Social Cost Analysis of Shared/Buy-in Computing Systems

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Shared/buy-in computing systems offer users with the option to select between buy-in and shared services. In such systems, idle buy-in resources are made available to other users for sharing. With strategic users, resource purchase and allocation in such systems can be cast as a non-cooperative game, whose corresponding Nash equilibrium does not necessarily result in the optimal social cost. In this study, we first derive the optimal social cost of the game in closed form, by casting it as a convex optimization problem and establishing related properties. Next, we derive a closed-form expression for the social cost at the Nash equilibrium, and show that it can be computed in linear time. We further show that the strategy profiles of users at the optimum and the Nash equilibrium are directly proportional. We measure the inefficiency of the Nash equilibrium through the price of anarchy, and show that it can be quite large in certain cases, e.g., when the operating expense ratio is low or when the distribution of user workloads is relatively homogeneous. To improve the efficiency of the system, we propose and analyze two subsidy policies, which are shown to converge using best-response dynamics.

CCS Concepts: • **Theory of computation** → **Algorithmic game theory**; **Quality of equilibria**.

Additional Key Words and Phrases: Computing clusters, social welfare, price of anarchy, aggregative games, pricing, efficiency

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1 INTRODUCTION

The shared/buy-in paradigm is being widely adopted by high-performance computing (HPC) clusters, especially among large academic institutions. Over 20 universities fully or partially use this paradigm to run their HPC clusters, e.g., Stanford University, the University of Illinois at Urbana-Champaign, the University of California, Berkeley, the University of California, San Diego, Boston University, and Rutgers University [3, 23, 27–30]. HPC clusters must cope with huge demand for computing resources. As a result, these clusters typically include hundreds of computing nodes. For instance, Sherlock at Stanford University maintains 1693 nodes used by 1092 research groups [27], while the Shared Computing Cluster at Boston University (BU SCC) maintains 835 nodes supporting 765 projects across 80 departments [3].

Under the shared/buy-in computing paradigm, users are able to choose between two tiers of services, namely: shared services and buy-in services. Shared services provide users with access to

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the shared resource pool for free, while buy-in services allow users to purchase additional buy-in resources in order to shorten job completion time. Crucially, buy-in resources are managed in a *semi-exclusive* manner, that is, when buy-in resources are idle, they are added to the shared resource pool and made available to all users. This policy is motivated by the observation that many users do not use their own buy-in resources all the time. Thus, by sharing idle resources that are temporarily left unused by their owners, all users can have access to more resources on demand.

A key concern for the provider of a shared/buy-in computing system is the *social cost*, typically captured by the sum of costs of the provider and the users. For example, while universities typically aim to provide as many computing resources as possible to their HPC users, the cost of operating servers must also be accounted for. The social cost is a metric that captures this trade-off. Social cost is also relevant to commercial computing clusters and cloud computing systems, such as Amazon AWS and Google Cloud Platform. Although revenue maximization is typically the first concern in those systems, social cost is also relevant [11, 24]. Indeed, while there exist different ways to increase revenue, some significantly degrade user experience, which in turn may result in users opting for other service providers. In such cases, keeping the social cost sufficiently low can help prevent users from leaving the system.

Shared/buy-in computing systems induce a subtle externality, whereby the utility of a user increases as buy-in resources are purchased by other users, since idle buy-in resources are made available in the shared resource pool. Therefore, as other users purchase more buy-in resources, a user may be less motivated to do so as well. As a result, the choice of each user of how much buy-in resources to purchase (or purchase nothing as a shared user) interacts with the choices of other users. This interaction gives rise to a non-cooperative game, in which the users, who are assumed to be rational and selfish, are the game players. The recent work by Shi et al. [25] formalizes this game, coining it a *shared/buy-in computing game*. This game model captures the strategic behavior of users when they can interact with each other through the sharing of idle buy-in resources. When making decisions on whether and how much to pay for buy-in resources, users need to consider the influence of available shared resources, which depends on the payments of other users. Shared/buy-in computing games exhibit interesting properties, including the existence of a unique Nash equilibrium and the convergence of best-response dynamics. However, the work in [25] does not address the key issue of social cost, except for briefly noting that it is sub-optimal at equilibrium.

Our contributions

This paper focuses on computing the social cost in shared/buy-in computing games. Our formulations consider an arbitrary number of users N with heterogeneous workloads. We aim to answer the following questions: What is the optimal social cost? What is the social cost at the Nash equilibrium, as a result of the non-cooperative manner in which users make their decisions? How far away is the social cost at equilibrium from the social optimum? What can be done to bridge the gap between the two? To answer these questions, we develop methods to efficiently compute and characterize the social cost both at the equilibrium and at the optimum. In order to measure the efficiency of the Nash equilibrium, we investigate the *price of anarchy* (PoA) of the game, namely the ratio of the social cost at the worst equilibrium to the optimal social cost. Our results suggest that, in order to approach the social optimum, users should purchase more buy-in resources than they do at the equilibrium. As ways to incentivize users to purchase more buy-in resources, we propose two subsidy policies and analyze their properties, including proving their convergence through best-response dynamics.

The model considered in this paper belongs to the class of *aggregative games* with strategic substitutes [6, 9], where the aggregative term is a linear sum of the strategies of all players (cf.

detailed discussion at the end of Section 2). In particular, since the computing resources in a shared/buy-in game are a public good, and buy-in users are contributing to this public good, our model can be viewed as a type of aggregative public good games [7]. We expect our results to be useful in other applications that involve a similar strategic interaction between the users.

In summary, our work makes the following contributions:

(1) We prove that the computation of the optimal social cost in shared/buy-in computing games can be cast as a constrained convex optimization problem. Furthermore, we derive a closed-form expression for the social cost at the optimum, which can be computed in linear time. This result is validated through a numerical comparison with coordinate descent methods.
(2) We derive a closed-form expression for the social cost at the Nash equilibrium, which can be computed in linear time. This complexity is much lower than first solving for the equilibrium and then computing the social cost, which is $O(N^4)$. To the best of our knowledge, this method is novel and potentially useful for the analysis of other game settings.
(3) We establish that a buy-in user at the Nash equilibrium will also be a buy-in user at the social optimum. Moreover, we show that the strategy profile vector (i.e., payment) of users at the optimum is directly proportional to that at the equilibrium.
(4) We derive the price of anarchy of the game in closed form.
    - For the special case where all the users are buy-in users, we further establish that the price of anarchy grows as $\Omega(\sqrt{N})$ in the worst-case.
    - Through numerical simulations, we show that for a fixed average workload, the price of anarchy decreases with the variance of the workload distribution (i.e., as the workload becomes more heterogeneous). Moreover, the price of anarchy can be quite large in some cases.
(5) We propose and analyze the impact of two subsidy policies. We prove that, under both policies, the game still has an equilibrium and converges through best-response dynamics (possibly under sufficient conditions). We also establish the relationship of the payments by users at the Nash equilibrium of the game under both subsidy policies. Moreover, numerical simulations show that both policies can significantly improve the social cost at the equilibrium.

The rest of the paper is organized as follows. We discuss related work in Section 2. In Section 3, we model shared/buy-in computing systems from a game-theoretic perspective, and introduce our main metrics, which are the social cost and the price of anarchy. We analyze the social cost at the optimum and at the equilibrium in Section 4, along with computing the price of anarchy. In Section 5, we propose two subsidy policies and characterize their properties. We conduct numerical simulations in Section 6 to validate and expand on our analytical results. The paper concludes in Section 7.

2 RELATED WORK

In a system where users interact with each other and try to maximize their own utilities, it is difficult to predict how the system will operate. In order to analyze such systems, game-theoretic models are often used since they are able to capture strategic interactions between the users [20]. For example, they are used to analyze network security [18], resource allocation [31], and advance reservation [26]. The problems from cloud economics also benefit from game-theoretic approaches. Abhishek et al. [1] investigated and compared two pricing schemes for cloud services, namely fixed and market-based pricing, with a game-theoretic model. Anselmi et al. [2] proposed a three-tier model for a cloud computing marketplace, based on which the market equilibria are characterized, and the impact of price competition is evaluated. Game-theoretic models answer the question of where the system operates by providing the Nash equilibrium (or equilibria) of the corresponding game.

After finding the Nash equilibrium (or equilibria) of the game, the next question is how well the system performs under the Nash equilibrium against the optimal case (i.e., optimal social cost). It is typical that the Nash equilibrium does not coincide with the optimum, thus the notion of price of anarchy has been proposed to evaluate the inefficiency of the Nash equilibrium (or Nash equilibria) due to the selfish behavior of the users [15]. Analysis of the price of anarchy can provide useful insights into the system [22]. Wu and Starobinski [32] analyzed the problem of server selection in content replication networks, showing that the price of anarchy increases as the server capacity becomes more heterogeneous, because selfish users avoid using slow servers. Similarly, the analysis by Chamberlain and Starobinski [5] suggests that, the price of anarchy in preemptive priority queues becomes larger as the service distribution gets more heterogeneous (i.e., the variance of the service distribution becomes larger).

However, it is not always true that heterogeneity results in a larger price of anarchy and makes the system perform worse. Korilis et al. [14] considered a routing game, where a manager attempts to steer the network into its social optimum by controlling part of the flow, while the rest of the flow is controlled by several selfish users. The results show that the "homogeneous" case, that is, the case of equal split of the total demand among all users, is the hardest case to reach the optimum and thus the least desired by the manager. In our work, we also find that if the average workload of users in the shared/buy-in computing system is fixed, the price of anarchy increases as the workload distribution becomes relatively more homogeneous.

An approach to reduce the degradation of system performance caused by selfish user behavior is to introduce subsidies. Buchbinder et al. [4] proposed a dynamic subsidy mechanism financed by taxes collected from the users in order to improve the performance of cost-sharing systems. Such subsidies help to keep the price of anarchy low, by collecting a small amount of taxes compared to the user payments. The work by Fang et al. [10] indicates that revenue of sharing economy platforms may be limited in practice, and subsidies can help encourage sharing and bring more revenue. In this paper, we also introduce subsidy policies to lower the price of anarchy and improve system performance (in our case, the social cost).

As mentioned in the introduction, the shared/buy-in computing paradigm is now common in many HPC clusters. The paradigm shares similarities at some level with proportional allocation mechanisms [12, 13, 16], where the bandwidth allocated to each user is proportional to its payment. The difference is that, in shared/buy-in computing systems, the total amount of resources is not fixed; instead, users contribute to the system by purchasing buy-in resources. As a result, it lacks the zero-sum nature of typical proportional allocation models. Liao et al. [17] conducted a statistical case study of the SCC cluster at Boston University. This work shows that buy-in resources are not fully utilized by their owners, and sharing idle buy-in resources indeed improves resource utilization. Such systems are formally modeled from a game-theoretic perspective in [25]. The analysis of the game shows that there exists a unique Nash equilibrium. Moreover, it is shown that, from any arbitrary initial state, the system always converges to the Nash equilibrium through best-response dynamics, possibly with users making decisions in a distributed manner (i.e., not having complete information on the decisions of other users). In the present study, we quantitatively investigate the inefficiency of the Nash equilibrium with respect to the social optimum. We explore what contributes to the inefficiency, by computing the exact social costs at the equilibrium and at the optimum.

Shared/buy-in computing games belong to the class of *aggregative games*, whereby the cost of each player depends on the aggregate strategies of all the other players, instead of the individual strategy of another player [6]. Specifically, the game is in the form of an aggregative public good game, where the computing resources in the shared pool are the public good, and buy-in users contribute to the public good with their own buy-in resources. Cornes and Hartley [7] analyzed aggregative public good games under a voluntary contribution model, where players can decide how much to contribute to the public good under their own budget constraints. In contrast, in shared/buy-in computing games, contributions to the public good are not entirely voluntary, since the idle buy-in resources are automatically made available to all users by the system. Our game also has the property of strategic substitutes, that is, when a player increases its contribution, other players tend to do the opposite [9, 25]. It is known that any aggregative game with strategic substitutes converges to one of its Nash equilibria through best-response dynamics. Furthermore, a Nash equilibrium is guaranteed to exist [9]. In this paper, we prove that after applying the subsidy policies, the resulting shared/buy-in computing games are aggregative games with strategic substitutes, which allows us to establish their convergence properties.

3 SYSTEM MODEL

In this section, we first formalize a game-theoretic model of a shared/buy-in computing system, and then introduce the notions of social cost and price of anarchy. We consider a shared/buy-in computing (SBC) game of the form of $(S, \{P_i\}_{i \in S,}, \{C_i\}_{i \in S})$, where S is the finite set of players, P_i is the non-empty strategy set of player i, and $C_i : P \longrightarrow \mathbb{R}$ is the cost of player i given a strategy profile of all players from the joint set $P = \prod_{i \in S} P_i$.

The SBC game has N players, who are the users of the system and belong to the set $S = \{i \mid 1 \leq i \leq N\}$. Since shared/buy-in computing systems are typically of large-scale and sustain many users, we assume that the number of users N is large (e.g., $N \geq 100$). Each player i has an average workload ω_i that has to be completed, and needs to decide upon a strategy $p_i \in P_i$, which is the payment for purchasing buy-in resources. The average workload can be estimated from the aggregate workload observed over a long time period. The strategy of player i has two levels, as follows: (a) the player will pay $p_i > 0$ if it decides to purchase buy-in resources or $p_i = 0$ otherwise, and (b) the value of p_i reflects the amount of purchased resources. Denote the payments made by all players by the vector $\mathbf{p} = [p_1, p_2, \ldots, p_N]^\intercal$. A player can utilize the idle buy-in resources of other players, thus each player is impacted by the strategies chosen by the other players, hence the cost of player i depends on $\mathbf{p}$.

In SBC games, players can only pay non-negative prices for buy-in resources, hence we have $p_i \geq 0, \forall i \in S$. Define the players that pay positive prices as *buy-in users*, and the players that pay nothing as *shared users*. Assume that there are n_1 buy-in users in the set $S_1 = \{i \mid p_i > 0,\ i \in S\}$, and n_2 shared users in the set $S_2 = \{i \mid p_i = 0,\ i \in S\}$, we have $n_1 + n_2 = N$. In the following, unless stated otherwise, we assume that all the subscripts used for distinguishing among players $\{i, j, \ell, m\}$ belong to the set S.

In our model, the amount of resources available to a player is measured by its computing rate, and the player's job completion time can be then computed as its workload divided by its computing rate. The computing rate of player i comes from two sources, namely: its own buy-in resources and other players' idle buy-in resources. We assume that, when a player i pays a price p_i for buy-in resources, it gets a computing rate of $k_b p_i$; alongside, it provides each of the other players, including both buy-in and shared users, with a computing rate $k_s p_i$. Thus, the overall computing rate of a player i is $k_b p_i + k_s \sum_{j \neq i} p_j$, where the first term is due to its own buy-in nodes, and the second term is from all the other buy-in nodes. In the following, we refer to k_b and k_s as the *buy-in factor* and the *shared factor* of the shared/buy-in computing system, respectively.

Intuitively, the buy-in factor k_b reflects the amount of buy-in resources that a player can get per unit of payment, a larger k_b implies that a player gets a higher computing rate from its own buy-in nodes. The shared factor k_s reflects the effect of sharing idle buy-in nodes. A larger k_s implies that a player gets a higher computing rate from other players' idle buy-in nodes.

We also make the reasonable assumption that $k_b > k_s$, that is, by paying a price p_i, player i gets a computing rate $k_b p_i$ that is larger than the computing rate $k_s p_i$ provided to another player, since player i has priority access to its own buy-in nodes. Intuitively, this motivates users to purchase their own buy-in nodes rather than just waiting for shared resources. It is also a necessary condition for the underlying game to admit a unique Nash equilibrium.

