Functional testing of computer hardware and data-transmission channels based on minimising the magnitude of undetected errors

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Abstract: The paper introduces a criterion for test generation based on minimising the expected magnitude of undetected errors. This criterion is used to develop a best strategy for testing, using the linear checks approach. The detailed analysis is carried out for single unidirectional and bidirectional errors and for multiple unidirectional errors. Specific results concerning the efficiency of the approach are given for basic arithmetical and logical instructions. This approach may be useful in the field testing of hardware which carries out data manipulation and in which small numerical errors can be tolerated; it may also be useful for testing digital transmission channels.

1 Introduction

With the advent of VLSI and the corresponding drastic increase in the density of gates on a chip, high-level functional testing is one of the most viable approaches to the testing of computer hardware. Two well known methods of functional testing are signature analysis [1] and syndrome check sums [2]. However, these methods have severe practical limitations, mainly owing to the considerable testing time and the difficulty in estimating the fault coverage.

Over the last few years, one of the authors [3–8] has developed another approach to the problem of high-level functional testing which, in many cases, gives better error-detecting and/or error-locating capability and, in general, requires less testing time. This approach provides a method for computing what are called 'linear check test sets'.

The approach is based on partitioning the set of all 2^n inputs into blocks (test sets), such that under fault-free conditions, the sum of the outputs for all inputs within a test set is the same constant C for every test set. The check as to whether this is indeed the case constitutes the error-detection method. Using several partitions, one can locate or even correct errors. These partitions depend, of course, on the function implemented by the device under the test.

Denote by $f(x)$, the output corresponding to the input signal $x = (x_1, \ldots, x_n)$ $(x_i \epsilon \{0, 1\})$. For example, $f(x)$ may be the data stored in a cell with the address $x = (x_1, \ldots, x_n)$.

Let G be the group of all binary n-vectors with respect to the operation $\oplus$ of componentwise addition mod 2. The method of error detection is then based on verification of a linear equality check

$$\sum_{\tau \epsilon T} f(t \oplus \tau) - C = 0 \quad \text{for every } t \epsilon \{0, 1, \ldots, 2^n - 1\} \quad (1)$$

where T is a 'check' subgroup in G, $t \oplus T = \{t \oplus \tau | \tau \epsilon T\}$ is a check set and C is a constant (we are using the same notation for a binary vector and its decimal equivalent).

For example, for a multiplier $f(x) = f(y, z) = y \cdot z(y, z \epsilon \{0, \ldots 2^r - 1;\ 2r = n\})$ [4], $T = \{0^{2r}, 0^r 1^r, 1^r 0^r, 1^{2r}\}$ $(a^i = \underbrace{a \ldots a}_{i}, a \epsilon \{0, 1\})$, and we have the following check:

$$yz + \bar{y}z + y\bar{z} + \bar{y}\bar{z} = (2^r - 1)^2$$

where $\bar{y} = y \oplus 1^r$ is the componentwise negation of a binary vector y.

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An optimal check set for the given function $f(x)$ is the one which minimises the cardinality $|T|$ of T. Methods for constructing check sets with the minimal cardinality and determining a constant C for a given function $f(x)$, implemented by a device under the test, are given in References 3 to 6.

It should be noted that, when T is chosen to be G, the check (eqn. 1) becomes the well known syndrome sum check [1].

In the above approach, one can choose any element t of the set G/T of coset representatives of subgroup T in G for testing by eqn. 1 [3]. If eqn. 1 is satisfied for this t, one can randomly choose another element of G/T and verify eqn. 1 for this new value of t. The process can be stopped when an error is detected or when one reaches the predetermined number of test sets. (If R is the number of tests sets, then $R|T|$ is the testing time.)

Instead of choosing the sequence of elements of G/T at random, one can choose them in a predetermined order, such that those elements which are chosen first will detect the most 'harmful' errors. It is a purpose of this paper to investigate this possibility.

The 'harmfulness' of an error $e(x)$ may be measured by its magnitude

$$|e(x)| = |\hat{f}(x) - f(x)|$$

where $\hat{f}(x)$ is the value of the output under fault conditions and $f(x)$ is the fault-free output.

In this paper, we shall propose a method for optimal ordering of test sets which will maximise the expected value of $|e(x)|$ for errors detected by a test when the number R of test sets (testing time) is given. This method will be useful only in the case when an output $f(x)$ has intrinsic numerical value, and errors in different components in a binary representation of $f(x)$ have different weights. We note that this approach, based on maximising the expected magnitude of errors detected by a test, is very similar to mean-square error-detection procedures widely used in data transmission [9, 10].

We shall give a solution of the problem of optimal ordering of test sets for the case of single symmetrical bidirectional errors and single and multiple unidirectional stuck-at errors. It is well known that many faults that commonly occur in VLSI circuits cause unidirectional errors [11]. Included here are those faults that affect the address decoders, word lines, power supply, read/write circuits [12–17], and also single stuck-at, cross-point, or bridging faults in programmable logic arrays and bursty errors that are due to failures in certain storage devices [11].

The approach described in this paper may be effectively used for maintenance (field) testing. Its efficiency increases with the increase in a number of bits n. (The time taken to generate the tests is proportional to 2^n, but this represents only a one-time cost.)

In Section 2, we introduce an objective or 'utility' function $W(t)$, which defines an expected value for the magnitude of errors detected by the test set $t \oplus T$. In the optimal test strategy, one chooses first $t \epsilon G/T$ for which $W(t)$ is maximum, followed by the element of G/T for which it has the next highest value, and so on.

In Section 3, we derive explicit expressions for the utility function $W(t)$ for single stuck-at errors. We also describe some analytical properties of the utility function $W(t)$ for single stuck-at unidirectional errors, and estimate the efficiency of our approach for testing basic computer instructions, both arithmetical and logical.

Generalisation of the results of Section 3 to the case of multiple unidirectional errors is given in Section 4.

Invariant properties of best test sequences and application of the approach to bridging faults and input errors are discussed in Section 5.

We conclude the introduction by pointing out the cases and conditions where the proposed testing approach may be efficient. The conditions are:

(*a*) where only input and output signals are available for testing, such as may be the case when the device is a part of a flight control system

(*b*) where exhaustive tests, verifying every value of $f(x)$, cannot be implemented in a reasonable time

(*c*) where a low expected magnitude of undetected errors at the output of the device can be tolerated, but where it is desirable to detect errors in and protect the more significant bits.

Some specific cases which come to mind are as follows:

(i) Testing of hardware for carrying out data manipulation, where small numerical errors can be tolerated, but where exhaustive testing is not possible because of time limitations. An example is a navigation system whose very precise determination of direction at a particular time may not be crucial. Another example is when the output of the computation is quantised into a range of intervals, and as long as the result is within a certain range, the action will be independent of the exact value.

(ii) Testing of complex systems having computer hardware at the input and at the output connected by a digital transmission channel ($f(x) = x$).

We must make it clear that the proposed approach does not apply to one of the most difficult-to-test part of a computer, the control unit; this can tolerate no errors.

2 Basic formulation

In this Section we define the utility function alluded to in Section 1 and describe the test strategy.

Let $\|f(x)\|$ be the number of 1s in the binary representation of the ouput $f(x)$ (the Hamming weight of $f(x)$). By definition, $n - \|f(x)\|$ is the number of 0s. In a program or a device, the presence of errors can corrupt any 1 into a 0 or vice-versa. Suppose that, as a result of an error, $f(x)$ is distorted to $\tilde{f}(x)$. The magnitude of the error in the computation of $f(x)$,

$$|e(x) = |\tilde{f}(x) - f(x)| \quad (2)$$

will depend on the type of error. For example, an error in the most significant (leftmost) bit may be more harmful than one in the least significant (rightmost) bit. Therefore, one may choose a test strategy which tends to emphasise large $|e(x)|$. However, although the impact of an error on $f(x)$ may be large, its probability may not be very high. Therefore, an alternative strategy may be to choose those tests which emphasise large expected values $E(x)$ of $|e(x)|$. That is, if p_e is the probability of a certain error $e \epsilon E$ (E is a set of all possible errors), and this error results in an error $e(x)$ in the computation of $f(x)$, one could use

$$E(x) = \sum_{e \epsilon E} |e(x)| p_e \quad (3)$$

for constructing the test strategy; i.e. one can choose that value of x for which $E(x)$ is maximum, followed by the value of x for which it has the next highest value, and so on.

To clarify the objective function $E(x)$ for choosing a test strategy, let us take a simple example of non-negative $f(x)$, which, for a certain value of x, is represented in the binary form as $f(x) = 1\ 0\ 1\ 1\ 1\ 0$.

Let us restrict ourselves to unidirectional $1 \to 0$ single errors and also assume that the probability of $1 \to 0$ is the same for all 1s and is equal to p. For this case, the following Table shows the various type of errors, their probabilities, and the errors in $f(x)$.

