International internship

1MSE15

Demand fulfilment probability under heavy-tailed demand

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Contents

Introduction

Inventory management literature is mainly built on the assumption that the demand for items is normally distributed. This assumption is made because of the ease of use of the normal distribution and convenient analytical results that it may produce.

In recent years, this simplifying normally distributed demand assumption has been questioned by researchers. In fact, there is growing evidence in the literature that some item demands, including book demand at Amazon Chevalier and Goolsbee, 2003, movie demand at Netflix Bimpikis and Markakis, 2016, and spare part demand of a European automobile manufacturer Natarajan et al., 2018 follow a heavy-tailed distribution.

In addition to the apparent widespread presence of Heavy-tailed distributions in item demands, the literature shows a significant impact of the demand shape on inventory management performance Ramaekers and Gerrit, 2008. The demand shape is crucial in determining inventory levels. Choosing a different demand shape than that of the actual demand can increase the inventory levels by more than 100%, depending on the coefficient of variation Ramaekers and Gerrit, 2008.

Because of the reasons mentioned above, it is crucial to investigate if a model can be built that assumes heavy-tailed distributed demand instead of normally distributed demand for determining optimal basestock levels.

This research study focuses on a multi-item, multi-period budget-constrained inventory optimization setting where the goal is to maximize customer service level subjected to a budget constraint on the total inventory investment. This inventory optimization problem has been solved in the literature under the assumption of normally distributed demand. Our goal is to investigate how this problem can be solved under the presence of heavy-tailed demand distributions. More specifically, we would like to answer the following questions:

- Can we formulate the optimization problem assuming the demand is Heavy-tailed?
- Can we solve the optimization problem assuming the demand is Heavy-tailed?
- What are the effects of the distributions shape on the optimal basestock levels?

In order to answer these questions, we will develop mathematical inventory models with heavy-tailed demands and conduct extensive numerical experiments to gain insights into the workings of these inventory models. Note that multiple different distributions can simulate heavy tail behavior. The literature indicates that the Pareto (power-law) distribution is widespread in inventory management of this set of heavy-tailed distributions. Therefore Pareto distributions are chosen to simulate heavy-tail behavior in this paper.

Section 1 summarizes the literature review performed for this research. Section 2 describes the inventory model, optimization problem, and the assumptions made in the model. The optimization problem is solved in Section 3. The section will present the results from the model for a hypothetical case. In this research the optimization problem is solved analytically. The literature indicates that there are other methods that can be used as well. These methods are described in Section 4. Finally, Section 5 summarizes the results from this research and adds recommendations for future research.

1 Literature review

This section reviews the relevant literature for our research. First, we review the literature that empirically proves the importance of heavy-tail distributions in inventory management. Second, the literature describing the impact of demand shape on inventory models is summarized. Then we review the literature covering papers that study heavy-tail distributions in inventory management. Finally, the literature that studies the budget-optimization problem is reviewed.

Restricting our focus to the base-stock policy in this article, we identify Oral, 1981 as the first to study a budget-constrained multi-item inventory system with a periodic-review base-stock inventory policy. Corlu et al., 2017 mention that the study of budget-constrained multi-item inventory systems dates back to the seminal work of Hadley and Whitin, 1963. Since then, there has appeared a significant body of research that can be mainly categorized based on the type of the inventory policy in place Schrijver et al., 2013.

1.1 Empirical evidence of heavy-tailed demand in inventory management

There is ample proof of the importance of heavy-tailed distributions in inventory management in the literature. Heavy-tailed distributions can model uncertainty when extreme events (large or small) are relatively likely to occur. Normal distributions, for instance, are not able to take these extreme events into account. Clauset et al., 2009 provides a method to determine if heavy-tail or light-tail distribution fits a particular data set best. The paper includes a list of applications of heavy-tail distributions on real-world data. The list is an indication of the widespread presence of heavy-tailed distributed processes in the real world. Furthermore, the importance of power-law distributions is expected to increase because internet commerce may boost sales generated from niche products, leading to long-tail or heavy-tail demands Brynjolfsson et al., 2011.

