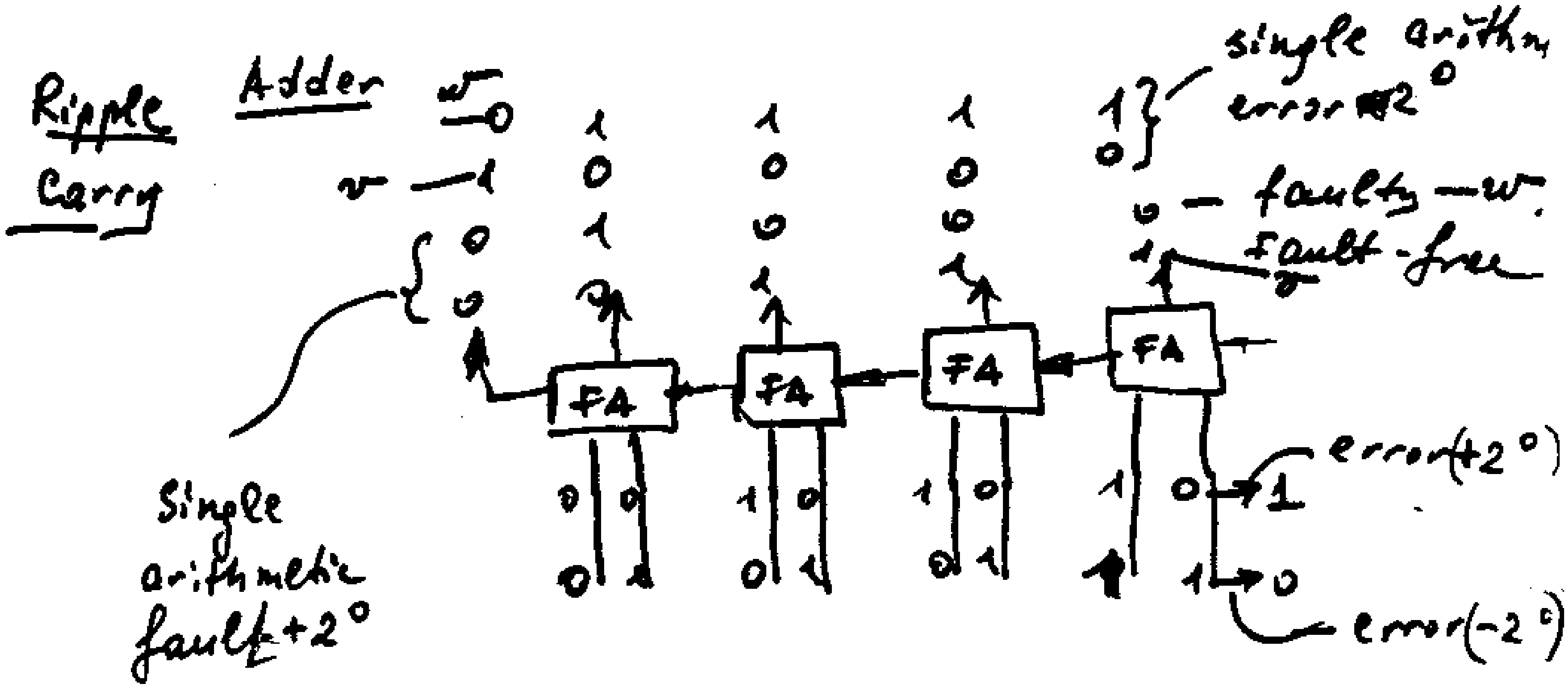
ERROR DETECTION FOR ALUs

Arithmetical Codes



Fault may happen in FA's at buses, registers storing operants, etc.

Similar for subtractors, counters, multipliers, etc

look-ahead adders, BCD adders

v - fault-free output number (in binary)

w - faulty ——————————————

$e = |v - w|$

<u>error</u> (symmetrical)