We distinguish between two types of jobs: buy-in jobs are those running on a user's own buy-in nodes, while public jobs are those running on other users' idle buy-in nodes. Then, the total computing rate for buy-in jobs is $k_b \sum_{i \in S} p_i$, whereas the total computing rate for public jobs is $(N-1)k_s \sum_{i \in S} p_i$. We note that the ratio between the two types of total computing rate can be expected to be roughly the same as that between the corresponding workloads. Indeed, the BU SCC data in [17] shows that, during 2015-2016, the buy-in workload was 1.42×10^7 CPU-hours, and the public workload was 7.51×10^6 CPU-hours. The ratio between the two types of total computing rate was $k_b \sum_{i \in S} p_i : (N-1)k_s \sum_{i \in S} p_i \approx 14.2 : 7.5$, which also justifies our assumption that $k_b > k_s$.

The two factors k_b and k_s serve as system parameters that can be adjusted by the provider. Both k_b and k_s are affected by the pricing of buy-in resources. A higher pricing of buy-in resources implies lower computing rate per unit of payment, thus lower k_b and k_s. Moreover, the provider can also adjust k_s by limiting the portion of idle buy-in resources that are available in the shared resource pool. When adjusting the system parameters k_b and k_s, the provider faces some constraints, which are discussed in more detail in Appendix A.

Remark 1. *A user is hardly affected by other users when using its own buy-in nodes, since it has priority access. Hence, one can assume that the buy-in factor k_b is the same for all users. On the other hand, the shared factors could be heterogeneous among the users (i.e., k_{si} for user i). Indeed, a user i might use its buy-in nodes for a longer time than others, resulting in less resources available for others and a lower k_{si}. Nonetheless, using the homogeneous shared factor k_s for all users provides a good approximation of the heterogeneous case, as shown by our numerical simulations in Section 6.3.*

We care about the *social cost* of the system, which is defined as the sum of costs of both the users (players) and the provider. The cost C_i of a player i is due to two components, namely time and money. The time cost is calculated by the job completion time T_i multiplied by cost per unit of time α_i. The monetary cost is simply the payment p_i. Thus, the cost of player i is

$$C_i(\mathbf{p}) = \alpha_i T_i(\mathbf{p}) + p_i.$$

The job completion time T_i of player i can be computed as its workload divided by its overall computing rate. As shown before, the overall computing rate of player i is $k_b p_i + k_s \sum_{j \neq i} p_j$, so its job completion time is

$$T_i(\mathbf{p}) = \frac{\omega_i}{k_b p_i + k_s \sum_{j \neq i} p_j}.$$

Remark 2. *Intuitively, α_i reflects the value of time for each player i. Indeed, some users might be more sensitive to the job completion time than other users. This way, our model can capture user heterogeneity, if needed.*

Player i aims to minimize its cost $C_i(\mathbf{p})$, that is, $p_i = \operatorname{argmin}_{p_i} C_i(\mathbf{p})$. Note that given the cost function $C_i(\mathbf{p})$, the optimal price paid by player i is implicitly upper-bounded by $\sqrt{\alpha_i \omega_i / k_b}$ [25]. Meanwhile, shared/buy-in computing systems are typically of large-scale. Thus, we assume that the system can always provide enough computing resources to each user.

The provider experiences some operational cost to run the system. In the meantime, it gets revenue from providing services to the users, that is, the sum of payments from users. We consider the net cost of the provider as the operational cost Θ minus the sum of payments from users.

Assume that the operational cost is positively correlated with the total amount of computing resources purchased with the payments (with more payment received, more servers need to run, which raises the operational cost). Thus, Θ is an increasing function of the users' strategy vector $\mathbf{p}$. As a result, the cost of the provider is:

$$C_{\text{provider}}(\mathbf{p}) = \Theta(\mathbf{p}) - \sum_{i \in S} p_i.$$

In our model, we assume that the operational cost is of a certain form. The model in [33] suggests that the operational cost is mainly due to power consumption, and in some simple implementations (e.g., without an optimization technique such as Dynamic Voltage Frequency Scaling, DVFS), the power consumption can be in a linear relationship with used resources. In typical shared/buy-in computing systems (e.g., Stanford Sherlock and BU SCC), when a user purchases buy-in nodes, the provider will use its payment to acquire additional nodes from external sources and manage the nodes for the user. In other words, the buy-in users "invest" in the system by adding more servers. As a result, we can use the overall payment by the users to estimate the total computing resources in the system. Thus, we assume $\Theta(\mathbf{p}) = \theta \sum_{i \in S} p_i$, where $0 < \theta < 1$. Note that $\theta = \Theta(\mathbf{p}) / \sum_{i \in S} p_i$, hence it can also be interpreted as the *operating expense ratio* of the system.

Remark 3. *We assume that the provider gets income only from the buy-in users, such that the operating expense ratio needs to satisfy $\theta < 1$ in order to keep the system running, otherwise the provider would have a negative net income (revenue). If the provider gets an additional external income (e.g., subsidy from the university) in the form of $v \sum_{i \in S} p_i$, where $0 < v < \theta$, we can define an equivalent operating expense ratio $\theta^* = \theta - v$. Such equivalent operating expense ratio satisfies $0 < \theta^* < 1$. Thus, instead of θ, one could use the equivalent operating expense ratio θ^* in the following analysis and the results would still hold.*

Next, we compute the social cost $C(\mathbf{p})$ as follows:

$$\begin{aligned} C(\mathbf{p}) &= \sum_{i \in S} C_i(\mathbf{p}) + C_{\text{provider}}(\mathbf{p}) \\ &= \sum_{i \in S} (\alpha_i T_i(\mathbf{p}) + p_i) + \Theta(\mathbf{p}) - \sum_{i \in S} p_i \\ &= \sum_{i \in S} \left(\frac{\alpha_i \omega_i}{k_b p_i + \sum_{j \neq i} k_s p_j} + \theta p_i \right). \end{aligned}$$

We are interested in deriving the value of the social cost under two scenarios, namely at the social optimum and at a Nash equilibrium.

At the *social optimum*, under the system parameters set by the provider, the players cooperate (or are centrally managed) such that the social cost is minimized. This is the ideal scenario from the social perspective.

Due to the selfish behavior of players, it is possible that the social optimum cannot be reached, instead, the system will end up at a *Nash equilibrium*, where a player cannot further lower its cost by unilaterally changing its own strategy.

The concept of the *price of anarchy* (PoA) has been introduced in order to quantify how bad the social cost may be at a Nash equilibrium, compared to the social optimum [15]. It is defined as the ratio between the social cost at the worst Nash equilibrium and the optimal social cost. In the best scenario, the worst Nash equilibrium coincides with the social optimum, which leads to a PoA

Notation	Description
S	Set of players (users).
S_1, S_2	Set of buy-in and shared players (users), respectively.
N	Number of players.
n_1, n_2	Number of buy-in and shared players, respectively.
i, j, ℓ, m	Index of players.
ω_i	Average workload of player i.
p_i	Strategy (payment for purchasing buy-in resources) of player i.
P_i	Strategy set of player i.
$\mathbf{p}$	Strategy profile of all players.
$C_i(\mathbf{p})$	Cost of player i given $\mathbf{p}$.
$C(\mathbf{p})$	Social cost given $\mathbf{p}$.
$U_i(\mathbf{p})$	Utility of player i given $\mathbf{p}$.
$U(\mathbf{p})$	Social welfare given $\mathbf{p}$.
α_i	Cost per unit of job completion time for player i.
θ	Operating expense ratio - operational cost of the system per unit of payment.
k_b	Buy-in factor - coefficient of proportionality between p_i and the corresponding computing rate it gets.
k_s	Shared factor - coefficient of proportionality between p_i and the corresponding computing rate it provides to another player.
$\Delta_i(\mathbf{p})$	Subsidy term - the amount of additional computing rate subsidized to player i given $\mathbf{p}$.
k_c	Subsidy factor - coefficient of proportionality between the subsidy term $\Delta_i(\mathbf{p})$ and p_i or p_i^2, depending on the form of subsidy.

Table 1. Notation summary.

equal to 1. A large PoA suggests that the system may badly suffer from the selfish behavior of its decision makers, and is far from the best possible scenario.

In the rest of the paper, we assume without loss of generality that players are labeled such that

$$\alpha_1\omega_1 \geq \alpha_2\omega_2 \geq \cdots \geq \alpha_N\omega_N.$$

The $\alpha_i\omega_i$ here can be interpreted as the adjusted workload of user i, considering the user's sensitivity to the job completion time.

In Table 1 we provide a summary of the notations employed in the paper.

4 SOCIAL COST AND PRICE OF ANARCHY

In this section, we analyze the social cost in general N-player SBC games. We first show that the social cost minimization problem is convex, such that it can be solved through constrained convex optimization techniques. We are also able to provide a closed-form expression for the optimal social cost, in the special case where all users are buy-in users. Then, we compute, in closed-form and linear time, the social cost at the Nash equilibrium. With these at hand, we evaluate the efficiency of the equilibrium by computing the price of anarchy.

4.1 Social cost at the optimum

In this subsection, we show that social cost minimization is a constrained convex optimization problem, and analyze the strategies of players at the social optimum. We also provide a closed-form expression of the optimal social cost.

Denote the payments made by all players through the vector $\mathbf{p} = [p_1, p_2, \ldots, p_N]^\intercal$. The following theorem establishes that the social cost $C(\mathbf{p})$ is convex in $\mathbf{p} = [p_1, p_2, \ldots, p_N]^\intercal$.

Theorem 4.1. *The following social cost function is (strictly) convex in* $\mathbf{p}$*:*

$$C(\mathbf{p}) = \sum_{i \in S} \left(\frac{\alpha_i \omega_i}{k_b p_i + \sum_{j \neq i} k_s p_j} + \theta p_i \right). \tag{1}$$

Proof. We prove the theorem by establishing that the Hessian matrix of $C(\mathbf{p})$ is positive definite. First, consider the gradient $\nabla C(\mathbf{p})$ of the social cost function, its i-th entry ∇C_i is

$$\nabla C_i = \theta - \frac{\alpha_i k_b \omega_i}{(k_b p_i + \sum_{\ell \neq i} k_s p_\ell)^2} - \sum_{m \neq i} \frac{\alpha_m k_s \omega_m}{(k_b p_m + k_s p_i + \sum_{\ell \neq i, \ell \neq m} k_s p_\ell)^2}.$$

Based on the gradient, the (i, i)-th entry of the Hessian matrix $\nabla^2 C(\mathbf{p})$ is

$$\nabla^2 C_{i,i} = \frac{2\alpha_i k_b^2 \omega_i}{(k_b p_i + \sum_{\ell \neq i} k_s p_\ell)^3} + \sum_{m \neq i} \frac{2\alpha_m k_s^2 \omega_m}{(k_b p_m + k_s p_i + \sum_{\ell \neq i, \ell \neq m} k_s p_\ell)^3}.$$

In addition, the (i, j)-th entry $(j \neq i)$ of the Hessian matrix $\nabla^2 C(\mathbf{p})$ is

$$\begin{aligned} \nabla^2 C_{i,j} =& \frac{2\alpha_i k_b k_s \omega_i}{(k_b p_i + \sum_{\ell \neq i} k_s p_\ell)^3} + \frac{2\alpha_j k_b k_s \omega_j}{(k_b p_j + k_s p_i + \sum_{\ell \neq i, \ell \neq j} k_s p_\ell)^3} \\ &+ \sum_{m \neq i, m \neq j} \frac{2\alpha_m k_s^2 \omega_m}{(k_b p_m + k_s p_i + \sum_{\ell \neq i, \ell \neq m} k_s p_\ell)^3}. \end{aligned}$$

For clarity, define

$$x_i^2 \triangleq \frac{2\alpha_i \omega_i}{(k_b p_i + \sum_{\ell \neq i} k_s p_\ell)^3}, \ \forall i,$$

then we can re-write the entries in the Hessian matrix $\nabla^2 C(\mathbf{p})$ as

$$\begin{aligned} \nabla^2 C_{i,i} =& k_b^2 x_i^2 + \sum_{m \neq i} k_s^2 x_m^2, \\ \nabla^2 C_{i,j} =& k_b k_s (x_i^2 + x_j^2) + \sum_{m \neq i, m \neq j} k_s^2 x_m^2, \ \forall j \neq i. \end{aligned}$$

Therefore, the Hessian matrix can be written as

$$\nabla^2 C(\mathbf{p}) = \mathbf{H}^\intercal \mathbf{H},$$

where

$$\mathbf{H} = \begin{bmatrix} k_b x_1 & k_s x_1 & \ldots & k_s x_1 \\ k_s x_2 & k_b x_2 & \ldots & k_s x_2 \\ \vdots & & \ddots & \vdots \\ k_s x_N & k_s x_N & \ldots & k_b x_N \end{bmatrix}.$$

If $\mathbf{H}$ is non-singular then the Hessian matrix $\nabla^2 C(\mathbf{p})$ is positive definite. Next, we show that $\mathbf{H}$ is indeed non-singular by proving that its columns are linearly independent.

Denoting $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_{m-1}]$, where $\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_{m-1}$ are columns vectors of $\mathbf{H}$, and let $\mathbf{h}_i(j)$ denote the j-th entry of vector $\mathbf{h}_i$, that is, $\mathbf{h}_i(j) = \mathbf{H}_{j,i}$. We have

$$\begin{aligned} \mathbf{h}_i(j) &= k_b x_i,\ j = i, \\ \mathbf{h}_i(j) &= k_s x_j,\ j \neq i. \end{aligned} \tag{2}$$

Assume by contradiction that the columns of $\mathbf{H}$ are linearly dependent. Then, there exists an i-th column that can be expressed as a linear combination of other columns:

$$\mathbf{h}_i = \sum_{\ell \neq i} \lambda_\ell \mathbf{h}_\ell,$$

which yields

$$\begin{aligned} \sum_{\ell \neq i} \lambda_\ell \mathbf{h}_\ell(i) &= \mathbf{h}_i(i), \\ \sum_{\ell \neq i} \lambda_\ell \mathbf{h}_\ell(j) &= \mathbf{h}_i(j),\ \forall\ j \neq i. \end{aligned} \tag{3}$$

Combining (2) with (3), and canceling out the x_i terms on both sides of the equations, we get

$$\sum_{\ell \neq i} \lambda_\ell k_s = k_b, \tag{4}$$

$$\lambda_j k_b + \sum_{\ell \neq i, \ell \neq j} \lambda_\ell k_s = \lambda_j (k_b - k_s) + \sum_{\ell \neq i} \lambda_\ell k_s = k_s,\ \forall\ j \neq i. \tag{5}$$

Substituting the term $\sum_{\ell \neq i} \lambda_i k_s$ in (5) with k_b by (4), we get

$$(1 + \lambda_j)(k_b - k_s) = 0,\ \forall\ j \neq i.$$

We already know that $k_b > k_s$, therefore it must be that

$$\lambda_j = -1,\ \forall\ j \neq i.$$

However, this contradicts (4), hence the assumption that $\mathbf{H}$ is column-dependent cannot hold.

Thus, we conclude that the columns of $\mathbf{H}$ are linearly independent, and the Hessian matrix $\nabla^2 C(\mathbf{p}) = \mathbf{H}^\intercal \mathbf{H}$ is positive definite. As a result, the social cost function $C(\mathbf{p})$ is strictly convex. □

Denote the gradient of the social cost function $C(\mathbf{p})$ by $\nabla C(\mathbf{p})$. The next lemma establishes the relationship between p_i and ∇C_i, which is the i-th entry of $\nabla C(\mathbf{p})$.

