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ON SUBSPACES CONTAINED IN SUBSETS OF FINITE HOMOGENEOUS SPACES

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Let $E(l)\subset E(l+1)\subset\cdots\subset E(n)$ be a system of finite sets and $H(l)\subset H(l+1)\subset\cdots\subset H(n)$ be a system of groups, where $H(k)$ is a transitive group of automorphisms of $E(k)$. Denote $G(k, E(n))=\{X\subset E(n)\colon \exists h\in H(n),\ h(E(k))=X\}$. We investigate the following problem: given n, $1\leqslant l\leqslant k\leqslant n$, $0<\lambda\leqslant|E(k)|$. What is the maximal cardinality $L(n, k, \lambda)$ of a set $M\subseteq E(n)$ such that for $\forall X\in G(k, E(n))$, $|X\cap M|<\lambda$? We shall establish an upper bound for $L(n, k, \lambda)$ and prove that for some important cases it will coincide with the lower bound for $L(n, k, \lambda)$. We shall consider the three special cases of our problem: linear spaces, Grassmann spaces, Turan's problem. For linear spaces, we obtain the exact formula for the maximal cardinality $L(n, k, q^k-1)$ of a subset M in a linear n-space E_q^n over $GE(q)$ such that M does not contain any k-subspace of E_q^n. We shall consider also some applications of this result.

1. Introduction

Let $E(l)\subset E(l+1)\subset\cdots\subset E(n)$ be a system of finite sets and $H(l)\subset H(l+1)\subset\cdots\subset H(n)$ be a system of groups, where $H(k)$ is a transitive group of automorphisms of $E(k)$. Denote $G(k, E(n))=\{X\subset E(n)\colon \exists h\in H(n),\ h(E(k))=X\}$.

We investigate the following problem: given n, $1\leqslant l\leqslant k\leqslant n$, $0<\lambda\leqslant|E(k)|$ ($|E(k)|$ is the cardinality of $E(k)$), what is the maximal cardinality $L(n, k, \lambda)$ of a set $M\subseteq E(n)$ such that for $\forall X\in G(k, E(n))$, $|X\cap M|<\lambda$? We shall consider particularly three special cases of this problem: linear spaces, Grassmann spaces, finite graphs.

(i) For linear spaces, we let $E(n)=E_q^n-0$ be a linear n-space over $GF(q)$ without 0. Let $\{e_i\}$ $(i=1,\ldots,n)$ be some basis in E_q^n and E_q^k be a linear span of $\{e_i\}$ $(i=1,\ldots,k)$. Then we set $E(k)=E_q^k-0$ and $H(k)=GL(n,k)$ is the group of linear automorphisms such that for $\forall h\in H(k)$, $h(e_i)=e_i$ $(i=k+1,\ldots,n)$. For linear spaces, we are interested in the case $l=1$, $\lambda=q^k-1$ and denote for this case $L(n, k, q^k-1)=L_q(n,k)$.

(ii) Using the same notations as in (i), let $H(k)=GL(n,k)$ $(k=l,\ldots,n)$ and $E(k)=G(l, E_q^k)$ be the set of all l-subspaces in E_q^k (Grassmann space). For Grassmann spaces, we set $\lambda=|G(l, E_q^k)|$ and denote $L(n,k,|G(l,E_q^k)|)=L_q(n,k,l)$.

(iii) Let G_n be a complete n-graph with vertices $e_1, \ldots, e_n$ and G_k ($k = l, \ldots, n$) be a complete subgraph of G_n with vertices $e_1, \ldots, e_k$. Then $E(k)$ is the set of edges of G_k, $H(k)$ is the group of automorphisms of vertices of G_k such that for $\forall h \in H(k)$, $h(e_i) = e_i$ ($i = k+1, \ldots, n$; $k = l, \ldots, n$).

For this case, we set $l = 1$ and $L(n, k, \lambda) + 1 = f_T(n, k, \lambda)$. We note that determination of $f_T(n, k, \lambda)$ is the well-known problem of Turan and values of $f_T(n, k, \lambda)$ are known only for some special cases [2, 6]. For example, $f_T(n, 4, 4)$ is an open problem [2].

