[redacted]

compared to the well known Walsh-Hadamard techniques. For almost all combinational networks the time complexity (time required to test a network) for detection of terminal faults with any multiplicity is shown to be $\lceil 1.25n \rceil$ and the storage required for storing the standard test data is $\lceil 0.75n \rceil$, where n is the number of arguments for the network and $\lceil m \rceil$ denotes the smallest integer greater or equal to m. It is also shown that for almost all networks the additional hardware required for testing is shown to be two T flip-flops, two 2-input AND gates and two $n/2$ -input NOR gates.

Some results on fault detection of internal s-a-fs using the RM transform technique are also presented. Upper bounds on the number of spectral coefficients that may be distorted by a stuck-at fault at the output of a gate realizing a product term is given. If any single internal fault distorts at most A spectral cofficients, then the minimum number of test patterns required to detect the fault is shown to be upper bounded by $\sum_{i=0}^{\lceil log_2(A+1) \rceil - 1} \binom{n}{i}$, where n is the number of input arguments to the network. Moreover the test patterns are independent of a function.

Time complexities for detection of input s-a-fs in some of the standard computer components are presented.

1 Introduction:

In recent years there has been a renewed interest in spectral techniques for Boolean functions and their applications [7-10, 15-23]. Many authors have focused their at-

[1]This work was supported by the National Science Foundation under the Grant DCR - 8317763.

tention on Walsh-Hadamard (WH) spectral techniques for fault detection in combinational networks. In this paper we consider a new spectral (transform) technique called the Reed-Muller (RM) spectral technique which is based on the Reed-Muller canonical (RMC) expansion of a Boolean function, where the spectra of a Boolean function is the set of coefficients of the Boolean function in the RM canonical expansion. The network constructed using the RM canonical expansion is called the RMC network. General structure of RMC network is shown in Fig 1.

Reddy [6] has shown that the RMC networks are easily testable. Many authors have studied the fault detection in RMC networks [6, 25]. We study the properties of the RM spectral coefficients and use these properties for fault detection in general combinational networks, sequential networks, PLA type networks and in some standard computer components.

Section 2 focuses on Reed-Muller transforms and their properties. Section 3 deals with the stuck-at-faults and bridging faults in general combinational networks. We describe the properties that must be satisfied by the spectral coefficients for detection of s-a-fs and bridging faults. Upper bounds on the number of spectral coefficients to be observed for detection of multiple terminal stuck-at-faults (s-a-fs) and single terminal bridging faults are shown to be n and $n-1$ respectively.

One of the drawbacks of the existing WH spectral techniques is that every spectral coefficient depends on the global behavior of a Boolean function ie., the values of the function at all the 2^n input vectors are required for computation of any single WH spectral coefficient. This results in enormous amount of computational and testing time. Unlike WH spectral coefficients RM spectral coefficients depend on a local behaviour of the function (ie., responses of f at only few input vectors out of the possible 2^n vectors). Another disadvantage of the WH spectra is that every spectral coefficient requires an n-bit counter and a coefficient selection network. Section 4 deals with hardware and time complexity for computing RM spectral coefficients. Hardware complexity required for computation of a single spectral coefficient is shown to be one T flip-flop, one two input AND gate and a NOR gate with at most n inputs.

Kuhl and Reddy [28] have shown that $2n + 4$ test patterns are sufficient to detect terminal s-a-fs of any multiplicity in a combinational network. Karpovsky and Levitin [27] have shown that there exist a standard universal test with $2n - 2$ test patterns to detect all multiple s-a-fs in almost all combinational networks. We show that at most $0.75n$ test patterns are sufficient for detection of all multiple s-a-fs and the time complexity is $1.25n$ for almost all combinational networks. (These are not standard but device oriented tests).

In section 5, a number of test patterns required for testing a general combinational network is shown to be upper bounded by $\sum_{i=0}^{r} \binom{n}{i}$, if a fault in the network causes distortion of at most $2^{r+1} - 1$ spectral coefficients. Upper bounds on a number of spectral coefficients that may be distorted by a s-a-f at the output of a gate realizing a product term is given.

In section 6, we estimate a number of spectral coefficients required and their time complexity for standard componets of computer systems which include both combinational and sequential circuits.

2 Reed-Muller Transforms And Their Applications

Fisher [3] has shown a relation between coefficients of generalized Reed-Muller (GRM) canonical expressions for a Boolean function f and values of f at the input vectors $X \in \{0,1\}^n$, where n is the number of input variables. Corresponding to the 2^n possible polarities, any function has the following GRM expansions.

$$f(x_0, ..., x_{n-1}) = \hat{f}_0(0) \oplus \hat{f}_0(1)x_{n-1} \oplus \hat{f}_0(2)x_{n-2} \oplus \ ... \ \oplus \hat{f}_0(2^n-1)x_0...x_{n-1} =$$

$$\hat{f}_k(0) \oplus \hat{f}_k(1)x_{n-1}^{k_{n-1}} \oplus \hat{f}_k(2)x_{n-2}^{k_{n-2}} \oplus \ ... \ \oplus \hat{f}_k(2^n-1)x_0^{k_0}...x_{n-1}^{k_{n-1}} =$$

$$\hat{f}_N(0) \oplus \hat{f}_N(1)\bar{x}_{n-1} \oplus \hat{f}_N(2)\bar{x}_{n-2} \oplus \ ... \ \oplus \hat{f}_N(N)\bar{x}_0...\bar{x}_{n-1} \quad (1)$$

where $x^0 = x, x^1 = \bar{x}$; $\hat{f}_k(W) \in \{0,1\}$ and $N = 2^n - 1$.

Let the binary representation of k and W be $(k_0, k_1, .., k_{n-1})$ and $(w_0, w_1, .., w_{n-1})$, then the variables associated with $\hat{f}_k(W)$ are all those variables which correspond to the 1's in $W = (w_0, w_1, .., w_{n-1})$.

Example 1 : *Function* $f(x_0, x_1) = x_0 \vee \bar{x}_1$ *has the following four GRM expansions.*

1. $f = 1 \oplus x_1 \oplus x_0x_1 \qquad (k=0)$,
2. $f = \bar{x}_1 \oplus x_0 \oplus x_0\bar{x}_1 \qquad (k=1)$,
3. $f = 1 \oplus \bar{x}_0x_1 \qquad (k=2)$,
4. $f = 1 \oplus \bar{x}_0 \oplus \bar{x}_0\bar{x}_1 \qquad (k=3)$.

Though relations between $f(W)$, and $\hat{f}_k(W)$ are given in [3], these relations are not convenient for mathematical manipulation. The following expressions are introduced.
Definition : Let $X = (x_0, x_1, .., x_{n-1})$ and $W = (w_0, w_1, .., w_{n-1})$. Then the Reed-Muller (RM) functions are defined in the following way:

$$R_X(W) = x_0^{w_0} x_1^{w_1} ... x_{n-1}^{w_{n-1}} = \prod_{i=0}^{n-1} x_i^{w_i} \quad (2)$$

where $0^0 = 1$, $1^0 = 1$, $0^1 = 0$.

2.1 Reed-Muller Transform

The relation between $f(X)$ and $\hat{f}_0(W)$ for 0^{th} polarity is given as

$$\hat{f}_0(W) = \bigoplus_{X=0}^{N} R_W(X)\, f(X)\,, \tag{3}$$

$$f(X) = \bigoplus_{W=0}^{N} R_X(W)\, \hat{f}_0(W)\,, \tag{4}$$

where $\bigoplus$ represents the modulo 2 addition and $N = 2^n - 1$. In the case of 0^{th} polarity '0' will be dropped from $\hat{f}_0(W)$. For the generalized RM trasform we have

$$\hat{f}_k(W) = \bigoplus_{X=0}^{N} R_W(X \oplus k)\, f(X)\,, \tag{5}$$

$$f(X) = \bigoplus_{W=0}^{N} R_{X\oplus k}(W)\, \hat{f}_k(W) \tag{6}$$

where $X \oplus k$ represents the componentwise (dyadic) addition, and k is the polarity.

Example 2 : *The following four matrices represent* $R_W(X\oplus k)$, *for* $k \in \{0,1,2,3\}$ *when* $n = 2$.

$X =$	00	01	10	11	00	01	10	11	00	01	10	11	00	01	10	11
$k =$	00				01				10				11			
$X \oplus k =$	00	01	10	11	01	00	11	10	10	11	00	01	11	10	01	00
$W \downarrow$																
00	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
01	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
10	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

It will be shown in the following sections that RM coefficients have similar properties as those of Walsh-Hadamard or Walsh-Chrestenson spectral coefficients [16, 22-24]. Hence, $\hat{f}(W)$ can be termed as RM spectral coefficients and the complete set of coefficients as the RM spectrum.