Lemma 4.2. *At the social optimum of an SBC game, if $p_i > 0$, we have $\nabla C_i = 0$; if $p_i = 0$, we have $\nabla C_i \geq 0$, where*

$$\nabla C_i = \theta - \frac{\alpha_i k_b \omega_i}{(k_b p_i + \sum_{\ell \neq i} k_s p_\ell)^2} - \sum_{m \neq i} \frac{\alpha_m k_s \omega_m}{(k_b p_m + k_s p_i + \sum_{\ell \neq i, \ell \neq m} k_s p_\ell)^2}.$$

Proof. The social cost minimization problem can be written as:

$$\begin{aligned} \min C(\mathbf{p}) &= \sum_{i \in S} \left(\frac{\alpha_i \omega_i}{k_b p_i + \sum_{j \neq i} k_s p_j} + \theta p_i \right), \\ s.t.\ &- p_i \leq 0,\ \forall i \in S. \end{aligned}$$

The Lagrangian function of the minimization problem above is

$$L(\mathbf{p}, \lambda) = C(\mathbf{p}) - \lambda^\intercal \mathbf{p}.$$

Since $C(\mathbf{p})$ is convex in $\mathbf{p}$, $L(\mathbf{p},\lambda)$ only has a unique global minimum. At the minimum, $\mathbf{p}$ and λ must satisfy the following KKT conditions for all i:

$$\begin{aligned}\nabla_{\mathbf{p}} L(\mathbf{p},\lambda)_i &= 0,\\ -\lambda_i p_i &= 0,\\ -p_i &\le 0,\\ \lambda_i &\ge 0.\end{aligned}$$

Considering player i at the optimum, by complementary slackness, if $p_i > 0$, we have $\lambda_i = 0$. Then we get

$$\nabla_{\mathbf{p}} L(\mathbf{p},\lambda)_i = \nabla C_i - \lambda_i = \nabla C_i = 0.$$

Moreover, if $p_i = 0$, we have $\lambda_i \ge 0$, thus $\nabla C_i \ge 0$ since

$$\nabla_{\mathbf{p}} L(\mathbf{p},\lambda)_i = \nabla C_i - \lambda_i = 0.$$

The lemma is thus established. □

The following lemma shows the relationship between the price paid by a player and its workload adjusted by cost per unit of time, namely, a player with a larger adjusted workload will pay no less than a player with a smaller adjusted workload.

Lemma 4.3. *At the social optimum of an SBC game, we have*

$$p_1 \ge p_2 \ge \cdots \ge p_N \ge 0.$$

Proof. Without loss of generality, we assume that i and j are ordered such that $\alpha_i\omega_i \ge \alpha_j\omega_j$. In the following, we will prove that $p_i \ge p_j$ always holds.

Considering the gradients of the costs of two players i and j, we have

$$\begin{aligned}\nabla C_i =&\theta - \frac{\alpha_i k_b \omega_i}{(k_b p_i + k_s p_j + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2}\\ &- \frac{\alpha_j k_s \omega_j}{(k_b p_j + k_s p_i + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2} - \sum_{m\neq i, m\neq j} \frac{\alpha_m k_s \omega_m}{(k_b p_m + \sum_{\ell\neq m} k_s p_\ell)^2},\\ \nabla C_j =&\theta - \frac{\alpha_j k_b \omega_j}{(k_b p_j + k_s p_i + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2}\\ &- \frac{\alpha_i k_s \omega_i}{(k_b p_i + k_s p_j + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2} - \sum_{m\neq i, m\neq j} \frac{\alpha_m k_s \omega_m}{(k_b p_m + \sum_{\ell\neq m} k_s p_\ell)^2}.\end{aligned}$$

And their difference is

$$\begin{aligned}&\nabla C_i - \nabla C_j\\ =&\frac{\alpha_j k_b \omega_j}{(k_b p_j + k_s p_i + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2} + \frac{\alpha_i k_s \omega_i}{(k_b p_i + k_s p_j + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2}\\ &- \frac{\alpha_i k_b \omega_i}{(k_b p_i + k_s p_j + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2} - \frac{\alpha_j k_s \omega_j}{(k_b p_j + k_s p_i + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2}\\ =&(k_b - k_s)\left(\frac{\alpha_j \omega_j}{(k_b p_j + k_s p_i + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2} - \frac{\alpha_i \omega_i}{(k_b p_i + k_s p_j + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2}\right).\end{aligned}$$

Note that $k_b - k_s > 0$ always holds since $k_b > k_s$.

Next, we consider four cases: (i) $p_i > 0$ and $p_j > 0$; (ii) $p_i > 0$ and $p_j = 0$; (iii) $p_i = 0$ and $p_j > 0$; (iv) $p_i = p_j = 0$.

(i) $p_i > 0$ and $p_j > 0$. We have $\nabla C_i = \nabla C_j = 0$, thus $\nabla C_i - \nabla C_j = 0$, from which we can get

$$\frac{\alpha_j\omega_j}{(k_b p_j + k_s p_i + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2} = \frac{\alpha_i\omega_i}{(k_b p_i + k_s p_j + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2}.$$

Since $\alpha_j\omega_j \leq \alpha_i\omega_i$, it must be that $k_b p_j + k_s p_i \leq k_b p_i + k_s p_j$, which implies that $p_i \geq p_j$.

(ii) $p_i > 0$ and $p_j = 0$. We have $\nabla C_i = 0$ and $\nabla C_j \geq 0$, thus $\nabla C_i - \nabla C_j \leq 0$, from which we can get

$$\frac{\alpha_j\omega_j}{(k_s p_i + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2} \leq \frac{\alpha_i\omega_i}{(k_b p_i + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2}.$$

Since $\alpha_j\omega_j \leq \alpha_i\omega_i$ and $k_s p_i < k_b p_i$, this case is possible.

(iii) $p_i = 0$ and $p_j > 0$. We have $\nabla C_i \geq 0$ and $\nabla C_j = 0$, thus $\nabla C_i - \nabla C_j \geq 0$, from which we can get

$$\frac{\alpha_j\omega_j}{(k_b p_j + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2} \geq \frac{\alpha_i\omega_i}{(k_s p_j + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2}.$$

However, since $\alpha_j\omega_j \leq \alpha_i\omega_i$ and $k_b p_j > k_s p_j$, the inequality never holds (the LHS is strictly smaller than the RHS), thus this case can be disregarded.

(iv) $p_i = p_j = 0$. We have $\nabla C_i \geq 0$ and $\nabla C_j \geq 0$. This case already satisfies $p_i \geq p_j$.

By considering all the four cases, we conclude that if $\alpha_i\omega_i \geq \alpha_j\omega_j$, it must be that $p_i \geq p_j$, which results in

$$p_1 \geq p_2 \geq \cdots \geq p_N \geq 0.$$

□

Next, based on the lemmas above, we derive the optimal social cost in closed form.

THEOREM 4.4. *At the social optimum of an SBC game, the social cost is*

$$C_{OPT}(\mathbf{p}) = \left(\frac{1}{\sqrt{\psi}} + \frac{\theta\sqrt{\psi}}{k_b + (n_1-1)k_s}\right)\sum_{i\in S_1}\sqrt{\alpha_i\omega_i} + \frac{(k_b + (n_1-1)k_s)}{k_s\sqrt{\psi}}\frac{\sum_{m\in S_2}\alpha_m\omega_m}{\sum_{i\in S_1}\sqrt{\alpha_i\omega_i}},$$

where

$$\psi = \frac{k_b + (n_1-1)k_s}{\theta}\left(1 + \frac{(k_b + (n_1-1)k_s)\sum_{m\in S_2}\alpha_m\omega_m}{k_s(\sum_{i\in S_1}\sqrt{\alpha_i\omega_i})^2}\right).$$

PROOF. Theorem 4.1 states that the social cost function $C(\mathbf{p})$ is strictly convex in $\mathbf{p}$, thus it has a unique optimum.

According to Lemma 4.2, at the social optimum, for each buy-in player $i \in S_1$, we have $p_i > 0$ and

$$\begin{aligned}\nabla C_i &= \theta - \frac{\alpha_i k_b \omega_i}{(k_b p_i + \sum_{\ell\in S_1,\ell\neq i} k_s p_\ell)^2} \\ &\quad - \sum_{j\in S_1, j\neq i}\frac{\alpha_j k_s\omega_j}{(k_b p_j + k_s p_i + \sum_{\ell\in S_1,\ell\neq i,\ell\neq j} k_s p_\ell)^2} - \sum_{m\in S_2}\frac{\alpha_m k_s \omega_m}{(\sum_{\ell\in S_1} k_s p_\ell)^2} \\ &= 0. \end{aligned} \quad (6)$$

For each shared player $m \in S_2$, we have $p_m = 0$.

Consider players i and j such that $i, j \in S_1$, we have

$$\begin{aligned}\nabla C_i - \nabla C_j &= (k_b - k_s)\left(\frac{\alpha_j\omega_j}{(k_b p_j + k_s p_i + \sum_{\ell\in S_1,\ell\neq i,\ell\neq j} k_s p_\ell)^2} - \frac{\alpha_i\omega_i}{(k_b p_i + k_s p_j + \sum_{\ell\in S_1,\ell\neq i,\ell\neq j} k_s p_\ell)^2}\right) \\ &= 0.\end{aligned}$$

Since $k_b \neq k_s$ (because our model assumes $k_b > k_s$), we deduce that

$$\frac{\alpha_j \omega_j}{(k_b p_j + k_s p_i + \sum_{\ell \in S_1, \ell \neq i, \ell \neq j} k_s p_\ell)^2} = \frac{\alpha_i \omega_i}{(k_b p_i + k_s p_j + \sum_{\ell \in S_1, \ell \neq i, \ell \neq j} k_s p_\ell)^2}, \ \forall i, j \in S_1.$$

As a result, for a given buy-in factor k_b, shared factor k_s, and users' adjusted workload $\alpha_i \omega_i$, there exists a constant ψ such that

$$\frac{\alpha_i \omega_i}{(k_b p_i + \sum_{\ell \in S_1, \ell \neq i} k_s p_\ell)^2} = \frac{1}{\psi}, \ \forall i \in S_1. \tag{7}$$

Note that the equations above are equivalent to

$$k_b p_i + \sum_{\ell \in S_1, \ell \neq i} k_s p_\ell = \sqrt{\psi \alpha_i \omega_i}, \ \forall i \in S_1. \tag{8}$$

Taking the sum of (8) for $i \in S_1$, we get

$$(k_b + (n_1 - 1)k_s) \sum_{i \in S_1} p_i = \sum_{i \in S_1} \sqrt{\psi \alpha_i \omega_i},$$

from which we derive the sum of buy-in players' payments:

$$\sum_{i \in S_1} p_i = \frac{\sum_{i \in S_1} \sqrt{\psi \alpha_i \omega_i}}{k_b + (n_1 - 1)k_s}. \tag{9}$$

Then, taking ψ in (7) back into equation (6), we get

$$\begin{aligned} \nabla C_i &= \theta - \frac{k_b}{\psi} - \sum_{\ell \in S_1, \ell \neq i} \frac{k_s}{\psi} - \sum_{m \in S_2} \frac{\alpha_m k_s \omega_m}{(\sum_{\ell \in S_1} k_s p_\ell)^2} \\ &= \theta - \frac{1}{\psi}(k_b + (n_1 - 1)k_s) - \sum_{m \in S_2} \frac{\alpha_m \omega_m}{k_s (\sum_{\ell \in S_1} p_\ell)^2} \\ &= 0. \end{aligned} \tag{10}$$

Next, we solve the optimal social cost in two cases: (i) all players are buy-in (i.e., $S_2 = \emptyset$); (ii) there exist both buy-in and shared players (i.e., $S_2 \neq \emptyset$). We shall prove that the optimal social cost in both cases satisfies the same expression.

(i) All players are buy-in, S_2 is empty. We have that the sum of the costs of shared users is $\sum_{m \in S_2} C_m(\mathbf{p}) = 0$. Moreover, from equation (10), we get

$$\psi = \frac{k_b + (n_1 - 1)k_s}{\theta}.$$

Combining it with (7) and (9), we get that the social cost is

$$
\begin{aligned}
C_{OPT}(\mathbf{p}) &= \sum_{i\in S_1} C_i(\mathbf{p}) + \sum_{m\in S_2} C_m(\mathbf{p}) + C_{\text{provider}}(\mathbf{p}) \\
&= \sum_{i\in S_1}\left(\frac{\alpha_i\omega_i}{k_b p_i + \sum_{\ell\neq i} k_s p_\ell} + p_i\right) + 0 + (\theta-1)\sum_{i\in S} p_i \\
&= \sum_{i\in S_1}\frac{\alpha_i\omega_i}{k_b p_i + \sum_{\ell\neq i} k_s p_\ell} + \theta\sum_{i\in S_1} p_i \\
&= \sum_{i\in S_1}\frac{\alpha_i\omega_i}{\sqrt{\psi\alpha_i\omega_i}} + \sum_{i\in S_1}\frac{\theta\sqrt{\psi\alpha_i\omega_i}}{k_b+(n_1-1)k_s} \\
&= \frac{2\sum_{i\in S_1}\sqrt{\theta\alpha_i\omega_i}}{\sqrt{k_b+(n_1-1)k_s}}. \qquad (11)
\end{aligned}
$$

Here, the fourth equality uses equations (7) and (9) to derive the first and second terms, respectively, and the fifth equality is derived by substituting ψ with $(k_b+(n_1-1)k_s)/\theta$.

(ii) There exist both buy-in and shared players, i.e., S_2 is nonempty. Considering (8) and (9), we get that the optimal social cost is

$$
\begin{aligned}
C_{OPT}(\mathbf{p}) &= \sum_{i\in S_1} C_i(\mathbf{p}) + \sum_{j\in S_2} C_j(\mathbf{p}) + C_{\text{provider}}(\mathbf{p}) \\
&= \sum_{i\in S_1}\left(\frac{\alpha_i\omega_i}{k_b p_i + \sum_{\ell\neq i} k_s p_\ell} + p_i\right) + \sum_{m\in S_2}\frac{\alpha_m\omega_m}{\sum_{\ell\in S_1} k_s p_\ell} + (\theta-1)\sum_{i\in S} p_i \\
&= \sum_{i\in S_1}\frac{\alpha_i\omega_i}{k_b p_i + \sum_{\ell\neq i} k_s p_\ell} + \theta\sum_{i\in S_1} p_i + \sum_{m\in S_2}\frac{\alpha_m\omega_m}{k_s\sum_{i\in S_1} p_i} \\
&= \sum_{i\in S_1}\frac{\alpha_i\omega_i}{\sqrt{\psi\alpha_i\omega_i}} + \sum_{i\in S_1}\frac{\theta\sqrt{\psi\alpha_i\omega_i}}{k_b+(n_1-1)k_s} + \frac{k_b+(n_1-1)k_s}{k_s\sqrt{\psi}}\frac{\sum_{m\in S_2}\alpha_m\omega_m}{\sum_{i\in S_1}\sqrt{\alpha_i\omega_i}} \\
&= \left(\frac{1}{\sqrt{\psi}} + \frac{\theta\sqrt{\psi}}{k_b+(n_1-1)k_s}\right)\sum_{i\in S_1}\sqrt{\alpha_i\omega_i} + \frac{(k_b+(n_1-1)k_s)}{k_s\sqrt{\psi}}\frac{\sum_{m\in S_2}\alpha_m\omega_m}{\sum_{i\in S_1}\sqrt{\alpha_i\omega_i}}. \qquad (12)
\end{aligned}
$$

Next, we obtain an expression for the constant ψ in (12). Note that equation (10) is equivalent to

$$
k_s(\sum_{i\in S_1} p_i)^2 = \frac{\sum_{m\in S_2}\alpha_m\omega_m}{\theta - \frac{1}{\psi}(k_b+(n_1-1)k_s)}.
$$

Combining it with (9), we get

$$
\frac{(k_b+(n_1-1)k_s)^2}{k_s} = \frac{(\sum_{i\in S_1}\sqrt{\psi\alpha_i\omega_i})^2}{\sum_{m\in S_2}\alpha_m\omega_m}(\theta - \frac{1}{\psi}(k_b+(n_1-1)k_s)),
$$

which is equivalent to

$$
\begin{aligned}
\theta &= \frac{k_b+(n_1-1)k_s}{\psi} + \frac{(k_b+(n_1-1)k_s)^2\sum_{m\in S_2}\alpha_m\omega_m}{\psi k_s(\sum_{i\in S_1}\sqrt{\alpha_i\omega_i})^2} \\
&= \frac{k_b+(n_1-1)k_s}{\psi}\left(1 + \frac{(k_b+(n_1-1)k_s)\sum_{m\in S_2}\alpha_m\omega_m}{k_s(\sum_{i\in S_1}\sqrt{\alpha_i\omega_i})^2}\right).
\end{aligned}
$$

From the equation above, ψ can be specified as

$$\psi = \frac{k_b + (n_1 - 1)k_s}{\theta}\left(1 + \frac{(k_b + (n_1 - 1)k_s)\sum_{m\in S_2}\alpha_m\omega_m}{k_s(\sum_{i\in S_1}\sqrt{\alpha_i\omega_i})^2}\right). \tag{13}$$

Combining (12) and (13), the closed-form optimal social cost in case (ii) is obtained.

