ERROR DETECTION IN DIGITAL DEVICES AND COMPUTER PROGRAMS WITH THE AID OF LINEAR RECURRENT EQUATIONS OVER FINITE COMMUTATIVE GROUPS

M. G. Karpovsky

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Error Detection in Digital Devices and Computer Programs with the Aid of Linear Recurrent Equations Over Finite Commutative Groups

MARK G. KARPOVSKY

Abstract—**A method is proposed for error detection in digital devices and computer programs calculating the values of functions $f(x)$, where $x \in G$ and G is a finite commutative group. For the case of network implementation of the method, "errors" are catastrophic structural failures; for the case of program implementation, they are errors in the text of the program.**

The method is based on finding, given the function, a linear "recurrent equation over G" with coefficients 0 or 1, of which f is a solution. The verification whether this is indeed the case constitutes an error detection method.

Implementation of the method requires only the operations of summation, delay, and the group operation in G.

The equation whose solution is the given function f will be sought using methods of abstract harmonic analysis on the group G.

Index Terms—**Error detection for digital devices and computer programs, error detection tests for digital devices, characters of commutative groups, Fourier transform over finite commutative groups, spectral and autocorrelation functions over finite commutative groups, fast Hadamard-Walsh transform.**

I. Introduction

SUPPOSE given a digital device or program for calculation of a function $f: G \to C$, where G is a finite commutative group and C is the field of complex numbers. (Examples of such devices are the blocks of the arithmetic unit of a computer, networks whose operation is described by two- or many-valued logical functions, devices operating in systems of residue classes, etc.) To detect errors, one can construct another device or program calculating the same $f(x)$; errors are detected by comparing the results of the calculations ("system redundancy method"). This method is widely used but is, generally speaking, highly uneconomical.

In this paper we propose another method of error detection, according to which, given the function f, one determines a linear recurrent equation over G:

$$\sum_{q \in G} a(q) f(x^* q^{-1}) = \Phi(x) \tag{1}$$

(* denotes the group operation in G, $q^{-1} \in G$ is the inverse of q) and checks the validity of (1) for given x.

Note that the meaning of the term "errors" depends on the context; errors in a digital device are catastrophic stable structural failures, and in programs they are errors in the texts of the programs. To simplify the error detection process, we shall consider the case $a(q) \in \{0,1\}$.

The proposed error detection method will not depend on the specific features of implementation of the device or program for calculating the function f; moreover, it will be universal in the sense that for any $f: G \to C$ there exists an equation (1) with $a(q) \in \{0,1\}$ and a "fairly simple" function $\Phi(x)$ (for example, $\Phi(x) = \text{const}$; see Section II, Theorem 1). It will also be seen that the above-mentioned "system redundancy method" is a special case of our method.

To search for the optimal checking equation (1) and to estimate the complexity of the equation we shall use methods of abstract harmonic analysis on G. The advantages and the limitations of this technique will be discussed in Section V.

Related questions, concerning the analysis, synthesis, and optimization of digital devices by methods of abstract harmonic analysis and generalized Fourier transforms, were dealt with in [1]–[5].

II. Error Detection by Linear Homogeneous Equations Over a Group

A. Let $f: G \to C$, where G is a finite commutative group and C the field of complex numbers; we denote the elements of G by $0, \cdots, g-1$ (g is the order of G); 0 is the identity element of G; * denotes the group operation in G.

Let f be a solution of a linear homogeneous equation over G with coefficients 0 or 1; i.e., for every $x \in G$,

$$f(x) + f(x^* q_1^{-1}) + \cdots + f(x^* q_{L(f)}^{-1}) = d,$$
$$q_1, \cdots, q_{L(f)} \in G, d \in C. \tag{2}$$

Equation (2) generates a simple method for detection of errors in the calculation of $f(x)$. A network interpretation of this method is illustrated in Fig. 1.

In the network of Fig. 1, signals corresponding to $x, x^* q_1^{-1}, \cdots, x^* q_{L(f)}^{-1}$ are applied at consecutive instants of time to the input of the network calculating $f(x)$; a nonzero signal at the output of the adder-accumulator with initial state $-d$ after $L(f)$ elementary additions may be used as an error signal.

The complexity of the network of Fig. 1 and the time required for error detection for fixed x depend only on the

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The author is with the Department of Mathematics, Tel-Aviv University, Ramat-Aviv, Tel-Aviv, Israel.