As in the case of WH matrices, RM matrices have recursive structure and can be represented as

$$[R]^{(n)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes [R]^{(n-1)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{(n)}, \tag{7}$$

where $\otimes$ is the Kronecker product and $[R]^{(n)}$ is the Reed-Muller matrix of 2^n th order, and n is the number of arguments. The fast RM transform based on this Kronecker factorization requires $n.2^{n-1}$ operations and 2^n memory cells.

Let

$$f = [f(0),\ f(1),\ \dots,\ f(N)]^T\ ,$$

$$\hat{f}_k = \left[\hat{f}_k(0),\ \hat{f}_k(1),\ \dots,\ \hat{f}_k(N)\right]^T\ ,$$

where T stands for matrix transpose. From (2) it can be easily shown that

$$\bigoplus_{W=0}^{N} R_X(W)\ R_W(Y) = \delta_{X,Y}\ , \tag{8}$$

where $\delta_{X,Y}$ is the Kronecker delta function. From (8)

$$[R]^{(n)} \cdot [R]^{(n)} = [I] \tag{9}$$

where $[I]$ is the identity matrix.

Besslich [9, 10] and Wu, et all [8] have shown that spectra $\hat{f}_0$ can be computed with at most $n.2^{n-1}$ modulo 2 operations and to obtain $\hat{f}_k$, $k \in \{0, 1, .., N\}$ only N iterations of 2^{n-1} modulo 2 additions need to be performed. However, their approach is based on folding of the Reed-Muller map introduced in [8]. In the following section it will be shown that spectra of any polarity can be obtained by a matrix multiplication.

2.2 Generation of $\hat{f}_k$ from $\hat{f}_0$ and $\hat{f}_0$ from $\hat{f}_k$

From (3), (5) and (9)

$$\hat{f}_0 = [R]_0^{(n)} . f = [R]_0 . f\ , \tag{10}$$

$$f = [R]_0^{(n)} . \hat{f}_0 = [R]_0 . \hat{f}_0\ , \tag{11}$$

$$\hat{f}_k = [R]_k^{(n)} . f = [R]_k . f\ , \tag{12}$$

where $[R]_k^{(n)} = [R]_k$ is the RM matrix of k^{th} polarity, n is the number of input arguments for the function and the order of $[R]_k$ is equal to 2^n. From (8) and (9)

$$\hat{f}_k = [R]_k . [[R]_0 . [R]_0] . f = [[R]_k . [R]_0] . [[R]_0 . f] = [[R]_k . [R]_0] . \hat{f}_0\ . \tag{13}$$

Whenever the polarity of the spectrum is 0, 0 will be dropped for convenience. The matrix $[[R]_k . [R]_0]$ is sparse and a number of operations required to compute $\hat{f}_k$ from $\hat{f}$ is very small. To illustrate this point $[[R]_k . [R]_0]$ for $k \in \{1, 2, 3\}$ are computed for the case $n = 2$.

$$[[R]_1 . [R]_0] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad [[R]_2 . [R]_0] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[[R]_3 \cdot [R]_0] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It can be shown that $[[R]_k \cdot [R]_0]^{(n)}$ are given by

$$[[R]_k \cdot [R]_0]^{(n)} = \begin{bmatrix} [[R]_k \cdot [R]_0]^{(n-1)} & 0 \\ 0 & [[R]_k \cdot [R]_0]^{(n-1)} \end{bmatrix} \quad if \;\; k < 2^{n-1},$$

$$[[R]_k \cdot [R]_0]^{(n)} = \begin{bmatrix} [A] & [A] \\ 0 & [A] \end{bmatrix} \quad if \;\; k \geq 2^{n-1}, \tag{14}$$

where $[A] = [R]_{k \oplus 10..0}^{(n-1)} \cdot [R]_0^{(n-1)}$.

Since $[R]_k$ and $[R]_0$ are upper triangle matrices with all diagonal elements equal to 1, the inverse of $[[R]_k \cdot [R]_0]$ is $[[R]_k \cdot [R]_0]$ itself. Hence

$$[[R]_k \cdot [R]_0] \cdot [[R]_k \cdot [R]_0] = [I], \tag{15}$$

and from (10), (11) and (13) one can obtain

$$\hat{f} = [R]_0^{(n)} \cdot f = [[R]_k \cdot [R]_0] \cdot \hat{f}_k. \tag{16}$$

Example 3 *: For the function given in Example 1, the four possible spectra are given below*

x_0	x_1	f	$\hat{f}_0$	$\hat{f}_1$	$\hat{f}_2$	$\hat{f}_3$
0	0	1	1	0	1	1
0	1	0	1	1	0	0
1	0	1	0	1	0	1
1	1	1	1	1	1	1

2.3 Some Properties Of RM Spectra

From now on we shall consider only 0^{th} polarity RM spectra. The following relationships can be easily verified from (2), (3) and (4):

$$\hat{f}(0) = f(0), \tag{17}$$

$$\hat{f}(N) = \bigoplus_{X=0}^{N} f(X), \tag{18}$$

$$R_X(0) = 1, \qquad \forall X, \tag{19}$$

$$R_W(0) = 1, \qquad \forall W, \tag{20}$$

$$R_N(W) = 1, \qquad \forall\, W\ , \tag{21}$$

$$R_0(W) = \delta_{W,0}\ , \tag{22}$$

$$\bigoplus_{W=0}^{N} R_X(W) = \delta_{X,0}\ , \tag{23}$$

$$\bigoplus_{X=0}^{N} R_X(W) = \delta_{W,N}\ , \tag{24}$$

$$R_X(W)R_X(\bar{W}) = \prod_{i=0}^{n-1} x_i = R_X(N)\ , \tag{25}$$

$$R_{X \bigwedge Y}(W) = R_X(W)R_Y(W)\ , \tag{26}$$

$$R_X(W \bigvee V) = R_X(W)R_X(V)\ , \tag{27}$$

where $\bigwedge$ and $\bigvee$ stands for componentwise AND and OR operations respectively. Using the above relations one can find composite spectra of $\bar{f}$, $f \oplus \phi$, $f \wedge \phi$, and $f \vee \phi$.

1. Let $\varphi(X) = \bar{f}(X)$, then

$$\hat{\varphi}(W) = \begin{cases} \hat{f}(W)\,, & if \quad W \neq 0\,; \\ 1 \oplus \hat{f}(W)\,, & if \quad W = 0\,. \end{cases} \tag{28}$$

2. Let $\varphi(X) = f(X) \oplus \phi(X)$, then

$$\hat{\varphi}(W) = \hat{f}(W) \oplus \hat{\phi}(W)\ . \tag{29}$$

   The above two relations are straightforword. The following relations are proved in the appendix.

3. Let $\varphi(X) = f(X) \wedge \phi(X)$, then

$$\hat{\varphi}(W) = \bigoplus_{U,V=0}^{N} \hat{f}(U).\hat{\phi}(V)\ \delta_{U \vee V, W}\ . \tag{30}$$

4. Let $\varphi(X) = f(X) \vee \phi(X)$, then

$$\hat{\varphi}(W) = \hat{f}(W) \oplus \hat{\phi}(W) \oplus \bigoplus_{U,V=0}^{N} \hat{f}(U)\ \hat{\phi}(V)\ \delta_{U \vee V, W}\ . \tag{31}$$

5. CONVOLUTION THEOREM: If $\hat{\varphi}(W) = \hat{f}(W) \wedge \hat{\phi}(W)$, then

$$\varphi(X) = \bigoplus_{Y,Z=0}^{N} f(Y).\phi(Z)\ \delta_{Y \vee Z, X}\ . \tag{32}$$

3 Fault Detection In Combinational Networks

Stuck-at-faults (s-a-f): A line h_i in a combinational network is said to be at s-a-0(1) if that line becomes permanently fixed as a result of the fault at the binary value 0 (1), and it is denoted as $h_i/0(1)$.

Bridging-fault: If two lines h_i and h_j are shorted and if the resulting signal after the short is AND operation then h_i and h_j are at AND bridging fault which is denoted as $(h_i h_j)_*$. Similarly one can introduce OR bridging $(h_i h_j)_+$.

From now on a faulty function is represented as f^*. The following theorem presents a condition on spectral coefficients for detection of s-a-fs at input lines.

