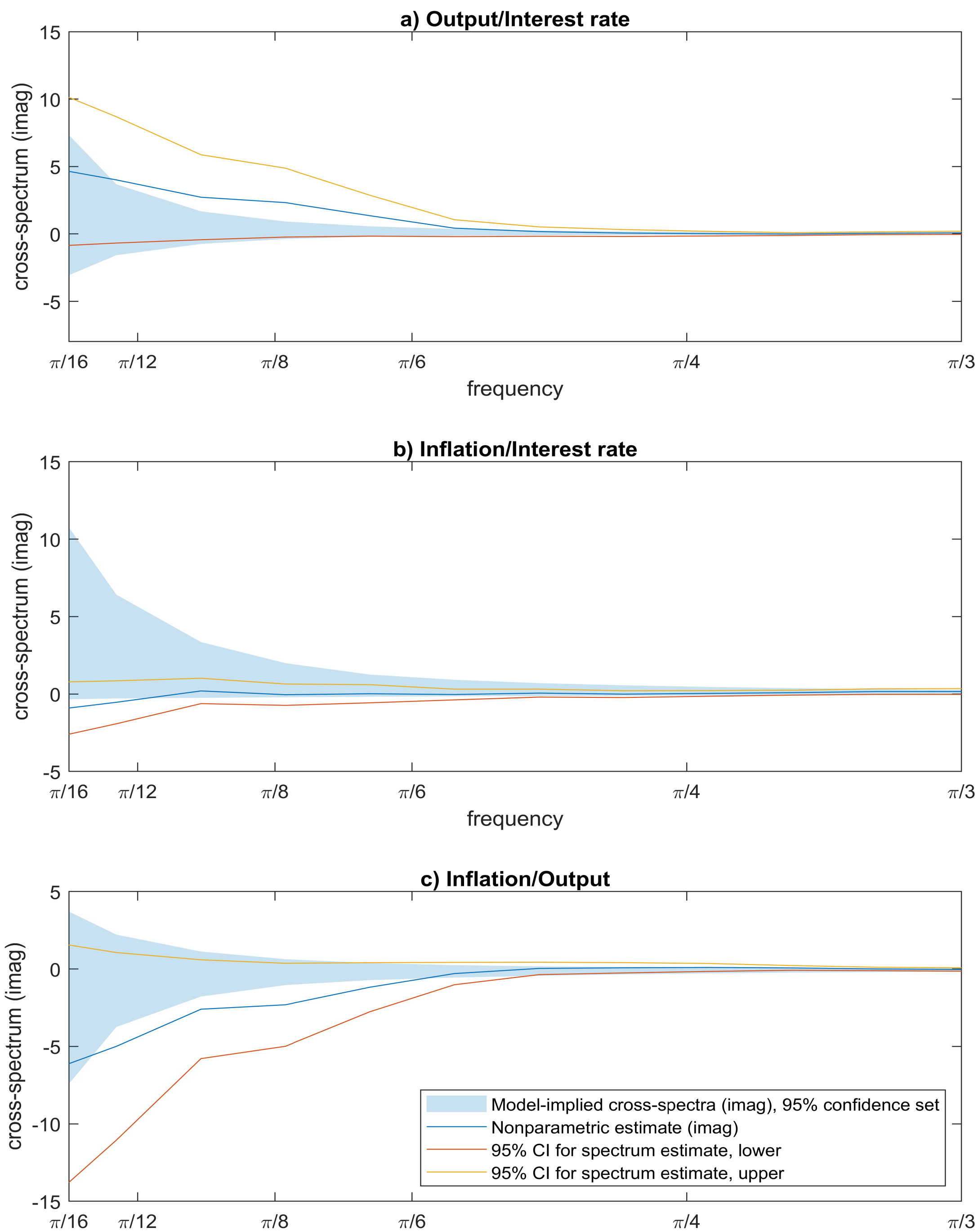


Figure A19: Log spectra of variables for the SW model: the extended sample.