Table 1: Errors

Location of error	$\tilde{f}(x)$						$\lvert e(x)\rvert$	Probability of error
Bit 1	0	0	1	1	1	0	2^5	$p(1-p)^3$
Bit 3	1	0	0	1	1	0	2^3	$p(1-p)^3$
Bit 4	1	0	1	0	1	0	2^2	$p(1-p)^3$
Bit 5	1	0	1	1	0	0	2^1	$p(1-p)^3$

Therefore, the expected value of $|e(x)|$, resulting from single unidirectional $1 \to 0$ errors, is

$$E(x) = p(1-p)^3(2^5 + 2^3 + 2^2 + 2^1) = p(1-p)^3 f(x) \quad (4)$$

In general, for single unidirectional $1 \to 0$ errors, and non-negative $f(x)$:

$$E(x) = \frac{p}{1-p} f(x)(1-p)^{\|f(x)\|} \quad (5)$$

In the context of error detection based on linear equality checks (see eqn. 1), instead of choosing $E(x)$ as the objective function for a testing strategy, one would choose the objective function $W(t)$ defined by

$$W(t) = \sum_{\tau \epsilon T} E(t \oplus \tau) \qquad t \epsilon G/T \quad (6)$$

For the simple case of unidirectional single error $1 \to 0$, for $f(x) \geqslant 0$, from eqn. 5:

$$W_1(t) = \frac{p}{1-p} \sum_{\tau \epsilon T} f(t \oplus \tau)(1-p)^{\|f(t \oplus \tau)\|} \quad (7)$$

or, in view of eqn. 1,

$$W_1(t) \simeq \frac{p}{1-p} \left\{ C - p \sum_{\tau \epsilon T} f(t \oplus \tau) \|f(t \oplus \tau)\| \right\} \quad \text{for small } p \quad (8)$$

(Here, we have added the subscript 1 to denote the multiplicity of the error, by which we mean the number of corrupted values of $f(x)$. This definition of multiplicity is natural if errors in $f(x)$ for different xs are independent, for example, when x is an address and $f(x)$ is the content of the memory location whose address is x).

$W(t)$ is the utility function mentioned in the Introduction. In the best test strategy, one will first choose the check set $t \oplus T$ for which $W(t)$ is maximum, followed by a check set for which $W(t)$ has next to the highest values, and so on.

The basic problem is then as follows: Given the function

$f(x)$, the type of error (e.g. multiplicity, unidirectional etc.), and the number R of tests $t \oplus T(t \epsilon G/T)$, what is the best sequence of tests $BT = (t_1^*, t_2^*, \ldots, t_R^*)$? The proposed optimal strategy is to choose BT such that

$$W(t_1^*) = \max_{t \epsilon G/T} W(t) \tag{9}$$

$$W(t_i^*) = \max_{t \epsilon G/T - \{t_1^*, \ldots, t_{i-1}^*\}} W(t) \quad (i = 2, \ldots, R) \tag{10}$$

The effectiveness of the proposed strategy will depend upon the variance in $W(t)$ i.e.

$$\text{Var } W(t) = \max_t W(t) - \min_t W(t). \tag{11}$$

The greater the variance, the more effective will be the strategy. This variance, in general, will depend on $f(x)$, the number of bits n, in the binary representation of $f(x)$, the probability p of errors, and the multiplicity of errors.

We also note that the same approach based on maximisation of $\sum_{i=1}^{R} W(t_i)$ $(t_i \epsilon G/T)$ may be used when one is given the maximum expected value W of magnitude of undetected errors, rather than the testing time $R|T|$. In this case, the number R of tests may be determined from the condition

$$\sum_{t \epsilon G/T} W(t) - \sum_{i=1}^{R} W(t_i^*) \leqslant W \tag{12}$$

where t_i^* $(i = 1, \ldots, R)$ is defined by eqn. 10. In view of eqn. 6, the condition (eqn. 12) can be rewitten as

$$\sum_{t \epsilon G/T} \sum_{\tau \epsilon T} E(t \oplus \tau) - \sum_{i=1}^{R} W(t_i^*)$$

$$= \sum_{t \epsilon G} E(t) - \sum_{i=1}^{R} W(t_i^*) \leqslant W \tag{13}$$

For example, for single unidirectional $1 \to 0$ errors and $f(x) = x$ (the case $f(x) = x$ may be important for testing communication channels and storage), we have from eqn. 5:

$$\sum_{t \epsilon G} \frac{p}{1-p} t(1-p)^{\|t\|} - \sum_{i=1}^{R} W(t_i^*)$$

$$= \frac{p}{1-p}(2^n - 1) \sum_{i=0}^{n-1} (1-p)^{i+1} \binom{n-1}{i} - \sum_{i=1}^{R} W(t_i^*)$$

$$= (2^n - 1)(2-p)^{n-1} p - \sum_{i=1}^{R} W(t_i^*) \leqslant W \tag{14}$$

where n is the number of bits in the binary representation of $f(x)$.

3 Single errors

In this Section, we derive the expression for the utility function for single errors ($1 \to 0$ or $0 \to 1$), using the general formalism of the preceding Section.

Let p be the probability of a $1 \to 0$ error and q the probability of a $0 \to 1$ error. These probabilities will be assumed to be the same for every bit.

For $f(x) \geqslant 0$ for all $x \epsilon G$, the probability of a *single $1 \to 0$* error in the binary representation of $f(x)$ is

$$p(1-p)^{\|f(x)\|-1}(1-q)^{n-\|f(x)\|} \tag{15}$$

Here the terms involving p represent the probability of a single error out of $\|f(x)\|$ '1 locations', and the term involving q represents the probability of no error at $(n - \|f(x)\|)$ '0 locations'. There are $\|f(x)\|$ possible locations for $1 \to 0$ errors and the sum of $|e(x)|$ for these locations is $f(x)$.

Similarly, the probability of a *single $0 \to 1$* error is

$$q(1-q)^{n-\|f(x)\|-1}(1-p)^{\|f(x)\|} \tag{16}$$

There are $n - \|f(x)\|$ possible locations for $0 \to 1$ errors and the sum of $|e(x)|$ is $2^n - 1 - f(x)$. Therefore, for *non-negative functions* and *single bidirectional errors* (either $1 \to 0$ or $0 \to 1$), the expected value of magnitude of an error in $f(x)$ is

$$E_1(x) = p(1-p)^{\|f(x)\|-1}(1-q)^{n-\|f(x)\|} f(x)$$
$$+ q(1-q)^{n-\|f(x)\|-1}(1-p)^{\|f(x)\|}\{2^n - 1 - f(x)\}$$
$$= (1-q)^n \left(\frac{p}{1-p} f(x) + \frac{q}{1-q}\{2^n - 1 - f(x)\} \right) \alpha^{\|f(x)\|}, \tag{17}$$

where

$$\alpha = \frac{1-p}{1-q} \tag{18}$$

is an asymmetry of errors parameter. Thus, the utility function for *non-negative* functions is

$$W_1(t) = (1-q)^n \left[\frac{p}{1-p} \sum_{\tau \epsilon T} f(t \oplus \tau) \alpha^{\|f(t \oplus \tau)\|} + \frac{q}{1-q} \sum_{\tau \epsilon T} \{2^n - 1 - f(t \oplus \tau)\} \alpha^{\|f(t \oplus \tau)\|} \right] \tag{19}$$

One can derive a similiar expression for *negative* functions, $f(x) < 0$ for all $x \epsilon G$, where in the error-free case the sign bit carries the value 1. For the representation of negative and arbitrary functions (with both positive and negative values) we use the additional sign bit (the 'sign and magnitude' representation). The probability of a *single $1 \to 0$* error is

$$p(1-p)^{\|Af(x)\|}(1-q)^{n-\|Af(x)\|} \tag{20}$$

where

$$Af(x) = |f(x)| \tag{21}$$

A $1 \to 0$ error at the sign bit location causes $|e(x)| = 2Af(x)$. For a $1 \to 0$ error at other locations, and there are $\|Af(x)\|$ such locations, the sum of $|e(x)|$ is equal to $Af(x)$.