In the literature, some papers evaluate the demand distribution of SKUs using real-world data. Chevalier and Goolsbee, 2003 have empirically proven that the demand for books at Amazon has a Pareto (power-law) tail with tail component $\alpha = 1.2$. Gaffeo et al., 2008 find that the Pareto (power-law) distribution with infinite variance represents a reasonable statistical model for fitting the correct tail of book sales distribution for the sales of books in Italy. The tail exponents lay in the interval $\alpha \subset (1.0, 1.4)$. Bimpikis and Markakis, 2016 analyzed the demand pattern of Netflix shows and found it to be Pareto (power-law) distributed with tail exponent $\alpha = 1.04$. Natarajan et al., 2018 noted that heavy-tail distributions provide the best fit for their data. In the paper, they use spare parts for car manufacturers as SKUs. Bimpikis and Markakis, 2016 have used the method described in Clauset et al., 2009 to fit the data on 626 similar products (sneaker shoes). The result leads to the rejection of normal and exponential distributions due to poor fit to the empirical distribution. The Pareto (power-law) distribution gave a reasonable description of the data and therefore is not rejected. Agrawal and Smith, 1996 found that the negative binomial distribution fits their data (sales from a major retailer with lost sales) significantly better than either the Poisson or the normal distributions. The negative binomial can represent the high variability in demand that occurs in retailing environments due to weather, competitors, promotions, and other random fluctuations than the Poisson or normal distributions. The paper also provides a method to estimate the parameters of the distribution using data.

1.2 Impact distribution assumption in inventory management

The literature shows a significant impact of the demand shape on inventory management performance Ramaekers and Gerrit, 2008, so demand shape is not a secondary factor in determining inventory levels. A different demand shape can increase average inventory levels by more than 100% according to Ramaekers and Gerrit, 2008, depending on the coefficient of variation. Therefore, it is essential to identify the demand shape and use the correct models to determine optimal basestock levels. According to Turrini and Meissner, 2019 the estimation on demand distribution impacts the performance of the inventory management system. An ill-suited hypothesized distribution may result in high preventable costs. The paper uses the Kolmogorov Smirnov (K-S) goodness-of-fit test to find the best-fitting distributions to data. The data comes from 4000 SKU's demand in the German renewable energy industry. After analyses, the paper concludes that fitting the proper distribution for demand is a critical issue in inventory management for spare parts. The importance of assuming the correct demand distribution is also highlighted in Syntetos et al., 2012. The validity of demand distribution assumptions (i.e., their goodness-of-fit) is distinguished from their utility (i.e., their real-world implications). The paper focuses on the demand for spare parts. Syntetos et al., 2012 found that assuming the correct demand distribution is essential for modeling optimal base-stock levels.

1.3 Heavy-tail distributions in inventory management

In the literature, several papers are focussing on the distributionally robust multi-item newsvendor problem. Das et al., 2018 suggests a robust model for newsvendor under different moments of α for heavy-tailed distributions. Das et al., 2021 expanded the model by including regularly varying distributions with parameter α. Das et al., 2021 proved that by assuming knowledge of the first and αth moment, the optimal order quantity is also optimal for a regularly varying distribution with tail index α. Natarajan et al., 2018 researched the distributionally robust multi-item newsvendor problem for data proven to be heavy-tailed. The paper adds the focus on asymmetry through second-order partitioned statistics to the robust newsvendor model.

The literature also covers the single-period, multi-location newsvendor models. Bimpikis and Markakis, 2016 consider a single-period, multi-location newsvendor model, where n different locations face independent and identically distributed demands and linear holding and backorder costs for heavy-tailed distributions. Bimpikis and Markakis, 2016 conclude that the heavy-tails significantly affect the optimal stock values. C. Yang et al., 2021 study a multi-location risk-averse newsvendor model, where a retailer owns the stores, and each location/store is operated by a manager who replenishes its stock to satisfy its own random demand. The paper concludes that under heavy tail distribution, centralization is preferred over inventory pooling.

H. Yang and Schrage, 2009 show that inventory pooling with heavy-tailed distributed demand can lead to overstocking of SKUs due to overcompensating the extreme events.