So far, we have derived the optimal social cost for all possible cases in (11)-(13). Moreover, we note that, when S_2 is empty, we have $\sum_{m\in S_2}\alpha_m\omega_m = 0$, such that (11) also satisfies (12) and (13). Thus, (12) and (13) represent the closed-form social cost at the optimum.

□

For the special case where all players are buy-in, the equation (11) in the proof of Theorem 4.4 gives the optimal social cost in a simpler form.

Corollary 1. *At the social optimum of an SBC game, if all players are buy-in users ($p_i > 0, \forall i \in S$), the social cost is*

$$C_{OPT}(\mathbf{p}) = \frac{2\sum_{i\in S}\sqrt{\theta\alpha_i\omega_i}}{\sqrt{k_b + (N-1)k_s}}.$$

Lemma 4.7 in the next subsection provides a way to identify the buy-in user group S_1 and shared user group S_2. Based on that result, one can then compute the optimal social cost in $O(N)$ time.

4.2 Social cost at Nash equilibrium

In this subsection, we analyze the strategies of players at the equilibrium, and specify the social cost in a closed form that incurs $O(N)$ computation time. Given the payments made by all players through the vector $\mathbf{p} = [p_1, p_2, \ldots, p_N]^\top$, each player i has the cost function

$$C_i(\mathbf{p}) = \frac{\alpha_i\omega_i}{k_b p_i + \sum_{j\neq i} k_s p_j} + p_i. \tag{14}$$

Meanwhile, we have the social cost function from (1) as

$$C(\mathbf{p}) = \sum_{i\in S}\left(\frac{\alpha_i\omega_i}{k_b p_i + \sum_{j\neq i} k_s p_j} + \theta p_i\right).$$

At a *Nash equilibrium*, each player employs its *best response*, which is defined as the player's optimal strategy given the strategies of all the other players. The best response strategy p_i of player i is to minimize its cost (14) given $\{p_j \mid j \in S, j \neq i\}$, namely:

$$p_i = \max\left(0, \sqrt{\frac{\alpha_i\omega_i}{k_b}} - \frac{k_s}{k_b}\sum_{j\neq i} p_j\right). \tag{15}$$

In the following, we use p_i^* to suggest that the strategy of player i is at a Nash equilibrium. Lemma 2 in [25] establishes that, at a Nash equilibrium, we have $p_1^* \geq p_2^* \geq \cdots \geq p_N^*$, that is, a player with a larger adjusted workload will pay no less than a player with a smaller adjusted workload. Therefore, we have $p_i^* > 0, \forall i \leq n_1$, and $p_i^* = 0, \forall i > n_1$. Moreover, note that, if $p_i^* = 0$, $\forall i > 1$, the cost of player 1 becomes

$$C_1([p_1, 0, \ldots, 0]^\top) = \frac{\alpha_1\omega_1}{k_b p_1} + p_1.$$

Hence, the best response strategy of player 1 will be $p_1^* > 0$. Thus, there exists at least one buy-in user ($n_1 \geq 1$).

Theorem 1 in [25] establishes that the game has a unique Nash equilibrium. Next, we give the closed-form social cost at the unique equilibrium.

Theorem 4.5. *At the Nash equilibrium of an SBC game, the social cost is*

$$C_{NE}(\mathbf{p}) = \left(\frac{1}{\sqrt{k_b}} + \frac{\theta\sqrt{k_b}}{k_b + (n_1-1)k_s}\right)\sum_{i\in S_1}\sqrt{\alpha_i\omega_i} + \frac{(k_b+(n_1-1)k_s)}{k_s\sqrt{k_b}}\frac{\sum_{j\in S_2}\alpha_j\omega_j}{\sum_{i\in S_1}\sqrt{\alpha_i\omega_i}}.$$

Proof. According to (15), the strategy of a buy-in user at the Nash equilibrium can be written as

$$p_i^* = \max\left(0, \sqrt{\frac{\alpha_i\omega_i}{k_b}} - \frac{k_s}{k_b}\sum_{j\neq i}p_j^*\right) = \sqrt{\frac{\alpha_i\omega_i}{k_b}} - \frac{k_s}{k_b}\sum_{j\in S_1, j\neq i}p_j^*,\ i\in S_1. \tag{16}$$

Note that the strategies of buy-in users are not affected by shared users. The strategies of all buy-in users can be solved through the following equations:

$$\begin{bmatrix} k_b & k_s & \dots & k_s \\ k_s & k_b & \dots & k_s \\ \vdots & & \ddots & \vdots \\ k_s & k_s & \dots & k_b \end{bmatrix}\begin{bmatrix} p_1^* \\ p_2^* \\ \vdots \\ p_{n_1}^* \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha_1 k_b\omega_1} \\ \sqrt{\alpha_2 k_b\omega_2} \\ \vdots \\ \sqrt{\alpha_{n_1} k_b\omega_{n_1}} \end{bmatrix}. \tag{17}$$

If we rewrite the equations above as $\mathbf{A}\mathbf{p}^* = \mathbf{b}$, and multiply by left with the vector $\mathbf{1}^\intercal = [1, 1, \dots, 1]$ on both sides, we get $\mathbf{1}^\intercal\mathbf{A}\mathbf{p}^* = \mathbf{1}^\intercal\mathbf{b}$, from which we can derive

$$(k_b + (n_1-1)k_s)\sum_{i\in S_1}p_i^* = \sum_{i\in S_1}\sqrt{\alpha_i k_b\omega_i}. \tag{18}$$

Combining the cost function (14) with the best response strategy (16) of a buy-in user, its cost at the Nash equilibrium can be specified as

$$C_i(\mathbf{p}^*) = \frac{\alpha_i\omega_i}{k_b p_i^* + \sum_{j\in S_1, j\neq i}k_s p_j^*} + p_i^* = \frac{\alpha_i\omega_i}{\sqrt{\alpha_i k_b\omega_i}} + p_i^*,\ i\in S_1. \tag{19}$$

A shared user pays 0 at the Nash equilibrium, and its cost can be computed by

$$C_i(\mathbf{p}^*) = \frac{\alpha_i\omega_i}{k_s\sum_{i\in S_1}p_i^*} + 0,\ i\in S_2. \tag{20}$$

The cost of the provider is

$$C_{\text{provider}}(\mathbf{p}^*) = \theta\sum_{i\in S}p_i^* - \sum_{i\in S}p_i^* = (\theta-1)\sum_{i\in S_1}p_i^*. \tag{21}$$

Therefore, from (18)-(21), we get that the social cost at the equilibrium is

$$\begin{aligned} C_{NE}(\mathbf{p}) &= \sum_{i\in S_1}C_i(\mathbf{p}^*) + \sum_{j\in S_2}C_j(\mathbf{p}^*) + C_{\text{provider}}(\mathbf{p}^*) \\ &= \sum_{i\in S_1}\left(\frac{\alpha_i\omega_i}{\sqrt{\alpha_i k_b\omega_i}} + (1+\theta-1)p_i^*\right) + \sum_{j\in S_2}\frac{\alpha_j\omega_j}{k_s\sum_{i\in S_1}p_i^*} \\ &= \sum_{i\in S_1}\frac{\sqrt{\alpha_i k_b\omega_i}}{k_b} + \sum_{i\in S_1}\frac{\theta\sqrt{\alpha_i k_b\omega_i}}{k_b+(n_1-1)k_s} + \frac{(k_b+(n_1-1)k_s)\sum_{j\in S_2}\sqrt{\alpha_j\omega_j}}{k_s\sqrt{k_b}\sum_{i\in S_1}\sqrt{\alpha_i\omega_i}} \\ &= \left(\frac{1}{\sqrt{k_b}} + \frac{\theta\sqrt{k_b}}{k_b+(n_1-1)k_s}\right)\sum_{i\in S_1}\sqrt{\alpha_i\omega_i} + \frac{(k_b+(n_1-1)k_s)}{k_s\sqrt{k_b}}\frac{\sum_{j\in S_2}\alpha_j\omega_j}{\sum_{i\in S_1}\sqrt{\alpha_i\omega_i}}. \end{aligned} \tag{22}$$

□

As a special case, we can get the social cost when all players are buy-in from Theorem 4.5.

Corollary 2. *At the Nash equilibrium of an SBC game, if all players are buy-in users ($p_i > 0$, $\forall i \in S$), the social cost is*

$$C_{NE}(\mathbf{p}) = \left(\frac{1}{\sqrt{k_b}} + \frac{\theta\sqrt{k_b}}{k_b + (n_1 - 1)k_s}\right)\sum_{i\in S}\sqrt{\alpha_i\omega_i}.$$

By comparing the social cost at the optimum and the Nash equilibrium, we can deduce that the strategies of players are only differentiated by a constant factor, which is formalized in the next lemma. This suggests that the optimum and the Nash equilibrium have the same sets of buy-in users S_1 and shared users S_2.

Lemma 4.6. *Denote by $\mathbf{p}^{OPT} = \left[p_1^{OPT}, p_2^{OPT}, \dots, p_N^{OPT}\right]^\top$ the strategy profile of the players at the social optimum, and $\mathbf{p}^{NE} = \left[p_1^{NE}, p_2^{NE}, \dots, p_N^{NE}\right]^\top$ the strategy profile of the players at the Nash equilibrium, we have*

$$\mathbf{p}^{NE} = \sqrt{\frac{k_b}{\psi}}\mathbf{p}^{OPT}.$$

Proof. We prove the lemma by construction, that is, we prove that if $\mathbf{p}^{OPT}$ is the unique optimum, then $\mathbf{p}^{NE} = \sqrt{\frac{k_b}{\psi}}\mathbf{p}^{OPT}$ is the unique Nash equilibrium.

Consider a buy-in player i and a shared player m at the optimum, where $p_i^{OPT} > 0$ and $p_m^{OPT} = 0$. From Lemma 4.2, we have $\nabla C_i = 0$ and $\nabla C_m \geq 0$, and the difference between ∇C_m and ∇C_i is

$$\nabla C_m - \nabla C_i = (k_b - k_s)\left(\frac{\alpha_i\omega_i}{(k_b p_i^{OPT} + \sum_{\ell\in S_1,\ell\neq i} k_s p_\ell^{OPT})^2} - \frac{\alpha_m\omega_m}{(\sum_{\ell\in S_1} k_s p_\ell^{OPT})^2}\right) \geq 0.$$

Combining the inequality above with (8), we get

$$\frac{\alpha_m\omega_m}{(\sum_{i\in S_1} k_s p_i^{OPT})^2} \leq \frac{1}{\psi}.$$

Equivalently,

$$\sqrt{\frac{\alpha_m\omega_m}{k_b}} \leq \frac{k_s}{k_b}\sum_{i\in S_1}\left(\sqrt{\frac{k_b}{\psi}}p_i^{OPT}\right) = \frac{k_s}{k_b}\sum_{i\in S_1} p_i^{NE},$$

which is the necessary and sufficient condition for $p_m^{NE} = 0$ according to Lemma 4 in [25]. Thus, the shared users at the optimum are also the shared users at the Nash equilibrium, and we have $p_m^{NE} = \sqrt{\frac{k_b}{\psi}}p_m^{OPT} = 0$ for a shared player m.

Next, consider buy-in players. Note that equations (8) and (17) are in the same form, except for their constant factors on the RHS. Thus, if $\left[p_1^{OPT}, p_2^{OPT}, \dots, p_{n_1}^{OPT}\right]^\top$ is the unique solution to (8), then $\sqrt{\frac{k_b}{\psi}}\left[p_1^{OPT}, p_2^{OPT}, \dots, p_{n_1}^{OPT}\right]^\top$ will solve equations (17). Therefore, the buy-in users at the optimum are also the buy-in users at the Nash equilibrium, and we have $p_i^{NE} = \sqrt{\frac{k_b}{\psi}}p_i^{OPT} > 0$ for a shared player i.

As a result, given the unique social optimum as $\mathbf{p}^{OPT} = \left[p_1^{OPT}, \dots, p_{n_1}^{OPT}, 0, \dots, 0\right]^\top$, we can construct the unique Nash equilibrium $\mathbf{p}^{NE} = \left[p_1^{NE}, \dots, p_{n_1}^{NE}, 0, \dots, 0\right]^\top$, where

$$p_i^{NE} = \sqrt{\frac{k_b}{\psi}}p_i^{OPT}, \ \forall i \in S.$$

□

Although Theorem 4.4 and Theorem 4.5 provide closed-form social cost at the optimum and the Nash equilibrium, respectively, we still need to know the sets S_1 and S_2 beforehand in order to compute the social cost. In other words, we need to distinguish buy-in users from shared users. We have already shown that $p_i^* > 0, \forall i \leq n_1$, and $p_i^* = 0, \forall i > n_1$. Thus we just need to identify n_1, i.e., the number of buy-in users. The following lemma provides a convenient way to do that, instead of computing the Nash equilibrium or the optimum directly.

Lemma 4.7. *At the Nash equilibrium or the social optimum of an SBC game, if there exist shared users ($n_1 < N$), then the strategy of player m is $p_m = 0$ if and only if*

$$\sqrt{\alpha_m \omega_m} \leq \frac{k_s \sum_{i<m} \sqrt{\alpha_i \omega_i}}{k_b + (m-2)k_s},$$

where $m \leq n_1 + 1$.

Proof. According to Lemma 4 in [25], at the Nash equilibrium, the strategy of player m is $p_m^* = 0$ if and only if

$$\frac{k_s}{k_b} \sum_{i<m} p_i^{**} \geq \sqrt{\frac{\alpha_m \omega_m}{k_b}}, \tag{23}$$

where $m \leq n_1 + 1$, and $\{p_i^{**} \mid i < m\}$ is the unique solution to

$$\begin{bmatrix} k_b & k_s & \dots & k_s \\ k_s & k_b & \dots & k_s \\ \vdots & & \ddots & \vdots \\ k_s & k_s & \dots & k_b \end{bmatrix} \begin{bmatrix} p_1^{**} \\ p_2^{**} \\ \vdots \\ p_{m-1}^{**} \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha_1 k_b \omega_1} \\ \sqrt{\alpha_2 k_b \omega_2} \\ \vdots \\ \sqrt{\alpha_{m-1} k_b \omega_{m-1}} \end{bmatrix}.$$

In other words, $\{p_i^{**} \mid i < m\}$ is the Nash equilibrium of the SBC game among just the players $\{1, 2, \dots, m-1\}$.

Multiplying the equation above from left with the vector $\mathbf{1}^\intercal = [1, 1, \dots, 1]$ on both sides, we get

$$(k_b + (m-2)k_s) \sum_{i<m} p_i^{**} = \sum_{i<m} \sqrt{\alpha k_b \omega_i}. \tag{24}$$

Combining (23) and (24), we get

$$\frac{k_s}{k_b} \times \frac{\sum_{i<m} \sqrt{\alpha_i k_b \omega_i}}{k_b + (m-2)k_s} \geq \sqrt{\frac{\alpha_m \omega_m}{k_b}},$$

which can be simplified as

$$\sqrt{\alpha_m \omega_m} \leq \frac{k_s \sum_{i<m} \sqrt{\alpha_i \omega_i}}{k_b + (m-2)k_s}.$$

According to Lemma 4.6, at the optimum, we also have $p_m = 0$ if and only if the condition above holds.