In Section 3, we shall obtain an upper bound for $L(n, k, \lambda)$ for the general case of homogeneous spaces which will coincide with lower bounds for $L(n, k, \lambda)$ for linear spaces, Grassmann spaces with $l = 1$ and some cases of Turan's problem. Since the case of linear spaces is the most interesting for us, we establish, in Section 2, the exact value of $L_q(n, k)$, and consider some corollaries from this result. Here we use the direct and shorter proof of the upper bound for $L_q(n, k)$ due to the referee of this paper.

We note that the main difference between our problem and analogous "Ramsey-type" problems (see, e.g. [3]) is that we are looking for elements from $G(k, E(n))$ in a given set $M \subseteq E(n)$ and not in one of two sets (M and $E(n) - M$).

2. Subspaces contained in subsets of linear spaces

In this section, we shall obtain the exact formula for the maximal cardinality $L_q(n, k)$ of a set $M \subseteq E_q^n - 0$ such that M does not contain any $X_q^k - 0$ where X_q^k is a k-subspace of E_q^n.

Theorem 2.1. *For* $1 \leqslant k \leqslant n$

$$L_q(n, k) = q^n - (q^{n-k+1} - 1)(q-1)^{-1} - 1. \tag{1}$$

Proof. *Lower bound.* We fix X_q^{n-k+1} and $M \subseteq E_q^n - 0$ such that for $\forall X_q^1 \subseteq X_q^{n-k+1}$, $|X_q^1 \cap M| < q - 1$. Then

$$|M| \leqslant q^n - (q^{n-k+1} - 1)(q-1)^{-1} - 1.$$

Now, let X_q^r be a subspace of maximal dimension such that $X_q^r - 0 \subseteq M$. If $r \geqslant k$, then $\exists X_q^1$ such that $X_q^1 \subseteq X_q^r \cap X^{n-k+1}$ and this contradicts the choice of M.

Upper bound. We use induction on k. For $k = 1$, the result is trivial as each $X_q^1 - 0$ has to contain an element that is not in M. Suppose, therefore, that

$$L_q(n, k-1) \leqslant q^n - (q^{n-k+2} - 1)(q-1)^{-1} - 1.$$

Suppose M contains no $X_q^k - 0$. If M contains no $X_q^{k-1} - 0$, we are done by induction. Let, therefore $V - 0 \subseteq M$, V is a $(k-1)$-dimensional subspace of E_q^n. Write $E_q^n = V + W$. (Direct sum of V and W.) For each 1-dimensional subspace T of W, choose $a_T \in T - 0$. Since M contains no $X_q^k - 0$, for each a_T, there must be a $v_T \in V$ and $\alpha_T \in GF(q) - 0$ such that $v_T + \alpha_T a_T \notin M$. If $v_T + \alpha_T a_T = v_{T'} + \alpha_{T'} a_{T'}$, then $v_T - v_{T'} = \alpha_{T'} a_{T'} - \alpha_T a_T$, $v_T = v_{T'}$, $\alpha_{T'} a_{T'} = \alpha_T a_T$, $T = T'$. Hence, there are $(q^{n-k+1}-1)(q-1)^{-1}$ distinct elements $v_T + \alpha_T a_T \notin M$ and since $0 \notin M$, we have (1).

This proof implies the following result.

Corollary 2.2. *Let $M \subset E_q^n - 0$ and M contains $X_q^{k-1} - 0$.*

If

$$|M| > q^n - (q^{n-k+1} - 1)(q-1)^{-1} - 1,$$

then there exists X_q^k, such that $X_q^k - 0 \subseteq M$ and $X_q^{k-1} \subseteq X_q^k$.

If $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r \subseteq E_q^n - 0$, we say that $M^{(r)} = (M_1, \ldots, M_r)$ is an r-flag of sets and if $X_q^{k_1} \subseteq \cdots \subseteq X_q^{k_r}$, we say that $X^{(r)} = (X_q^{k_1}, \ldots, X_q^{k_r})$ is an r-flag of spaces. We write $X^{(r)} - 0 \subseteq M^{(r)}$ if $X_q^{k_i} - 0 \subseteq M_i (i = 1, \ldots, r)$.

