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COORDINATE DENSITY OF SETS OF VECTORS

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Let $n_0=0, n_1\geq 1,\ldots,n_s\geq 1$ be given natural numbers,

$$J_i=\left\{\sum_{t=0}^{i-1} n_t+1,\ldots,\sum_{t=0}^{i} n_t\right\}\quad (i=1,\ldots,s)$$

and

$$\prod_{i=1}^{s} E_{q_i}^{n_i}=\left\{(x^{(i)},\ldots,x^{(n)}):\quad n=\sum_{i=1}^{s} n_i \text{ and if } r\in J_i, \text{ then } x^{(r)}\in\{0,\ldots,q_i-1\}\right\}.$$

A set $R\subseteq\prod_{i=1}^{s} E_{q_i}^{n_i}$ is said to be $(m_1,\ldots,m_s)$-dense $(1\leq m_i\leq n_i)$ if there exist $I_i\subseteq J_i$ such that $|I_i|=m_i$ $(i=1,\ldots,s)$ and $|P^{(I)}(R)|=\prod_{i=1}^{s} q_i^{m_i}$, where $P^{(I)}(R)$ is the projection of R on the coordinate axes whose indices lie in $I=\bigcup_{i=1}^{s} I_i$.

In this paper we establish necessary and sufficient conditions for an arbitrary set $R\subseteq\prod_{i=1}^{s} E_{q_i}^{n_i}$ with given $|R|$ to be $(m_1,\ldots,m_s)$-dense.

1.

Given natural numbers $n_0=0, n_1\geq 1,\ldots,n_s\geq 1$, put

$$J_i=\left\{\sum_{t=0}^{i-1} n_t+1,\ldots,\sum_{t=0}^{i} n_t\right\}$$

and let $\prod_{i=1}^{s} E_{q_i}^{n_i}$ denote the set of n-vectors $(x^{(1)},\ldots,x^{(n)})$, where $n=\sum_{i=1}^{s} n_i$, and if $r\in J_i$, then $x^{(r)}\in\{0,\ldots,q_i-1\}$. Let $I\subseteq\{1,\ldots,n\}$. For any $x\in\prod_{i=1}^{s} E_{q_i}^{n_i}$ let $P^{(I)}(x)$ denote the projection of x on the coordinate axes whose indices lie in I. If $I=\emptyset$ is the empty set, then $P^{(\emptyset)}(x)=(0,\ldots,0)$ for all $x\in\prod_{i=1}^{s} E_{q_i}^{n_i}$. If $R\subseteq\prod_{i=1}^{s} E_{q_i}^{n_i}$, then $P^{(I)}(R)=\{P^{(I)}(x): x\in R\}$.

A set $R\subseteq\prod_{i=1}^{s} E_{q_i}^{n_i}$ is said to be $(m_1,\ldots,m_s)$-dense $(1\leq m_i\leq n_i, i=1,\ldots,s)$ if there exist $I_i\subseteq J_i$ such that $|I_i|=m_i (i=1,\ldots,s)$ and $|P^{(I)}(R)|=\prod_{i=1}^{s} q_i^{m_i}$, where $I=\bigcup_{i=1}^{s} I_i$ (throughout, $|A|$ denotes the cardinality of the set A).

In this paper we shall establish necessary and sufficient conditions for an arbitrary set $R\subseteq\prod_{i=1}^{s} E_{q_i}^{n_i}$ of prescribed cardinality to be $(m_1,\ldots,m_s)$-dense. Note that when $s=1$ and $q_1=2$ this problem is equivalent to a problem suggested by Erdös in the language of systems of subsets of a certain set and solved in [2] and also in [3]. (In [2] the maximum m for which $R \subseteq E_2^n$ is m-dense is called the "density" of R.)

We shall present a solution of the problem for arbitrary s and $q_1, \ldots, q_s$. It should also be noted that for the case $s=1$, when E_q^n is linear n-space over GF(q), a necessary and sufficient condition was proved in [1] for an arbitrary set $R \subseteq E_q^n$ of given cardinality to contain a subspace E_q^m $(1 \leqslant m \leqslant n)$ (such that $|E_q^m| = q^m$).

2.