□

With the lemma above, we can identify n_1 in $O(N)$ time, such that we can distinguish buy-in users from shared users, as follows: iterate from player 1 to player N, until we find the player m such that $p_m^* = 0$, n_1 can be then computed by $n_1 = m - 1$. Note that, if we maintain the sum $\sum_{i<m} \sqrt{\alpha_i \omega_i}$ for all players that have been iterated, it takes just $O(1)$ time to update the checking condition in each iteration. Thus, the time needed to check all players is $O(N)$.

Although an algorithm to compute the Nash equilibrium is given in [25], using it in order to compute the Nash equilibrium explicitly and afterwards compute the corresponding social cost would incur $O(N^4)$. Here, in contrast, we manage to give the closed-form social cost without explicitly computing the Nash equilibrium, incurring a time complexity of just $O(N)$. More specifically, we first distinguish buy-in users from shared users by Lemma 4.7, then compute the social cost at the Nash equilibrium based on Theorem 4.5, both of which can be completed in $O(N)$ time. Similarly, the optimal social cost can be computed in $O(N)$ time, based on Lemma 4.7 and Theorem 4.4.

4.3 Price of anarchy

In [25], the convergence to the unique Nash equilibrium of the SBC game through best-response dynamics is established. However, the Nash equilibrium might not (and typically does not) coincide with the social optimum. As explained, the gap between the two is captured through the *price of anarchy*, namely:

$$PoA = \frac{C_{\text{worst\_NE}}(\mathbf{p})}{C_{\text{OPT}}(\mathbf{p})}.$$

It is shown in [25] that the game has a unique Nash equilibrium, hence the Nash equilibrium that we compute is necessarily the worst (i.e., has the largest social cost among all equilibria). A larger PoA indicates that the Nash equilibrium is less efficient.

According to Theorem 4.4 and Theorem 4.5, we can solve the price of anarchy in closed form.

Lemma 4.8. *The price of anarchy of an SBC game is*

$$PoA = \sqrt{\frac{\psi}{k_b}} \times \frac{(\theta k_b + k_b + (n_1 - 1)k_s)k_s(\sum_{i\in S_1}\sqrt{\alpha_i\omega_i})^2 + (k_b + (n_1-1)k_s)^2\sum_{m\in S_2}\alpha_m\omega_m}{(\theta\psi + k_b + (n_1 - 1)k_s)k_s(\sum_{i\in S_1}\sqrt{\alpha_i\omega_i})^2 + (k_b + (n_1-1)k_s)^2\sum_{m\in S_2}\alpha_m\omega_m},$$

where

$$\psi = \frac{k_b + (n_1-1)k_s}{\theta}\left(1 + \frac{(k_b + (n_1-1)k_s)\sum_{m\in S_2}\alpha_m\omega_m}{k_s(\sum_{i\in S_1}\sqrt{\alpha_i\omega_i})^2}\right).$$

Proof. Combine (12)-(13) in the proof of Theorem 4.4 and (22) in the proof of Theorem 4.5, the price of anarchy is

$$\begin{aligned} PoA =& \frac{C_{\text{NE}}(\mathbf{p})}{C_{\text{OPT}}(\mathbf{p})} \\ =& \frac{\left(\frac{1}{\sqrt{k_b}} + \frac{\theta\sqrt{k_b}}{k_b+(n_1-1)k_s}\right)\sum_{i\in S_1}\sqrt{\alpha_i\omega_i} + \frac{(k_b+(n_1-1)k_s)}{k_s\sqrt{k_b}}\frac{\sum_{m\in S_2}\alpha_m\omega_m}{\sum_{i\in S_1}\sqrt{\alpha_i\omega_i}}}{\left(\frac{1}{\sqrt{\psi}} + \frac{\theta\sqrt{\psi}}{k_b+(n_1-1)k_s}\right)\sum_{i\in S_1}\sqrt{\alpha_i\omega_i} + \frac{(k_b+(n_1-1)k_s)}{k_s\sqrt{\psi}}\frac{\sum_{m\in S_2}\alpha_m\omega_m}{\sum_{i\in S_1}\sqrt{\alpha_i\omega_i}}} \\ =& \sqrt{\frac{\psi}{k_b}} \times \frac{(\theta k_b + k_b + (n_1 - 1)k_s)k_s(\sum_{i\in S_1}\sqrt{\alpha_i\omega_i})^2 + (k_b + (n_1-1)k_s)^2\sum_{m\in S_2}\alpha_m\omega_m}{(\theta\psi + k_b + (n_1 - 1)k_s)k_s(\sum_{i\in S_1}\sqrt{\alpha_i\omega_i})^2 + (k_b + (n_1-1)k_s)^2\sum_{m\in S_2}\alpha_m\omega_m}. \end{aligned}$$

□

In the following, we provide the price of anarchy in closed form for the special case where all players are buy-in users. The price of anarchy in this case is simply the ratio between the social cost in Corollary 1 and that in Corollary 2. In Section 6.1, we also present numerical results of the PoA in the general case.

Corollary 3. *When all players are buy-in users, the price of anarchy of an SBC game is*

$$PoA = \frac{1}{2}\left(\sqrt{\frac{k_b + (N-1)k_s}{\theta k_b}} + \sqrt{\frac{\theta k_b}{k_b + (N-1)k_s}}\right).$$

As the operating expense ratio θ increases, the price of anarchy decreases (note that we have $\theta < 1$). Alongside, as the ratio k_s/k_b increases, the price of anarchy increases.

If the number of users N is large, the price of anarchy becomes arbitrarily large. Specifically, the price of anarchy grows in the order of $\Omega(\sqrt{N})$ in this special case.

In Section 6, we show through numerical simulations that, the price of anarchy actually decreases as the user workload distribution becomes more heterogeneous. As a result, the worst-case price of anarchy seems to occur when all users have homogeneous workloads, in which case they are all buy-in users.

5 SUBSIDY POLICIES

In this section, we investigate one possible way to lower the price of anarchy, namely by subsidizing users according to their payments. We first introduce the subsidy term into our model, then propose two subsidy policies and analyze their effects.

In [25], it is shown that, in order to reach the social optimum, users need to pay more than the prices they pay at the Nash equilibrium. We are thus motivated to come up with policies that incentivize users to purchase buy-in resources. One such policy (adopted by BU SCC) is that users get some credits when their idle buy-in nodes are used by others, and those credits can be used in order to acquire more resources on demand. In the following, we will change the cost function of each user (14), to take the effects of such a subsidy policy into account, and investigate how the outcome of the game changes consequently.

The credits are awarded to users only when their buy-in resources are used by others, which can be translated into an additional computing rate $\Delta_i(\mathbf{p})$ in the cost function.

As a result, the new cost function of each player i becomes:

$$C_i(\mathbf{p}) = \frac{\alpha_i \omega_i}{k_b p_i + \sum_{j \neq i} k_s p_j + \Delta_i(\mathbf{p})} + p_i. \tag{25}$$

Define the utility of player i as the gross payoff $\Gamma(\omega_i)$ for completing its workload ω_i, minus its own cost. The utility function can be expressed by:

$$U_i(\mathbf{p}) = \Gamma(\omega_i) - \frac{\alpha_i \omega_i}{k_b p_i + \sum_{j \neq i} k_s p_j + \Delta_i(\mathbf{p})} - p_i.$$

For each player i, the best response strategy is still the one that can minimize its own cost $C_i(\mathbf{p})$ (or equivalently, maximize its utility $U_i(\mathbf{p})$), which is given by $p_i = \operatorname{argmax}_{p_i} U_i(\mathbf{p}) = \operatorname{argmin}_{p_i} C_i(\mathbf{p})$.

The next task is to state the exact form of $\Delta_i(\mathbf{p})$. There are many possibilities for determining $\Delta_i(\mathbf{p})$, yet we prefer a simple yet effective one, so that not only the outcome (e.g., social cost) is easier to predict, but it is also easier for users to make decisions. With that in mind, we compare between the following two forms of subsidy within our model: $\Delta_i(\mathbf{p}) = k_c p_i$ (linear form) and $\Delta_i(\mathbf{p}) = k_c p_i^2$ (quadratic form), where k_c is a coefficient of proportionality.

Note that, by choosing a quadratic form for $\Delta_i(\mathbf{p})$, heavy users get a larger marginal increase in their computing rate. Considering the fact that light users benefit relatively more in the game previously considered in [25], the quadratic form makes the game fairer, compared with a linear form.

Next, we investigate the properties of the SBC game with the two different subsidy policies.

Subsidy of the form of $\Delta_i(\mathbf{p}) = k_c p_i$. The next theorem states that, with this form of subsidy policy, the game satisfies all the properties of regular SBC games in terms of the Nash equilibria.

THEOREM 5.1. *An SBC game with subsidy of the form of* $\Delta_i(\mathbf{p}) = k_c p_i$ *has the following properties:*

(i) At a Nash equilibrium of the game,

$$p_1 \geq p_2 \geq \cdots \geq p_N \geq 0.$$

(ii) The game has a unique Nash equilibrium.
(iii) The game is submodular.
(iv) Through best-response dynamics, the game converges to its unique Nash equilibrium from all possible initial states.

Proof. For each player i, the cost function is

$$\begin{aligned} C_i(\mathbf{p}) &= \frac{\alpha_i \omega_i}{k_b p_i + \sum_{j \neq i} k_s p_j + k_c p_i} + p_i \\ &= \frac{\alpha_i \omega_i}{(k_b + k_c) p_i + \sum_{j \neq i} k_s p_j} + p_i \\ &= \frac{\alpha_i \omega_i}{k_b^* p_i + \sum_{j \neq i} k_s p_j} + p_i, \end{aligned}$$

where $k_b^* = k_b + k_c$. Consider k_b^* as the equivalent k_b in a regular SBC game, the theorem then follows Lemma 1, Theorem 1, Lemma 7, and Theorem 2 in [25]. □

Note that the new cost function for each player still satisfies the definition of an SBC game, except that the buy-in factor is larger. Therefore, all the conclusions in previous sections are applicable to an SBC game with subsidy of the form of $\Delta_i(\mathbf{p}) = k_c p_i$.

Lemma 5.2. *For an SBC game with subsidy of the form of $\Delta_i(\mathbf{p}) = k_c p_i$, we have:*
(i) The social cost at the optimum is

$$C_{OPT}(\mathbf{p}) = \left(\frac{1}{\sqrt{\psi}} + \frac{\theta \sqrt{\psi}}{k_b^* + (n_1 - 1) k_s} \right) \sum_{i \in S_1} \sqrt{\alpha_i \omega_i} + \frac{(k_b^* + (n_1 - 1) k_s)}{k_s \sqrt{\psi}} \frac{\sum_{m \in S_2} \alpha_m \omega_m}{\sum_{i \in S_1} \sqrt{\alpha_i \omega_i}},$$

(ii) The social cost at Nash equilibrium is

$$C_{NE}(\mathbf{p}) = \left(\frac{1}{k_b^*} + \frac{\theta}{k_b^* + (n_1 - 1) k_s} \right) \sqrt{k_b^*} \sum_{i \in S_1} \sqrt{\alpha_i \omega_i} + \frac{(k_b^* + (n_1 - 1) k_s)}{k_s \sqrt{k_b^*}} \frac{\sum_{j \in S_2} \alpha_j \omega_j}{\sum_{i \in S_1} \sqrt{\alpha_i \omega_i}},$$

(iii) The price of anarchy is

$$PoA = \sqrt{\frac{\psi}{k_b^*}} \times \frac{(\theta k_b^* + k_b^* + (n_1 - 1) k_s) k_s (\sum_{i \in S_1} \sqrt{\alpha_i \omega_i})^2 + (k_b^* + (n_1 - 1) k_s)^2 \sum_{m \in S_2} \alpha_m \omega_m}{(\theta \psi + k_b^* + (n_1 - 1) k_s) k_s (\sum_{i \in S_1} \sqrt{\alpha_i \omega_i})^2 + (k_b^* + (n_1 - 1) k_s)^2 \sum_{m \in S_2} \alpha_m \omega_m},$$

where

$$\begin{aligned} k_b^* &= k_b + k_c, \\ \psi &= \frac{k_b + k_c + (n_1 - 1) k_s}{\theta} \left(1 + \frac{(k_b + k_c + (n_1 - 1) k_s) \sum_{m \in S_2} \alpha_m \omega_m}{k_s (\sum_{i \in S_1} \sqrt{\alpha_i \omega_i})^2} \right). \end{aligned}$$

Proof. In the proof of Theorem 5.1, it is shown that an SBC game with subsidy of the form of $\Delta_i(\mathbf{p}) = k_c p_i$ is also an SBC game with buy-in factor $k_b^* = k_b + k_c$. Thus, Theorem 4.4 and Theorem 4.5 still apply to the game. Replacing the buy-in factor by $k_b + k_c$, we get the closed-form social cost at the optimum and the Nash equilibrium in (i) and (ii), respectively. The price of anarchy in (iii) can be computed accordingly as their ratio. □

Subsidy of the form of $\Delta_i(\mathbf{p}) = k_c p_i^2$. We next obtain several properties of the game, by establishing a connection to aggregative games. Based on these properties, we then derive a convergence result for the best-response dynamics of the game. The convergence result enables us to compute the Nash equilibrium by running best-response dynamics. As a result, we can get numerical solutions to the social cost and price of anarchy, and evaluate the impact of the subsidy even though closed-form solutions are difficult to derive.

Lemma 5.3. *An SBC game with subsidy of the form* $\Delta_i(\mathbf{p}) = k_c p_i^2$ *admits a Nash equilibrium under the following sufficient condition*

$$k_c \leq \frac{\alpha_i k_b^2 \omega_i}{2k_s \sum_{j\neq i} p_j}, \ \forall i. \tag{26}$$

Moreover, the utility function $U_i(\mathbf{p})$ *of player i is strictly concave in* p_i*, given* $\{p_j | j \neq i\}$.

Proof. The idea of the proof is to find out the sufficient condition such that $U_i(\mathbf{p})$ is concave with respect to p_i, as concave games always admit a Nash equilibrium following from Theorem 1 in [21].

For each player i, the best response is lower-bounded by 0. Taking the partial derivative of $U_i(\mathbf{p})$ in terms of p_i, we get

$$\frac{\partial U_i(\mathbf{p})}{\partial p_i} = \frac{(k_b + 2k_c p_i)\alpha_i \omega_i}{(k_b p_i + k_c p_i^2 + \sum_{\ell\neq i} k_s p_\ell)^2} - 1.$$

Note that, if p_i is greater than a large enough $p_{i-upper}$, the partial derivative $\partial U_i(\mathbf{p})/\partial p_i$ will be less than 0, which indicates that the utility decreases as p_i increases, hence the best response must be upper-bounded by $p_{i-upper}$. As a result, the best response p_i can take any value from $[0, p_{i-upper}]$, which implies that P_i is compact and convex.