1.4 Inventory optimization under budget constraint

The problem of inventory optimization under budget constraints for multiple SKUs has been solved in Hausman et al., 1998. This paper uses a multivariate normal distribution to model the demand. Hausman et al., 1998 studies the problem of maximizing the joint demand fulfillment probability and discusses a heuristic approach in which equal safety factors (equal fractiles) are specified for all items. Similarly, L. Yang et al., 2020 focuses on maximizing order fulfillment under budget constraints for normally distributed demands. The suggested model allows firms to assess whether the current inventory performance is Pareto optimal, quantify the trade-offs between various performance measures, and identify the right inventory level according to the firm's strategic goals.

Korevaar et al., 2007 solve a single-echelon inventory problem with a system-wide service level. The paper maximizes the service level under budget constraints. The optimal base-stock level is determined by using an EOQ-model. The model deals with intermittent demand and uses normal, Gaussian, or poison distributions to model the demand.

Bera et al., 2009 develop an algorithm that uses fuzzy chance-constraints programming techniques to solve a multi-item mixture inventory model in which both demand and lead times are random. The model looks for the optimal order quantity and safety stock without assuming any distribution on demand during lead time. The model includes a budget constraint.

Recently Qiu et al., 2021 proposed a DRO approach to solving a worst-case expected profit model with budget constraints. The model does not make assumptions on the data distribution but instead focuses on maximizing the worst-case profit.

1.5 Conclusion of literature review

Heavy-tail distributions are relevant in inventory management, and their importance is expected to increase in the near future. Currently, there is no literature on solving the optimization problem with budget constraints for SKUs with heavy-tailed distributed demands in the literature.

2 Inventory model

We consider a P-item inventory setting, where item demands are independent of each other and are Pareto distributed. Each item operates under a periodic-review base-stock policy. In each period, the inventory is reviewed, ordering decision is made, replenishment orders are received, and finally item demands arrive. We assume that unsatisfied demands are backlogged. We use the following notation throughout the paper:

$D_{p,t}$, item p demand faced by the firm in period t, $p = 1, 2, \dots, P$

$A_{p,t}$, item p replenishment coming in period t, $p = 1, 2, \dots, P$

$X_{p,t}$, net inventory level at the end of period t, $p = 1, 2, \dots, P$

I_p, item p inventory target, $p = 1, 2, \dots, P$

c_p, item p unit investment cost, $p = 1, 2, \dots, P$

L_p, item p deterministic lead time, $p = 1, 2, \dots, P$

k, response time window satisfying $L_p \geq k$, $p = 1, 2, \dots, P$

2.1 General optimization model

We measure the customer service level in this inventory system via demand fulfillment probability, which is defined as the probability that the customer orders in a period are filled within a time frame of k periods. The probability that demands in period t are satisfied within k periods is characterized as:

$$\Pr\left\{X_{p,t} + \sum_{\ell=t+1}^{t+k} A_{p,\ell} \geq 0, p = 1, 2, \dots, P\right\}$$

Notice that $X_{p,t} + \sum_{\ell=t+1}^{t+k} A_{p,\ell}$ is equivalent to $X_{p,t+k} + \sum_{\ell=t+1}^{t+k} D_{p,\ell}$. Using this relationship, this probability can be written as:

$$\Pr\left\{X_{p,t+k} + \sum_{\ell=t+1}^{t+k} D_{p,\ell} \geq 0, p = 1, 2, \dots, P\right\}$$

Using the relationship $X_{p,t} = I_p - \sum_{\ell=t-L_p}^{t} D_{p,\ell}$ in Hadley and Whitin, 1963, we can replace $X_{p,t+k} + \sum_{\ell=t+1}^{t+k} D_{p,\ell}$ by $I_p - \sum_{t-L_p+k}^{t} D_{p,\ell}$. This yields the following characterization for the demand fulfillment probability (hausman1998joint, Lemma 1):

$$\Pr\left\{\sum_{\ell=t-L_p+k}^{t} D_{p,\ell} \leq I_p, p = 1, 2, \dots, P\right\}$$