Corollary 2.3. *If $M^{(r)} = (M_1, \ldots, M_r)$ is an r-flag of sets and*

$$|M_i| > q^n - (q^{n-k_i+1} - 1)(q-1)^{-1} - 1,$$

then there exists an r-flag of spaces $X^{(r)} = (X_q^{k_1}, \ldots, X_q^{k_r})$ such that $X^{(r)} - 0 \subseteq M^{(r)}$.

The proof follows immediately from Corollary 2.2.

Let E_q^∞ be a linear infinite-dimensional space over $GF(q)$ consisting of all the finite sequences $E_q^\infty = \{\alpha_1, \ldots, \alpha_n, 0, 0, \ldots): \alpha_i \in GF(q);\ i = 1, \ldots, n;\ n = 1, 2, \ldots)$, and $E_q^n = \{\alpha_1, \ldots, \alpha_n, 0, 0, \ldots): \alpha_i \in GF(q);\ i = 1, \ldots, n\}$ be an n-dimensional subspace of E_q^∞.

Corollary 2.4. *If $M_\infty \subseteq E_q^\infty$ is such that $0 \in M_\infty$ and $\lim_{n\to\infty} \gamma_n = 1$, where $\gamma_n = q^{-n} |M_\infty \cap E_q^n|$ $(n = 1, 2, \ldots)$, then there exists an infinite-dimensional space E such that $E \subset M_\infty$.*

Proof. It is sufficient to prove that there exists a sequence of numbers $n_k (n_k < n_{k+1})$ and a sequence of subspaces $X_q^k \subset E_q^\infty$ $(k = 1, 2, \ldots)$, such that $X_q^k \subseteq (M_\infty \cap E_q^{n_k})$ and $X_q^k \subset X_q^{k+1}$. Then we may put $E = \bigcup_{k=1}^\infty X_q^k$. We construct $\{n_k\}$ and $\{X_q^k\}$ by induction on k. Choose n_1 such that $\gamma_{n_1} > 1 - (q-1)^{-1} + q^{-n_1}(q-1)^{-1}$. Then by Theorem 2.1, there exists X_q^1 such that $X_q^1 \subseteq (M_\infty \cap E_q^{n_1})$. Suppose we have found $n_1 < n_2 < \cdots < n_{k-1}$ and $X_q^1 \subset \cdots \subset X_q^{k-1}$ such that $X_q^i \subset (M_\infty \cap E_q^{n_i})$ $(i = 1, \ldots, k-1)$. Choose $n_k > n_{k-1}$ from the condition

$$\gamma_{n_k} > 1 - q^{-k+1}(q-1)^{-1} + q^{-n_k}(q-1)^{-1}$$

and by Theorem 2.1, there exists X_q^k such that $X_q^{k-1} \subset X_q^k$ and $X_q^k \subseteq (M_\infty \cap E_q^{n_k})$.

Let us note that no analogue of this corollary for the case of fields R or C is known though it is of a great interest in functional analysis; while Ramsey analogues of Theorem 2.1 are well-known for these fields [4, 5].

3. Subspaces contained in subsets of finite homogeneous spaces

In this section, we investigate $L(n, k, \lambda)$ for the general case of homogeneous spaces (see Section 1).

Lemma 3.1. *For any $M \subseteq E(n)$ and $1 \leq t \leq n$*

$$\frac{|M|}{|E(n)|} = \frac{1}{|G(t, E(n))|\,|E(t)|} \sum_{X \in G(t, E(n))} |X \cap M|. \tag{2}$$

Proof. In the sum on the right side of (2), each element of M is counted as many times as there are $X \in G(t, E(n))$ which contain this element, i.e. $|E(t)|\,|E(n)|^{-1}\,|G(t, E(n))|$ times.

We note that left and right sides of (2) are equal to the invariant normalized measure of $M \subseteq E(n)$; and (2) follows from uniqueness of this measure on $E(n)$.

Corollary 3.2. *For $1 \leq k \leq n$, $1 \leq \lambda \leq |E(k)|$*

$$L(n, k, \lambda) \leq (\lambda - 1)\,|E(n)|\,|E(k)|^{-1}. \tag{3}$$

The proof follows from Lemma 3.1 with $t = k$.

By $N(m, k, \lambda)$, we denote a number of $X \in G(k, E(n))$ such that for any $M \subseteq E(n)$ with $|M| = m$, we have $|X \cap M| \geq \lambda$ and we denote $\mu(m, k, \lambda) = N(m, k, \lambda)\,|G(k, E(n))|^{-1}$.