To clarify the exposition, we first consider the case $s=1$. Thus, let $E_q^n = \{x : x = (x^{(1)}, \ldots, x^{(n)}),\ x^{(r)} \in \{0, \ldots, q-1\},\ r = 1, \ldots, n\}$, $R \subseteq E_q^n$, $1 \leqslant m \leqslant n$. Our aim is to find necessary and sufficient conditions for an arbitrary set $R \subseteq E_q^n$ of given cardinality $|R|$ to be m-dense.

We shall say that M has property $A_q(n, m)$, writing $M \in A_q(n, m)$, iff for any $I \subseteq \{1, \ldots, n\}$ such that $|I| = m$ there exists $x \in P^{(I)}(M)$ such that, for any $y \in E_q^n$, if $P^{(I)}(y) = x$, then $y \in M$.

Lemma 2.1. *A set $R \subseteq E_q^n$ is not m-dense iff*

$$R^c = E_q^n - R \in A_q(n, m) \quad (1 \leqslant m \leqslant n).$$

Proof. A set $R \subseteq E_q^n$ is not m-dense iff for any $I \subseteq \{1, \ldots, n\}$ such that $|I| = m$ there exists $x \in P^{(I)}(\prod_{i=1}^{s} E_{q_i}^{n_i})$ such that $P^{(I)}(y) \neq x$ for all $y \in R$. It follows that if $P^{(I)}(y) = x$ for some $y \in E_q^n$, then $y \in R^c$ and so $R^c \in A_q(n, m)$.

Now let $H_q(n, m)$ denote any minimal subset of E_q^n such that $H_q(n, m) \in A_q(n, m)$ and put $h_q(n, m) = |H_q(n, m)|$.

Lemma 2.2. *For any n and $1 \leqslant m \leqslant n$,*

$$h_q(n, m+1) \geqslant \sum_{r=1}^{n-m} (q-1)^{r-1} h_q(n-r, m). \tag{1}$$

Proof. We fix $H_q(n, m+1)$ and construct a set $D^{(r)} \subseteq E_q^{n-r}$ $(r = 1, \ldots, n-m)$ as follows: $x \in D^{(r)}$ iff there exists $\psi^{(r)}(x) \in \{0, \ldots, q-1\}$ such that for any $y^{(1)}, \ldots, y^{(r-1)} \in \{0, \ldots, q-1\}$,

$$(y^{(1)}, \ldots, y^{(r-1)}, \psi^{(r)}(x), x) \in H_q(n, m+1). \tag{2}$$

(Given x, there may be several $\psi^{(r)}(x)$ satisfying (2). In that case we fix one of them arbitrarily.)

We claim that $D^{(r)} \in A_q(n-r, m)$. Indeed, let us fix $r, i_1, \ldots, i_m$ $(r < i_1 < \cdots < i_m \leqslant n)$ arbitrarily. Since $H_q(n, m+1) \in A_q(n, m+1)$, there exist

$$x^{(r)}, x^{(i_1)}, \ldots, x^{(i_m)} \in \{0, \ldots, q-1\}$$

such that, for any $y=(y^{(1)},\ldots,y^{(n)})\in E_q^n$, if

$$y^{(r)}=x^{(r)},\ y^{(i_1)}=x^{(i_1)},\ldots,y^{(i_m)}=x^{(i_m)},$$

then $y\in H_q(n,m+1)$. Consequently, by the definition of $D^{(r)}$, we have for any $z=(z^{(1)},\ldots,z^{(n-r)})$ with $z^{(i_t-r)}=x^{(i_t)}(t=1,\ldots,m)$ that $z\in D^{(r)}$. Since $i_1<i_2<\cdots<i_m$ were chosen arbitrarily, we obtain $D^{(r)}\in A_q(n-r,m)$ and

$$|D^{(r)}|\geqslant h_q(n-r,m). \tag{3}$$

Now let $T^{(r)}(r=1,\ldots,n-m)$ denote the set of all vectors $(y^{(1)},\ldots,y^{(r-1)},\psi^{(r)}(x),x)$, where $x\in D^{(r)}$, and in addition

$$y^{(r-1)}\neq\psi^{(r-1)}(\psi^{(r)}(x),x), \tag{4a}$$

if $(\psi^{(r)}(x),x)\in D^{(r-1)}$, and

$$y^{(i)}\neq\psi^{(i)}(y^{(i+1)},\ldots,y^{(r-1)},\psi^{(r)}(x),x)\quad(i=r-2,\ldots,1) \tag{4b}$$