Our goal is to maximize the demand fulfillment probability subject to a total inventory investment budget B where the demand is Pareto distributed:

$$\text{maximize}_{I_1, I_2, \dots, I_P} \Pr\left\{\sum_{\ell=t-L_p+k}^{t} D_{p,\ell} \leq I_p,\ p = 1, 2, \dots, P\right\} \quad (1)$$

$$\text{subject to} \sum_{p=1}^{P} c_p I_p \leq B, \quad (2)$$

$$I_p \geq 0,\ p = 1, 2, \dots, P. \quad (3)$$

2.2 Optimization function when demand is Pareto distributed

In this section, we assume that item p demands in period t, $D_{p,t}$, are Pareto distributed with shape parameter α_p, $p = 1, 2, \dots, P$, and scale parameter β_p, $p = 1, 2, \dots, P$. The PDF of the Pareto distribution used is denoted by:

$$f(I_p) = \frac{\alpha}{\beta}\left(\frac{\beta}{I_p+\beta}\right)^{\alpha+1}, \quad I_p, \alpha, \beta > 0 \tag{4}$$

$\alpha > 0$ **but is not a positive integer**.

The figure below shows the behavior of this formula for different α values, β is the scale parameter and determines the start of the distribution. In this case $\beta = 1$ so the PDF starts at 1.

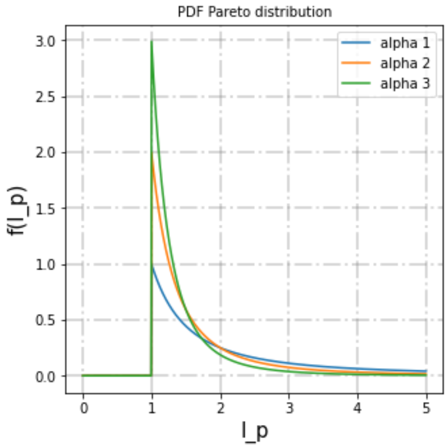

Figure 1: PDF of Pareto distribution

Ramsay, 2008 formulates a method to analytically sum Pareto distributions. Using the formulas proposed in this paper allows us to derive the proposition below:

We derive that:

$$F_{L_p-k+1}(I_p),\ p=1,2,\ldots,P = \Pr\left\{\sum_{\ell=t-L_p+k}^{t} D_{p,\ell} \le I_p,\ p=1,2,\ldots,P\right\}$$

So the objective function of the optimization problem when the demand follows a Pareto distribution is denoted by:

$$\prod_{p=1}^{P} F_{L_p-k+1}(I_p),\ p=1,2,\ldots,P \tag{5}$$

Where:

$$n = \textit{amount of periods to consider} = L_p - k + 1, \quad n \ge 1$$

$$F_n(I_p) = \int_0^\infty \frac{\left(1-\mathrm{e}^{-I_p x/\beta_p}\right)}{x}\chi_n(x,\alpha_p)\mathrm{d}x \tag{6}$$

$$\chi_n(x,\alpha_p) = \sum_{r=0}^{\lfloor (n-1)/2\rfloor} \frac{(-1)^r}{\pi}\begin{pmatrix} n \\ 2r+1\end{pmatrix}(R(x,\alpha_p))^{n-2r-1}(I(x,\alpha_p))^{2r+1} \tag{7}$$

$$R(x,\alpha_p) = 1 + \sum_{r=1}^{\infty}\frac{x^r}{(\alpha_p-1)\cdots(\alpha_p-r)} - \frac{\pi x^{\alpha_p}\mathrm{e}^{-x}}{\Gamma(\alpha_p)}\cot(\pi\alpha_p)$$

$$I(x,\alpha_p) = \frac{\pi x^{\alpha_p}\mathrm{e}^{-x}}{\Gamma(\alpha_p)}$$

Note that $R(x, \alpha)$ can be rewritten to:

$$R(x, \alpha_p) = M(1; 1 - \alpha_p; -x) - \frac{\pi x^{\alpha_p} \mathrm{e}^{-x}}{\Gamma(\alpha_p)} \cot(\pi \alpha_p)$$

Where $M(a; b; x)$ is the Kummer function given by:

$$M(a; b; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}$$

The proposition above assumes that, when looking at each individual item p, the demand for each period in n is independent and identically distributed. Also the lead time is assumed to be deterministic.

3 Solve optimization problem

To solve the optimization problem from equation 1, a Wolfram Mathematica notebook is created. The notebook can solve the optimization problem for any values of $\alpha, \beta, n \geq 0$ and any number of items in P.