Theorem 1 : *A stuck-at fault at input line x_i of a combinational network implementing $f(x_0,\ x_1,\ ...,\ x_{n-1})$ is detected if and only if for the faulty network $\hat{f}^*(W) = 0$, for all W such that $W = (w_0 .. w_{i-1}\ 1\ w_{i+1} .. w_{n-1})$.*

Proof: From (3)

$$\hat{f}^*(W) = \bigoplus_{X=0}^{N} R_W(X) f^*(X) \ = \ \bigoplus_{X=0}^{N} w_0^{x_0} ... 1^{x_i} ... w_{n-1}^{x_{n-1}}\ f^*(X).$$

Since

$$(w_0^{x_0}, .., w_{i-1}^{x_{i-1}}, 1^0, w_{i+1}^{x_{i+1}}, .., w_{n-1}^{x_{n-1}}) \ = \ (w_0^{x_0}, .., w_{i-1}^{x_{i-1}}, 1^1, w_{i+1}^{x_{i+1}}, .., w_{n-1}^{x_{n-1}})$$

and

$$f^*(x_0, .., x_{i-1}, 0, x_{i+1}, .., x_{n-1}) \ = \ f^*(x_0, .., x_{i-1}, 1, x_{i+1}, .., x_{n-1}),$$

the above summation mod 2 over all X will be zero.

From Theorem 1 it is clear that a s-a-f at input line x_i can be detected by verifying $\hat{f}(W)$ iff $\hat{f}(W) = 1$ for a fault-free device and $w_i = 1$ ($W = (w_0, ..., w_{n-1})$).

Corollary 1 : *In a combinational network the multiple input s-a-f at $x_{i_1}, ..., x_{i_l}$ is detected if and only if for the faulty network $\hat{f}^*(W) = 0$, for all $W = (w_0, ..., w_{n-1})$ such that $w_{i_1} = \ ... \ = w_{i_l} = 1$.*

From the above theorem and corollary the following theorem is obvious.

Theorem 2 : *In any combinational network all single or multiple terminal s-a-fs can be detected by observing at most n spectral coefficients, where n is the number of input arguments to the network.*

Corollary 2 : *In a combinational network if $\hat{f}(N) = 1$, then all single and multiple terminal s-a-fs can be detected by verification of only one spectral coefficient $\hat{f}^*(N)$.*

It can be readily seen from (5) that the above theorems and corollary are true for spectral coefficients with any polarity.

The following theorem provides a necessary and sufficient condition for detection of AND bridging fault between two input lines x_i and x_j.

Theorem 3 : *The AND bridging fault* $(x_i x_j)_*$ *between two input lines* x_i *and* x_j *of combinational network implementing* $f(x_0, x_1, ..., x_{n-1})$ *is detected if and only if for the faulty network* $\hat{f}^*(W) = 0$, *for all* $W = (w_0, .., w_i, .., w_j, .., w_{n-1})$ *such that* $w_i \neq w_j$.

Proof: From (3)

$$\hat{f}^*(W) = \bigoplus_{X=0}^{N} R_W(X) f^*(X) \;=\; \bigoplus_{X=0}^{N} w_0^{x_0}...w_i^{x_i}...w_j^{x_j}...w_{n-1}^{x_{n-1}}\, f^*(X).$$

Since $w_i \neq w_j$, let $w_i = 0$ and $w_j = 1$. Then

$$\hat{f}^*(W) = \bigoplus_{X=0}^{N} w_0^{x_0}...0^{x_i}...1^{x_j}...w_{n-1}^{x_{n-1}}\, f^*(X) = 0$$

since

$$(w_0^{x_0}, .., \overset{i}{0^0}, .., \overset{j}{1^1}, .., w_{n-1}^{x_{n-1}}) = (w_0^{x_0}, .., \overset{i}{0^0}, .., \overset{j}{1^0}, .., w_{n-1}^{x_{n-1}}) \, ,$$

and after the fault

$$f^*(x_0, .., \overset{i}{0}, .., \overset{j}{0}, .., x_{n-1}) \;=\; f^*(x_0, .., \overset{i}{0}, .., \overset{j}{1}, .., x_{n-1}).$$

From Theorem 3 it is clear that a bridging fault between input lines x_i and x_j can be detected by verifying $\hat{f}(W)$ iff $\hat{f}(W) = 1$ for a fault-free device and $w_i \neq w_j$ $(W = (w_0, ..., w_{n-1}))$.

The next theorem follows immediately from Theorem 3.

Theorem 4 : *In a combinational network implementing* $f(x_0,\ x_1,\ ...,\ x_{n-1})$ *all single and multiple input AND bridging faults can be detected by observing at most* $n - 1$ *spectral coefficients.*

Example 4 : *Let* $f(x_0 x_1 x_2) = x_0 x_1 \vee x_1 x_2 \vee x_0 x_2$, *then the fault free and faulty spectra of the function for various faults are shown below.*

x_0	x_1	x_2	f	$\hat{f}$	$f^*(x_1/0)$	$\hat{f}^*(x_1/0)$	$f^*((x_0x_1)_*)$	$\hat{f}^*((x_0x_1)_*)$
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	0	0
1	1	0	1	1	0	0	1	1
1	1	1	1	0	1	0	1	0

It can be easily seen from (5) that the above two theorems are valid for spectra with any polarity.

Combining Theorems 2 and 4 we have the following result.

Theorem 5 : *For any irredundant combinational network implementing $f(x_0, x_1, ..., x_{n-1})$ there exists a set of n spectral coefficients, which detect all detectable single or multiple terminal s-a-fs and single input AND bridging faults.*

Internal faults in combinational networks will be considered in Section 5.

4 Hardware And Time Complexity for Testing

4.1 Hardware complexity

Scheme 1: Testing of a network for any fault involves computing a set of spectral coefficients called spectral signature [19] and each spectral coefficient must be equal to 1 for the fault-free network. Since $\hat{f}^*(W)$ for any W depends on the modulo 2 sum of the output responses of the network it requires only one T flip-flop. However, an input to this T flip-flop depends on the spectral coefficient being computed. Inputs to the T flip-flop can be controlled by a NOR gate whose inputs can be determined as follows. We have

$$\hat{f}(W) = \bigoplus_{X=0}^{N} w_0^{x_0} ... w_i^{x_i} ... w_{n-1}^{x_{n-1}} \, f(X)$$

but

$$w_i^{x_i} = w_i \vee \bar{x}_i = \begin{cases} \bar{x}_i \, , & if \quad w_i = 0 \, ; \\ 1 \, , & if \quad w_i = 1 \, . \end{cases}$$

Therefore

$$\hat{f}(W) = \bigoplus_{X=0}^{N} \prod_{w_i=0} \bar{x}_i \, f(X).$$

For example, let $n = 3$ then

$$\hat{f}(010) = \bigoplus_{X=0}^{N} \bar{x}_0 \bar{x}_2 \, f(X) = \bigoplus_{X=0}^{N} \overline{x_0 \vee x_2} \, f(X)$$

hence the inputs to the NOR gate are x_0 and x_2. The test circuit for the general case is shown in Fig 2.

If more than one spectral coefficient have to be computed, then the corresponding NOR gates and T flip-flops have to be added. Instead of a n bit counter and a network for coefficient selection as in the case of Walsh-Hadamard spectral coefficients, Reed-Muller spectral coefficients require only one T flip-flop and a couple of gates.

Scheme 2: Another scheme for testing which can be used in real time is to store all the responses of the function for computing spectral coefficients of a signature in a shift register as shown in Fig 3. This scheme can be useful only if $\| W \|$ is small, where $\| W \|$ is the number of 1's in W, in each spectral coefficient. Each spectral coefficient is computed by hard-wired EX-OR gates. For a fault-free network these spectral coefficients should be 1. All the spectral coefficients are then ANDed together to provide the error output. If this output is 1 then the network is fault-free otherwise it is faulty.

Example 5 : *For $n = 4$ and if $\hat{f}(1000) = 1$, $\hat{f}(0100) = 1$, $\hat{f}(0011) = 1$, then the network in Fig 3 which computes $\hat{f}(1000), \hat{f}(0100)$, and $\hat{f}(0011)$, can be used for testing.*

In this case the following sequence of test patterns should be applied

$$T = (0000,\ 1000,\ 0100,\ 0001,\ 0010,\ 0011).$$

Fig 3 shows the hardware implementation details.

In general the additional hardware required for the second scheme amounts to one shift register of length $(L+1)$, where

$$L = \sum_{W_i \in S} (2^{\|W_i\|} - 1),$$

L 2-input EX-OR gates and $(|S| - 1)$ 2-input AND gates, where $|S|$ is the number of coefficients in the spectral signature S.

The following theorem provides an estimation on a number of spectral coefficients required for detection of all single and multiple s-a-fs for almost all networks.