In the case of *bidirectional errors* ($1 \to 0$ and $0 \to 1$) and *negative functions* the probability of a single $0 \to 1$ error is

$$q(1-p)^{\|Af(x)\|+1}(1-q)^{n-\|Af(x)\|-1} \tag{22}$$

and it can occur at any of $n - \|Af(x)\|$ locations. The sum of $|e(x)|$ for all these locations is $\{2^n - 1 - Af(x)\}$. Therefore, for negative functions and single errors (either $1 \to 0$ or $0 \to 1$), the expected value of magnitude of an error in $f(x)$ is

$$E_1(x) = p(1-p)^{\|Af(x)\|}(1-q)^{n-\|Af(x)\|}\{2Af(x) + Af(x)\}$$
$$+ q(1-p)^{\|Af(x)\|+1}(1-q)^{n-\|Af(x)\|-1} \times \{2^n - 1 - Af(x)\}$$
$$= (1-q)^n [3pAf(x) + q\alpha\{2^n - 1 - Af(x)\}]\alpha^{\|Af(x)\|} \tag{23}$$

Therefore, the utility function for negative functions is

$$W_1(t) = (1-q)^n \left[3p \sum_{\tau \epsilon T} Af(t \oplus \tau) \alpha^{\|Af(t \oplus \tau)\|} + q\alpha \sum_{\tau \epsilon T} \{2^n - 1 - Af(t \oplus \tau)\} \alpha^{\|Af(t \oplus \tau)\|} \right] \quad (24)$$

For an *arbitrary function*, where the value of $f(x)$ may be both positive and negative, one can obtain the utility function by using eqn. 24 and a slightly modified form of eqn. 19. The modification is required to reflect two extra features: (i) the number of 0s in the binary representation of $f(x)$ is now $n - \|f(x)\| + 1$ rather than $n - \|f(x)\|$; (ii) an $0 \to 1$ error at the sign-bit location causes $|e(x)| = 2Af(x)$. The first feature is incorporated by multiplying the first and second terms in eqn. 19 by $(1-q)$ and the second feature is incorporated by adding $2Af(t \oplus \tau)$ within the bracket in the second term (causing $-f(t \oplus \tau)$ to become $+Af(t \oplus \tau)$). Combining this modified form of eqn. 19 with eqn. 24, we get the utility function for arbitrary $f(x)$ as

$$W_1(t) = (1-q)^n \sum_{\tau \epsilon T} [p\alpha^{-1}\{1 + 2sf(t \oplus \tau)\}Af(t \oplus \tau) + q\{2^n - 1 + f(t \oplus \tau)\}]\, \alpha^{\|Af(t \oplus \tau)\| + sf(t \oplus \tau)}, \quad (25)$$

where the sign function s is defined by

$$sf(x) = \begin{cases} 0, & f(x) \geqslant 0 \\ 1, & f(x) < 0 \end{cases} \quad (26)$$

eqn. 25 can be programmed on a computer for a given $f(x)$, p and q to calculate the values of the utility function $W_1(t)$ for various check sets $t \oplus T$, and then one could determine the best R check sets corresponding to the highest R values of the utility function.

We will now give some general results showing the limitations and the effectiveness of the basic testing strategy introduced in this paper.

3.1 Symmetrical errors

For such errors, by definition $p = q, \alpha = 1$. From eqn. 15,

$$W_1(t) = p(1-p)^n \sum_{\tau \epsilon T} [\{1 + 2sf(t \oplus \tau)\}Af(t \oplus \tau) + \{2^n - 1 + f(t \oplus \tau)\}]$$

$$= p(1-p)^n \{|T|(2^n - 1) + 2 \sum_{\tau \epsilon T} Af(t \oplus \tau)\} \quad (27)$$

If $f(x) \geqslant 0$ for all $x (sf = 0)$ or $f(x) < 0$ for all $x (sf = 1)$, then, from eqn. 1, we have:

$$\sum_{\tau} Af(t \oplus \tau) = (-1)^{sf} C \quad (28)$$

Thus, for *non-negative* (or *negative*) functions,

$$W_1(t) = p(1-p)^n \{|T|(2^n - 1) + 2(-1)^{sf} C\} = \underset{t}{\text{const}} \quad (29)$$

and all test sets $t \oplus T$ are equally good.

However, this is not the case for *arbitrary* functions. Denote

$$f^+(x) = \begin{cases} f(x) & f(x) \geqslant 0 \\ 0 & f(x) < 0 \end{cases}$$

$$f^-(x) = \begin{cases} 0 & f(x) \geqslant 0 \\ f(x) & f(x) < 0 \end{cases} \quad (30)$$

Then

$$f(x) = f^+(x) + f^-(x) \qquad Af(x) = f^+(x) - f^-(x)$$

and, from eqn. 1,

$$\sum_{\tau \epsilon T} Af(t \oplus \tau) = 2 \sum_{\tau \epsilon T} f^+(t \oplus \tau) - C \quad (31)$$

Thus, we have for the utility function in the case of *single symmetrical errors and arbitrary functions*:

$$W_1(t) = p(1-p)^n (|T|(2^n - 1) + 4 \sum_{\tau \epsilon T} f^+(t \oplus \tau) - 2C) \quad (32)$$

Denote

$$K(t) = \sum_{\tau \epsilon T} f^+(t \oplus \tau) \quad (33)$$

It follows now, from eqn. 32, in view of eqns. 9 and 10, that the best test

$BT = (t_1^*, t_2^*, \ldots)$ may be determined from the conditions

$$K(t_1^*) = \max_{t \epsilon G/T} K(t)$$

$$K(t_i^*) = \max_{t \epsilon G/T - \{t_1^*, \ldots, t_{i-1}^*\}} K(t) (i = 2, 3, \ldots) \quad (34)$$

Formulas 27 and 32 can provide analytical solutions to the problem of optimally ordering the tests for single symmetrical errors. This is illustrated by Table 2. In this Table, check sets T, utility functions W_1 and best sequences $BT = (t_1^*, t_2^*, \ldots)$ ($t_i^* \epsilon G/T$) of the tests are given for some analytical functions in the case of single symmetrical errors. The following notations have been used in this Table:

$BT = (a_1, \ldots, a_K, \forall)$ means that the best test sequence is $a_1, \ldots, a_K$, and after a_K any sequence of remaining elements of G/T may be chosen in any order

$$x, y \epsilon \{0, 1, \ldots, N\}, N = 2^n - 1$$

$$\Delta(x, a) = \begin{cases} x & x \leqslant a \\ a & x > a \end{cases} \qquad s(x-a) = \begin{cases} 1 & x < a \\ 0 & x \geqslant a \end{cases}$$

$[a]$ is the greatest integer less than or equal to a

$$\delta_{r,t} = \begin{cases} 1 & r = t \\ 0 & r \neq t \end{cases}$$

3.2 Unidirectional errors

For $1 \to$ errors, $q = 0$ and eqn. 25 becomes

$$W_1(t) = p \sum_{\tau \epsilon T} (1-p)^{-\{1 - sf(t \oplus \tau)\}} \times \{1 + 2sf(t \oplus \tau)\}Af(t \oplus \tau)(1-p)^{\|Af(t \oplus \tau)\|} \quad (35)$$

Table 2: Utility functions W_1 and best test sequences BT for single symmetrical errors ($N = 2^n - 1$)

no.	f	T	G/T	$0.5p^{-1}(1-p)^{-n}W_1$	BT
1	$f(x) = x - 1$	$\{0^n, 1^n\}$	$\{0, 1, \ldots, 2^{n-1} - 1\}$	$W_1(t) = 2\{N - \Delta(t, 1)\}$	$(0, \forall)$
2	$f(x) = x - a,\ 0 < a < 2^n$	$\{0^n, 1^n\}$	$\{0, 1, \ldots, 2^{n-1} - 1\}$	$W_1(t) = 2(N - \Delta\{t, \min([a], \overline{[a]})\}$	$\{0, 1, \ldots, \min([a], \overline{[a]}), \forall\}$
3	$f(x) = ax + b,$ $0 < a^{-1}b < 2^n.$	$\{0^n, 1^n\}$	$\{0, 1, \ldots, 2^{n-1} - 1\}$	$W_1(t) = N + a[N - 2\Delta\{t, \min([a^{-1}b], \overline{[a^{-1}b]})\}]$	$\{0, 1, \ldots, \min([a^{-1}b], \overline{[a^{-1}b]}), \forall\}$
4	$f(x) = \sum_{i=0}^{n-1} x_i(-2)^i$ $\left(x = \sum_{i=0}^{n-1} x_i 2^i, x_i \in \{0, 1\}\right)$	$\{0^n, 1^n\}$	$\{0, 1, \ldots, 2^{n-1} - 1\}$	$W_1(t) = N + \lvert f(t) - f(\bar{t})\rvert$	$(t_1^*, t_2^*, \ldots)$ where $f(t_1^*) = \max_{t \in G} f(t)$; $f(t_i^*) = \max_{t \in G - \{t_1^*, \ldots, t_{i-1}^*\}} f(t) \quad (i = 2, 3, \ldots)$
5	$f(x) = \begin{cases} (-1)^x & x < a \\ 0 & x > a \end{cases}$ $(a \in \{1, 3, \ldots, 2^n - 1\})$	$\{0^n, 0^{n-1}1\}$	$\{0, 2, \ldots, 2^n - 2\}$	$W_1(t) = N + 2a(t - a)$	$(0, 2, 4, \ldots, 0.5(a - 1), \forall)$
6	$f(x) = (-1)^x x$	$\{0^n, 0^{n-1}1\}$	$\{0, 2, \ldots, 2^n - 2\}$	$W_1(t) = N + 2t + 1$	$(N - 1, N - 3, N - 5, \ldots)$
7	$f(x, y) = x - y$	$\{0^{2n}, 1^{2n}\}$	$\{0, 1, \ldots, 2^n - 1\} \times \{0, 1, \ldots, 2^{n-1} - 1\}$	$W_1(t, r) = 2^{2n} - 1 + 2\lvert t - r\rvert$	$\{(N, 0), (N - 1, 0), (N - 1, 1), (N - 2, 0), (N - 1, 1), (N, 2), \ldots\}$
8	$f(x, y) = (-1)^{x+y} xy$	$\{0^n, 0^{n-1}1\} \times \{0^n, 0^{n-1}1\}$	$\{0, 2, \ldots, 2^n - 2\} \times \{0, 2, \ldots, 2^n - 2\}$	$W_1(t, r) = 2(2^{2n} - 1) + (2t + 1)(2r + 1)$	$\{(N - 1, N - 1), (N - 1, N - 3), (N - 3, N - 1), (N - 3, N - 3), (N - 3, N - 5), (N - 5, N - 3), \ldots\}$
9	$f(x, y) = (x - 1)(y - 1)$	$\{0^n, 1^n\} \times \{0^n, 1^n\}$	$\{0, 1, \ldots, 2^{n-1} - 1\} \times \{0, 1, \ldots, 2^{n-1} - 1\}$	$W_1(t, r) = 2(2^{2n} - 1) + (N - 1)^2 + (2N + 3)\delta_{t,0}\delta_{r,0}$	$\{(0, 0), \forall\}$