In this section a hypothetical two item case is optimized to demonstrate the capabilities of the notebook. The two item case contains the following two items:

- Item 1 where $p = 1, \alpha = 5.5, \beta = 1, n = 2$
- Item 2 where $p = 2, \alpha = 0.5, \beta = 1, n = 2$

The notebook works in two steps. The first step is to create a CDF for each item p summed over n periods (subsection 3.1). When the summed CDFs are known, the model formulates the optimization function and solves it (subsection 3.2).

3.1 Generate summed CDF for each item p over n periods

The first step in solving the optimization problem is formulating the optimization equation 5 for a specific set of items. To formulate equation 5 we need the summed CDFs for each item p summed over n periods. The model generates the summed CDFs over n periods with equation 6. Below the results from the notebook for both items are presented.

Item 1: $\alpha_1 = 5.5, \beta_1 = 1, n = 2, p = 1$

The summed CDF function is denoted by:

$$\begin{aligned} F_2(I_1) =& 1 + \frac{2.03883 \times 10^{-11}}{(2+I_1)^{11}} - \frac{1}{(1+I_1)^{15.5}\left(\frac{2+1I_1}{1+I_1}\right)^{11}} 2 \\ & \left(1024 + 10240 I_1 + 32000 I_1^2 + 47786.7 I_1^3 + 39040 I_1^4 + 18688 I_1^5 + \right. \\ & \left. 5834.67 I_1^6 + 1389.71 I_1^7 + 225.714 I_1^8 + 22.2222 I_1^9 + I_1^{10}\right) \end{aligned} \tag{8}$$

Resulting in the graph below.

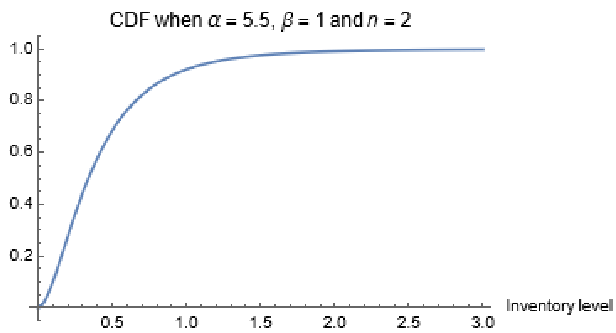

Figure 2: CDF case 1

Item 2: $\alpha_2 = 0.5, \beta_2 = 1, n = 2, p = 2$

The summed CDF function is denoted by:

$$F_2(I_2) = \frac{2 + I_2 - 2(1+I_2)^{0.5}}{2 + I_2} \tag{9}$$

Resulting in the graph below.

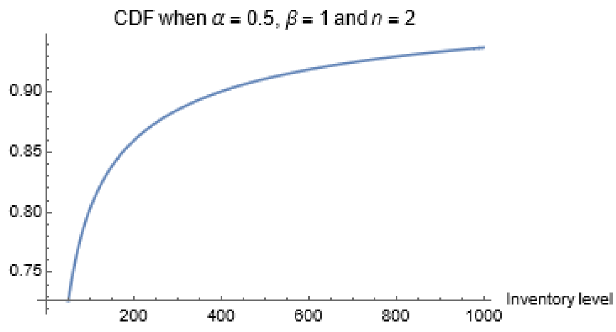

Figure 3: CDF case 2

The notebooks results for summing CDFs are identical to the results from Ramsay, 2008.

3.2 Find optimal solution

The optimization function takes both Equation 8 and Equation 9 as inputs.

$$F_2(I_1, I_2) = \prod_{p=1}^{P} F_n(I_p) = F_2(I_1) * F_2(I_2)$$

The graph below depicts the corresponding solutions space. One of the characteristics of a CDF is that it is continuously increasing over its input variable. That means that the solution space knows no local optimal and minimal values.

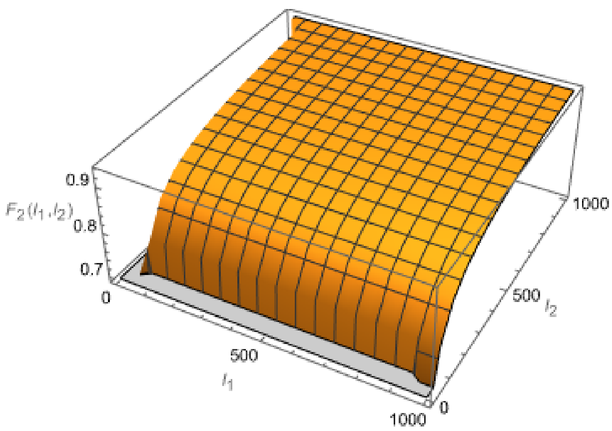

Figure 4: Solution space

To solve the optimization problem we need to define some more variables. In the two product case we are solving in this section $c_1 = 25, c_2 = 10$ *and* $B = 1000$.