Corollary 3.3. *For $1 \leq m \leq |E(n)|$, $1 \leq k \leq n$, $1 \leq \lambda \leq |E(k)|$*

$$(|E(k)|\,|E(n)|^{-1} m - \lambda + 1) \leq \mu(m, k, \lambda) \leq |E(k)|\,|E(n)|^{-1} m \lambda^{-1}. \tag{4}$$

Proof. By definition of $\mu(m, k, \lambda)$ and (2) with $t = k$,

$$\begin{aligned} \lambda\mu(m, k, \lambda) &\leq |G(k, E(n))|^{-1} \sum_{X \in G(k, E(n))} |X \cap M| = m\,|E(k)|\,|E(n)|^{-1} \\ &\leq (1 - \mu(m, k, \lambda))(\lambda - 1) + \lambda\mu(m, k, \lambda) \end{aligned} \tag{5}$$

and (4) follows from (5).

Theorem 3.4. *If for some function $f(n, k, \lambda)$*

$$]|E(k)|\,|E(k+1)|^{-1}] \cdots]|E(n-1)|\,|E(n)|^{-1} f(n, k, \lambda)[\cdots [[\geq \lambda, \tag{6}$$

then

$$L(n, k, \lambda) \leq f(n, k, \lambda) - 1. \tag{7}$$

(]a[*denotes the least integer* $\geq a$).

Proof. Let $M \subseteq E(n)$ and $|M| = f(n, k, \lambda)$. Then by Lemma 3.1 with $t = n - 1$, $\exists X^{n-1} \in G(n-1, E(n))$ such that

$$]|E(n-1)|\,|E(n)|^{-1} f(n, k, \lambda)[\leq |X^{n-1} \cap M|. \tag{8}$$

Next, using (8) and Lemma 3.1 for $n-1$, $M_1 = X^{n-1} \cap M$, $t = n-2$, $\exists X^{n-2} \subseteq X^{n-1}$

$(X^{n-2} \in G(n-2, E(n)))$ such that

$$]|E(n-2)|\,|E(n-1)|^{-1}]|E(n-1)|\,|E(n)|^{-1}f(n, k, \lambda)[[$$
$$\leqslant]|E(n-2)|\,|E(n-1)|^{-1}\,|M_1|\,[\leqslant |X^{n-2} \cap M_1| \leqslant |X^{n-2} \cap M|.$$

Continuing this procedure for all t down to $t=k$, we find that $\exists X^k \in G(k, E(n))$ such that

$$]|E(k)|\,|E(k+1)|^{-1}]\cdots]|E(n-1)|\,|E(n)|^{-1}f(n, k, \lambda)[\cdots[[\leqslant |X_n^k \cap M|.$$

Hence by (6) and definition of $L(n, k, \lambda)$, we have (7).

We note that the upper bound from Theorem 2.1 immediately follows from Theorem 3.4 in a view of

$$\left]\frac{q^k-1}{q^{k+1}-1}\right]\cdots\left]\frac{q^{n-1}-1}{q^n-1}\left(q^n-\frac{1}{q-1}(q^{n-k+1}-1)\right)\right[\cdots\left[\ =q^k-1\right.$$
$$(q \geqslant 2,\ i \leqslant k \geqslant n-1).$$

For the case (ii) of Grassmann spaces $E(n)=G(l, E_q^n)$ (see Section 1), we have the following bounds for $L_q(n, k, l)$.

Corollary 3.5. *For any* $1 \leqslant l \leqslant k \leqslant n$

$$|G(l, E_q^n)|-|G(l, E_q^{n-k+l})| \leqslant L_q(n, k, l) \leqslant |G(l, E_q^n)|-|G(l, E_q^n)|\,|G(l, E_q^k)|^{-1} \tag{9}$$

where $|G(l, E_q^t)| = [{t \atop l}]_q$ *is the q-binomial coefficient,*

$$|G(l, E_q^t)| = [{t \atop l}]_q = \prod_{i=0}^{l-1}(q^t-q^i)(q^l-q^i)^{-1}. \tag{10}$$

Proof. Define $M=G(l, E_q^n)-G(l, E_q^{n-k+l})$. Then $\dim(X_q^k \cap E_q^{n-k+l}) \geqslant l$, for every $X_q^k \in G(k, E_q^n)$. So there exists $X_q^l \subseteq E_q^{n-k+l}$, $X_q^l \subseteq X_q^k$. But then $X_q^l \not\subset M$ and we have the lower bound from (9). Upper bound follows from Corollary 3.2 with $E(k)=G(l, E_q^k)$ and $\lambda = |G(l, E_q^k)|$.