For $0 \to 1$ errors, $p = 0$ and eqn. 25 becomes

$$W_1(t) = q \sum_{\tau \epsilon T} \{2^n - 1 + f(t \oplus \tau)\} \times (1-q)^{n - \|Af(t \oplus \tau)\| - sf(t \oplus \tau)} \quad (36)$$

The equations can be used to analyse the effectiveness of the test strategy proposed in this paper for the case of unidirectional errors. When the device or a computer program for a function $f(x)$ is highly reliable, $(1-p) \simeq (1-q) \simeq 1$, $W(t) \simeq const$ for all the check sets, and the strategy is not very effective. As p increases, $W(t)$s will tend to be widely dispersed and the effectiveness of the strategy will increase.

To illustrate the effectiveness of the strategy, we now present analytical estimates of the utility functions in the case of single $1 \to 0$ errors and basic arithmetic and logical instructions for selected check sets. These estimations were, in most part, made for $t \epsilon \{0,\ 2^{n-i}-2,\ 2^{n-i}-1,\ 2^{n-i},\ 2^{n-i}+1,\ 2^{n-i}+2\} (i \ll n)$ and are given in Tables 3 to 7. In the estimation procedure, we assume that $(1-p)^i \simeq 1$ but that $(1-p)^n = a \neq 1$.

Thus, for a highly reliable device or program, i is restricted to $\ll n$, for example, $i \simeq \log_2 n$. As an example, for $a = 1/2$, $p = 0.0108$ ($n = 64$), for $i = \log_2 64 = 6$, $(1-p)^i \simeq 0.94$.

Before we discuss these Tables, for pedagogical reasons, we provide below some details of the estimation:

$$(a) f(x) = x, f(x) = x+1, f(x) = x-1, x \epsilon \{0,1\}^n$$

For all of these cases, $T = \{0^n, 1^n\}$, i.e. the check set consists of x and $\bar{x} = 2^n - x - 1$ [4]. For $f(x) = x$, from eqn. 35,

$$W_1(t) \simeq p\{t(1-p)^{\|t\|} + (2^n - 1 - t)(1-p)^{\|2^n - 1 - t\|}\}$$
$$= p\{t(1-p)^{\|t\|} + (2^n - 1 - t)(1-p)^{n - \|t\|}\}$$

As an example,

$$W_1(2^{n-i}-1) = p\{(2^{n-i}-1)(1-p)^{n-i} + (2^n - 2^{n-i})(1-p)^i\}$$
$$\simeq p\{2^{n-i}a + (2^n - 2^{n-i})\}$$
$$= p2^n\{1 + (a-1)2^{-i}\}$$

For $f(x) = x$, the utility functions for $t = 2^{n-i}-1$ and $2^{n-i}-2$, for $i = \log_2 n$ is about $1/a$ times more than those for $t = 0, 2^{n-i}, 2^{n-i}+1, 2^{n-i}+2$. Thus, the former two should be chosen in preference to the latter ones for linear check tests. However, for $f(x) = x - 1$, $t = 2^{n-i}-1$ check set is no longer desirable. $t = 2^{n-i}-2$ is desirable for all three cases.

$$(b) f(x,y) = x + y, f(x,y) = x - y (x, y \epsilon \{0,1\}^n)$$

For both these cases, $T = \{0^{2n}, 1^{2n}\}$; i.e. the check set consists of $x + y$ and $\bar{x} + \bar{y}$ for $f(x,y) = x + y$, and $x - y$ and $\bar{x} - \bar{y}$ for $f(x,y) = x - y$ [4]. As an example, for $f(x,y) = x + y$,

$$W(2^{n-i}-2, 2^{n-j}-2) \simeq p\{(2^{n-i} - 2 + 2^{n-j} - 2)a + (2^n - 1 - 2^{n-i} + 2 + 2^n - 1 - 2^{n-j} + 2)\}$$
$$\simeq p(2^{n-i} + 2^{n-j})a + (2 \times 2^n - 2^{n-i} - 2^{n-j})$$
$$\simeq p2^n\{2 + (a-1)(2^{-i} + 2^{-j})\}$$

Here again, the ratio of maximum to minimum for the utility function approaches $1/a$ when $n \to \infty$ and $i = j = \log_2 n$. In Table 4, expressions are given for the utility functions for $x + y$.

For $f(x,y) = x - y$, one can derive an analytical expression for W_1. Recalling that $\bar{x} - \bar{y} = -(x - y)$, from eqn. 35, we have:

Table 3: Approximate values of the utility function for $f(x) = x, \bar{x}, x + 1, \bar{x} + 1, x - 1$ and single unidirectional errors (in units of $p2^n$), $l \equiv 2^{-i}$, $(1-p)^n \equiv a, i \ll n$

t	$f(x) = x$, $f(x) = \bar{x}$	$f(x) = x + 1$, $f(x) = \bar{x} + 1$	$f(x) = x - 1$
0	a	1	a
$2^{n-i} - 2$	$1 + (a-1)l$	$1 + (a-1)l$	$1 + (a-1)l$
$2^{n-i} - 1$	$1 + (a-1)l$	1	a
2^{n-i}	$a + (1-a)l$	1	a
$2^{n-i} + 1$	$a + (1-a)l$	$a + (1-a)l$	$a + (1-a)l$
$2^{n-i} + 2$	$a + (1-a)l$	$a + (1-a)l$	$a + (1-a)l$
$\lim_{\substack{i \to \infty \\ i = o(n)}} \frac{W_{max}}{W_{min}}$	$\frac{1}{a}$	$\frac{1}{a}$	$\frac{1}{a}$

$$W_1(t,r) = p\{(1 + 2s(t-r)\}|t-r| + \{1 + 2s(\bar{t}-\bar{r})|\bar{t}-\bar{r}|\}(1-p)^{\|A(t-r)\|}$$
$$= 4p|t-r|(1-p)^{\|A(t-r)\|}$$

For $t = r = 0$, $W_1 = 0$, and for $r = 0, t = 2^n - 1$, $\|A(t-r)\| = n$, $W_1 \simeq p2^{n+2}a$. The best check set is $(2^n - 1, 0)$ and the worst one $(0, 0)$ and the ratio of maximum to minimum for the utility function approaches infinity. Thus for $f(x,y) = x - y$, our testing strategy is very effective.