$v = 7$, $w = 8 \Rightarrow e = 1$

<u>multiplicity of error</u> (symmetrical) $\|e\|_A$ -

ARITHMETICAL WEIGHT

min # of terms $\pm 2^i$ to represent e

$(i = 0, 1, \ldots, m-1)$

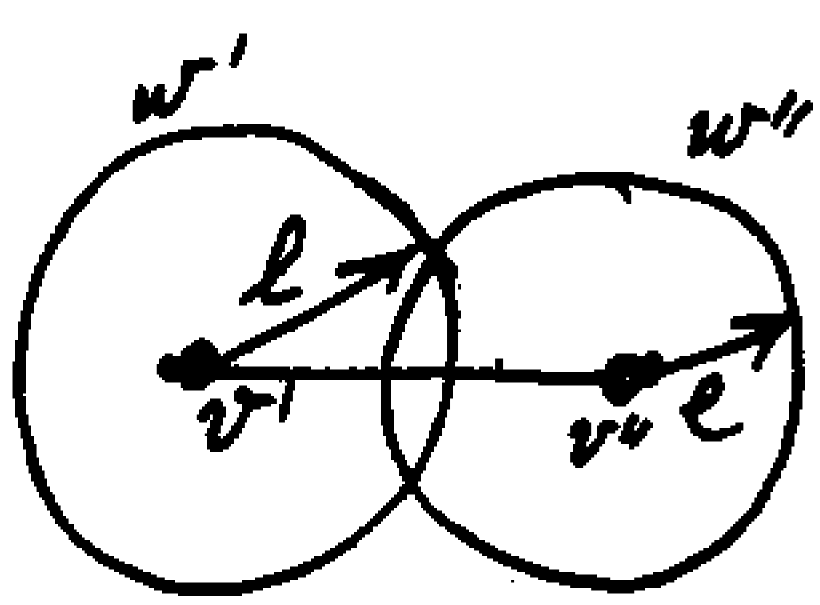
<u>Ex.</u> $v = 0$, $w = 7$ $(m=4) \Rightarrow$

$e = 7$ $\|e\|_A = 2$

$7 = +2^3 - 2^0$ double error

Geometrical Interpretation

Error detection



l - multiplicity of errors

$$\boxed{d_A(v', v'') \geq l+1}$$

$v', v'' \in C$ - code

Arithmetical distance $d_A(x,y) = \| x-y \|_A$

Ex. 1 $\begin{matrix} x = 0 \\ y = 7 \end{matrix} \Rightarrow \overset{m>3}{d_A(x,y) = 2}$

Ex 2 $x = 24$, $y = 11$, $x - y = 13 \Rightarrow \| x-y \|_A = 3$

$d(24, 11) = 3$

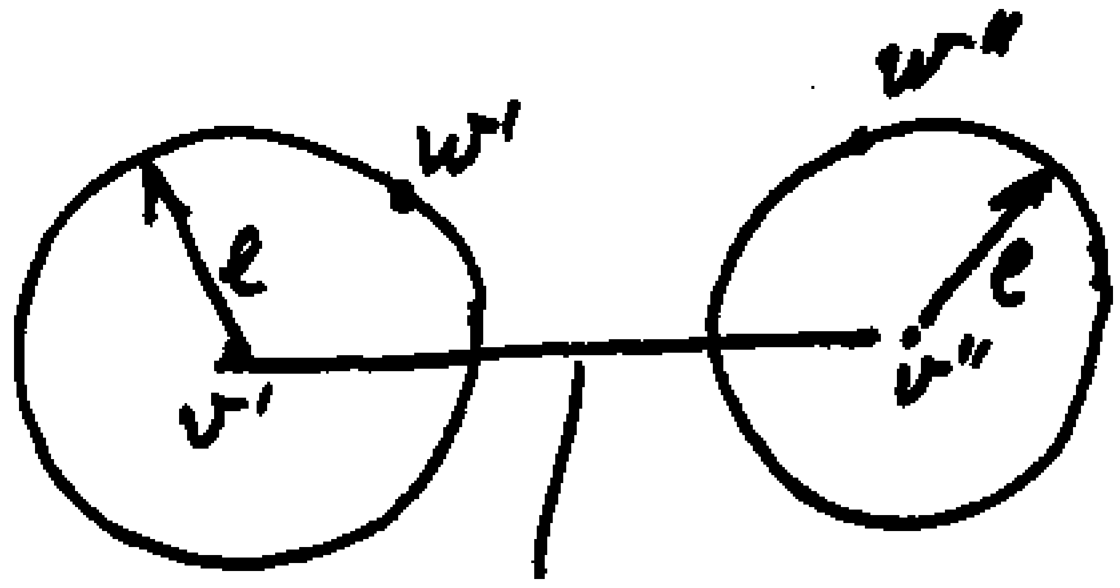
$d_A(x,y) = d_A(y,x)$, $d_A(x,x) = 0$ $\quad 0 \leq d_A(x,y) \leq m$

$(x, y \leq 2^m - 1)$

$d_A(x,y) + d_A(y,z) \geq d(x,z)$

$\forall x, y, z$.