Theorem 6 : *For almost all combinational networks there exists two spectral coefficients which detect all terminal s-a-fs with any multiplicity.*

Proof: Consider any coefficient W_i, then

$$prob\ \{\ \hat{f}(W_i) = 1\} \ = \ prob\ \{\hat{f}(\bar{W}_i) \ = \ 1\} \ = 0.5.$$

Thus

$$prob\ \{\ \hat{f}(W_i) \neq 1 \text{ or } \hat{f}(\bar{W}_i) \ \neq \ 1\ \} \ = 0.75\ ,$$

$$prob\ \{\ \hat{f}(W_i) \neq 1 \text{ or } \hat{f}(\bar{W}_i) \ \neq \ 1 \quad \text{for all } i\ \} = \left(\frac{3}{4}\right)^{2^{n-1}} .$$

Now consider the case when $\| W_i \| = \lfloor n/2 \rfloor$. Then

$$prob\ \{\ (\ \hat{f}(W_i) \neq 1 \text{ or } \hat{f}(\bar{W}_i) \ \neq \ 1),\ \ \| W_i \| = \frac{n}{2}\ \} = \left(\frac{3}{4}\right)^{\frac{1}{2}\cdot\binom{n}{\lfloor \frac{n}{2} \rfloor}} ,$$

which goes to zero as n increases. Hence only two coefficients are sufficient for almost all combinational networks.

From the above theorem it is evident that the hardware complexity for testing of almost all combinational networks for input s-a-fs of any multiplicity amounts to two T flip-flops, two $\lfloor n/2 \rfloor$ input NOR gates and two 2-input AND gates only. When compared to the n bit counter and a coefficient selection network required to compute every WH spectral coefficient, hardware required by the RM spectral technique is very small.

4.2 Time Complexity

Let the time complexity for testing a network be defined as the total time required for computing all spectral coefficients of a spectral signature. The time complexity depends on the spectral coefficients being computed, since

$$\hat{f}(W) = \bigoplus_{X=0}^{N} R_W(X) f(X) = \bigoplus_{X \subseteq W} f(X) ,$$

where $X \subseteq W$ means $x_i \leq w_i$ for all i (we call these X's descendants of W). For example, the descendants of (101) are 000, 001, 100, 101 . Therefore, the fewer 1's in W less the time required to compute W, and the time complexity to compute $\hat{f}(W)$ is

$$time\ complexity \ = \ 2^{\|W\|} . \tag{33}$$

Time cmplexities for testing of input faults for some standard cmponents of computer systems are presented in Section 6. The following theorem provides an estimation on a time complexity required for testing, for almost all combinational networks.

Theorem 7 *: In almost all combinational networks the number of test patterns required to detect all multiple terminal s-a-fs by Reed-Muller transforms is equal to 0.75n and the time complexity is equal to 1.25n where n is the number of input variables.*

Proof is given in the appendix.

We have performed a statistical analysis using computer simulation to check the applicability of the above theorem and found that the results of the simulation are very close to the results presented in Theorem 7. In the following table we summarize results of simulation.

n	*Av. time*	*std. div*
8	11.96	1.28
16	22.54	1.66
24	32.12	1.33
32	42.22	1.56

where *Av. time* is the average time complexity required for 50 randomly generated networks with n arguments.

If the second testing scheme is used for testing in real time, from the above theorem and Example 5 it is evident that the additional hardware for testing consists of one shift register of length $\lceil 1.25n \rceil$ bits, $\lceil 1.25n \rceil$ 2-input EX-OR gates and $\lceil 3n/4 \rceil$ 2-input AND gates.

5 Detection of Internal Stuck-at-faults

This section deals with detection of internal s-a-fs in combinational networks implemented by two-level AND-OR structures. In the previous section it was shown that at most n spectral coefficients are sufficient to detect all multiple terminal s-a-fs in any combinational network. If a network has an AND-OR structure, then all multiple termianl s-a-fs and all multiple internal s-a-1 faults are detected since any internal s-a-1 fault results in $\hat{f}^*(W) = 0$, for all $W \neq (0, 0, ..., 0)$. The only s-a-fs to be detected are the s-a-0 faults at the outputs of AND gates. The following theorem deal with internal faults in fanout-free networks.

Theorem 8 : *For any fanout-free network there exists a set of n spectral coefficients which detects all multiple terminal s-a-fs and all single internal s-a-fs, where n is the number of input arguments for the network.*

Proof: Since a network is fanout-free there are exactly n paths from inputs to the output. Consider a path p_i starting from input line x_i. Let X_i be the input vector which sensitizes path p_i such that $\| X_i \|$ is minimal and $f(X_i) = 1$. Clearly $\hat{f}(X_i) = 1$ and if any line h_i on the path p_i is at s-a-f then $\hat{f}^*(X_i) = 0$ detecting the fault. Here there is no assumption made on the method of implementation of the fanout-free network.

Lemma 1 : *If in an AND-OR network all AND gates have n inputs, then only one spectral coefficient $\hat{f}(11...1)$ is sufficient for detection of all single internal s-a-fs.*

Proof: Clearly

$$\hat{f}(11..1) = \bigoplus_{X=0}^{N} f(X)$$

is a parity function and any single internal s-a-0 fault changes the parity detecting the fault.

Example 6 : *If $f(X) = \sum(0, 3, 6, 9, 12, 15)$ mod 3 checker for $n = 4$, then $\hat{f}(W) = 1$ where $W \in \{0, 1, 2, 4, 5, 7, 8, 10, 11, 13, 14\}$ and $\hat{f}^*(1, 1, ..., 1) = 1$ for any single internal s-a-f, detecting the fault.*

Theorem 9 : *If $f = f_1 \vee ... \vee f_r$ where $f_i = f_i(X_i)$, $X_i \wedge X_j = \emptyset$,*

$$\bigcup_{i=1}^{r} X_i = X = \{x_0, x_1, ..., x_{n-1}\},$$

and every product term of subfunctions f_i depend on all of its arguments X_i, then a number of spectral coefficients required to detect all internal s-a-fs is at most r.

Proof: It suffices to note that every product term of each subfunction is a minterm and by Lemma 1 the theorem is true.

Theorem 10 : *For any threshold function implemented by a two level AND-OR network only one spectral coefficient is sufficient to detect all single internal s-a-fs.*

Proof: The threshold function is given as

$$f(X) = \begin{cases} 1, & if \ \sum_{i=0}^{n-1} x_i \geq t; \\ 0, & if \ \sum_{i=0}^{n-1} x_i < t, \end{cases}$$

where t is the threshold. Therefore, function f is a sum of product terms consisting of exactly t variables. If any AND gate in an AND-OR structure realizing a product term say, $p_i = x_1x_2...x_t$ is s-a-0 then only one minterm $X_i = (x_1, x_2, .., x_t, 0, .., 0)$ and the corresponding spectral coefficient $\hat{f}^*(X_i)$ will be effected. Hence $\hat{f}(1,1,...,1)$ would detect all single s-a-0 faults.

Theorem 11 *: For any unate fuction if every product term depends exactly on $(n-i)$ variables and the total number of product terms is equal to*

$$\binom{n-i}{\lfloor (n-i)/2 \rfloor}, \tag{34}$$

then $\hat{f}(1,1,...,1)$ is sufficient to detect all single internal s-a-fs, where $\lfloor m \rfloor$ is the integer part of m.

Proof: It was shown in [26] that the maximum number of product terms in any unate function with exactly $(n-i)$ variables in each product term is given by (34). Hence as in the previous theorem any single internal s-a-0 fault effects only one minterm. Therefore $\hat{f}(1,1,...,1)$ detects all single internal s-a-fs.

A relation between the number of spectral coefficients that may be effected by a fault and the minimum number of test patterns required to detect the fault will be presented by Theorem 12. Now the following lemmas give an upper bound on the number of spectral coefficients that may be effected by an internal stuck-at fault.

Lemma 2 *: Suppose $f(x_0, ..., x_{n-1})$ is implemented by a 2-level AND-OR network. Then if the output of an AND gate implementing a product term $P = \dot{x}_0, ..., \dot{x}_{n-1}$, $\dot{x}_i \in \{x_i, \bar{x}_i, d\}$ is s-a-0, then the number A_k of spectral coefficients in $\hat{f}_k$, $k \in \{0, 1, ..., N\}$, that may be effected by the s-a-0 fault is at most*

$$A_k \leq 2^{n - \|p \oplus k\|}, \tag{35}$$

where $k = (k_0, ..., k_{n-1})$ is the polarity of the RM transform and p is obtained from P by replacing x_i by 1 and $\bar{x}_i$ by 0 in P, d is the don't care variable, $d \oplus k_j = 0$, $k_j \in \{0,1\}$ and $\| p \oplus k \|$ is the number of 1's in $p \oplus k$.