The notebook uses non-linear optimization methods to solve the optimization problem from equation 1. The answer to the optimization problem, in this case, is $F_2(2.77, 93.08) = 0.79$.

3.3 Experimentation

The budget constraint, α, and β all influence the model's outcome in the 2 item case with $n = 2$. This section numerically charts the impact of these parameters on the outcome of the model.

Budget constraint
The table below shows the effect of the budget constraint on the optimal solution. All parameters are set equal to the hypothetical two product case from previously solved in Section 3.2 and Section 3.1.

Table 1: Optimal value for increasing budget constraint B

B	I_1	I_2	$F_2(I_1, I_2)$
50	0.98	3.3	0.18
100	1.05	7.37	0.36
150	1.3	11.74	0.46
200	1.5	16.26	0.53
300	1.78	25.54	0.62
400	2	34	0.67
500	2.18	44.56	0.71
1000	2.77	93.08	0.79
2000	3.43	191.41	0.86
3000	3.87	290.34	0.88
3000	3.87	290.34	0.88
5000	4.68	488.83	0.91

As expected, the optimal value increases when the allowed budget increases. In this example, the model selects a higher inventory position for item 2.

The most obvious reason is that the price set for item 2 is lower than for item 1. Another contributor is that item 2 has a lower α value than item 1. In figure 1 it is clear that a lower α makes the distribution more heavy-taled. When the tail is heavier, a higher inventory position is needed to increase the demand fulfillment probability of an item.

Alpha
To analyse the effect of α on the models outcome, the following 2 item case will be analysed:

- Item 1 where $p = 1, \alpha_1 = Variable, \beta_1 = 1, n = 2, c_1 = 20$
- Item 2 where $p = 2, \alpha_2 = 1.001, \beta_2 = 1, n = 2, c_2 = 20$

$Budget(B) = 1000$

The table below shows the results from the model.

Table 2: Optimal value for increasing α_1

α_1	I_1	I_2	$F_2(I_1, I_2)$
0.1	36.89	13.11	0.07
0.25	35.32	14.68	0.29
0.5	32.24	17.76	0.59
0.75	28.71	21.29	0.76
1.001	25	25	0.84
2.001	13.4	36.6	0.93
5.001	3.95	46.05	0.95
10.001	1.64	48.36	0.96

The increase of α_1 leads to a lower allocation of I_1 and lower optimal value.

Increasing α_1 leads to a heavier tailed demand. This means that for the same demand fulfillment probability, a higher inventory position is needed. That is why $F_2(I_1, I_2)$ is lower for lower values of α_1.

The model generates a higher inventory for item 1 for low values of α_1 because a lower alpha value generates a heavier tail. So the CDF of this item converges slower to 1, which means that it pays off to invest more in the inventory of this item since the CDF does not stagnate as quickly compared to CDFs with a high α.

Beta
To analyse the effect of β on the models outcome, the following 2 item case will be analysed:

- Item 1 where $p = 1, \alpha_1 = 1.001, \beta_1 = Variable, n = 2, c = 20$
- Item 2 where $p = 2, \alpha_2 = 1.001, \beta_2 = 1, n = 2, c = 20$

$Budget(B) = 1000$

The table below shows the results from the model.

Table 3: Optimal value for increasing β_1

β_1	I_1	I_2	$F_2(I_1, I_2)$
0.1	11.80	38.20	0.93
1	25	25	0.84
2	29.33	20.67	0.77
3	31.71	18.29	0.72
4	33.28	16.72	0.67
5	34.42	15.58	0.63

Table 3 shows two main trends. The inventory allocation of item 1 increases as β_1 increases. Also $F_2(I_1, I_2)$ decreases as β_1 increases.

Increasing β_1 like mentioned under Equation 4 translates the entire PDF to the right.