Next, observe that $k_b > 0$, $k_s > 0$, and $p_i \geq 0$ for all i, thus the payoff $U_i(\mathbf{p})$ of player i is continuous in $\mathbf{p}$. Moreover, taking the second derivative of the payoff function $U_i(\mathbf{p})$ with respect to p_i, we obtain

$$\frac{\partial^2 U_i(\mathbf{p})}{\partial p_i^2} = \frac{2k_c(\sum_{j\neq i} k_s p_j) - \alpha_i\omega_i(k_b^2 + 2k_b k_c p_i + 2k_c^2 p_i^2)}{(k_b p_i + k_c p_i^2 + \sum_{j\neq i} k_s p_j)^3}.$$

The condition (26) is equivalent to

$$2k_c(\sum_{j\neq i} k_s p_j) - \alpha_i k_b^2 \omega_i < 0, \ \forall i.$$

Thus we have

$$\begin{aligned}
\frac{\partial^2 U_i(\mathbf{p})}{\partial p_i^2} &= \frac{2k_c(\sum_{j\neq i} k_s p_j) - \alpha_i\omega_i(k_b^2 + 2k_b k_c p_i + 2k_c^2 p_i^2)}{(k_b p_i + k_c p_i^2 + \sum_{j\neq i} k_s p_j)^3} \\
&= \frac{(2k_c(\sum_{j\neq i} k_s p_j) - \alpha_i k_b^2 \omega_i) - 2\alpha_i\omega_i(k_b k_c p_i + k_c^2 p_i^2)}{(k_b p_i + k_c p_i^2 + \sum_{j\neq i} k_s p_j)^3} \\
&< 0,
\end{aligned}$$

which implies that $U_i(\mathbf{p})$ is strictly concave in p_i given fixed $\{p_j \mid j \neq i\}$. The lemma then follows from Theorem 1 in [21].

□

Lemma 5.4. *At a Nash equilibrium of an SBC game with subsidy of the form* $\Delta_i(\mathbf{p}) = k_c p_i^2$*, we have*

$$p_1 \geq p_2 \geq \cdots \geq p_N \geq 0,$$

under the following sufficient condition

$$k_c \leq \min\left\{\frac{\alpha_i k_b^2 \omega_i}{2k_s \sum_{j\neq i} p_j}, \frac{k_s(k_b - k_s)\sum_{j\in S} p_j}{\alpha_i \omega_i}\right\}, \ \forall i. \tag{27}$$

Proof. In order to prove the lemma we shall show that, given $\alpha_i\omega_i \geq \alpha_j\omega_j$, we have $p_i \geq p_j$ at the Nash equilibrium, for all $i, j \in S$. We prove this statement by contradiction, that is, given $\alpha_i\omega_i \geq \alpha_j\omega_j$, we show that $p_i < p_j$ cannot hold.

First, we investigate the conditions that p_i and p_j must satisfy at the Nash equilibrium. At the equilibrium, player i maximizes its payoff $U_i(\mathbf{p})$ given other players' payments. In Lemma 5.3, we have proven that the payoff function $U_i(\mathbf{p})$ of player i is strictly concave in p_i, given $\{p_\ell | \ell \neq i\}$, that is, $\partial^2 U_i(\mathbf{p})/\partial p_i^2 < 0$. As a result, the partial derivative of the payoff function $\partial U_i(\mathbf{p})/\partial p_i$ decreases with respect to p_i. Let p^* denote the solution to the equation

$$\frac{\partial U_i(\mathbf{p})}{\partial p_i} = \frac{(k_b + 2k_c p_i)\alpha_i\omega_i}{(k_b p_i + k_c p_i^2 + \sum_{\ell\neq i} k_s p_\ell)^2} - 1 = 0.$$

Consider the constraint that $p_i \geq 0$, the payoff function $U_i(\mathbf{p})$ reaches its optimum at $p_i = p^*$, if $p^* > 0$; or $p_i = 0$, if $p^* \leq 0$. Therefore, at the Nash equilibrium, p_i satisfies

$$\begin{cases} p_i = p^* > 0, \text{ iff } (k_b + 2k_c p_i)\alpha_i\omega_i = (k_b p_i + k_c p_i^2 + \sum_{\ell\neq i} k_s p_\ell)^2, \\ p_i = 0 \geq p^*, \text{ iff } (k_b + 2k_c p_i)\alpha_i\omega_i \leq (k_b p_i + k_c p_i^2 + \sum_{\ell\neq i} k_s p_\ell)^2. \end{cases}$$

Next, assume that given $\alpha_i\omega_i \geq \alpha_j\omega_j$, we have $p_i < p_j$, which has two cases: $0 = p_i < p_j$, and $0 < p_i < p_j$. In both cases, p_i and p_j satisfy

$$(k_b + 2k_c p_i)\alpha_i\omega_i \leq (k_b p_i + k_c p_i^2 + k_s p_j + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2, \tag{28}$$

$$(k_b + 2k_c p_j)\alpha_j\omega_j = (k_b p_j + k_c p_j^2 + k_s p_i + \sum_{\ell\neq i,\ell\neq j} k_s p_\ell)^2. \tag{29}$$

We show the contradiction by proving the equations above cannot hold if $\alpha_i\omega_i \geq \alpha_j\omega_j$ and $p_i < p_j$.

Take the difference between (28) and (29), we get

$$\begin{aligned} &k_b(\alpha_i\omega_i - \alpha_j\omega_j) + 2k_c(\alpha_i\omega_i p_i - \alpha_j\omega_j p_j) \\ \leq &(p_i - p_j)(k_b - k_s + k_c(p_i + p_j))((k_b + k_s)(p_i + p_j) + k_c(p_i^2 + p_j^2) + 2\sum_{\ell\neq i,\ell\neq j} k_s p_\ell) \\ = &(p_i - p_j)(k_b - k_s + k_c(p_i + p_j))(k_b(p_i + p_j) + k_c(p_i^2 + p_j^2) + 2\sum_{i\in S} k_s p_\ell). \end{aligned} \tag{30}$$

The LHS of (30) can be re-written as

$$k_b(\alpha_i\omega_i - \alpha_j\omega_j) + 2k_c p_j(\alpha_i\omega_i - \alpha_j\omega_j) + 2k_c\alpha_i\omega_i(p_i - p_j). \tag{31}$$

Move the last term of (31) to the RHS of (30), we get

$$\begin{aligned}
&k_b(\alpha_i\omega_i - \alpha_j\omega_j) + 2k_c p_j(\alpha_i\omega_i - \alpha_j\omega_j)\\
\leq&(p_i - p_j)[(k_b - k_s + k_c(p_i + p_j))(k_b(p_i + p_j) + k_c(p_i^2 + p_j^2) + 2\sum_{i\in S} k_s p_\ell) - 2k_c\alpha_i\omega_i]\\
=&(p_i - p_j)[(2(k_b - k_s)\sum_{i\in S} k_s p_\ell - 2k_c\alpha_i\omega_i) + 2k_c(p_i + p_j)\sum_{i\in S} k_s p_\ell\\
&+ (k_b - k_s + k_c(p_i + p_j))(k_b(p_i + p_j) + k_c(p_i^2 + p_j^2))]. \qquad (32)
\end{aligned}$$

When k_c satisfies the constraint (27), we have

$$2(k_b - k_s)\sum_{i\in S} k_s p_\ell - 2k_c\alpha_i\omega_i \geq 0.$$

Under the assumption that $0 \leq p_i < p_j$, we get that the RHS of (32) is strictly less than 0. However, from $\alpha_i\omega_i \geq \alpha_j\omega_j$ we get that the LHS of (32) is no less than 0, such that the inequality (32) cannot hold. Hence, $\alpha_i\omega_i \geq \alpha_j\omega_j$ and $p_i < p_j$ cannot the hold at the same time, and the lemma is proven. □

Define a game $(S, \{P_i\}_{i\in S,}, \{U_i\}_{i\in S})$ as an *aggregative game* [6], if for each player i, its payoff U_i is a function of p_i and $\sum_{j\in S} p_j$, i.e., $U_i(\mathbf{p}) = U_i(p_i, \sum_{j\in S} p_j)$. In an aggregative game, the payoff of player i depends only on its own strategy p_i and the aggregate of all players' strategies $\sum_{j\in S} p_j$.

Lemma 5.5. *An SBC game with subsidy of the form $\Delta_i(\mathbf{p}) = k_c p_i^2$ is an aggregative game.*

Proof. Note that in an SBC game with subsidy of the form $\Delta_i(\mathbf{p}) = k_c p_i^2$, the payoff function U_i of player i can be written as

$$\begin{aligned}
U_i(\mathbf{p}) =&\Gamma(\omega_i) - \frac{\alpha_i\omega_i}{k_b p_i + \sum_{j\neq i} k_s p_j + k_c p_i^2} - p_i\\
=&\Gamma(\omega_i) - \frac{\alpha_i\omega_i}{(k_b - k_s)p_i + k_c p_i^2 + k_s\sum_{j\in S} p_j} - p_i\\
=&U_i^*(p_i, \sum_{j\in S} p_j).
\end{aligned}$$

The lemma then follows. □

Lemma 5.6. *An SBC game with subsidy of the form $\Delta_i(\mathbf{p}) = k_c p_i^2$ is submodular.*

Proof. The strategy set P_i of player i is continuous by assumption and lower-bounded by 0. Moreover, it has an inherent upper-bound since when the strategy p_i is above a threshold, the cost of will increase monotonically with p_i. Therefore, P_i is a compact subset of $\mathbb{R}$.

Given a twice continuously-differentiable function $f : \mathbf{X} \longrightarrow R$, f has decreasing difference in if and only if

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \leq 0, \ \forall\, i \neq j.$$

Consider the payoff function $U_i(p)$ of player i. We have

$$\frac{\partial^2 U_i(\mathbf{p})}{\partial p_i \partial p_j} = -\frac{\alpha_i k_s(k_b + 2k_c p_i)\omega_i}{(k_b p_i + k_c p_i^2 + \sum_{\ell\neq i} k_s p_\ell)^3} < 0, \ \forall\, j \neq i.$$

Therefore, the payoff function $U_i(\mathbf{p})$ has decreasing difference in (p_i, p_{-i}), where p_{-i} denotes the strategies of players other than player i. Furthermore, note that $k_b > 0$, $k_s > 0$, $p_i \geq 0$ and $p_{-i} \geq 0$

for all i, from which we get that $U_i(\mathbf{p})$ is continuous in (p_i, p_{-i}). Thus, we deduce that an SBC game with subsidy in the form of $\Delta_i(\mathbf{p}) = k_c p_i^2$ is submodular. □

LEMMA 5.7. *Given the other players' strategies, the best response of player i in an SBC game with subsidy in the form of $\Delta_i(\mathbf{p}) = k_c p_i^2$ is single-valued, except when it has a specific workload ω_i^0, in which case there exist two best response strategies, one of them being 0. Moreover, the best response is a non-decreasing function of ω_i.*

PROOF. Note that $p_i \geq 0$. Taking the partial derivative of $C_i(\mathbf{p})$, we get

$$\frac{\partial C_i(\mathbf{p})}{\partial p_i} = \frac{(k_b p_i + k_c p_i^2 + \sum_{j\neq i} k_s p_j)^2 - \alpha_i \omega_i (k_b + 2k_c p_i)}{(k_b p_i + k_c p_i^2 + \sum_{j\neq i} k_s p_j)^2}.$$

Note that the numerator can be written as $f_i(p_i) - \alpha_i k_b \omega_i$, where $f_i(p_i)$ is defined as

$$f_i(p_i) = (k_b p_i + k_c p_i^2 + \sum_{j\neq i} k_s p_j)^2 - 2\alpha_i k_c \omega_i p_i. \tag{33}$$

As a result, to solve $\partial C_i(\mathbf{p})/\partial p_i = 0$, we only need to compare $\alpha_i k_b \omega_i$ with $f_i(p_i)$. Specifically,
if $f_i(p_i) < \alpha_i k_b \omega_i$, we have $\partial C_i(\mathbf{p})/\partial p_i < 0$;
if $f_i(p_i) = \alpha_i k_b \omega_i$, we have $\partial C_i(\mathbf{p})/\partial p_i = 0$;
if $f_i(p_i) > \alpha_i k_b \omega_i$, we have $\partial C_i(\mathbf{p})/\partial p_i > 0$.
Next, in order to compare $\alpha_i k_b \omega_i$ with $f_i(p_i)$, we take the derivative of $f_i(p_i)$, such that we can find out how it changes with p_i.
The first derivative of $f_i(p_i)$ is

$$\begin{aligned}\frac{\mathrm{d}f_i(p_i)}{\mathrm{d}p_i} =& 2(k_b p_i + k_c p_i^2 + \sum_{j\neq i} k_s p_j)(2k_c p_i + k_b) - 2\alpha_i k_c \omega_i \\ =& 2\left[2k_c^2 p_i^3 + 3k_b k_c p_i^2 + (2k_c k_s \sum_{j\neq i} p_j + k_b^2) p_i + k_b k_s \sum_{j\neq i} p_j\right] - 2\alpha_i k_c \omega_i,\end{aligned}$$

which is an increasing function of p_i when $p_i \geq 0$. Moreover, note that when $p_i = 0$, we have

$$\frac{\mathrm{d}f_i(0)}{\mathrm{d}p_i} = 2k_b k_s \sum_{j\neq i} p_j - 2\alpha_i k_c \omega_i.$$

Next, consider the following two cases: (a) $\frac{\mathrm{d}f_i(0)}{\mathrm{d}p_i} \geq 0$; and (b) $\frac{\mathrm{d}f_i(0)}{\mathrm{d}p_i} < 0$. We use p_i^n to denote specific values such that $f_i(p_i^t) = \alpha_i k_b \omega_i$, where $t = 0, 1, 2, 3, \ldots$ is used to number those values. The minimum of $f_i(p_i)$ is denoted by $f_i^{\min}$, and the maximum is denoted by $f_i^{\max}$.

(a) $\frac{\mathrm{d}f_i(0)}{\mathrm{d}p_i} \geq 0$, equivalently, $k_b k_s \sum_{j\neq i} p_j \geq \alpha_i k_c \omega_i$. In such a case, $f_i(p_i)$ increases as p_i increases. There are two sub-cases (a-i) and (a-ii), according to the value of ω_i:
  - (i) $f_i(0) \geq \alpha_i k_b \omega_i$, so $f_i(p_i) > \alpha k_b \omega_i$ for all $p_i > 0$. Then we get that C_i is an increasing function of p_i, thus, $p_i = 0$ is the best response.
  - (ii) $f_i(0) < \alpha_i k_b \omega_i$, then there exists p_i^0, such that $f_i(p_i^0) = \alpha_i k_b \omega_i$, $f_i(0) < \alpha_i k_b \omega_i$ for $p_i < p_i^0$, and $f_i(0) > \alpha_i k_b \omega_i$ for $p_i > p_i^0$. Thus, C_i decreases within interval $[0, p_i^0]$, increases within interval (p_i^0, ∞), and $p_i = p_i^0$ is the best response. Furthermore, note that $f_i(p_i)$ increases with p_i, so p_i^0 increases as ω_i increases.

(b) $\frac{\mathrm{d}f_i(0)}{\mathrm{d}p_i} < 0$, equivalently, $k_b k_s \sum_{j\neq i} p_j < \alpha_i k_c \omega_i$. In such a case, $f_i(p_i)$ first decreases to $f_i^{\min}$, then increases as p_i increases. There are three sub-cases (b-i), (b-ii), and (b-iii), according to the value of ω_i:

(i) $\alpha_i k_b \omega_i < f_i^{\min}$, so $f_i(p_i) > \alpha_i k_b \omega_i$ for all $p_i > 0$. Then we get that C_i is an increasing function of p_i, thus, $p_i = 0$ is the best response.

(ii) $f_i^{\min} \leq \alpha_i k_b \omega_i < f_i(0)$, then there exists p_i^1 and p_i^2, such that $f_i(p_i^1) = f_i(p_i^2) = \alpha_i k_b \omega_i$. Moreover, C_i increases within the interval $[0, p_i^1)$, decreases within the interval $[p_i^1, p_i^2)$, increases within interval (p_i^2, ∞). To find the best response, we need to compare between $C_i(p_i^2)$ and $C_i(0)$. Let δ_1 denote the increase of C_i within interval $[0, p_i^1)$, and δ_2 denote the decrease of C_i within interval $[p_i^1, p_i^2)$. We then have $C_i(p_i^2) = C_i(0) + \delta_1 - \delta_2$.
Consider the case where ω_i is relatively small (close to $f_i^{\min}$), p_i^1 and p_i^2 are very close to each other, so that $\delta_1 > \delta_2$, and $C_i(0) < C_i(p_i^2)$. Therefore, $p_i = 0$ is the best response. As ω_i becomes larger, p_i^1 goes towards 0, and p_i^2 goes towards ∞, so that δ_1 decreases and δ_2 increases, as a result, $C_i(p_i^2)$ decreases, eventually becomes less than $C_i(0)$, so $p_i = p_i^2$ becomes the best response. A special case is when $\delta_1 = \delta_2$ given a workload ω_i^0, where $C_i(0) = C_i(p_i^2)$, and both 0 and p_i^2 are the best responses.