Proof: From (5)

$$\hat{f}_k(W) = \bigoplus_{X=0}^{N} R_W(X \oplus k)\, f(X) = \bigoplus_{X \oplus k \subseteq W} f(X). \tag{36}$$

Clearly if $\| X \oplus k \| = m$, then there exists 2^{n-m} $W's$ which satisfy $X \oplus k \subseteq W$. Now if $\hat{f}_k(W) \neq \hat{f}^*_k(W)$ for a given W, then there exists a X such that $X \oplus k \subseteq W$, $P(X) = 1$. Without loss of generality, suppose that

$$P = x_0x_1...x_{i-1}\bar{x}_i...\bar{x}_{j-1} \implies p = \overbrace{11...1}^{i}\overbrace{00...0}^{j-i}\overbrace{dd...d}^{n-j}.$$

If $k_s \oplus p_s = 0$, then all $W = (w_0, w_1, ..., w_{n-1})$ with $w_s = 0$ and $w_s = 1$ may be distorted by P s-a-0. If $k_s \oplus p_s = 1$ then only $W = (w_0, w_1, ..., w_{n-1})$ with $w_s = 1$ may be distorted by P s-a-0. If $s > j-1$ then for all $W = (p_0 \oplus k_0 , p_1 \oplus k_1 , ..., p_{j-1} \oplus k_{j-1} , w_j, ..., w_{n-1})$ with $w_s = 0$ and $w_s = 1$ may be distorted by P s-a-0. Thus, we have for a total number of spectral coefficients that may be effected by this fault

$$A_k \leq 2^{j-\|p\oplus k\|+n-j} = 2^{n-\|p\oplus k\|}$$

In particular if $k = 0$, a number of spectral coefficients that may be effected by a s-a-0 fault at the output of an AND gate realizing P is upper bounded by $2^{n-\|P\|}$, where $\| P \|$ is the number of variables appearing in P in their true form.

As we will see below an upper bound on the number of test patterns increases monotonically with A_k. Since the number of spectral coefficients that may be effected depends on the polarity of spectrum k, we will now describe a simple procedure to minimize an upper bound on A_k given by (35).

Let

$$f = \bigvee_{i=1}^{N_p} P_i \ ,$$

where N_p is the number of product terms in f. Consider minimal weight vectors b_i such that $P_i(b_i) = 1$. Form a matrix $B = (b_{ij})$, with b_i's as rows. Then the elements of $k = (k_0, \ ..., \ k_{n-1})$ are found from the matrix B as shown below

$$k_j = \begin{cases} 0, & if \ \sum_{i=1}^{N_p} b_{ij} \geq \lfloor N_p/2 \rfloor \ ; \\ 1, & otherwise. \end{cases}$$

From now on only $k = 0$ will be considered (results may be easily generalized for any k). The following lemma is obvious.

Lemma 3 *: In a network implementing $f = \bigvee_{i=1}^{N_p} P_i$ if P_i s-a-0, then none of the spectral coefficients W's such that $b_i \not\subseteq W$, are effected by the s-a-0 fault (where b_i is the minimal vector such that $P_i(b_i) = 1$).*

Example 7 *: Let $P = \bar{x}_0 x_1 x_2 d$, $b = 0110$ and $W = 0101$. Then the descendants of W are (0000, 0001, 0100, 0101), none of them would make $P = 1$, and hence any s-a-0 at P does not effect W.*

Lemma 4 *: In a network implementing $f = \bigvee_{i=1}^{N_p} P_i$ if P_i s-a-0 then for all W's such that $b_i \subset W$, $(b_i \neq W)$ and for all $b_s (s \neq i)$, $b_s \not\subseteq W$, $\hat{f}(W) = \hat{f}^*(W)$. The only spectral coefficient that is effected by the s-a-0 fault is $W = b_i$, $(\hat{f}(b_i) \neq \hat{f}^*(b_i))$.*

Proof: From (3)

$$\hat{f}(W) = \bigoplus_{X=0}^{N} R_W(X)\, f(X) \ = \ \bigoplus_{X \subseteq W} f(X). \tag{37}$$

Consider W such that $b_s \not\subseteq W$ for all s, and $b_i \subset W$, then the number of descendants of W at which the value f equalsto 1 is even and hence $\hat{f}(W) = 0$. Similarly, f^* is 0 for all the descendants of W and $\hat{f^*}(W) = 0$. Therefore none of the W's such that $b_i \subset W$ and for all s $b_s \not\subseteq W$ are effected. However, if $W = b_i$ clearly $\hat{f}(b_i) \neq \hat{f}^*(b_i)$.

In the next couple of paragraphs we remind some results from coding theory which we will be using for testing of networks for internal faults.

Consider the vectors $v_0, v_1, v_2, ..., v_n$ of length 2^n and all products of r or fewer vectors from $\{v_1, v_2, ..., v_n\}$ at a time which form the basis for Reed-Muller codes of r^{th} order [29]. They can be arranged in a matrix form V, with $\sum_{i=0}^{r} \begin{pmatrix} n \\ i \end{pmatrix} = M$ rows and 2^n columns. It can be easily seen that the rows of V are same as some of the rows of $[R]_0$ (see (7), (10) - (12)). Hence, from (4), components of $V.\hat{f}$ are values of f(X) for $\| X \| \geq n - r$.

For example, for $n = 4$ and $r = 2$

$$V.\hat{f} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_1 v_2 \\ v_1 v_3 \\ v_1 v_4 \\ v_2 v_3 \\ v_2 v_4 \\ v_3 v_4 \end{bmatrix} .\hat{f} = \begin{bmatrix} 1111 & 1111 & 1111 & 1111 \\ 1111 & 1111 & 0000 & 0000 \\ 1111 & 0000 & 1111 & 0000 \\ 1100 & 1100 & 1100 & 1100 \\ 1010 & 1010 & 1010 & 1010 \\ 1111 & 0000 & 0000 & 0000 \\ 1100 & 1100 & 0000 & 0000 \\ 1010 & 1010 & 0000 & 0000 \\ 1100 & 0000 & 1100 & 0000 \\ 1010 & 0000 & 1010 & 0000 \\ 1000 & 1000 & 1000 & 1000 \end{bmatrix} \begin{bmatrix} \hat{f}(0000) \\ \hat{f}(0001) \\ \hat{f}(0010) \\ \hat{f}(0011) \\ \hat{f}(0100) \\ \hat{f}(0101) \\ \hat{f}(0110) \\ \hat{f}(0111) \\ \hat{f}(1000) \\ \hat{f}(1001) \\ \hat{f}(1010) \\ \hat{f}(1011) \\ \hat{f}(1100) \\ \hat{f}(1101) \\ \hat{f}(1110) \\ \hat{f}(1111) \end{bmatrix} = \begin{bmatrix} f(1111) \\ f(0111) \\ f(1011) \\ f(1101) \\ f(1110) \\ f(0011) \\ f(0101) \\ f(0110) \\ f(1001) \\ f(1010) \\ f(1100) \end{bmatrix}$$

Remark 1: It is known from coding theory [30] that any $2^{r+1} - 1$ columns in V are linearly independent, where V is the matrix whose rows are the basis vectors of r^{th} order Reed-Mulller code.

The following theorems give estimations on a number of test patterns required to test a network.

Theorem 12 : *If any fault in a network implementing f causes a distortion of at most A spectral coefficients in the RM expansion with $k = 0$, then a minimal number of test patterns required to detect all single faults in the network is upper bounded by*

$$\sum_{i=0}^{\lceil log_2(A+1) \rceil - 1} \begin{pmatrix} n \\ i \end{pmatrix}. \tag{38}$$

Proof: Let us take as test patterns all X such that $\| X \| \geq n - r$, where $r = \lceil log_2(A+1) \rceil - 1$. From Remark 1, since the number of linearly independent columns in V is $2^{r+1} - 1$, that many errors in $\hat{f}$ can be detected by the code with check matrix V [29]. Computation of r and the number of rows in V yields (38).

Example 8 *: Let f is a 9 out of 12 function, ie.,*

$$f(X) = \begin{cases} 1, & if \quad \| X \| = 9\,; \\ 0, & otherwise. \end{cases}$$

It is clear that every product term consists of nine variables. If any single fault effects at most one product term then by Lemma 2, with $k = 0$, $\| b_i \| = 9$, we have $A = 2^{12-9}$, $r = 3$ and a number of test patterns required to test the network is at most

$$M = \sum_{i=0}^{3} \binom{n}{i} = 299.$$

Definition: Let $T_M = \{X_1, ..., X_M\}$ be the set of test patterns such that $\| X_i \| \geq n - r$. Denote for a given

$$f = \bigvee_{i=1}^{N_p} P_i,$$

by Q_j a number of $X's$, $X \in T_M$, such that X has exactly j product terms which are proper descendants of X.