For $f(x,y) = x - y$, the utility function for $t = 2^{n-i}$ is, within our approximation, the same as for $t = 2^{n-i} - 2$. The same is true for $t = 2^{n-i}$, $t = 2^{n-i} + 1$, and $t = 2^{n-i} + 2$. Therefore, in Table 5, we have given utility functions only for $t, r = 2^{n-i} - 1$ and 2^{n-i}:

$$(c) f(x,y) = x \oplus y, f(x,y) = x \vee y, f(x,y) = x \wedge y; (x,y) \epsilon \{0,1\}^n$$

(Symbols $\vee$ and $\wedge$ stand for logical addition and multiplication, respectively.) For all these logical functions, $T = \{0^{2n}, 0^n1^n, 1^n0^n, 1^{2n}\}$ [4]. For example, for $f(x,y) = x \vee y$, the check set consists of points $x \vee y, \bar{x} \vee y, x \vee \bar{y}$, and $\bar{x} \vee \bar{y}$. For these functions also, $W(2^{n-i}-1, r) \simeq W(2^{n-i}-2, r)$ and $W(2^{n-i}, r) \simeq W(2^{n-i}+1, r) \simeq W(2^{n-i}+2, r)$. Therefore, in Tables 6 to 8, we have given utility functions only for $t, r = 2^{n-i} - 1$ and 2^{n-i}. However, $W(t, r)$ depends on whether $t \geq r$. Once again, the ratio of maximum to minimum for the utility function approaches $1/a$ for $f(x,y) = x \oplus y$ and $x \wedge y$, but approaches only $(2/3 + 1/3a)$ for $f(x,y) = x \vee y$.

$$(d) f(x,y) = xy; x, y \epsilon \{0,1\}^n$$

Here, as for logical functions, $T = \{0^{2n}, 0^n1^n, 1^n0^n, 1^{2n}\}$ [4]. In this case (Table 9), $\|f(x,y)\|$ can be as large as $2n$, the expression for the utility function can have a term proportional to a^2, and the ratio of its maximum to its minimum value will approach $1/a^2$.

In Table 10, we have summarised the results in terms of the best tests $t^* \epsilon G/T$ and the corresponding values of the utility function $W(t^*)$. In this Table, we also give the asymptotic ratio of maximum $W(t)$ to minimum $W(t)$. For the 'subtract' instruction, our approach is most effective, followed by 'multiply'. Also, its effectiveness increases as n increases or p decreases.

We conclude this Section by pointing out that another measure of the effectiveness of the approach could be the ratio of max $W(t)$ to $W(t)$ for a randomly selected $t \epsilon G/T$ (which is the average value of $W(t)$). This ratio will of course be less than that of max $W(t)$ to min $W(t)$. For example, for $f(x) = x$,

Table 4: Approximate values of the utility function for $f(x, y) = x + y$ (in units of $p2^n$) for single unidirectional errors; $(1-p)^n \equiv a$, $2^{-i} \equiv I$, $2^{-j} \equiv J$, $L \equiv I - J$, $M \equiv I + J$; $i, j \ll n$

r \ t	0	$2^{n-j}-2$	$2^{n-j}-1$	2^{n-j}	$2^{n-j}+1$	$2^{n-j}+2$	$2^n-2^{n-j}+1$	2^n-2^{n-j}	$2^n-2^{n-j}-1$	$2^n-2^{n-j}-2$	$2^n-2^{n-j}-3$	2^n-1
0	$2a$	$2+(a-1)J$	$2a$	$2a+(1-a)J$	$2a+(1-a)J$	$2a+(1-a)J$	$(1+a)+(a-1)J$	$(1+a)+(a-1)J$	$2a$	$(1+a)+(1-a)J$	$(1+a)+(1-a)J$	$2a$
$2^{n-i}-2$	$2+(a-1)I$	$2+(a-1)M$	$2+(a-1)M$	$2+(a-1)M$	$2a$	$2a+(1-a)M$	$2a$	$(1+a)+$ $+(a-1)L$	$(1+a)+(a-1)L$	$(1+a)+(a-1)L$	$(1+a)+(a-1)L$	$(1+a)+(a-1)I$
$2^{n-i}-1$	$2a$	$2+(a-1)M$	$2+(a-1)M$	$2a$	$2a+(1-a)M$	$2a+(1-a)M$	$(1+a)+(1-a)L$	$2a$	$(1+a)+(a-1)L$	$(1+a)+(a-1)L$	$(1+a)+(a-1)L$	$(1+a)+(a-1)I$
2^{n-i}	$2a+(1-a)I$	$2+(a-1)M$	$2a$	$2a+(1-a)M$	$2a+(1-a)M$	$2a+(1-a)M$	$(1+a)+(1-a)L$	$1+a+(1-a)L$	$2a$	$2a$	$2a$	$(1+a)+(1-a)I$
$2^{n-i}+1$	$2a+(1-a)I$	$2a$	$2a+(1-a)M$	$2a+(1-a)M$	$2a+(1-a)M$	$2a+(1-a)M$	$(1+a)+(1-a)L$	$1+a+(1-a)L$	$2a$	$2a$	$2a$	$(1+a)+(1-a)I$
$2^{n-i}+2$	$2a+(1-a)I$	$2a+(1-a)M$	$2a+(1-a)M$	$2a+(1-a)M$	$2a+(1-a)M$	$2a+(1-a)M$	$(1+a)+(1-a)L$	$1+a+(1-a)L$	$2a$	$2a$	$2a$	$(1+a)+(1-a)I$

Table 5: Approximate values of the utility function for $f(x, y) = x - y$ (in units of $p2^{n+2}$) for single unidirectional errors: $(1-p)^n \equiv a$, $L = I - J$, $M = I + J; i, j \ll n$

$r \backslash t$	0	$2^{n-j}-1$	2^{n-j}	$2^n - 1$	$2^n - 2^{n-j}$	$2^n - 2^{n-j} - 1$
0	0	aJ	J	a	$(1-J)$	$(1-J)a$
$2^{n-i}-1$	aI	$\|L\|$	$\|L\|a$	$(1-I)$	$(1-M)$	$(1-M)a$
2^{n-i}	I	$\|L\|a$	$\|L\|$	$(1-I)a$	$(1-M)$	$(1-M)a$

Table 6: Approximate values of the utility function for $f(x, y) = x \oplus y$ (in units of $p2^{n+1}$) for single unidirectional errors; $(1-p)^n \equiv a$, $L = I - J$, $M = I + J; i, j \ll n$

$r \backslash t$	0	$2^{n-j}-1$			2^{n-j}		
		$i<j$	$i=j$	$i>j$	$i<j$	$i=j$	$i>j$
0	a	$1+(a-1)J$	$1+(a-1)J$	$1+(a-1)J$	$a+(1-a)J$	$a+(1-a)J$	$a+(1-a)J$
$2^{n-i}-1$	$1+(a-1)I$	$a+(1-a)L$	$a+(1-a)L$	$a-(1-a)L$	$1+(a-1)L$	$1+(a-1)M$	$1+(a-1)M$
2^{n-i}	$a+(1-a)I$	$1+(a-1)M$	$1+(a-1)M$	$1+(1-a)L$	$a+(1-a)M$	$a+(1-a)L$	$a+(1-a)M$

Table 7: Approximate values of the utility function for $f(x, y) = x \vee y$ (in units of $p2^n$) for single unidirectional errors; $(1-p)^n \equiv a$, $2^{-i} \equiv I$, $2^{-j} \equiv J$, $L = I - J$, $M = I + J; i, j \ll n$

$r \backslash t$	0	$2^{n-j}-1$			2^{n-j}		
		$i<j$	$i=j$	$i>j$	$i<j$	$i=j$	$i>j$
0	$3a$	$(1+2a) + (a-1)J$	$(1+2a) + (a-1)J$	$(1+2a) + (a-1)J$	$3a+(1-a)I$	$3a+(1-a)I$	$3a+(1-a)I$
$2^{n-i}-1$	$(1+2a) + (a-1)I$	$(1+2a) + (a-1)J$	$(1+2a) + (a-1)J$	$(1+2a) + (a-1)I$	$(1+2a) + (a-1)L$	$(1+2a) + (a-1)I$	$(1+2a) + (a-1)M$
2^{n-i}	$3a+(1-a)I$	$(1+2a) + (a-1)M$	$(1+2a) + (a-1)J$	$(1+2a) - (a-1)L$	$3a+(1-a)M$	$3a+(1-a)I$	$3a+(1-a)M$

Table 8: Approximate values of the utility function for $f(x, y) = x \wedge y$ (in units of $p2^n$) for single unidirectional errors; $(1-p)^n \equiv a$, $2^{-i} \equiv I$, $2^{-j} \equiv J$, $L = I - J$, $M = I + J; i, j \ll n$

$r \backslash t$	0	$2^{n-j}-1$			2^{n-j}		
		$i<j$	$i=j$	$i>j$	$i<j$	$i=j$	$i>j$
0	a	$1+(a-1)J$	$1+(a-1)J$	$1+(a-1)J$	$a+(1-a)J$	$a+(1-a)J$	$a+(1-a)J$
$2^{n-i}-1$	$1+(a-1)I$	$1+(a-1)J$	$1+(a-1)J$	$1+(a-1)I$	$1+(a-1)I$	$1+(a-1)J$	$1+(a-1)M$
2^{n-i}	$a+(1-a)I$	$1+(a-1)M$	$1+(a-1)J$	$1+(a-1)J$	$1+(1-a)M$	$a+(1-a)J$	$a+(1-a)M$