if $(y^{(i+1)},\ldots,y^{(r-1)},\psi^{(r)}(x),x)\in D^{(i)}$. Then, by the definition of $T^{(r)}$,

$$T^{(i)}\cap T^{(j)}=\emptyset \tag{5}$$

(where $\emptyset$ is the empty set), and since $T^{(r)}\subseteq H_q(n,m+1)$ $(r=1,\ldots,n-m)$,

$$\bigcup_{r=1}^{n-m}T^{(r)}\subseteq H_q(n,m+1). \tag{6}$$

By the definition of $T^{(r)}$ and $D^{(r)}$, it follows from (3), (4a and b) that

$$|T^{(r)}|\geqslant|D^{(r)}|(q-1)^{r-1}\geqslant(q-1)^{r-1}h_q(n-r,m). \tag{7}$$

In combination with (5) and (6), this inequality yields (1) and proves Lemma 2.2.

As will follow from Theorem 2.4(i) below, equality holds in (1) for any n and $1\leqslant m\leqslant n$.

We shall need the following numerical identity:

Lemma 2.3. *For any n and $2\leqslant m<n$,*

$$\sum_{r=1}^{n-m+1}\sum_{t=0}^{m-2}(q-1)^{n-t-1}\binom{n-r}{t}=\sum_{j=0}^{m-1}(q-1)^{n-j}\binom{n}{j}-q^{m-1}(q-1)^{n-m+1}. \tag{8}$$

Proof. Setting $i=n-r$, $j=t+1$, we have

$$\begin{aligned}\sum_{r=1}^{n-m+1}\sum_{t=0}^{m-2}(q-1)^{n-t-1}\binom{n-r}{t}&=\sum_{t=0}^{m-2}(q-1)^{n-t-1}\sum_{i=m-1}^{n-1}\binom{i}{t}\\&=\sum_{t=0}^{m-2}(q-1)^{n-t-1}\left(\sum_{i=t}^{n-1}\binom{i}{t}-\sum_{i=t}^{m-2}\binom{i}{t}\right)\\&=\sum_{j=0}^{m-1}(q-1)^{n-j}\binom{n}{j}-(q-1)^{n-m+1}\sum_{j=0}^{m-1}(q-1)^{m-j-1}\binom{m-1}{j}\\&=\sum_{j=0}^{m-1}(q-1)^{n-j}\binom{n}{j}-q^{m-1}(q-1)^{n-m+1}.\end{aligned}$$

Theorem 2.4. (i) *For any t such that*

$$t \leq \sum_{j=0}^{m-1} (q-1)^{n-j}\binom{n}{j} \tag{9}$$

there exists $R \subseteq E_q^n$ such that $|R| = t$ and R is not m-dense.

(ii) *If $R \subseteq E_q^n$ and*

$$|R| > \sum_{j=0}^{m-1} (q-1)^{n-j}\binom{n}{j}, \tag{10}$$

then R is m-dense.

Proof. (i) We construct R as follows. Let $S_q(n, n-m)$ be the sphere in E_q^n of radius $n-m$ in the Hamming metric, i.e., the set of q-ary n-vectors with at most $n-m$ nonzero components. Then for any $y \in S_q^c(n, n-m) = E_q^n - S_q(n, n-m)$ the number of zero components in y is at most $m-1$, and so $S_q^c(n, n-m)$ is not m-dense. Since

$$|S_q^c(n, n-m)| = q^n - \sum_{j=0}^{n-m} (q-1)^{n-j}\binom{n}{j}$$

it follows that for any t satisfying (9) there exists $R \subseteq S_q^c(n, n-m)$ such that $|R| = t$ and R is not m-dense.

(ii) Suppose that (10) is true but R is not m-dense. Then by Lemma 2.1, $R^c \in A_q(n, m)$ and by Lemma 2.2

$$|R^c| \geq h_q(n, m) \geq \sum_{r=1}^{n-m+1} (q-1)^{r-1} h_q(n-r, m-1). \tag{11}$$

We now show by induction on m that (11) implies

$$h_q(n, m) \geq q^n - \sum_{j=0}^{m-1} (q-1)^{n-j}\binom{n}{j}. \tag{12}$$

Since $|R^c| \geq h_q(n, m)$, this will contradict (10), implying that R must be m-dense.