Error correction



$$\boxed{d_A(v', v'') \geq 2l+1}$$

The same as for algebraic codes

but different errors, distances

and multiplicities

AN+B codes

$C = \{B, A+B, 2A+B, 3A+B, \dots\}$

$v \in C \Leftrightarrow v = AN+B$

in many cases $B=0 \Rightarrow$

all multiples of A in C

algebraic codes $\quad H \cdot v = 0 \Leftrightarrow v \in C \pmod 2$

arithmetic codes $\quad v \equiv 0 \pmod A \quad v \in C$

$S_w \equiv w \pmod A$ – syndrome

$0 \leq s < A$

$v \in C \Leftrightarrow S=0$

Example

1) $A = 7$, $B = 1$, $m = 6$

$C = \{1, 8, 15, 22, 29, 36, 43, 50, 57\}$

9 out of 2^6 possible numbers

2) $B = 0$

$C = \{0, 7, 14, 21, 28, 35, 42, 49, 56, 63\}$

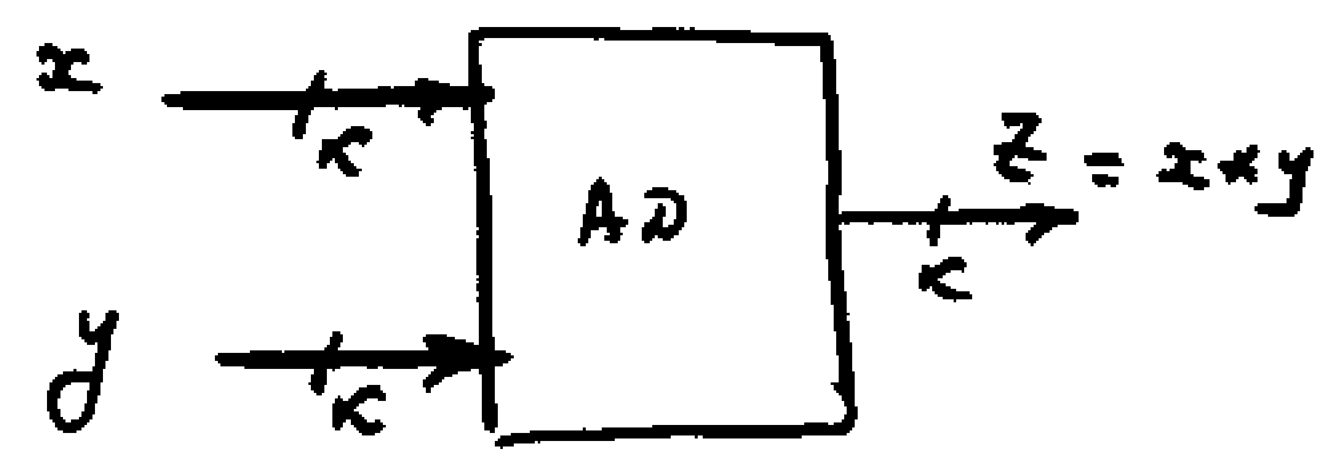
for $w = 31 \Rightarrow S_w \equiv w \pmod 7$