The difference between the lower and upper bounds (9) is not very large. For example, in the case $k \to \infty$, $n-k \to \infty$ from (10) follows that for a large l and q

$$\lim_{k\to\infty, n-k\to\infty} |G(l, E_q^n)|\,|G(l, E_q^k)|^{-1}\,|G(l, E_q^{n-k+l})|^{-1}$$
$$= \prod_{i=0}^{l-1}(1-q^{-l+i}) \approx \exp(-(q-1)^{-1}). \tag{11}$$

We note also that by the lower bound (9) and the upper bound from Theorem 3.4, we can establish sometimes the exact value for $L_q(n, k, l)$. For example

$$L_q(n, k, 1) = |G(1, E_q^n)|-|G(1, E_q^{n-k+1})| = (q^n-q^{n-k+1})(q-1)^{-1}. \tag{12}$$

We consider now the case (iii) when $E(n)$ is the set of edges of a complete n-graph (problem of Turan, see Section 1). Turan [6] proved that if $n = c(k-1)+r$ ($c \in \{0, 1, \ldots\}$, $r \in \{0, \ldots, k-2\}$), then

$$f_T\left(n, k, \binom{k}{2}\right) = 0.5(k-2)(k-1)^{-1}(n^2 - r^2) + \binom{r}{2} + 1. \tag{13}$$

We note here that the exact upper bound for $f_T(n, k, \binom{k}{2})$ follows immediately from Theorem 3.4 since

$$\left]\binom{k}{2}\binom{k+1}{2}^{-1}\right] \cdots \left]\binom{n-1}{2}\binom{n}{2}^{-1}\left(\frac{k-2}{2(k-1)}(n^2-r^2) + \binom{r}{2} + 1\right)\right[\cdots \left[\left[= \binom{k}{2}.\right.\right. \tag{14}$$

Other examples are given by formulas [4]:

$$f_T(n, k, [0.25k^2] + u) = [0.25n^2] + u \quad (u \leqslant [0.25(k+1)]), \tag{15}$$

$$f_T(n, k, [0.25k^2] + [0.5(k-1)]) = [0.25n^2] + [0.5(n-1)] \quad (k > 4) \tag{16}$$

($[a]$ the greatest integer $\leqslant a$). The exact upper bounds from (15) and (16) also follow from Theorem 3.4.

We shall give now one more corollary from Theorem 3.4 which may be useful for determination of $f_T(n, k, \lambda)$.

Corollary 3.6. *Let* $f(n, k, \lambda) = L(n, k, \lambda) + 1$. *For any* $1 \leqslant s \leqslant k \leqslant n$, $0 < \lambda \leqslant |E(s)|$

$$f(n, k, f(k, s, \lambda)) \geqslant f(n, s, \lambda) \tag{17}$$

and, if for all $t \in \{k+1, \ldots, n\}$

$$]|E(t-1)|\,|E(t)|^{-1} f(t, s, \lambda)[= f(t-1, s, \lambda), \tag{18}$$

then

$$f(n, k, f(k, s, \lambda)) = f(n, s, \lambda). \tag{19}$$

Proof. Formula (17) follows from the definition of $f(n, k, \lambda)$. Let $M \subseteq E(n)$ and $|M| = f(n, s, \lambda)$. Then as it was shown in the proof of Theorem 3.4, $\exists X^k \in G(k, E(n))$ such that

$$]|E(k)|\,|E(k+1)|^{-1}] \cdots]|E(n-1)|\,|E(n)|^{-1} f(n, s, \lambda)[\cdots [[\leqslant |X^k \cap M|$$

and in a view of (18)

$$f(k, s, \lambda) \leqslant |X^k \cap M|.$$

Hence,

$$f(n, k, f(k, s, \lambda)) \leqslant f(n, s, \lambda)$$

and by (17) we have (19).