Indeed, for $m = 1$, we have $h_q(n-r, 0) = q^{n-r}$ and from (11) we get

$$h_q(n, 1) \geq \sum_{r=1}^{n} (q-1)^{r-1} q^{n-r} = q^n - (q-1)^n.$$

Thus (12) holds for $m = 1$ for all $n \geq 1$. Let (12) holds for $m-1 (m \geq 2)$ and all

$n \geq m$. Then by (11), (12), using Lemma 2.3, we obtain

$$\begin{aligned} h_q(n, m) &\geq \sum_{r=1}^{n-m+1} \left(q^{n-r}(q-1)^{r-1} - \sum_{j=0}^{m-2} (q-1)^{n-j-1} \binom{n-r}{j} \right) \\ &= q^n - q^{m-1}(q-1)^{n-m+1} - \sum_{r=1}^{n-m+1} \sum_{j=0}^{m-2} (q-1)^{n-j-1} \binom{n-r}{j} \\ &= q^n - \sum_{j=0}^{m-1} (q-1)^{n-j} \binom{n}{j}. \end{aligned}$$

This proves (12) and completes the proof of part (ii).

Theorem 2.4 yields the following exact formula for the function $h_q(n, m)$:

$$h_q(n, m) = q^n - \sum_{j=0}^{m-1} (q-1)^{n-j} \binom{n}{j} = \sum_{j=0}^{n-m} (q-1)^j \binom{n}{j}. \tag{13}$$

Remark 2.5. It follows from the proof of Theorem 2.4 that $S_q(n, n-m)$ (the sphere of radius $n-m$ in the Hamming metric) is a minimal set with property $A_q(n, m)$, and $S_q^c(n, m) \subseteq E_q^n$ is a maximal set such that $S_q^c(n, m)$ is not m-dense. However, $S_q(n, m)$ is not the only set with these extremal properties. For example, for $q = 2$, $m = 1$, the set $\{(0, \ldots, 0), (0, \ldots, 0, 1), (0, \ldots, 0, 1, 1), \ldots, (1, \ldots, 1)\}$ has the same properties.

Theorem 2.6. *Let R be the set of infinite-dimensional q-ary vectors. Then, either R is an m-dense set for any m or there exists M such that for any $m > M$ and any $I = \{i_1, \ldots, i_m\}$ $(i_1 < i_2 < \cdots < i_m)$*

$$|P^{(I)}(R)| \leq \sum_{j=0}^{M} (q-1)^{m-j} \binom{m}{j}. \tag{14}$$

Proof. If there exists m such that R is not m-dense, let M denote the largest m such that R is m-dense. Then if

$$|P^{(I)}(R)| \leq \sum_{j=0}^{M} (q-1)^{m-j} \binom{m}{j}$$

it follows from Theorem 2.4 that $P^{(I)}(R)$ is $(M+1)$-dense, so that R is $(M+1)$-dense, contradicting the choice of M.

3.

We now consider the case $s > 1$. In this section we shall establish necessary and sufficient conditions for an arbitrary $R \subseteq \prod_{i=1}^{s} E_{q_i}^{n_i}$ of given cardinality to be $(m_1, \ldots, m_s)$-dense $(1 \leq m_i \leq n_i, i = 1, \ldots, s)$.

As before, let

$$J_i=\left\{\sum_{t=0}^{i-1} n_t+1,\ldots,\sum_{t=0}^{i} n_t\right\} \quad (n_0=0, n_1\geq 1,\ldots,n_s\geq 1, i=1,\ldots,s).$$

Let $M\subseteq E$. We shall write $M\in A_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1,\ldots,m_s)$ iff, for any $I_1,\ldots,I_s$ such that $I_i\subseteq J_i$ and $|I_i|=m_i (i=1,\ldots,s)$, there exists $x\in P^{(I)}(M)(I=\bigcup_{i=1}^s I_i)$ such that for any $y\in\prod_{i=1}^s E_{q_i}^{n_i}$, if $P^{(I)}(y)=x$, then $y\in M$.