Theorem 13 *: In any 2-level AND-OR network all internal single s-a-0 faults can be detected by at most*

$$\sum_{i=0}^{r} \binom{n}{i} - Q_0 - Q_1 \tag{39}$$

test patterns, where $r = \max_i(n - \| P_i \|)$, $\| P_i \|$ denotes the number of variables appearing in their true form in P_i.

Proof: Consider the test patterns from $T_M = \{X_1, ..., X_M\}$ ($\| X_i \| \geq n - r$). Then, the first two terms in (39) are obtained directly from Theorem 12 and Lemma 3. Now, let b_i corresponding to P_i be the vector such that $b_i \subset X_j$ and $b_l \not\subseteq X_j$, for all l. Since $b_i \subset X_j$, we have $\| b_i \| < \| X_j \|$, and

$$n - \| X_j \| < n - \| b_i \| \leq r.$$

Now by definition of T_M, there exists $X_i \in T_M$ such that $X_i = b_i$. Then both X_i and X_j would detect a s-a-0 fault in P_i, but any one of these test patterns is sufficient for detection of this fault. Hence, Q_1 redundant test patterns are subtracted.

It is possible to select a polarity of the spectrum such that the r in the above theorem is small. For example, if f is unate then the number of test patterns required to test all internal s-a-0 faults is given by

$$\sum_{i=0}^{r} \binom{n}{i}$$

where $r = \max_i(n - | P_i |)$, and $| P_i |$ number of variables in P_i ($| P_i | \geq \| P_i \|$).

Example 9 *: Let* $f = x_0x_1 \vee x_2x_3 \vee x_0x_3$*, then* $b_1 = 1100$, $b_2 = 0011$, $b_3 = 1001$, *for all* i, $\| P_i \| = 2$, $r = n - \| P_i \| = 2$,

$$T_M = (1111, 0111, 1011, 1101, 1110, 0011, 0101, 0110, 1001, 1010, 1100)$$

and $Q_0 = 2$ *since the test patterns (0101, 0110, 1010) have no product terms of f as their descendants. Similarly* $Q_1 = 2$ *since each one of the test patterns (0111, 1110) has exactly one product term as its descendant. And by Theorem 13, a number of test patterns is at most 6.*

6 Time Complexities For Detection Of Input Faults In Standard Components

In this section, time complexities for detection of input s-a-fs by Reed-Muller spectral techniques for some of the standard computer hardware components are presented. In the case when the function has multiple outputs the EX-OR sum of all the outputs have been used for observation.

Time complexities For Standard Components

S.No.	*Function*	*time complexity*
1.	*parity checker*	$2n$
2.	*OR function*	$2n$
3.	*NOR function*	$2n$
4.	*Quadratic function*	$2n$
5.	*Match detector*	$2n$
6.	*Threshold detector* $(t = 2)$	$2n$
7.	$f(X) = S_2(X) = 1 \; iff \; \| X \| = 2$	$2n$
8.	*Threshold function* $t = c$	$\lceil \frac{n}{c} \rceil 2^c$
9.	$f(X) = S_c(X) = 1 \; iff \; \| X \| = c$	$\lceil \frac{n}{c} \rceil 2^c$
10.	*AND function*	2^n
11.	*NAND function*	2^n
12.	*comparator* $X \leq Y \Rightarrow 1$	$3n$
13.	*Multiplexer*	$\sum_{i=0}^{n-1} \binom{n}{i} 2^{i+1}$
14.	$f(X) = x^i$	$n + 1$
15.	$f(X) = X^i + 1$	$n + 2$
16.	$f(X) = X + 1$	$n + 2$
17.	$f(X, Y) = X + Y$ *(adder)*	$n + 1$
18.	$f(X, Y) = X.Y$ *(multiplier)*	$1.5n + 1$
19.	$\odot$ *shift left/right*	$n + 1$
20.	*shift left/right*	$n + 2$

where $\lceil m \rceil$ denotes the least integer greater than m.

APPENDIX

1. Let $\varphi(X) = f(X) \wedge \phi(X)$ then

$$\hat{\varphi}(W) = \bigoplus_{U,V=0}^{N} \hat{f}(U)\hat{\phi}(V)\, \delta_{U\vee V,W}\,.$$

Proof: From (3), (4) and (13) we have

$$\hat{\varphi}(W) = \bigoplus_{X=0}^{N} R_W(X) f(X)\phi(X) =$$

$$\bigoplus_{X=0}^{N} R_W(X) \bigoplus_{U=0}^{N} R_X(U)\hat{f}(U) \bigoplus_{V=0}^{N} R_X(V)\hat{\phi}(V) =$$

$$\bigoplus_{X=0}^{N} R_W(X) \bigoplus_{U,V=0}^{N} R_X(U)R_X(V)\hat{f}(U)\hat{\phi}(V) =$$

$$\bigoplus_{X=0}^{N} R_W(X) \bigoplus_{U,V=0}^{N} R_X(U\vee V)\hat{f}(U)\hat{\phi}(V)\,.$$

Interchanging the order of mod 2 addition and from (15) we have

$$\hat{\varphi}(W) = \bigoplus_{U,V=0}^{N} \hat{f}(U)\hat{\phi}(V)\, \delta_{U\vee V,W}\,.$$

2. Let $\varphi(X) = f(X) \vee \phi(X)$. Then

$$\varphi(X) = f(X) \oplus \phi(X) \oplus f(X)\phi(X)\,.$$

From the previous result we obtain

$$\hat{\varphi}(W) = \hat{f}(W) \oplus \hat{\phi}(W) \oplus \bigoplus_{U,V=0}^{N} \hat{f}(U)\hat{\phi}(V)\, \delta_{U\vee V,W}\,.$$

3. Let $\hat{\varphi}(W) = \hat{f}(W)\hat{\phi}(W)$. Then

$$\varphi(X) = \bigoplus_{W=0}^{N} R_X(W)\hat{f}(W)\hat{\phi}(W)\,.$$

Proof is similar to case 1.

4. **Proof of Theorem 7:** The proof consists of two parts.
i). It will be shown that there exist B_1 and B_2 such that

$$\| \bigvee_{W\in B_1\cup B_2} W \| = n\,,$$

$$B_1 = \{W \mid \| W \| = 1,\ W_i \wedge W_j = (00...0),\ \| \bigvee_{W \in B_1} W \| = \lfloor n/2 \rfloor\},$$

$$prob\ \{\hat{f}(W) = 1\ \ for\ all\ W \in B_1\} \longrightarrow 1\ as\ n \longrightarrow \infty$$

and

$$B_2 = \{W \mid \| W \| = 2,\ W_i \wedge W_j = (00...0),\ \| \bigvee_{W \in B_2} W \| = \lceil n/2 \rceil\},$$

$$prob\ \{\hat{f}(W) = 2\ \ for\ all\ W \in B_2\} \longrightarrow 1\ as\ n \longrightarrow \infty$$

ii). From i) it will be shown that Theorem 7 is true.

i): First we note

$$E\ \{W_i \mid \hat{f}(W_i) = 1,\ \| W_i \| = 1\ \} = n/2$$

From this it is clear that with the probability converging to 1 there exists a set B_1 of $\lfloor n/2 \rfloor$ spectral coefficients which are sufficient for detection of stuck-at faults at the corresponding $\lfloor n/2 \rfloor$ input lines. (If $\| W_i \| = 1$, $W_i = (00...0\ \overset{i}{1}\ 0...0)$, $\hat{f}(W_i) = 1$ the stuck-at fault at input line x_i can be detected by verification of $\hat{f}^*(W_i)$). Stuck-at faults at the remaining $\lceil n/2 \rceil$ input lines will be detected by verification of $\hat{f}^*(W)$ where $\| W \| = 2$. It will now be shown that such a set of W exists with probability converging to 1 as $n \longrightarrow \infty$. Let us denote $\lceil n/2 \rceil = m$. We note that

$$prob\ (\hat{f}(W_i) = 1, \| W_i \| = 2) = 0.5\ .$$

Consider B_2 such taht for any $W \in B_2$, $\| W \| = 2$,
$\| \bigvee_{W \in B_2} W \| = m$ and for any $W_i, W_j \in B_2$, $W_i \wedge W_j = (00...0)$.
Then

$$prob\ \{\hat{f}(W) = 1\ for\ all\ W \in B_2\} \geq 2^{-m/2}$$

Let $P = \{B_2\}$, the set of all B_2 satisfying the above definition. Then

$$prob\ \{\exists\ B_2 \in P\} \geq 1 - (1 - 2^{-m/2})^Q$$

where $Q = | P |$, and

$$Q \leq \left(\lfloor \frac{m}{2} \rfloor !\right)^{-1} \binom{m}{2} \binom{m-2}{2} ... \binom{2}{2} = \left(\lfloor \frac{m}{2} \rfloor !\right)^{-1} \frac{m!}{2^{m/2}}.$$