Table 9: Approximate values of the utility function for $f(x, y) = xy$ (in units of $p2^{2n}$) for single unidirectional errors; $(1-p)^n \equiv a$, $2^{-i} \equiv I$, $2^{-j} \equiv J$; $i, j \ll n$

$r \backslash t$	0	$2^{n-j}-2$	$2^{n-j}-1$	2^{n-j}	$2^{n-j}+1$	$2^{n-j}+2$
0	a	a	a	a	a	a
$2^{n-i}-2$	a	$1+(a^2-1)I + (a^2-1)J + (1+a-2a^2)IJ$	$1+(a-1)I + (a^2-1)J + (1-a^2)IJ$	$a^2+a(1-a)I + (1-a^2)J - (1-a^2)IJ$	$a^2+a(1-a)I + (1-a^2)J - C_1 IJ$	$a^2+a(1-a)I + (1-a^2)J - C_2 IJ$
$2^{n-i}-1$	a	$1+(a-1)J + (a^2-1)I + (1-a^2)IJ$	$1+(a-1)I + (a-1)J + (1-a)IJ$	$a+(a-1)I + (1-a)J + (a-1)IJ$	$a+(1-a)J - C_3 IJ$	$a+(1-a)J - C_4 IJ$
2^{n-i}	a	$a^2+a(1-a)J + (1-a^2)I - (1-a^2)IJ$	$a+(a-1)J + (1-a)I + (a-1)IJ$	$a+(1-a)IJ$	$a+(1-a)IJ$	$a+(1-a)IJ$
$2^{n-i}+1$	a	$a^2+a(1-a)I + (1-a^2)J - C'_1 IJ$	$a+(1-a)J - C'_3 IJ$	$a+(1-a)IJ$	$a+(1-a)IJ$	$a+(1-a)IJ$
$2^{n-i}+2$	a	$a^2+a(1-a)I + (1-a^2)J - C'_2 IJ$	$a+(1-a)J - C'_4 IJ$	$a+(1-a)IJ$	$a+(1-a)IJ$	$a+(1-a)IJ$

$$C_1 = \begin{cases} 1-a^2, & j > i+1 \\ 1+a-2a^2, & \text{otherwise} \end{cases} \quad C_2 = \begin{cases} 1-a^2, & j > i \\ 1+a-2a^2, & \text{otherwise} \end{cases} \quad C_3 = \begin{cases} 1-a, & j > i \\ 1-a^2, & \text{otherwise} \end{cases} \quad C_4 = \begin{cases} 1-a, & j > i-1 \\ 1-a^2, & \text{otherwise} \end{cases}$$

$$C'_1 = \begin{cases} 1-a^2, & i > j+1 \\ 1+a-2a^2, & \text{otherwise} \end{cases} \quad C'_2 = \begin{cases} 1-a^2, & i > j \\ 1+a-2a^2, & \text{otherwise} \end{cases} \quad C'_3 = \begin{cases} 1-a, & i > j \\ 1-a^2, & \text{otherwise} \end{cases} \quad C'_4 = \begin{cases} 1-a, & i > j-1 \\ 1-a^2, & \text{otherwise} \end{cases}$$

from eqn. 14, the average value of $W(t)$ is

$$\frac{p(2^n-1)}{2^{n-1}}\sum_{i=0}^{n-1}(1-p)^i\binom{n-1}{i}\simeq 2^n(1-p/2)^{n-1}p$$

Therefore, for $f(x)=x$,

$$\frac{\max W(t)}{\text{average } W(t)}\simeq\frac{1+\{(1-p)^n-1\}/\log_2 n}{(1-p/2)^{n-i}} \tag{37}$$

For $p=0.0108$, $n=64$, this is equal to 1.38 as compared to the value 2 for max $W(t)$/min $W(t)$.

For *negative* functions, recalling that in the fault-free case the sign bit has the value 1, the probability of $r(0\leqslant r\leqslant l)$ errors is $p^r(1-p)^{\|Af(x)\|-r+1}$. When the sign bit is correct, the number of possible locations of such errors is $\binom{\|Af(x)\|}{r}$, and the sum of $|e(x)|$ in these locations is $\binom{\|Af(x)\|-1}{r-1}Af(x)$. However, when the sign bit is corrupted, the number of possible combinations for the remaining $r-1$ errors is $\binom{\|Af(x)\|}{r-1}$, and the sum of $|e(x)|$ in these locations is

$$\binom{\|Af(x)\|}{r-1}+\binom{\|Af(x)\|-1}{r-1}Af(x)\qquad (r\leqslant\|Af(x)\|+1) \tag{40}$$

The expected value of $|e(x)|$ is therefore

$$E_l(x)=\left(\sum_{r=1}^{l}\left\{\binom{\|Af(x)\|}{r-1}+2\binom{\|Af(x)\|-1}{r-1}\right\}\times\left(\frac{p}{1-p}\right)^r\right)Af(x)(1-p)^{\|Af(x)\|+1}\times$$
$$(1\leqslant l\leqslant\|Af(x)\|+1) \tag{41}$$

Thus, the utility function for *negative* functions for unidirectional *1 → 0 errors of multiplicity l* is

$$W_l(t)=\sum_{\tau\epsilon T}Af(t\oplus\tau)(1-p)^{\|Af(t\oplus\tau)\|+1}\sum_{r=1}^{l}\left\{\binom{\|Af(t\oplus\tau)\|}{r-1}+2\binom{\|Af(t\oplus\tau)\|-1}{r-1}\right\}\left(\frac{p}{1-p}\right)^r \tag{42}$$

One can combine eqns. 39 and 42 to get the utility function for *arbitrary* $f(x)$ as

$$W_l(t)=\sum_{\tau\epsilon T}Af(t\oplus\tau)(1-p)^{\|Af(t\oplus\tau)\|+sf(t\oplus\tau)}\times\sum_{r=1}^{l}\left\{(1+sf(t\oplus\tau))\binom{\|Af(t\oplus\tau)\|-1}{r-1}+sf(t\oplus\tau)\binom{\|Af(t\oplus\tau)\|}{r-1}\right\}\left(\frac{p}{1-p}\right)^r \tag{43}$$

where the sign function s is defined by eqn. 26.

Table 10: Best tests; effectiveness of the testing strategy for various algebraic and logic functions $a\equiv(1-p)^n$, $l=2^{-i}$. For best tests, $l\simeq\log_2 n$, $l\simeq 1/n$

Instruction	Function	Best test		Asymptotic value of max $W(t)$/min $W(t)$
		t^*	$p^{-1}2^{-n}W(t^*)$	
Transfer x	x	$2^{n-i}-2$	$1+(a-1)l$	$1/a$
Increase x	$x+1$	$2^{n-i}-2$	$1+(a-1)l$	$1/a$
Decrease x	$x-1$	$2^{n-i}-2$	$1+(a-1)l$	$1/a$
Logical multiplication (AND)	$x\wedge y$	$(2^{n-i}-2, 0)$	$1+(a-1)l$	$1/a$
Logical addition (OR)	$x\vee y$	$(2^{n-i}-2, 0)$	$1+2a+(a-1)l$	$2/3+1/3a$
Exclusive OR	$x\oplus y$	$(2^{n-i}-2, 0)$	$1+(a-1)l$	$1/a$
Addition	$x+y$	$(2^{n-i}-2, 0)$	$2+(a-1)l$	$1/a$
Subtraction	$x-y$	$(2^n-2^{n-i}, 0)$	$1-l$	∞
Multiplication	xy	$(2^{n-i}-2, 2^{n-i}-1)$	$\{1+(a+a^2)l+(1-a^2)l^2\}2^n$	$1/a^2$

4 Multiple errors

The formalism developed in the preceding Sections can be generalised for the case of unidirectional errors of multiplicity l.

4.1 Unidirectional $1\to 0$ errors $(q=0)$

For $f(x)\geqslant 0$, *for all* $x\epsilon G$, the probability of $r(0\leqslant r\leqslant l)$ errors in the binary representation of $f(x)$ is $p^r(1-p)^{\|f(x)\|-r}$. The number of possible combinations of such errors is $\binom{\|f(x)\|}{r}$ and the sum of $|e(x)|$ in these combinations is

$$\binom{\|f(x)\|-1}{r-1}f(x)$$

Therefore, the expected value of $|e(x)|$ is

$$E_l(x)=\left(\sum_{r=1}^{l}\binom{\|f(x)\|-1}{r-1}\left(\frac{p}{1-p}\right)^r\right)f(x)\times(1-p)^{\|f(x)\|}(1\leqslant l\leqslant\|f(x)\|) \tag{38}$$

Thus, the utility function for *non-negative* functions for unidirectional *1 → 0 errors of multiplicity l* is

$$W_l(t)=\sum_{\tau\epsilon T}f(t\oplus\tau)(1-p)^{\|f(t\oplus\tau)\|}\times\sum_{r=1}^{l}\binom{\|f(t\oplus\tau)\|-1}{r-1}\left(\frac{p}{1-p}\right)^r \tag{39}$$

For $l=n$ (unidirectional errors of arbitrary unlimited multiplicity), $W_n(t)=p\sum_{\tau\epsilon T}f(t\oplus\tau)=pC$, and all test sets are equally good.