Let $H_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1,\ldots,m_s)$ denote any minimal subset of $\prod_{i=1}^s E_{q_i}^{n_i}$ such that

$$H_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1,\ldots,m_s)\in A_{q_1,\ldots,q_s}(n_1,\ldots,n_s;\,m_1,\ldots,m_s),$$

and put

$$h_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1,\ldots,m_s)=|H_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1,\ldots,m_s)|.$$

Lemma 3.1. *For any $q_i, n_i, 1<m_i\leq n_i (i=1,\ldots,s)$,*

$$h_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1+1,m_2,\ldots,m_s)$$
$$\geq\sum_{r=1}^{n_1-m_1}(q_1-1)^{r-1}h_{q_1,\ldots,q_s}(n_1-r,n_2,\ldots,n_s;m_1,\ldots,m_s). \quad (15)$$

Proof. We fix $H_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1+1,m_2,\ldots,m_s)$ and construct

$$D^{(r)}\subseteq E_{q_1}^{n_1-r}\times\prod_{i=2}^{s}E_{q_i}^{n_i} \quad (r=1,\ldots,n_1-m_1)$$

as follows: $x\in D^{(r)}$ iff there exists $\psi^{(r)}(x)\in\{0,\ldots,q_1-1\}$ such that for any $y^{(1)},\ldots,y^{(r-1)}\in\{0,\ldots,q_1-1\}$

$$(y^{(1)},\ldots,y^{(r-1)};\psi^{(r)}(x),x)\in H_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1+1,\ldots,m_s).$$

Then, as in the proof of Lemma 2.2, we can easily see that $D^{(r)}\in A_{q_1,\ldots,q_s}(n_1-r,n_2,\ldots,n_s;m_1,\ldots,m_s)$. Therefore,

$$|D^{(r)}|\geq h_{q_1,\ldots,q_s}(n_1-r,n_2,\ldots,n_s;m_1,\ldots,m_s).$$

Proceeding by analogy with (4), we introduce sets $T^{(r)}(r=1,\ldots,n_1-m_1)$

$$\left(T^{(i)}\cap T^{(j)}=\emptyset,\ \bigcup_{r=1}^{n_1-m_1}T^{(r)}\subseteq H_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1+1,m_2,\ldots,m_s)\right).$$

Then

$$|T^{(r)}|\geq|D^{(r)}|(q_1-1)^{r-1}$$

and

$$h_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1+1,m_2,\ldots,m_s)\geq\sum_{r=1}^{n_1-m_1}(q_1-1)^{r-1}|D^{(r)}|$$
$$\geq\sum_{r=1}^{n_1-m_1}(q_1-1)^{r-1}h_{q_1,\ldots,q_s}(n_1-r,n_2,\ldots,n_s;m_1,\ldots,m_s).$$

Theorem 3.2. (i) *For any t such that*

$$t \leq \prod_{i=1}^{s} q_i^{n_i} - \prod_{i=1}^{s} \sum_{j=0}^{n_i-m_i} (q_i-1)^j \binom{n_i}{j} \tag{16}$$

there exists $R \subseteq \prod_{i=1}^{s} E_{q_i}^{n_i}$ such that $|R| = t$ and R is not $(m_1, \ldots, m_s)$-dense $(1 \leq m_i \leq n_i;\, i = 1, \ldots, s)$.

(ii) *If $R \subseteq \prod_{i=1}^{s} E_{q_i}^{n_i}$ and*

$$|R| > \prod_{i=1}^{s} q_i^{n_i} - \prod_{i=1}^{s} \sum_{j=0}^{n_i-m_i} (q_i-1)^j \binom{n_i}{j}, \tag{17}$$

then R is $(m_1, \ldots, m_s)$-dense.

Proof. (i) Let $S_{q_i}(n_i, n_i - m_i)$ be the sphere of radius $n_i - m_i$ in $E_{q_i}^{n_i}$ in the Hamming metric and $S_q(n, n-m) = \prod_{i=1}^{s} S_{q_i}(n_i, n_i - m_i)$. Then if

$$S_q^c(n, n-m) = \prod_{i=1}^{s} E_{q_i}^{n_i} - \prod_{i=1}^{s} S_{q_i}(n_i, n_i - m_i)$$

we see, as in the proof of part (i) of Theorem 2.4, that $S_q^c(n, n-m)$ is not $(m_1, \ldots, m_s)$-dense. But

$$|S_q^c(n, n-m)| = \prod_{i=1}^{s} q_i^{n_i} - \prod_{i=1}^{s} \sum_{j=0}^{n_i-m_i} (q_i-1)^j \binom{n_i}{j}.$$

Therefore, for any t satisfying (16), there exists $R \subseteq S_q^c(n, n-m)$ such that $|R| = t$ and R is not $(m_1, \ldots, m_s)$-dense.