$S_w = 3$ $\quad w \notin C$

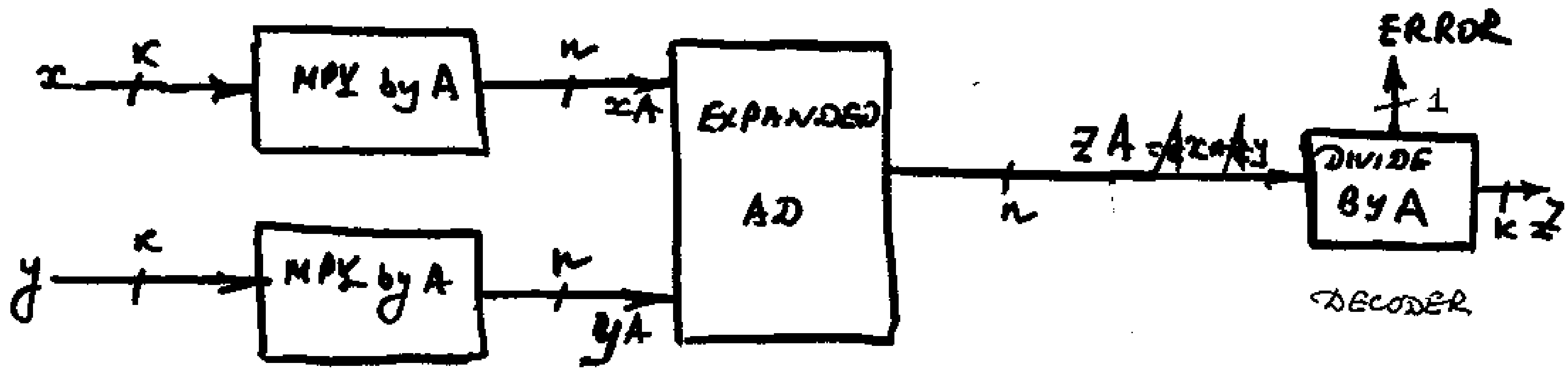
syndrome

SELF ERROR DETECTION IN ARITHMETICAL DEVICE (AD) BY MODULO A CHECKS

ORIGINAL DEVICE



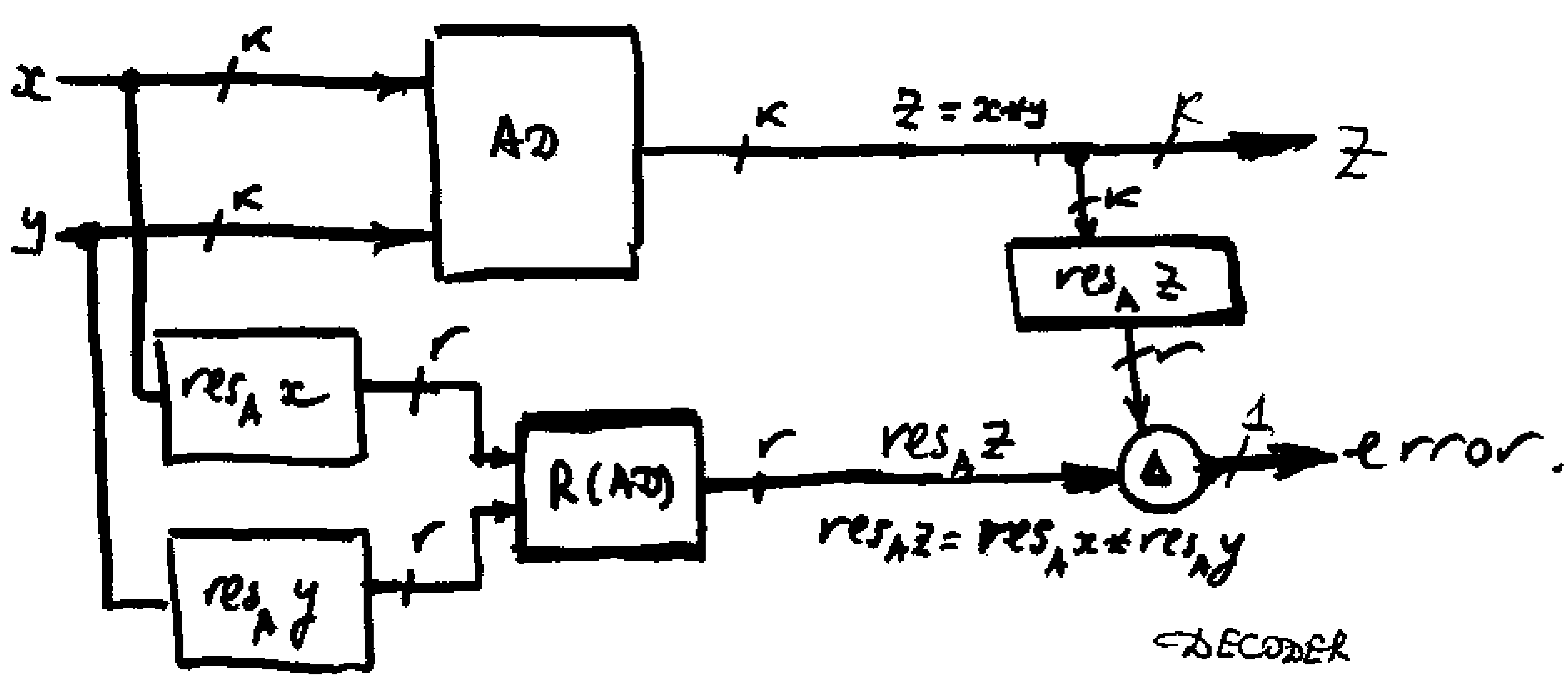
Redundand device (VERSION I) nonsystematic