For unidirectional $1 \to 0$ errors of *arbitrary unlimited multiplicity*, $l \leqslant n+1$, (recalling that the maximum number of $1 \to 0$ errors, including at the sign bit is $n+1$, when $f(x)$ has positive and negative values), we have:

$$W_{n+1}(t) = p \sum_{\tau \epsilon T} \{1 + 2(1-p)sf(t \oplus \tau)\} Af(t \oplus \tau) \tag{44}$$

and all tests are not equally good.

4.2 *Unidirectional $0 \to 1$ errors ($p = 0$)*

This case can be analysed in a manner similar to that of Section 4.1. For *non-negative functions*, the expression for the utility function can be obtained by replacing p by q, $\|f(t \oplus \tau)\|$ by $n - \|f(t \oplus \tau)\|$ and $f(t \oplus \tau)$ by $\{2^n - 1 - f(t \oplus \tau)\}$. That is, for non-negative functions,

$$W_l(t) = \sum_{\tau \epsilon T} \{2^n - 1 - f(t \oplus \tau)\}(1-q)^{n - \|f(t \oplus \tau)\|} \sum_{r=1}^{l} \binom{n - \|f(t \oplus \tau)\| - 1}{r-1} \left(\frac{q}{1-q}\right)^r \tag{45}$$

Table 11: Utility functions when $f(x)$ is non-negative ($N \equiv 2^n - 1$)

no.	Type of error	Utility function $W(t)$
1	Single unidirectional $1 \to 0$	$\frac{p}{1-p} \sum_{\tau \epsilon T} f(t \oplus \tau)(1-p)^{\|f(t \oplus \tau)\|}$
2	Single unidirectional $0 \to 1$	$\frac{q}{1-q} \sum_{\tau \epsilon T} \{N - f(t \oplus \tau)\}(1-q)^{n - \|f(t \oplus \tau)\|}$
3	Single bidirectional ($p \neq q$)	$(1-q)^n \left[\frac{p}{1-p} \sum_{\tau \epsilon T} f(t \oplus \tau)\, \alpha^{\|f(t \oplus \tau)\|} + \frac{q}{1-q} \sum_{\tau \epsilon T} \{N - f(t \oplus \tau)\} \alpha^{\|f(t \oplus \tau)\|}\right]$
4	Single symmetrical ($p = q$)	$q(1-q)^n \lvert T \rvert N + 2C)$
5	Multiple unidirectional $1 \to 0$	$\sum_{\tau \epsilon T} f(t \oplus \tau)(1-p)^{\|f(t \oplus \tau)\|} \sum_{r=1}^{l} \binom{\|f(t \oplus \tau)\|}{r-1} \left(\frac{p}{1-p}\right)^r$
6	Multiple unidirectional $0 \to 1$	$\sum_{\tau \epsilon T} \{N - f(t \oplus \tau)\}(1-q)^{n - \|f(t \oplus \tau)\|} \sum_{r=1}^{l} \binom{n - \|f(t \oplus \tau)\| - 1}{r-1} \left(\frac{q}{1-q}\right)^r$

Table 12: Utility functions when values of $f(x)$ may be both positive and negative ($f(x)$ is represented in the sign and magnitude form; $N \equiv 2^n - 1$)

no.	Type of error	Utility function $W(t)$
1	Single unidirectional $1 \to 0$	$\frac{p}{1-p} \sum_{\tau \epsilon T} \{1 + 2sf(t \oplus \tau)\} Af(t \oplus \tau)(1-p)^{\|Af(t \oplus \tau)\| + sf(t \oplus \tau)}$
2	Single unidirectional $0 \to 1$	$q \sum_{\tau \epsilon T} \{N + f(t \oplus \tau)\}(1-q)^{n - \|f(t \oplus \tau)\| - sf(t \oplus \tau)}$
3	Single bidirectional ($p \neq q$)	$(1-q)^n \sum_{\tau \epsilon T} \frac{p}{\alpha} \{1 + 2sf(t \oplus \tau)\} Af(t \oplus \tau) + q(N + f(t \oplus \tau))\, \alpha^{\|f(t \oplus \tau)\| + sf(t \oplus \tau)}$
4	Single symmetrical ($p = q$)	$p(1-p)^n \{N \lvert T \rvert + 2 \sum_{\tau \epsilon T} Af(t \oplus \tau)\}$
5	Multiple unidirectional $1 \to 0$	$\sum_{\tau \epsilon T} Af(t \oplus \tau)(1-p)^{\|Af(t \oplus \tau)\| + sf(t \oplus \tau)} \sum_{r=1}^{l} \left[\{1 + sf(t \oplus \tau)\} \binom{\|Af(t \oplus \tau)\| - 1}{r-1} + sf(t \oplus \tau) \binom{\|Af(t \oplus \tau)\|}{r-1}\right] \left(\frac{p}{1-p}\right)^r$
6	Multiple unidirectional $0 \to 1$	$\sum_{\tau \epsilon T} (1-q)^{n - \|Af(t \oplus \tau)\| + 1 - sf(t \oplus \tau)} \sum_{r=1}^{l} \left[2\{1 - sf(t \oplus \tau)\} \binom{n - \|Af(t \oplus \tau)\|}{r-1} Af(t \oplus \tau) + \left[\binom{n - \|Af(t \oplus \tau)\| - 1}{r-1} - \{1 - sf(t \oplus \tau)\} \binom{n - \|Af(t \oplus \tau)\| - 1}{r-2}\right] \{N - Af(t \oplus \tau)\}\right] \left(\frac{q}{1-q}\right)^r$
7	unlimited unidirectional $1 \to 0$	$p \sum_{\tau \epsilon T} \{1 + 2(1-p)sf(t \oplus \tau)\} Af(t \oplus \tau)$
8	unlimited unidirectional $0 \to 1$	$q\ \ N \lvert T \rvert - \sum_{\tau \epsilon T} \{1 - 2sf(t \oplus \tau)\} Af(t \oplus \tau)$

For errors of arbitrary multiplicity, this reduces to

$$W_n(t) = q \sum_{\tau \epsilon T} \{2^n - 1 - f(t \oplus \tau)) = q\{(2^n - 1)|T| - C\} \quad (46)$$

and all tests are equally good.

For *negative* functions, since no corruption of the sign bit is allowed, the expression for the utility function will be the same as for the non-negative function, i.e. eqn. 46, except that $f(t \oplus \tau)$ is to be replaced by $Af(t \oplus \tau)$.

For an *arbitrary* function, when the value of the (fault-free) function is positive, the 0 in the sign-bit position could be corrupted. One can analyse this possibility similarly to the case of Section 4.1, and the resulting expression is

$$W_l(t) = \sum_{\tau \epsilon T} (1-q)^{n - \|Af(t \oplus \tau)\| + 1 - sf(t \oplus \tau)} \times \sum_{r=1}^{l} \{2(1 - sf(t \oplus \tau)\} \binom{n - \|Af(t \oplus \tau)\|}{r-1} Af(t \oplus \tau) + \left[\binom{n - \|Af(t \oplus \tau)\| - 1}{r-1} - \{1 - sf(t \oplus \tau)\} \times \binom{n - \|Af(t \oplus \tau)\| - 1}{r-2}\right] \{2^n - 1 - Af(t \oplus \tau)\} \times \left(\frac{q}{1-q}\right)^r \quad (47)$$

For errors of an *arbitrary unlimited multiplicity*, it simplifies to

$$W_{n+1}(t) = q \sum_{\tau \epsilon T} 2^n - 1 + \{1 - 2sf(t \oplus \tau)\}Af(t \oplus \tau)$$
$$= q\left[(2^n - 1)|T| - \sum_{\tau \epsilon T} \{1 - 2sf(t \oplus \tau)\}Af(t \oplus \tau)\right] \quad (48)$$

and all tests are not equal.

For the convenience of the reader, we have assembled the expressions for the utility functions for various types of errors in Tables 11 and 12.