Using Sterling's formula one can get

$$Q \simeq m^{m/2} e^{-m/2} \sqrt{2},$$

and

$$\lim_{m \to \infty} (1 - 2^{-m/2})^Q = 0\ \ iff\ \ \frac{Q}{2^{m/2}} \to \infty\ as\ \ m \to \infty\ .$$

Clearly

$$\frac{Q}{2^{m/2}} = \frac{m^{m/2}\sqrt{2}}{e^{m/2}\,2^{m/2}} \to \infty \; as \;\; m \to \infty \, .$$

This implies that there exists at least one set B_2 which is sufficient for detection of stuck-at faults at the remaining $\lceil n/2 \rceil = m$ input lines.

ii). The total number of coefficients to be observed is $\mid B_1 \mid + \mid B_2 \mid \leq \lfloor n/2 \rfloor + \lceil \frac{1}{2} \lceil n/2 \rceil \rceil \simeq 3n/4$. Hence $3n/4$ memory cells are sufficient to store these coefficients. The time complexity required to compute these spectral coefficients can be estimated as follows. Suppose

$$W_i = (00...1...0). \, Then \; \hat{f}(W_i) = \begin{cases} f(0...1...0), & if \;\; f(00...0) = 0; \\ 1 \oplus f(0...1...0), & if \;\; f(00...0) = 1. \end{cases}$$

In both cases only $(0... \overset{i}{1} ...0)$ has to be applied to the network and the ouput of the T flip flop or its complement is taken as the $\hat{f}(0...1...0)$. Since there are $\lfloor n/2 \rfloor$ spectral coefficients with $\| W \| = 1$ only $\lfloor n/2 \rfloor$ steps are required. Similarly for the case when $\| W \| = 2$, 3n/4 steps are required. In total the time complexity for detection of all multiple terminal s-a-fs is 1.25n .

REFERENCES:

[1]. D.E.Muller, "Application of Boolean Algebra to switching circuit design and to error detection," IRE Trans. Electron. Computers., vol. EC-3, pp 6-12, Sep. 1954.

[2]. I.S.Reed, "A class of multiple error correcting codes and the decoding scheme," IRE Trans. Inform. Theory, vol. IT-4, pp 38-49, Sep. 1954.

[3]. L.T.Fisher, "Unateness properties of AND-EXCLUSIVE-OR Logic Circuits," IEEE Trans. Computers, vol. c-23, no.2, Feb 1974, pp 166-172.

[4]. M.Davio, "Ring-Sum expansions of Boolean functions," in Proc. Symp. on Computers and Automata, Polytech. Inst. of Brooklyn Symp. Proc., vol. 21, pp. 411-418, April 1971.

[5]. C.V.Ramamoorthy, "Procedures for minimisation of 'Exclusive-OR' and 'Logical-Equivalence' switching circuits," in 1965 IEEE Symp. on Switching Theory and Logical Design (Ann Arbor, Mich., Oct 6-8), pp. 143-149.

[6]. S.M.Reddy, "Easily testable realisations for logic functions," IEEE Trans. Computers vol. c-21, pp. 1183-1188, Nov 1972.

[7]. Chen.X, "The mapping of spectral coefficients and its application," Computer Electronic Engineering 9, pp 167-180 (1982).

[8]. Wu.X., Chen.X and Hurst.S.L, "Mapping of Reed-Muller coefficients and the minimisation of Exclusive-OR switching functions," Proc. IEE E-129, pp. 15-20 (1982).

[9]. Ph.W.Besslich, "Spectral Processing of Switching Functions using Signal-flow Transformations," in M.G.Karpovsky (Ed), Spectral Techniques and Fault Detection. Academic Press, Inc. 1985.

[10]. Ph.W.Besslich, "Efficient computer method for Ex-OR logic design," IEE Proc., vol. 130, part E, 1983, pp. 203-206.

[11]. K.L.Kodandapani and R.V.Setlur., "A note on minimal Reed-Muller Canonical forms of switching functions," IEEE Trans. on Computers, Mar 77, c-26, pp 310-313.

[12]. Even.S., Kohavi.I. and Paz.A., "On minimal modulo-2 sums of products for switching functions," IEEE Trans. EC-16, pp 671-674, 1967.

[13]. Mukhopadhyay.A., and Schmitz.G., "Minimisation of Exclusive-OR and logic equivalence switching circuits," IEEE Trans. c-19, pp. 132-140, 1970.

[14]. Bioul.G., Davio.D., and Deschamps.J.P., "Minimisation of Ring-Sum expansions of Boolean functions," Philips Research Reports 1973, 28, pp. 17- 36.

[15]. Zhang.Y.Z., and Rayner.P.J.W., "Minimisation of Reed-Muller polynomials with fixed polarity," Proc. IEE. vol E-131, No.5 Sep 1984, pp.177-186.

[16]. Miller.D.M., and Muzio.J.C., "Spectral Techniques for fault detection in Combinational logic," in M.G.Karpovsky (Ed.), Spectral Techniques and Fault Detection., Academic Press, Inc. 1985.

[17]. Miller.D.M., and Muzi0.J.C.,"Spectral techniques for Fault detection," Proc. 12th Intl. Symp. on Fault Tolerant Computing, 1982, pp.152-158.

[18]. Muzio.J.C., and Miller.D.M., "Spectral Fault signatures for internally unate combinational networks," in IEEE Trans. on Computers V. C-32, 1983, pp.1058-1062.

[19]. Miller.D.M. and Muzio.J.C., "Spectral fault signatures for single stuck-at-fault signatures in combinational networks," IEEE Trans. on Computers V C-33, pp.765-769, 1984.

[20]. Susskind.A.K., "Testing by verifying Walsh coefficients, " IEEE Trans. on Computers V. C-32, pp.198-201, 1983.

[21]. Muzio.J.C., "Composite spectra and the analysis of switching circuits," IEEE Trans, on Computers, V. C-29, pp.750-753, 1980.

[22]. Hurst.S.L., Miller.D.M., and Muzio.J.C., Spectral Techniques in Digital Logic, Academic Press, 1985.

[23]. M.G.Karpovsky (Ed.), Spectral Techniques and Fault Detection, Academic Press, 1985.

[24]. M.G.Karpovsky, Finite Orthogonal Series in the Design of Digital Devices, John Wiley, N.Y. 1976.

[25]. Saluja.K.K., and S.M.Reddy," Fault Detecting Test Sets For Reed-Muller Canonic Networks," IEEE Trans. on Computers C-24, (1975), pp.995-998.

[26]. S.Muroga, " Threshold Logic And Its Applications," Wiley-Interscience, 1971, New York.

[27]. M. Karpovsky and L.B. Levitin., " Universal testing, " International Symp. on Fault Detection and Spectral Techniques, Boston University, Boston, Oct 1983.

[28]. Kuhl, J.G. and Reddy, S.M., " On the detection of stuck-at-faults, " IEEE Trans. on Computers, C-27, (May 1978), pp. 467-469.

[29]. W.W. Peterson and E.J. Weldon, Jr., " Error Correcting Codes, " The MIT Press, Cambridge.

[30]. J.H.van Lint, " Introduction to Coding Theory, " Springer- Verlog, New York, 1982.