$n = k + r$

$r = \lceil \log_2 A \rceil$

VERSION II systematic



Detection of single errors

$d_A = 2$

$w = v \pm 2^i \qquad e = \pm 2^i \qquad \ell = 1$

$S_w \equiv \pm 2^i \pmod{A} \quad \Rightarrow$

Condition for A:

$\pm 2^i \not\equiv 0 \pmod{A} \quad \Rightarrow$

$\pm 2^i$ is not divisible by A for any $i = 0, 1, \dots, m-1$

$\Rightarrow A$ odd

take $A = 3$ Modulo 3 checks

Code:

$\{0, 3, 6, 9, \dots\}$.

A is independent on m

$A = 3$ is an analog of parity checking

<u>Error detection</u> $s \neq 0$

<u>Error correction</u> for two errors

e_1 and e_2 $\Rightarrow$ $s_1 \neq s_2$

Total # of errors <u>(single)</u> $l = 1$ $\Rightarrow$

$2m$

Total # of different syndromes A

$$\boxed{A \geq 2m+1}$$

Analog of the Hamming bound for correction of single errors

$A = 2m+1$ - for perfect single error correcting codes

Correction of Single Errors:

Detection of double errors

Condition for A:

$$\pm 2^i \not\equiv \pm 2^j \quad (\mathrm{mod}\ A)$$

for Any $i, j = 0, 1, \ldots, m-1 \quad (i \neq j)$

Example $A = 7$

$2^0 \equiv 1$, $2^1 \equiv 2$, $2^2 \equiv 4$, | $2^3 \equiv 1$

$-2^0 \equiv 6$, $-2^1 \equiv 5$, $-2^2 = 3$.

$m = 3$.

$A = 2m+1$ perfect code

Example $A = 9$

$2^0 \equiv 1$, $2^1 = 2$, | $2^3 \equiv 8 \equiv -2^0$ (mod 9)

$-2^0 \equiv 8$, $-2^1 = 7$,

$m = 3$ not a perfect code

Correction of Single Errors

Example $A=11$

(mod 11) $+2^0 \equiv 1$, $+2^1 \equiv 2$, $+2^2 \equiv 4$, $2^3 \equiv 8$, $2^4 \equiv 5$

$-2^0 \equiv 10$, $-2^1 \equiv 9$, $-2^2 \equiv 7$, $-2^3 \equiv 3$, $-2^4 \equiv 6$

for $m=5$ (5-bit output)

all single errors are corrected $\Rightarrow$ produce different syndromes

$\boxed{A = 2m+1 \quad \text{perfect code}}$

Ex. If syndrome is $=7$ then there is an error (-2^2) (1→0 error in the bit number 2)

Most popular AN codes for $A = 2^a - 1$ – "low-cost codes"

Example JPL STAR computer $A=15$

tables of perfect arithmetic single symmetric error - correcting codes in W. W. Peterson "Error Correcting Codes"

see also
D. Sewiorek, Swarz "Theory and Practice of Reliable System Design".

Let M is a ~~max~~ member that ~~any~~

all $\pm 2^i$ are different mod. A $i = 0, 1, \ldots, \lfloor \log_2 AM \rfloor$

M is a number of fault-free outputs vectors in a

best code correcting single arithmetical errors.

$$M \le \frac{1}{A} \left(2^{\frac{A-1}{2}} + A\right)$$

A	M
11	3
13	5
19	27
23	89
29	565
37	7085
47	178,481
53	1,266,205

If. $M = \left(2^{\frac{A-1}{2}} + A\right) A^{-1}$

then the corresponding

AN code is perfect

LOW-COST ARITHMETICAL CODES

Minimal complexity of decoding (COMPUTING $res_A\, x$)