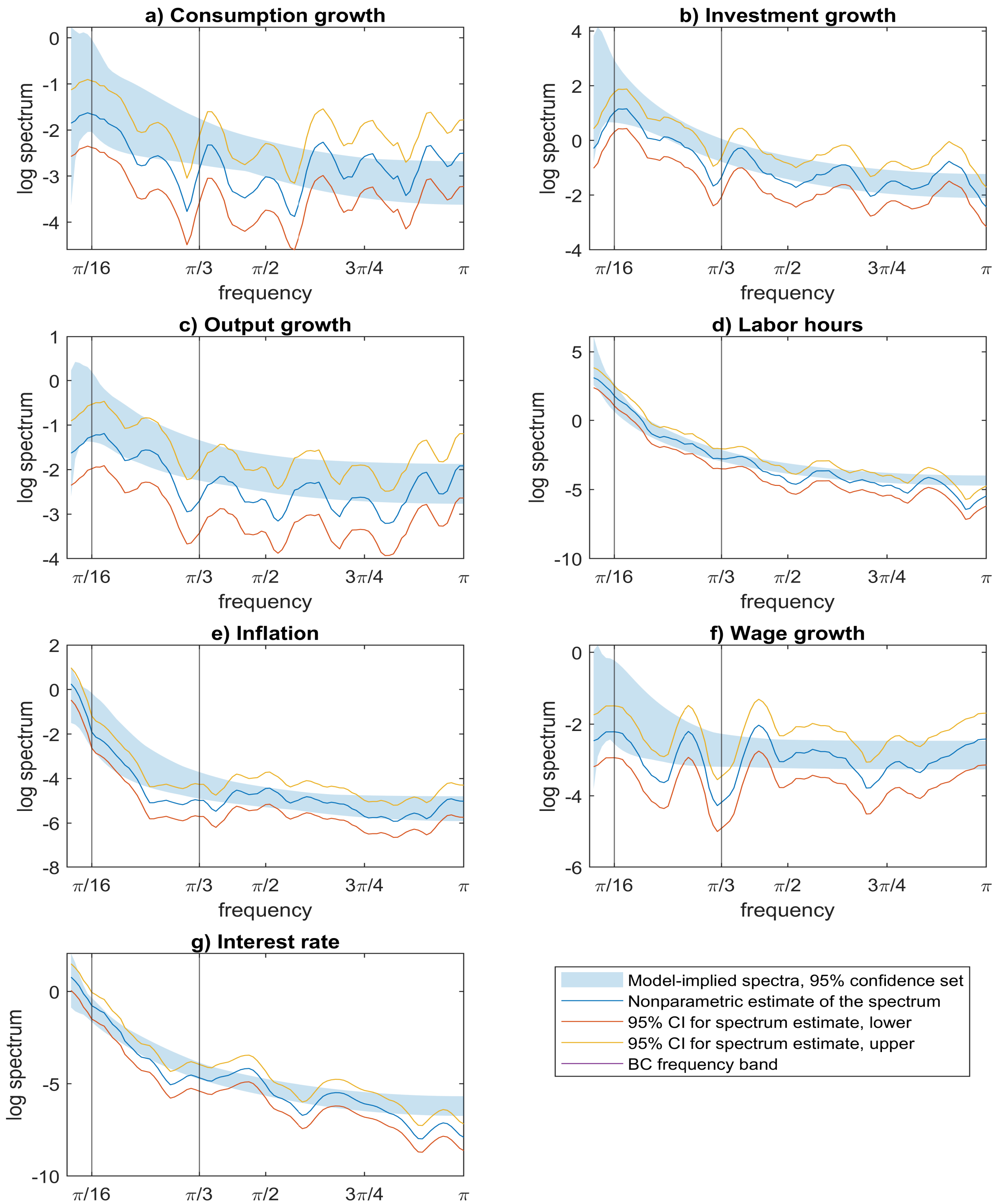


Figure A20: Log spectra of variables for the SW model: the original sample.