We note that the case of unidirectional $1 \to 0$ or $0 \to 1$ errors of an arbitrary unlimited multiplicity is important from the practical point of view (see e.g. Reference 11). Comparing the utility function for symmetric single errors (eqn. 27) with those for unidirectional unlimited errors (eqns. 44 and 48), we can see that the method of constructing the best test based on eqns. 31 to 34 can also be used for unidirectional errors, and best tests $(t_1^*, \ldots, t_R^*)$ for symmetric single errors and unlimited unidirectional errors coincide. For example, all best tests (BT) represented in Table 1 are also best tests for unlimited unidirectional ($1 \to 0$ or $0 \to 1$) errors.

5 Properties of best tests, conclusions

In general, the best test sequence depends on the function implemented by the fault-free device, and for certain functions there is no best test sequence; that is to say, all tests detect the same expected magnitude of errors. We have already presented some results in the preceding Sections for various types of errors and various functions. In the following text, we shall discuss some invariant properties of best test sequences.

First we note that, for *non-negative* functions, it follows from eqns. 7, 8, 39 and 45 that if $\|f(x)\| = \text{const}$, then for single unidirectional and bidirectional errors and for multiple unidirectional errors, $W(t) = \text{const}$ and so all tests are equal.

We note also that, if $f(x)$ is a Boolean function ($f(x) \epsilon \{0, 1\}$), then $\sum_{\tau \epsilon T} f(t \oplus \tau)\, \alpha^{\|f(t \oplus \tau)\|} = \sum_{\tau \epsilon T} \alpha f(t \oplus \tau) = \alpha C$ for any α, t and τ, and for single unidirectional and bidirectional errors, eqns. 6 and 19 again show $W(t)$ to be a constant, and all tests are equal.

For an *arbitrary* function (which has both positive and negative values), if $b \neq 0$ is a constant, then, from eqns. 27, 44 and 48, for single symmetrical errors and unidirectional errors of an unlimited multiplicity, the best tests for $f(x)$ and $bf(x)$ coincide ($BT(f) = BT(bf)$). This is not true, however, for non-negative functions, since $\|f(x)\| \neq \|bf(x)\|$. We note also that, for $b \neq 0$, $BT(f) \neq BT(b + f)$. For example, for single symmetrical errors, if $f(x) = x - 1$, then from Table 2 $\max_t W_1(t) = W_1(0)$ and $W_1(0) > W_1(t)$ for all $t \neq 0^n, 1^n$; but for $f(x) = x$, $W(t) = const$ for all t.

The previous remarks show that there is no simple relationship between best tests for $f(x)$ and its linear transform $af(x) + b$ (a, b are constants), but this is not true in the case of the ***linear (affine) transform of arguments over GF(2)***. In this case, we shall introduce the following notations. Let σ be an ($n \times n$)-binary matrix, nonsingular over GF(2), σx be a product over GF(2) of σ and a binary column vector x, 'a' be some binary column vector and

$$\phi(x) = f(\sigma x \oplus a) \quad (49)$$

Then, for any function f and any type of errors, if $(t_1^*, \ldots, t_R^*)$ is the best test for f, then $(\sigma^{-1}(t_1^* \oplus a), \ldots, \sigma^{-1}(t_R^* \oplus a))$ is the best test for ϕ or

$$BT(\phi) = \sigma^{-1}(BT(f) \oplus a) \quad (50)$$

Here σ^{-1} represents the inverse of σ over GF(2)). To prove eqn. 50, we denote check sets for f and ϕ as T_f and T_ϕ; utility functions for f and ϕ we denote as W_f and W_ϕ. Then

$$W_f(t) = \sum_{\tau \epsilon T_f} E\{f(t \oplus \tau), \|Af(t \oplus \tau)\|\} \quad (51)$$

where E is some function which depends on the type of the errors (see Tables 11 and 12), and

$$W_\phi(t) = \sum_{q \epsilon T_\phi} E\{\phi(t \oplus q), \|A\phi(t \oplus q)\|\}$$
$$= \sum_{q \epsilon T_\phi} E\{f(\sigma(t \oplus q) \oplus a\}, \|Af\{\sigma(t \oplus q) \oplus a)\|\} \quad (52)$$

Denote $\sigma q = \tau$, $\sigma t \oplus a = r$. Since [7] $T_f = \sigma^{-1} T_\phi$, we have, from eqns. 49 to 52,

$$W_\phi(t) = \sum_{\tau \epsilon \sigma^{-1} T_\phi} E\{f(r \oplus \tau), \|Af(r \oplus \tau)\|\} = W_f(r)$$
$$= W_f(\sigma t \oplus a) \quad (53)$$

Formula 50 follows now immediately from eqn. 53.

In the preceding sections, we have considered stuct-at errors. However, the approach can easily be generalised to other types of errors. For example, in the case of memory testing, let $f(x)$ represent the data in a memory location (row), whose address is x. Then, for *OR bridging* between corresponding cells in the rows whose addresses are x and y, the expected value of the resulting error in the computation of

$f(x)$ is

$$E(x) = \sum_{y \epsilon G} |\{f(x) \vee f(y)\} - f(x)| p(x,y) \qquad (54)$$

where $p(x, y)$ is the probability of bridging occurring, and $\vee$ stands for componentwise logical addition.

The corresponding utility function is, as before,

$$W(t) = \sum_{\tau \epsilon G} E(t \oplus \tau) = \sum_{\tau \epsilon T} \sum_{y \epsilon G} |(f(t \oplus \tau) \vee f(y))$$

$$-f(t \oplus \tau)| p(t \oplus \tau, y) \qquad (55)$$

When bridging can occur only between corresponding cells in neighbouring rows, i.e.

$$p(x,y) = 0 \quad \text{for } y \neq x \pm 1 \qquad (56)$$

the utility function is

$$W(t) = p \sum_{\tau \epsilon T} (|[f(t \oplus \tau) \vee f\{(t \oplus \tau) + 1)\} - f(t \oplus \tau)|$$

$$+ |[f(t \oplus \tau) \vee f\{(t \oplus \tau) - 1)\}] - f(t \oplus \tau)|) \qquad (57)$$

where p is the probability of bridgings between any neighbouring rows. For a given function $f(x)$, this expression can be analysed to determine best test sequences.

Similiar expressions can be derived for column bridgings in memory and for cross-talk errors.

For *input errors* (e.g. *wrong decoding of memory* addresses), instead of errors occurring directly in $f(x)$, they occur in the value of x, leading to an erroneous output $f(x)$. The utility function is given by

$$W(t) = \sum_{\tau \epsilon T} \sum p_{t,\tilde{t}} |f(t \oplus \tau) - f(\tilde{t} \oplus \tau)| \qquad (58)$$

where $\tilde{t}$ is the corrupted value of the variable t and $p_{t,\tilde{t}}$ is the probability of this corruption taking place. For single bidirectional symmetrical input errors, the corruption can occur in any of the bits with equal probability, i.e. $p_{t,\tilde{t}} = 1$ when $\tilde{t}$ differs from t in only one bit position; = 0 otherwise.

For this case, $W(t) = p \sum_{\tau \epsilon T} \sum_{i=1}^{n} |f(t \oplus \tau) - f(t \oplus 0^{i-1} 1\, 0^{n-i} \oplus \tau)|$, and for functions such as x, $x+1$, $x-1$, $x+y$, $x-y$, xy, $x \wedge y$, $x \vee y$, $x \oplus y$ $W(t)$ is a constant, and all tests are equally good.

In conclusion, in this paper we have introduced a criterion for test generation based on minimising the expected magnitude of undetected errors. This should be contrasted with the usual criteria, where the probability of undetected error is minimised. This criterion has been used to develop a best test strategy using the linear checks approach. The tests applied represent functional tests, in that they are independent of the internal structure of the device under the test. The expressions for the utility function are easy to program, and for networks of a practical size (where n may be in the region of 16 to 24), the best test strategy can be determined in a reasonable amount of computer time. The approach becomes more efficient as n increases.

We note that the approach can also be used for testing the software for computing numerical functions. In the case of noninteger computations ($f(x)$ is a real (noninteger) number for some x), the linear equality checks represented by eqn. 1 have to be replaced by the inequality checks

$$|f(t \oplus \tau) - C| \leqslant \epsilon \qquad \text{for every } t \epsilon \{0, 1, \ldots, 2^n - 1\} \qquad (59)$$

where, as in eqn. 1, T is a check subgroup in G, $t \oplus T$ is a check set, C is a constant and $\epsilon \geqslant 0$ is a small constant. (The check eqn. 1 is a special case of eqn. 59 with $\epsilon = 0$; methods for constructing optimal inequality checks and complexity estimations for these checks are given in References 18 and 19.) For the linear inequality checks in eqn. 59 one can use the same approach for ordering the test sets as was used for the equality checks in eqn. 1.

It is also easy to generalise this approach if the distribution of errors in the bits of $f(x)$ is known *a priori* and/or the cost function for errors in different bits is given by the user. Finally, we note again that this approach is useful for field-testing systems in which small errors may be tolerated, and the weights of errors in different components are different.

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