$A = 2^a - 1$

<u>Example</u> $a = 4$, $A = 15$

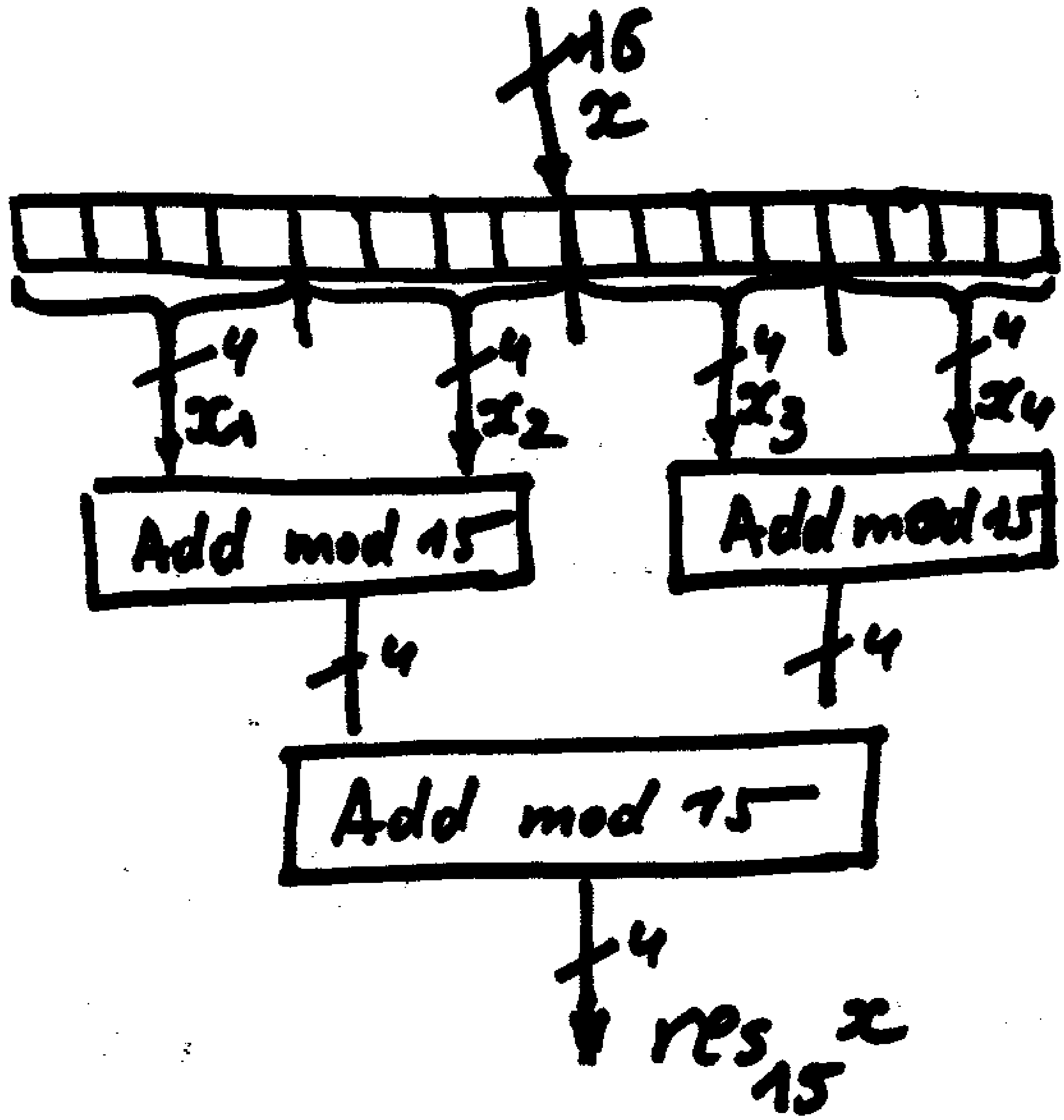
$m = 16$

Any 16-bit binary number x can be represented as

$$x = x_0 + 2^4 x_1 + 2^8 x_2 + 2^{16} x_3$$

$$0 \le x_0, x_1, x_2, x_3 < 2^4$$

$$x \equiv x_0 + x_1 + x_2 + x_3 \pmod{15}$$



For a low-cost AN code with $A = 2^a - 1$ all unidirectional errors with multiplicity $\ell \leq a-1$ are detected

Detection of Unidirectional Arithmetical Errors

$e = +2^{i_1} + 2^{i_2} + \dots + 2^{i_l}$ — l- errors

$\|e\|_A = l$

Condition for detection:

$$res_A\,(2^{i_1} + 2^{i_2} + \dots + 2^{i_l}) \neq 0$$

for all $0 \le i_1 < i_2 < \dots < i_l \le m-1$

$A \to \min$

Hamming norm of A (number of ones in binary representation of A) should be greater than l.

$$\min A = 2^{l+1} - 1$$

$(2^{l+1}-1)N$ codes detect all unidirectional arithmetical errors for any length m

Redundancy $r = \lceil \log_2(A+1) \rceil$

$r = \ell + 1$

These are the <u>best</u> codes to detect unidirectional errors.