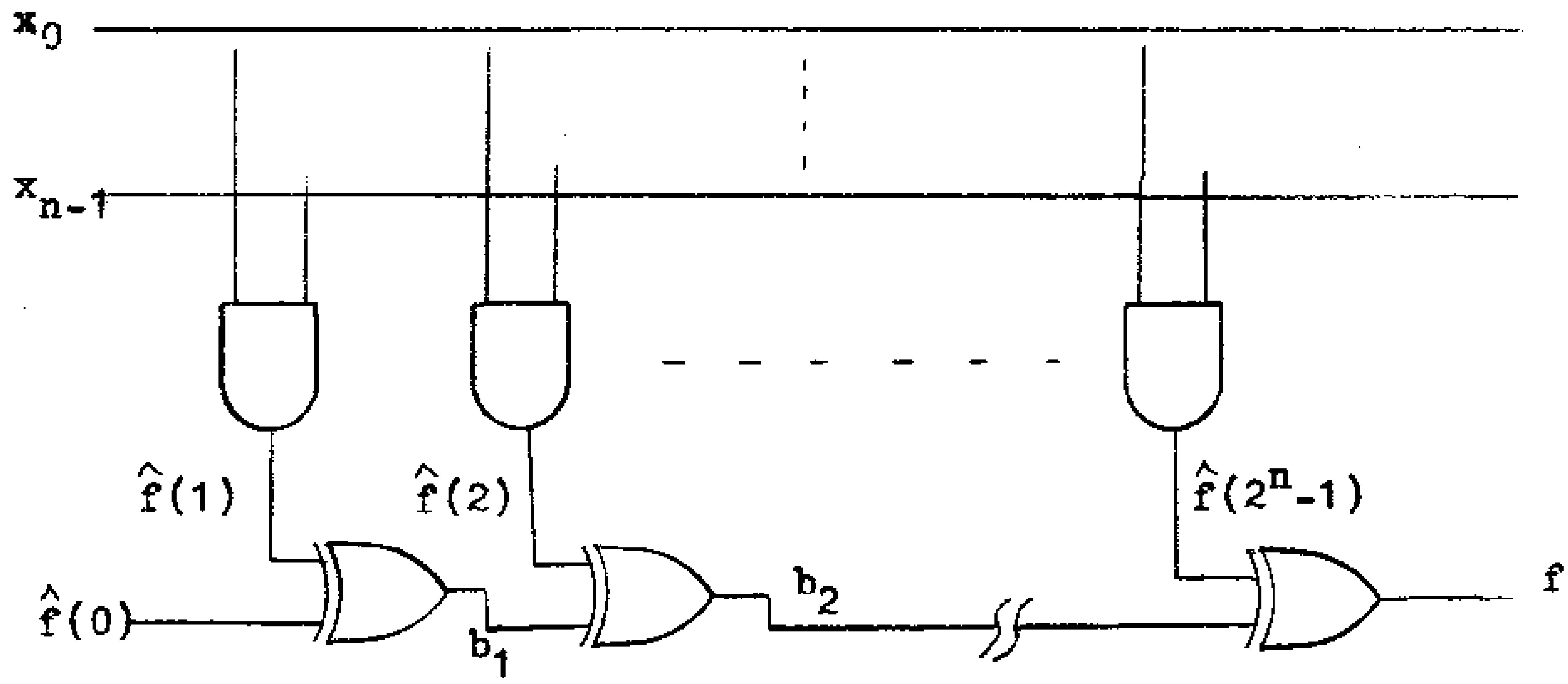

Fig 1. Reed - Muller Canonical Network.

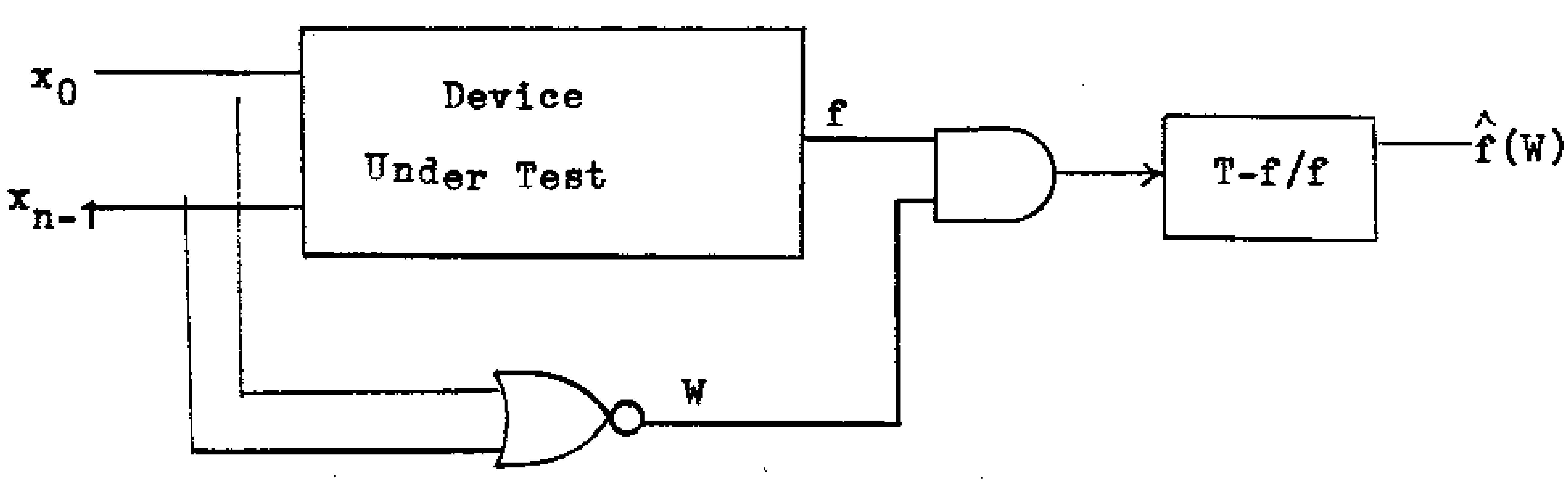

Fig 2. Network for computing a RM spectral coefficient.

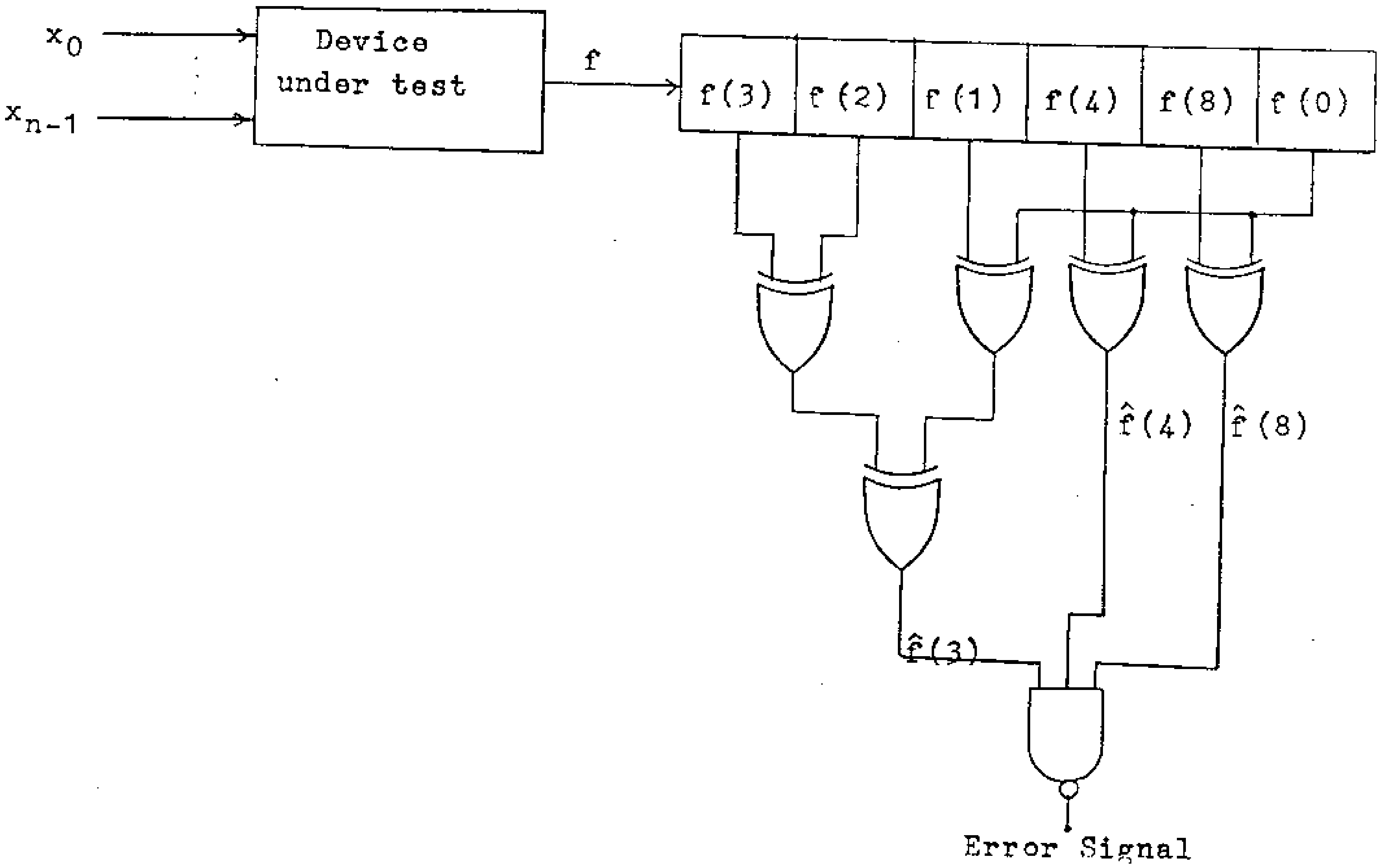

Fig 3. Testing by scheme 2 for problem in Example 5.