By Corollary 3.6 with $s=4$, $\lambda=6$ and (13) we have, for example,

$$f_T(n,5,9)=f_T(n,6,13)=f_T(n,7,17)=3^{-1}(n^2-r^2)+\binom{r}{2}+1 \tag{20}$$

where $0\leqslant r\leqslant 2$, $n\equiv r \pmod 3$.

4. Sufficient conditions for existence of linear error-correcting codes with given parameters

Let $\rho(x, y)$ be a metric in E_q^n. A linear (n, k)-code with base q and distance d in the metric ρ is defined as a a subspace X_q^k such that $\min_{a,b\in X_q^k}\rho(a,b)=d$.

Theorem 4.1. *For any n, $k<n$, and any $Q\subseteq E_q^n (0\in Q)$, a sufficient condition for the existence of an (n, k)-code X_q^k with base q, distance d in the metric ρ, such that $X_q^k\subseteq Q$, is that*

$$|Q-\{a:\|a\|_\rho<d\}|\geqslant q^n-(q-1)^{-1}(q^{n-k+1}-1) \quad (\|a\|_\rho=\rho(a,0)). \tag{21}$$

Proof. A subspace X_q^k is an (n, k)-code with distance d in the metric ρ iff $X_q^k\cap\{a:\|a\|_\rho<d\}=0$. We therefore set $M=Q-\{a:0<\|a\|_\rho<d\}$. Then $|M|=1+|Q-\{a:\|a\|_\rho<d\}|$, and by Theorem 2.1, if (21) holds, there exists an (n, k)-code $X_q^k\subseteq Q$ with distance d.

Theorem 4.1 yields sufficient conditions for (n, k)-codes X_q^k satisfying constraints of the type $X_q^k\subseteq Q$. For example, let $Q=\{a:\|a\|_\rho\leqslant d+\varepsilon\}$, where ρ is the Hamming metric [1].

Corollary 4.2. *For any n and $k<n$, a sufficient condition for the existence of an (n, k)-code X_q^k with base q and distance d in the Hamming metric ρ, such that*

$$\max_{a,b\in X_q^k-0}|(\|a\|_\rho-\|b\|_\rho)\leqslant\varepsilon,$$

is that

$$\sum_{i=d}^{d+\varepsilon}\binom{n}{i}(q-1)^i\geqslant q^n-(q^{n-k+1}-1)(q-1)^{-1}. \tag{22}$$

Proof. Set $Q=\{a:\|a\|_\rho\leqslant d+\varepsilon\}$; then

$$|Q-\{a:\|a\|_\rho<d\}|=|\{a:d\leqslant\|a\|_\rho\leqslant d+\varepsilon\}|=\sum_{i=d}^{d+\varepsilon}\binom{n}{i}(q-1)^i,$$

and by Theorem 4.1 there exists an (n, k)-code $X_q^k\subseteq Q$ with distance d, such that for any $a\in X_q^k$ we have $d\leqslant\|a\|_\rho\leqslant d+\varepsilon$.

Note that in the case $\varepsilon=n-d$ the condition (22) is very close to the well-known Varshamov-Gilbert bound [1].

Note added in proof

It came to our attention that the upper bound for $L_q(n, k)$ also appeared in the paper by M. Deza and F. Hoffman, IEEE Int. Trans. (July 1977) 517–518.

References

[1] E.R. Berlekamp, Algebraic Coding Theory (McGraw-Hill, New York 1968).
[2] P. Erdős, Extremal problems in graph theory, in: Proc. Symp. on The Theory of Graph and its Applications, Smolenice, June (1963).
[3] R.L. Graham and B.L. Rothschild, Ramsey's theorem for n-parameter sets, Trans. Am. Math. Soc. 159 (1971) 257–292.
[4] V.D. Milman, A new proof of the theorem of A. Dvoretzky on sections of convex bodies, Functional Anal. Appl. 5 (1971) 28–37 (translated from Russian).
[5] V.D. Milman, Asymptotic property of function of several variables, defined on homogeneous spaces, Dokl. Acad. Sci. USSR 199 (1971). (translated in: Soviet Math. Dokl. 12 (1971) 1277–1281).
[6] P. Turan, On the Theory of Graphs, Colloq. Math. 3 (1954) 19–30.