(iii) $f_i(0) \leq \alpha_i k_b \omega_i$, then there exists p_i^3, such that $f_i(p_i^3) = \alpha_i k_b \omega_i$, $f_i(0) < \alpha_i k_b \omega_i$ for $p_i < p_i^3$, and $f_i(0) > \alpha_i k_b \omega_i$ for $p_i > p_i^3$. Thus, C_i decreases within interval $[0, p_i^3]$, increases within interval (p_i^3, ∞), and $p_i = p_i^3$ is the best response. Furthermore, note that $f_i(p_i)$ increases with p_i, so p_i^3 increases as ω_i increases.

We thus conclude that, in all cases, the best response of a player is either a unique positive value $p_i^\star > 0$, or $\{0, p_i^\star\}$. This suggests that Lemma 5.7 holds. □

We are ready to prove that, if k_c satisfies the condition in Lemma 5.3, the game can always converge to an equilibrium under some reasonable assumptions.

THEOREM 5.8. *Assume that if a player has two best response strategies that yield the same minimal cost, one of which is to pay* 0*, then the player prefers not to pay. Under this assumption, through best-response dynamics, an SBC game with subsidy of the form of* $\Delta_i(\mathbf{p}) = k_c p_i^2$ *converges to its Nash equilibrium (or equilibria) from all possible initial states under the sufficient condition*

$$k_c \leq \frac{\alpha_i k_b^2 \omega_i}{2k_s \sum_{j \neq i} p_j}, \ \forall i. \tag{34}$$

PROOF. By Lemma 5.3, condition (34) guarantees that there exists at least one Nash equilibrium in the game. The proof then follows essentially the same as the proof of Theorem 2 in [25], as we have shown that the game is aggregative and submodular, and has single-valued best response. The best response is single-valued under the assumption that that players prefer not to pay if they can minimize costs, which addresses the only case where there exist two best responses and one of them is 0. □

6 NUMERICAL RESULTS

In this section, we present numerical results to validate and expand on our theoretical analysis. We first provide numerical results about the social cost both at equilibrium and at optimum, and the corresponding price of anarchy. Next, we evaluate the subsidy policies introduced in Section 5 and show that they are indeed able to significantly lower the social cost at the Nash equilibrium. We then simulate the game with heterogeneous shared factors, in order to show that the assumption of a homogeneous shared factor provides a satisfactory approximation.

6.1 Social cost and price of anarchy

In this subsection, we numerically evaluate the social costs at equilibrium and at optimum, and then compute the corresponding price of anarchy. We consider shared/buy-in computing games modeled by heterogeneously distributed workloads observed in real-world systems, and investigate the influence of different parameters on the social cost and price of anarchy.

Several studies have shown that job completion time (makespan) in large-scale clouds can be modeled with a log-normal distribution [8, 19]. This is also true for shared/buy-in clusters [17]. The formula for the log-normal distribution is

$$P(W \le \omega) = \frac{1}{2} + \frac{1}{2}\text{erf}\left[\frac{\ln \omega - \nu}{\sigma}\right],$$

where W denotes the random variable, which is workload in our model, and erf$[\cdot]$ stands for the error function

$$\text{erf}[z] = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

The log-normal distribution we use has two parameters ν and σ, which are respectively the mean and the standard deviation of the random variable's natural logarithm.

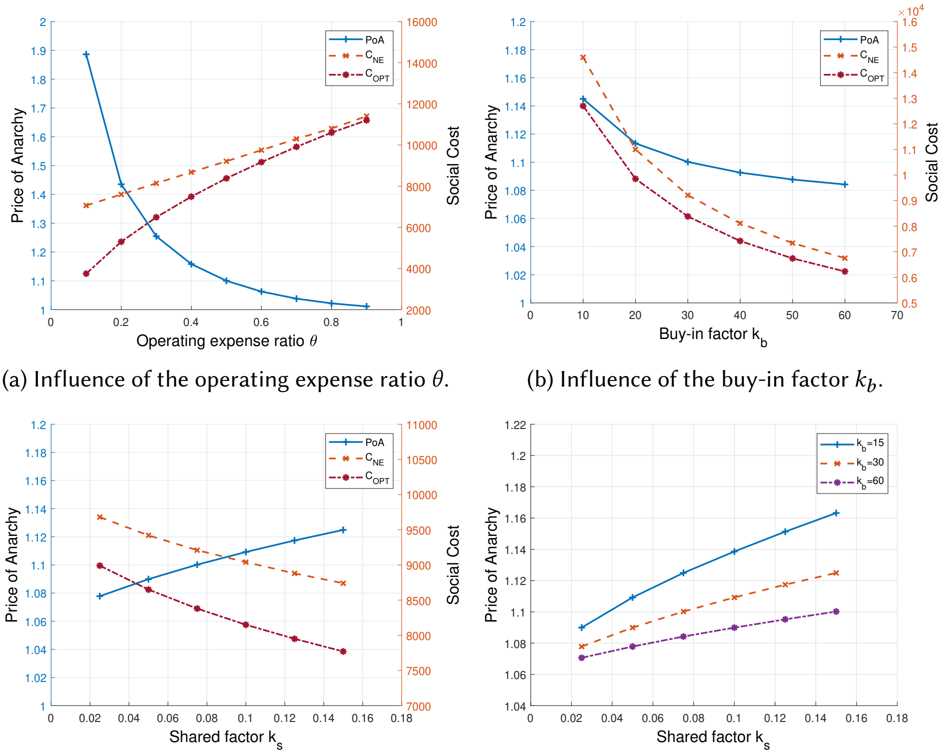
Unless stated otherwise, in our simulations, we set $\nu = 7.37$ and $\sigma = 5.69$ for the base case, similar to [25]. We discard very large random samples, since real-world shared/buy-in computing systems cannot sustain workloads that exceed some threshold. We set this threshold to $\omega = 5 \times 10^6$ (CPU-hours). Since the workloads are generated randomly from a log-normal distribution, each run of simulations yields different results, even under the same parameters. Therefore, for each setting of log-normal distribution parameters, we generate 100 groups of workloads and use them to run 100 individual simulations.

We assume that the system has $N = 200$ users, and for each of them, the cost per unit of time is $\alpha = 1$. For the base case, we set $k_b = 30$ and $k_s = 0.075$, such that the ratio between the two types of resources used $k_b \sum_{i \in S} p_i / (N-1) k_s \sum_{i \in S} p_i$, is roughly the same as the ratio observed in the BU SCC data [17]. The operating expense ratio of the system is set to $\theta = 0.5$, for the base case.

First, we validate our closed-form social cost at the optimum, by comparing it with the results obtained through coordinate descent methods. We have proven in Theorem 4.1 that the social cost function is strictly convex in the payments of players. Since the social cost function (1) is continuous and differentiable, a simple approach to solving this constrained optimization problem is by employing a coordinate descent method, which minimizes along the coordinate directions until it reaches a local optimum. To implement coordinate descent, we adopt an approach similar to best-response dynamics: each player updates its strategy in turn, so as to minimize the social cost function (1) given the other players' strategies, until no one needs to change their strategy anymore.

We find that the numerical results by coordinate descent are identical to the closed-form social cost (within the range of error). Specifically, for the base case described above, the simulated social cost and our closed-form social cost are both $C_{OPT} = 8,749$ (CPU-hours) on average. The ratio between their difference and the simulated social cost has an average of only 2.71×10^{-7}, and a standard deviation of 4.01×10^{-8}. We obtained similar results for different combinations of the parameters θ, k_b, and k_s. This serves as a validation of the closed-form optimal social cost given in Theorem 4.4.

We further compute the player strategies at the Nash equilibrium to validate the closed-form social cost at the equilibrium, provided by Theorem 4.5. We find that the results indeed coincide, with $C_{NE} = 9,628$ (CPU-hours). Furthermore, when numerically computing the ratio of the player strategies at the equilibrium to the player strategies at the optimum, we find that their ratio is identical to the value given by Lemma 4.6, with an average of 0.6411 and a standard deviation of 0.0036.



(a) Influence of the operating expense ratio θ.

(b) Influence of the buy-in factor k_b.

(c) Influence of the shared factor k_s.

(d) Influence of k_s under different settings of k_b.

Fig. 1. Influence of different system parameters on the social cost and price of anarchy. The operating expense ratio θ has the most significant influence.

We next compare the average values of the social cost at the optimum and the equilibrium, and compute the price of anarchy. For the base case described above, the social cost at the Nash equilibrium is $C_{NE} = 9,628$ (CPU-hours), and the social cost at the optimum is $C_{OPT} = 8,749$ (CPU-hours), which results in price of anarchy equal to 1.1005. Therefore, in this case, the social cost of the Nash equilibrium is about 10% higher than optimal.

Next, we vary the parameters θ, k_b, and k_s, to investigate how the social cost and price of anarchy are influenced by these different factors. The results are depicted in Fig. 1. We find that the operating expense ratio θ has the most significant influence on the price of anarchy: with other parameters kept unchanged, when θ decreases toward 0, the price of anarchy significantly increases. Recall that θ is the ratio between the operational cost and the payment by users. Thus, the Nash equilibrium of the system can be pretty inefficient if the provider has a low operating expense ratio, i.e., it gets a relatively higher net revenue from the same overall payment. On the other hand, while the price of anarchy also depends on the buy-in factor k_b and shared factor k_s, the influence of these parameters is not as significant as θ. Moreover, Fig. 1(d) shows that, no matter what the values of k_b and k_s are, as long as the ratio k_s/k_b is fixed, the price of anarchy remains the same. When the ratio k_s/k_b increases, the price of anarchy increases as well. Intuitively, users benefit more from the shared resources as k_s/k_b grows larger, such that they tend to pay less and rely on the shared resources. However, when everyone pays less for buy-in resources, the overall amount of available

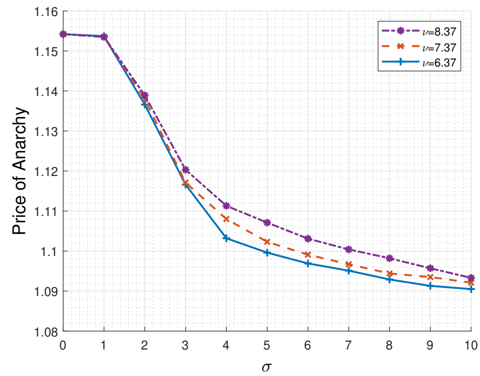


Fig. 2. Influence of the distribution parameter σ under different settings of ν (ν and σ^2 are the mean and variance of the workload's natural logarithm, respectively). The PoA decreases as the workload becomes more heterogeneous.

resources decreases, hence the system becomes less efficient in completing all the workloads. It is also worth noting that the social cost does decrease as k_b and k_s increase.

The distribution of user workloads also affects the social cost and price of anarchy. Fig. 2 evaluates the impact of different settings of the ν and σ parameters. We find that as the distribution parameter ν increases, the price of anarchy increases slightly. This implies that as the total user workload increases, the system operates at a less efficient state. The distribution parameter σ has the opposite and more significant influence: the price of anarchy decreases as σ increases, that is, as the distribution of user workloads becomes more heterogeneous. We also note that, if all the other system parameters are kept fixed, the price of anarchy reaches its maximum of 1.1542 when $\sigma = 0$, no matter what the value of ν is. In fact, the case where $\sigma = 0$ indicates that every user has the same workload, hence each user will be a buy-in user with the same payment. In Corollary 3, we have given the closed-form price of anarchy in the case where all users are buy-in users, and the result is precisely 1.1542 for our setting of system parameters k_b, k_s, and θ. Moreover, the analytical results in Lemma 3 show that the price of anarchy is not influenced by the user workloads if all the users are buy-in users, which explains why the price of anarchy is the same when $\sigma = 0$ under different values of ν. Note that the worst price of anarchy is still not very large in this example; this is due to the fact that, in order to guarantee the benefit of buy-in users, the shared factor k_s is in practice typically much smaller than the buy-in factor k_b.

In the simulations above, the parameters change around the parameters in the base case. In the base case, we have that $k_b \sum_{i \in S} p_i$ and $(N-1)k_s \sum_{i \in S} p_i$ are relatively comparable, which implies that N and k_s cannot be very large at the same time. However, in other scenarios, where θ is relatively small, and both N and k_s are relatively large, we can find that the price of anarchy becomes even larger. For example, consider the same parameters as in the base case, except that we let k_s range from 2 to 10, and θ ranges from 0.1 to 0.3. In such a case, Fig. 3 shows the price of anarchy can be close to 4, which indicates that the system is highly inefficient.

6.2 Impact of subsidy policies

In this subsection, we present numerical results regarding the price of anarchy and social cost when employing subsidy policies of the two forms elaborated in Section 5.

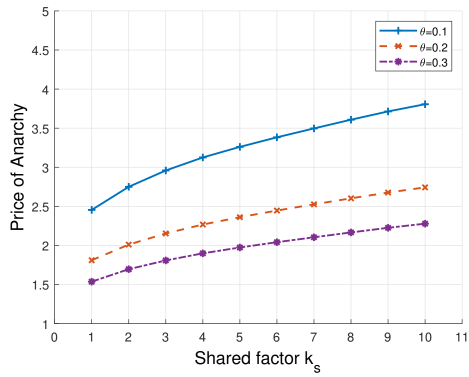


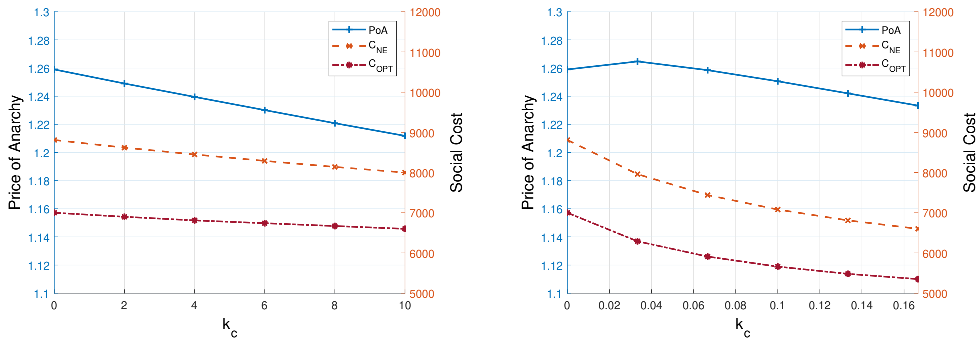
Fig. 3. Influence of the operating expense ratio θ and the shared factor k_s on the price of anarchy. When θ is small and k_s is large, the price of anarchy can be as large as 4 in this example.

We employ the subsidy policies on the base case of Subsection 6.1, except that $\theta = 0.3$. We assume that the total amount of computing resources is fixed, and the subsidy resources come from the shared resource pool. In the simulations, when we increase the subsidy term $\Delta_i(\mathbf{p})$, we decrease k_s such that the total computing rate of all users $\sum_{i\in S}[(k_b + (N-1)k_s)p_i + \Delta_i(\mathbf{p})]$ is fixed. For the linear form of subsidy $\Delta_i(\mathbf{p}) = k_c p_i$, we can keep the total computing rate exactly the same when we change k_c. For the quadratic form of subsidy $\Delta_i(\mathbf{p}) = k_c p_i^2$, due to the quadratic term, we can only make the total computing rate stay approximately the same. The impact of employing subsidy policies is illustrated in Fig. 4. Note that the price of anarchy in Fig. 4(b) is a lower-bound, since we have not formally proven that coordinate descent converges to the global minimum for the quadratic form of subsidies. Nevertheless, in our simulations, coordinate descent always yields the same minimum, irrespective of the initial state. It is also worth mentioning that, the values of the parameter of subsidy k_c shown in the two figures are not directly comparable, as k_c in Fig. 4(b) is smaller, so as to keep the total computing rate of all users fixed; instead, the same positions on the x-axes of the two figures share approximately the same amount of subsidy resources $\sum_{i\in S}\Delta_i(\mathbf{p})$.