(ii) Suppose that R is not $(m_1, \ldots, m_s)$-dense. Then by the definition of $A_{q_1, \ldots, q_s}(n_1, \ldots, n_s; m_1, \ldots, m_s)$, we see as in Lemma 2.1, that $R^c \in A_{q_1, \ldots, q_s}(n_1, \ldots, n_s; m_1, \ldots, m_s)$ and by Lemma 3.1.

$$\begin{aligned} |R^c| &\geq h_{q_1, \ldots, q_s}(n_1, \ldots, n_s; m_1, \ldots, m_s) \\ &\geq \sum_{r=1}^{n_1-m_1+1} (q_1-1)^{r-1} h_{q_1, \ldots, q_s}(n_1 - r, n_2, \ldots, n_s; m_1 - 1, m_2, \ldots, m_s). \end{aligned} \tag{18}$$

We now show that (18) implies

$$h_{q_1, \ldots, q_s}(n_1, \ldots, n_s; m_1, \ldots, m_s) \geq \prod_{i=1}^{s} h_{q_i}(n_i, m_i). \tag{19}$$

In view of (13), it will follow from (19) that

$$h_{q_1, \ldots, q_s}(n_1, \ldots, n_s; m_1, \ldots, m_s) \geq \prod_{i=1}^{s} \sum_{j=0}^{n_i-m_i} (q_i-1)^j \binom{n_i}{j}. \tag{20}$$

Since $|R^c| \geq h_{q_1, \ldots, q_s}(n_1, \ldots, n_s; m_1, \ldots, m_s)$, inequality (20) contradicts (17) and so R must be $(m_1, \ldots, m_s)$-dense.

It remains only to prove inequality (19). We first observe that

$$h_{q_1,\ldots,q_s}(n_1,\ldots,n_s;0,\ldots,0)=\prod_{i=1}^{s} q_i^{n_i}=\prod_{i=1}^{s} h_{q_i}(n_i,0). \tag{21}$$

Thus (19) is true for all $n_1,\ldots,n_s$ when $m_1=m_2=\cdots=m_s=0$.

Suppose now that (19) is true for all $n_1,\ldots,n_s$ and some $m_1-1, m_2,\ldots,m_s$ ($m_1\geqslant 1$). We claim that then (19) is true for all $n_1,\ldots,n_s$ and $m_1,m_2,\ldots,m_s$. It follows from Theorem 2.4 that

$$h_{q_i}(n_i,m_i)=\sum_{r=1}^{n_i-m_i+1}(q_i-1)^{r-1}h_{q_i}(n_i-r,m_i-1). \tag{22}$$

Then, by (18), (19) and (22), we have

$$\begin{aligned}&h_{q_1,\ldots,q_s}(n_1,\ldots,n_s,m_1,\ldots,m_s)\\&\geqslant\sum_{r=1}^{n_1-m_1+1}(q_1-1)^{r-1}h_{q_1}(n_1-r,m_1-1)\prod_{i=2}^{s}h_{q_i}(n_i,m_i)=\prod_{i=1}^{s}h_{q_i}(n_i,m_i).\end{aligned} \tag{23}$$

We have thus proved (19) for arbitrary m_1 when $m_2=\cdots=m_2=0$. But it follows from the definition of $h_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1,\ldots,m_s)$ that

$$\begin{aligned}&h_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1,\ldots,m_s)\\&=h_{q_{\sigma(1)},\ldots,q_{\sigma(s)}}(n_{\sigma(1)},\ldots,n_{\sigma(s)};m_{\sigma(1)},\ldots,m_{\sigma(s)})\end{aligned}$$

where $(\sigma(1),\ldots,\sigma(s))$ is an arbitrary permutation of $1,\ldots,s$. Therefore, interchanging the indices 1 and 2 and using (18), (22), we prove (19) for arbitrary m_1,m_2 when $m_3=\cdots=m_s=0$, and so on. This completes the proof of part (ii) and of Theorem 3.2.

We note that Theorem 3.2 implies the following formula for $h_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1,\ldots,m_s)$:

$$\begin{aligned}&h_{q_1,\ldots,q_s}(n_1,\ldots,n_s;m_1,\ldots,m_s)\\&=|S_q(n,n-m)|=\prod_{i=1}^{s}\sum_{j=0}^{n_i-m_i}(q_i-1)^j\binom{n_i}{j}.\end{aligned} \tag{24}$$

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