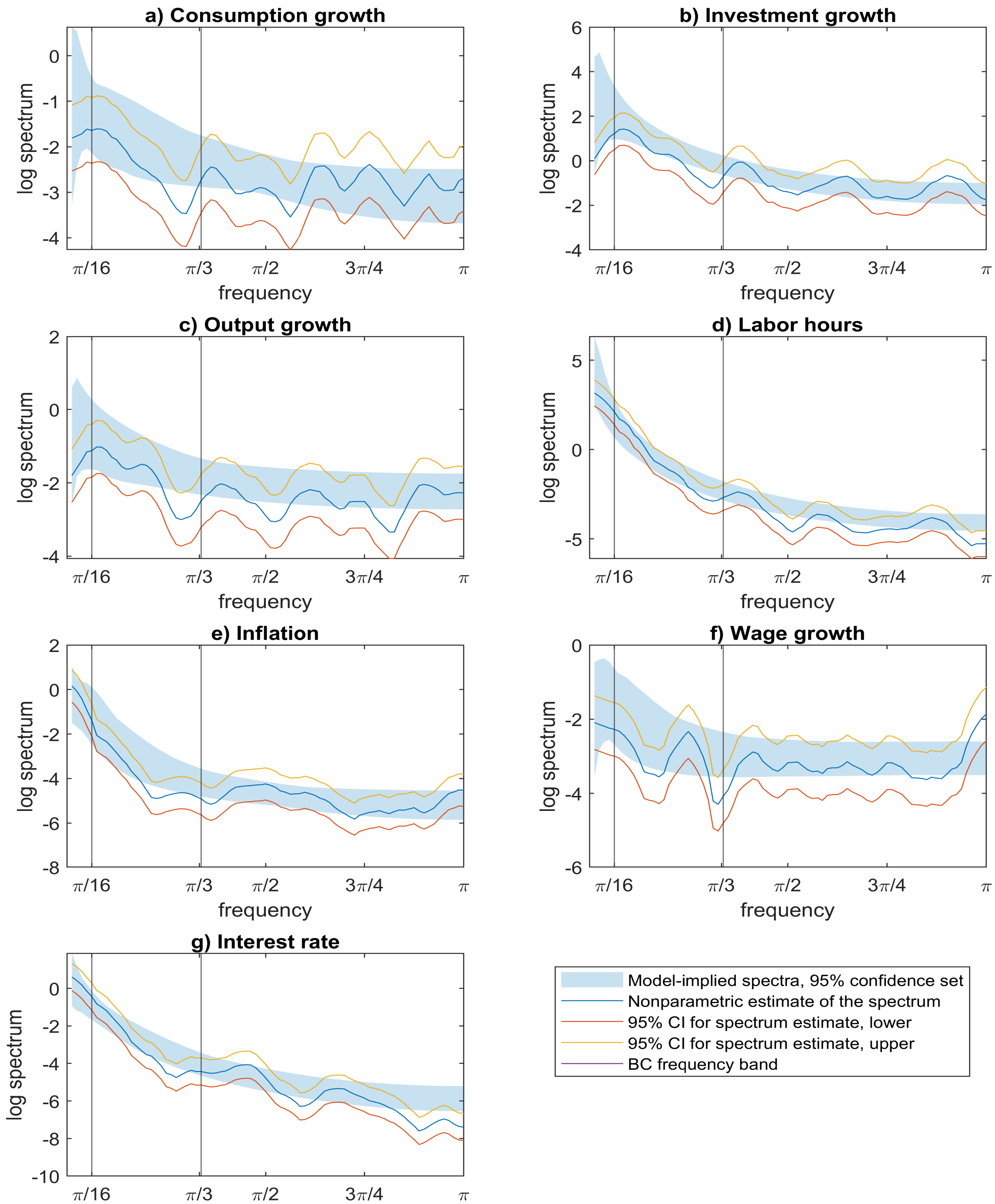
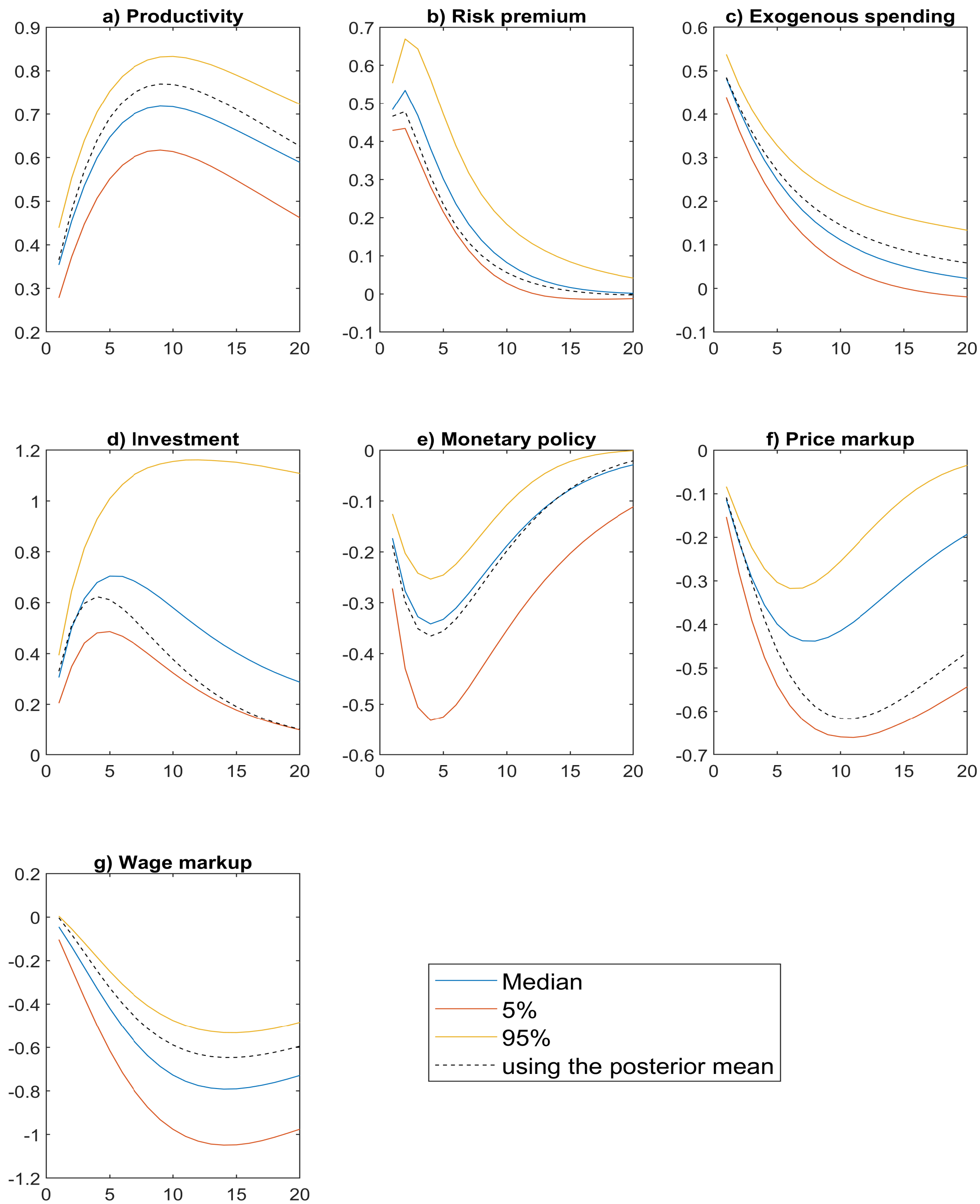


Figure A21: Impulse responses of output to seven shocks: the extended sample.



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