The translation of the PDF explains that $F_2(I_1, I_2)$ decreases when β_1 increases since more inventory is needed to increase the demand fulfillment probability for a particular item.

The translation of the PDF also explains why the allocation of inventory for I_1 increases with β_1. Since large β_1 means that the model needs to select higher inventory to raise the value of $F_2(I_1, I_2)$, due to Pareto's shape, the first product contributes more to the demand fulfillment probability than the final products, thus increasing the value of I_1 with β_1.

3.4 Model limitations

As mentioned earlier, the model can solve the optimization problem for any values of $\alpha, \beta, n \geq 0$ (where α is not an integer) and any number of items in P.

The three major assumptions used in the model are:

- Demand for each item is independent, and Pareto distributed.
- Lead time is deterministic.
- When summing the Pareto distributions for n periods, the model assumes that the Pareto distributions are independent and identically distributed.

To improve the model, these assumptions can be relaxed in future research, see Section 4.

Another limitation of the current model is that when n increases, the optimization function's complexity also increases significantly. The main reason for this is Equation 7. When n increases, the summation's upper limit increases as well, generating more and more complex functions as n increases. So for large instances of n ($n \geq 5$), the model becomes computationally expensive. Section 4 indicate multiple methods to decrease the required computational power for the model.

4 Additional methods for solving optimization problem

In the research, the optimization problem is solved analytically. There are several different options to solve the optimization problem in the literature. This section elaborates on these options from the literature.

4.1 Monte Carlo simulation

Yao et al., n.d. indicates that Monte Carlo simulation can derive the sum of any amount Pareto distributions.

In Yao et al., n.d. a multivariate Pareto distribution is formulated. The multivariate Pareto distribution can relax the assumption of independent demand. In reality, demand is likely dependent, so this would be an improvement over the current model.

Using Monte Carlo, it is possible to relax the three main assumptions of the current model:

- Lead time is assumed to be deterministic.
- Looking at each item p, the demand for each period in n is independent and identically distributed.
- Demand is dependent for each item.

Note that even do Monte Carlo seems very promising, there are some major drawbacks in using this method:

- May be costly and time-consuming to build the simulation.
- Easy to misuse simulation by “stretching” it beyond the limits of credibility.
- Monte Carlo simulation usually requires several (perhaps many) runs at given input values and is therefore computationally expensive.

4.2 Series expansion

Quang et al., 2013 provides a method called series expansion. This paper formulates a closed-form expression to approximate the sum of Pareto distributions. The paper indicates that the method is computationally less expensive than the Monte Carlo simulation and generates near identical results.

The proposed function is the survival function for the sum of Pareto distributions depicted below.

$$
\begin{aligned}
P\left(X_1+X_2+X_3>s\right)=&\sum_{l=0}^{\infty}\beta_1^{\alpha_1}h\left(\alpha_1,l\right)\left(\sum_{j=0}^{l}\frac{l!}{j!\left(l-j\right)!}\frac{\alpha_2\beta_2^j}{\alpha_2-j}\frac{\alpha_3\beta_3^{l-j}}{\alpha_3-l+j}\right)s^{-\alpha_1+l}\\
&+\sum_{l=0}^{\infty}\beta_2^{\alpha_2}h\left(\alpha_2,l\right)\left(\sum_{j=0}^{l}\frac{l!}{j!\left(l-j\right)!}\frac{\alpha_1\beta_1^j}{\alpha_1-j}\frac{\alpha_3\beta_3^{l-j}}{\alpha_3-l+j}\right)s^{-\alpha_2+l}\\
&+\sum_{l=0}^{\infty}\beta_3^{\alpha_3}h\left(\alpha_3,l\right)\left(\sum_{j=0}^{l}\frac{l!}{j!\left(l-j\right)!}\frac{\alpha_1\beta_1^j}{\alpha_1-j}\frac{\alpha_2\beta_2^{l-j}}{\alpha_2-l+j}\right)s^{-\alpha_3+l}\\
&+\beta_1^{\alpha_1}\beta_2^{\alpha_2}c\left(\alpha_1,\alpha_2\right)\sum_{k=0}^{\infty}h\left(\alpha_1+\alpha_2,k\right)w\left(\left(\alpha_3,\beta_3\right),k\right)s^{-\left(\alpha_1+\alpha_2+k\right)}\\
&+\beta_1^{\alpha_1}\beta_3^{\alpha_3}c\left(\alpha_1,\alpha_3\right)\sum_{k=0}^{\infty}h\left(\alpha_1+\alpha_3,k\right)w\left(\left(\alpha_2,\beta_2\right),k\right)s^{-\left(\alpha_1+\alpha_3+k\right)}\\
&+\beta_2^{\alpha_2}\beta_3^{\alpha_3}c\left(\alpha_2,\alpha_3\right)\sum_{k=0}^{\infty}h\left(\alpha_2+\alpha_3,k\right)w\left(\left(\alpha_1,\beta_1\right),k\right)s^{-\left(\alpha_2+\alpha_3+k\right)}\\
&+\beta_1^{\alpha_1}\beta_2^{\alpha_2}\beta_3^{\alpha_3}c\left(\alpha_1,\beta_1\right)c\left(\alpha_1+\alpha_2,\alpha_3\right)s^{-\left(\alpha_1+\alpha_2+\alpha_3\right)}
\end{aligned}
\tag{10}
$$