We consider the Nash equilibrium of the game under the two subsidy policies. We find that, under both policies, the game exhibits two identical properties. First, we simulate the convergence through the best-response dynamics of the game, and find that the game always converges to a unique Nash equilibrium from random initial states. Second, we note that the Nash equilibrium satisfies $p_1 \geq p_2 \geq \cdots \geq p_N \geq 0$, that is, the prices paid by users decrease with their adjusted workloads ($\alpha_i\omega_i$). Note that the simulations of the game with the quadratic subsidy policy suggest more general properties than those established analytically in Section 5.

We note that both subsidy policies lower the social cost at equilibrium. However, the change is more significant for the quadratic form of subsidy $\Delta_i(\mathbf{p}) = k_c p_i^2$. Intuitively, with the quadratic form of subsidy, heavy users with large workloads get higher overall computing rates per unit of payment, thus they benefit more from the subsidy policy. Since the social cost is dominated by the costs of heavy users, subsidizing mainly the heavy users works better than subsidizing all users equally.

It is also worth noting that the subsidy policies improve both the social cost and the price of anarchy, except when k_c is close to 0 with quadratic subsidy. Since we can only make the total computing rate stay approximately the same when simulating the quadratic subsidy, the reason for this exception could be that the total computing rate with quadratic subsidy is slightly smaller



(a) Impact of the linear form of subsidy $\Delta_i(\mathbf{p}) = k_c p_i$.

(b) Impact of the quadratic form of subsidy $\Delta_i(\mathbf{p}) = k_c p_i^2$.

Fig. 4. Impact of subsidy policies on the social cost and price of anarchy. The linear subsidy policy primarily impacts the PoA, while the quadratic subsidy policy primarily impacts the social cost at equilibrium and at optimum.

than in the base case. More specifically, we find that the linear subsidy lowers the social cost at equilibrium by 9.2%, the social cost at optimum by 5.7%, and the price of anarchy by 4.8%; in comparison, the quadratic subsidy lowers the social cost at equilibrium by 25.1%, the social cost at optimum by 23.5%, and the price of anarchy by 2.0%. Considering the changes in the optimal social cost and price of anarchy, we find that the linear subsidy is preferable for lowering the price of anarchy, while the quadratic subsidy is preferable for lowering the optimal social cost.

Intuitively, under the linear subsidy policy, every user gets the same amount of subsidy per unit of payment, while the quadratic subsidy case is akin to a "differential" subsidy, in the sense that heavy users get more subsidy per unit of payment. In practice, the provider may prefer such a differential subsidy policy, as our simulations show that it is better at lowering social cost at equilibrium than a linear subsidy.

6.3 Simulations with heterogeneous shared factors

In this subsection, we run two series of numerical simulations for the case where each user i has a heterogeneous shared factor k_{si}. The goal is to verify whether the assumption of a homogeneous shared factor k_s for all users, as used in our analysis, provides a satisfactory approximation of the actual case.

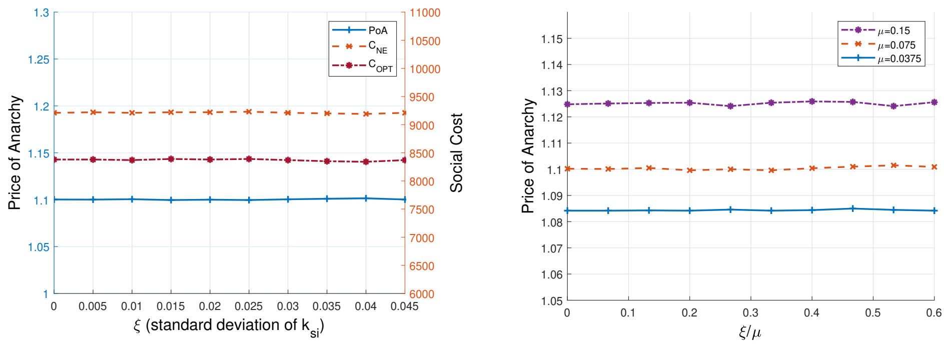
In the first series of simulations, we assume that k_{si} follows a normal distribution $\mathcal{N}(\mu, \xi^2)$, with mean $\mu = k_s$. Since the shared factor must be non-negative (the buy-in resources owned by a user can only benefit other users), we discard the sample of k_{si} if $k_{si} < 0$. In order to ensure that only a small fraction of samples will be discarded, we set ξ such that $\mu - 2\xi > 0$.

The user workloads are generated from the same log-normal distribution introduced in Section 6.1. For the base case, the other parameters are also the same as in Section 6.1: $N = 200$, $k_b = 30$, $k_s = 0.075$, and $\alpha = 1$ for all users. We distinguish between two cases in terms of the shared factors: the homogeneous case where $k_{si} = k_s$ for all the users, and the heterogeneous case where k_{si} generated from $\mathcal{N}(\mu, \xi^2)$. Note that, when $\xi = 0$, every user has the same shared factor, hence yielding the homogeneous case.

First, we compare the Nash equilibria and convergence properties of the homogeneous case versus the heterogeneous case. We find that, in both cases, the game always converges to a unique

Standard deviation (ξ)	0	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045
Rounds	7.75	7.78	7.79	7.72	7.84	7.78	7.75	7.87	7.85	7.91

Table 2. Average number of rounds needed for convergence vs. standard deviation ξ of the shared factors.



(a) Influence of the standard deviation ξ.

(b) Influence of ξ under different means μ.

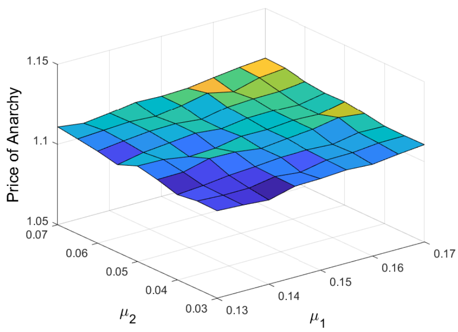
Fig. 5. The distribution of shared factors (k_{si} for user i) has little influence on the price of anarchy and social cost. μ and ξ are the mean and standard deviation of the shared factors, respectively.

Nash equilibrium, regardless of the initial state. Moreover, the game converges within a few rounds, where a round consists of having each player update its strategy per its best response. In Table 2, we show the relationship between the average number of rounds needed for convergence and the standard deviation ξ of the shared factors, by simulating the best-response dynamics from a random initial state 1,000 times. We find that the convergence rounds are basically not affected by the heterogeneity of the shared factors. We also find that the Nash equilibria for all ξ are very close to the one in the homogeneous case. For example, the average difference of the prices paid by a player between the homogeneous case and the heterogeneous case for $\xi = 0.025$ is only 0.1557, which is very small, as the average payment at equilibrium is 21.0750.

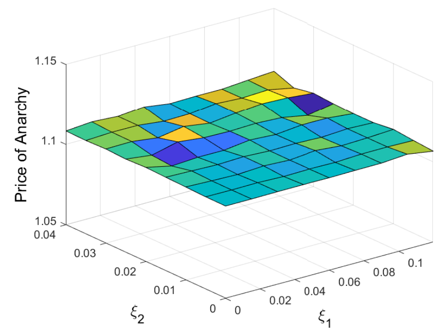
Next, we investigate the influence of the distribution of the shared factors k_{si} on the price of anarchy and social cost of the game. The results are shown in Fig. 5. The user workloads are set in a similar manner to Section 6.1. We generate five groups of workloads from the log-normal distribution, which are used in five individual simulations with the same distribution parameters of the shared factors, and we generate the shared factors from the Gaussian distribution.

We first fix the mean of the shared factors ($\mu = k_s = 0.075$), and change the heterogeneity of the shared factors by changing its standard deviation from 0 to 0.045. Fig. 5(a) shows that both the price of anarchy and social cost stay at the same level, regardless of the heterogeneity of the shared factors. Then, we try different values of the mean of the shared factors, from 0.0375 to 0.15. The result in Fig. 5(b) shows that the price of anarchy increases with the mean, and the standard deviation of the shared factors has little influence on the price of anarchy under different means. Note that, when $\xi = 0$, the result in Fig. 5(b) is in accordance with Fig. 1(c), where the price of anarchy increases with the homogeneous shared factor k_s.

Next, we conduct another series of simulations, where the shared factors exhibit even more heterogeneity. We assume that the users are divided into two groups, namely frequent users and occasional users. Frequent users tend to use their buy-in nodes very often, resulting in a low k_s; occasional users, on the other hand, tend to have a high k_s. We then randomly sample the shared factors out of two normal distributions $\mathcal{N}_1(\mu_1, \xi_1^2)$ and $\mathcal{N}_2(\mu_2, \xi_2^2)$, with 50% probability on each. In



(a) Influence of the means μ_1 and μ_2 of shared factors. The price of anarchy increases with $\mu_1+\mu_2$. When $\mu_1+\mu_2$ is fixed, $|\mu_1-\mu_2|$ has little influence on the price of anarchy.



(b) Influence of the standard deviations ξ_1 and ξ_2 of shared factors. ξ_1 and ξ_2 have little influence on the price of anarchy.

Fig. 6. Increasing the heterogeneity of shared factors has little influence on the price of anarchy. Two normal distributions $\mathcal{N}_1(\mu_1, \xi_1^2)$ and $\mathcal{N}_2(\mu_2, \xi_2^2)$ are used to simulate shared factors of the occasional and frequent user groups, respectively. The heterogeneity becomes higher when ξ_1, ξ_2, or $|\mu_1-\mu_2|$ increases.

other words, half of the users follow $\mathcal{N}_1(\mu_1, \xi_1^2)$ and the other half follows $\mathcal{N}_2(\mu_2, \xi_2^2)$. Moreover, their means satisfy $(\mu_1+\mu_2)/2 = k_s$. Again, we discard the negative samples of shared factors, and ascertain that sure $\mu_1 - 2\xi_1 > 0$ and $\mu_2 - 2\xi_2 > 0$.

We find that even as the heterogeneity increases (i.e., ξ_1 or ξ_2 increases, or μ_1 becomes further away from μ_2), the game still converges within a few rounds, and yields Nash equilibria similar to the homogeneous case. Moreover, the social cost and the price of anarchy stay roughly at the same level as the heterogeneity changes. Fig. 6 illustrates how the price of anarchy is influenced by the heterogeneity of the shared factors. In Fig. 6(a), we fix $\xi_1 = 0.06$ and $\xi_2 = 0.02$, and change μ_1 and μ_2. We find that the price of anarchy increases with $\mu_1 + \mu_2$ in general, which coincides with the results in Fig. 1(c). We also note that, when $\mu_1 + \mu_2$ is fixed, the price of anarchy experiences little change as μ_1 and μ_2 become closer to or further away from each other, which corresponds to a lower or higher heterogeneity, respectively. In Fig. 6(b), we fix $\mu_1 = 0.15$ and $\mu_2 = 0.05$, such that $k_s = (\mu_1+\mu_2)/2 = 0.075$, which is the same as the base case. Fig. 6(b) shows that changing either ξ_1 or ξ_2 has little influence on the price of anarchy. We also run simulations with different θ and k_b, and still find that the heterogeneity of shared factors has little influence on the price of anarchy.

Based on the simulation results, we conclude that assuming that all users have a homogeneous shared factor k_s is a reasonable approximation of the actual heterogeneous case. In particular, the variance of the shared factors has little influence on the Nash equilibrium, social cost, and price of anarchy. Intuitively, even if the variance of the shared factors is large, the aggregative term $\sum_{i\in S} k_{si}p_i$ stays roughly the same (assuming the mean of the shared factors is the same). As a result, the computing rate and cost function experienced by each user do not change much, and users will make similar decisions even though the variance of the shared factors changes.

7 CONCLUSION

This paper investigates the social cost and price of anarchy in shared/buy-in computing games. We consider a system with an arbitrary number of users and general heterogeneously distributed workloads, and establish methods for efficiently computing the optimal social cost and the (closed-form) social cost at equilibrium. For the special case where all users are buy-in users, we also derive closed-form expressions for the optimal social cost and the price of anarchy. The closed-form price of anarchy in this case can be arbitrarily large as the number of users N grows. We further show that the price of anarchy can be arbitrarily large as the operating expense ratio θ tends to 0, which indicates high inefficiency of the Nash equilibrium in that case. However, under practical settings of system parameters, the price of anarchy tends to lie within a certain range and is much smaller than the theoretical worst-case. In order to lower the social cost at equilibrium, we propose and analyze two subsidy policies. Numerical simulations show how the social cost and the price of anarchy are influenced by different factors, and demonstrate the significant effectiveness of the proposed subsidy policies.

Our results shed light on what factors contribute to the inefficiency of shared/buy-in computing systems, and provide insights into how to improve the social cost at the equilibrium. Thus, since the price of anarchy decreases as the user workload distribution gets more heterogeneous, the system should aim to diversify its users, for instance, adjusting the pricing of resources to attract more light users. Since the price of anarchy increases significantly as the operating expense ratio θ gets smaller, the provider needs to be judicious on how much revenue it makes from the payment of users, so as to keep the social cost low and maintain the attractiveness of the system. The shared factor k_s also plays an important role in the system. Although increasing k_s can result in a larger price of anarchy, sharing idle buy-in resources benefits the system as a whole by enhancing the utilization of system resources and attracting light users.

As one of the initial works to investigate the social cost and price of anarchy in shared/buy-in computing systems, this study can be further extended in many aspects. For instance, one may want to consider other functions for the operational cost, which not be directly proportional to user payments. The cost model of users can also take other forms, for example, the workload must be completed before a specific deadline, otherwise the cost increases sharply. Likewise, further investigating the design and analysis of subsidy policies, including determining when the Nash equilibrium is unique in the case of quadratic subsidy, represents an interesting area for further work.

ACKNOWLEDGMENTS

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A SUPPLEMENTARY DISCUSSION OF THE SYSTEM MODEL

Constraints on setting buy-in and shared factors

Here, we consider the constraint faced by the provider when it adjusts the system parameters k_b and k_s. Given the total payment $\sum_{i\in S} p_i$ from all players, there exists a maximum amount of resources that can be provided by the system. As a result, there also exists an upper-bound Ψ on the total computing rate of all players.

$$\Psi \geq \sum_{i\in S}(k_b p_i + k_s \sum_{j\neq i} p_j)$$
$$= \sum_{i\in S}(k_b + (N-1)k_s)p_i.$$

Assuming that the maximum computing rate is proportional to the total payment, we define $\kappa \triangleq \Psi/\sum_{i\in S} p_i$, where κ represents the maximum computing rate that can be provided per unit of payment. Replace Ψ with $\kappa \sum_{i\in S} p_i$, we get

$$\kappa \sum_{i\in S} p_i \geq \sum_{i\in S}(k_b + (N-1)k_s)p_i,$$

which can be simplified as

$$k_b + (N-1)k_s \leq \kappa.$$

Intuitively, this constraint implies that the provider cannot provide more than the maximum possible resources given the total payment.

When we consider the game with subsidy policies, the total computing rate of all players includes the subsidy term. Similarly, the total computing rate of all players must not be larger than the maximum computing rate that can be provided by the system, that is:

$$\kappa \sum_{i\in S} p_i \geq \sum_{i\in S}[(k_b + (N-1)k_s)p_i + \Delta_i(\mathbf{p})].$$