This function can be expanded using vectoring to any amount of Pareto distributions. Even though series expansion is an approximation method, it might generate more accurate results than the current analytical model because one can relax the assumption that the Pareto distributions are independent and identically distributed over the n amount of periods.

4.3 Use of stable distribution

Stable distributions can be used to approximate Pareto distributions. Stable distributions have the property that when summed, the new distributions still have a stable distribution Zaliapin et al., 2005. The resulting distribution will be an approximation and, therefore, less accurate than the analytical model. The only advantage this method has over the analytical model is that it is less complex and reduces the required computation time.

4.4 Bounded Pareto sums

In theory, infinite large demands are possible under Pareto demands, making it challenging to sum Pareto distributions. It is possible to bound the Pareto distributions to a certain maximum. Grassi and Coluccia, n.d. derives a method for summing bounded Pareto distributions.

4.5 Use stochastic period lengths

Ramsay, 2009 derives an analytical method to relax the assumption that the lead time is deterministic. Future research can use this conclusion to extend the current model further.

4.6 Neural network

Neural networks can be used to estimate complex functions. A neural network can thus be used to to estimate the optimization function taking into account inter-dependencies between item demands. So using neural networks the assumption of independent demand can be relaxed.

5 Conclusion

We consider a multi-item, multi-period budget-constrained inventory optimization setting. The goal is to maximize customer service level subject to a budget constraint on the total inventory investment for heavy-tailed demand.

The research formulates and solves the optimization problem analytically using a mathematical model. The proposed model uses analytical expressions to find the optimal basestock levels for each item in a set of items P. Each item has its Pareto distribution with parameters $\alpha \ and \ \beta$. The model allows looking n amount of periods ahead by summing the Pareto demands over these n amounts of periods.

Note that the following assumptions hold for the model:

- Demand for each item is independent, and Pareto distributed.
- Lead time is deterministic.
- When summing the Pareto distributions for n periods, the model assumes that the Pareto distributions are independent and identically distributed.

Another limitation of the current model is that when n increases, the optimization function's complexity also increases significantly.

Experimentation with the model gave some insight into the effects of the parameters $\alpha, \beta, Budget\ constraint$ on the optimal basestock levels and correspond demand fulfillment probability.

- Demand fulfillment probability increases as budget constraint increases.
- Demand fulfillment probability increases as alpha increases for an item in P.
- Demand fulfillment probability decreases as beta increases for an item in P.

These insights can all be explained by the analytical formulation of the optimization problem.

The results from this research model have not been compared to the results under the assumption of normal distributions. This comparison should be the first step in future research since it is crucial to know the difference the assumption of demand distribution makes on the performance of the inventory policy.

In future research, assumptions made for this model can be relaxed. The assumption of independent demand for each item in P can be relaxed by using Monte Carlo Simulation or neural networks to estimate the optimization function. The assumption that lead time is deterministic can be relaxed in the current model by applying the findings in Ramsay, 2009. In the model, the assumption is made that the Pareto distribution for an item in P is independent and identically distributed. By applying series expansion, this assumption can be relaxed as well.

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