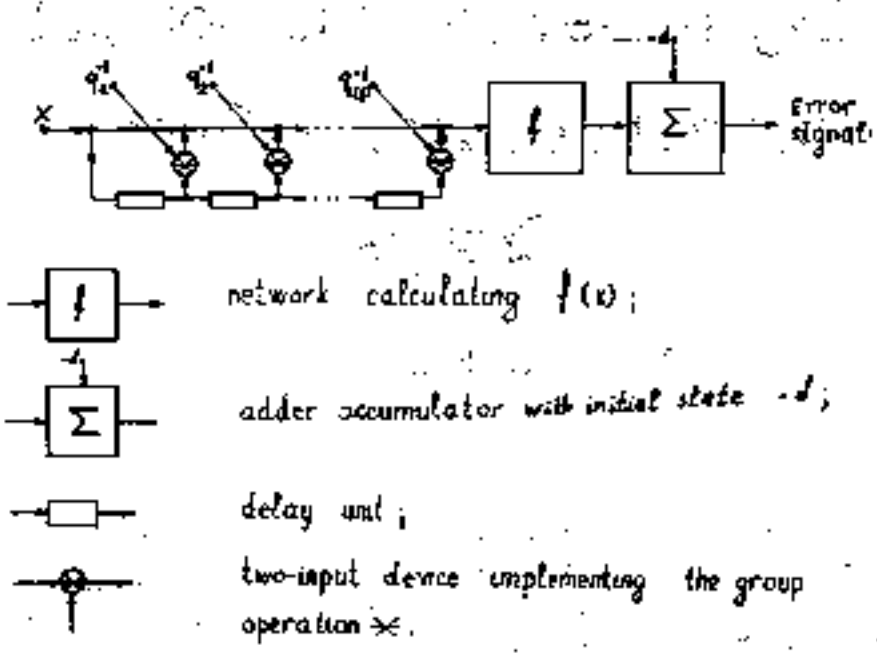


Fig. 1. Error detection by a linear homogeneous equation.

number $L(f)$ in the checking equation (2) ($1 \leqslant L(f) \leqslant g-1$).

B. We now consider the problem of finding a linear equation (2), given f.

For the solution of this problem and the other problems considered below, we use methods of abstract harmonic anslysis. We recall the main definitions.

A character of a finite commutative group G is defined as a homomorphism of G into the multiplicative group of complex numbers. The set of characters of G is a multiplicative group isomorphic to G [6, p. 367]. The character mapped onto an element $\omega \in G$ under this isomorphism will be denoted by $\chi_\omega(x) (x \in G)$.

Express G as a direct product of cyclic subgroups $G = \prod_{s=0}^{m-1} G_s$. Let $\xi_s \in G_s$ denote a generator of G_s, p_s — the order of G_s where p_s is power of prime. Then [6, p. 367]

$$\chi_\omega(x) = \exp\left(2\pi i \sum_{s=0}^{m-1} \frac{\omega_s x_s}{p_s}\right) \tag{3}$$

where

$$x = *_{s=0}^{m-1} \xi_s x_s \qquad \omega = *_{s=0}^{m-1} \xi_s \omega_s$$
$$x_s, \omega_s \in \{0, 1, \cdots, p_s - 1\} \qquad i = \sqrt{-1},$$
$$\xi_s r = \underbrace{\xi_s * \xi_s * \cdots * \xi_s}_{r}, \quad \xi_s 0 = 0.$$

Due to the orthogonality and completeness of the set of characters $\{\chi_\omega\}$, one can use this set as a complete orthogonal basis in the space of functions mapping G into the field C of complex numbers. Thus if $f: G \to C$, then

$$f(x) = \sum_{\omega \in G} S_f(\omega) \chi_\omega(x) \tag{4}$$

where

$$S_f(\omega) = 1/g \sum_{x \in G} f(x) \overline{\chi}_\omega(x) \tag{5}$$

and $\overline{\chi}_\omega$ is the character complex-conjugate to χ_ω.

Equations (4) and (5) define the generalized Fourier transform F_G and the inverse generalized Fourier transform F_G^{-1} on G; each function f is associated with its spectrum S_f.

Theorem 1: For any $f: G \to C$ there exist $a: G \to \{0,1\}$ ($a \neq 0$) and $d \in C$ such that for every $x \in G$

$$\sum_{q \in G} a(q) f(x * q^{-1}) = d \tag{6}$$

and

$$\sum_{q \in G} a(q) = g/g_a \tag{7}$$

where g_a is the order of an arbitrary subgroup G_a of G such that if $\omega \in G_a$, and $\omega \neq 0$ then $S_f(\omega) = 0$.

Proof: The left-hand side of (6) is the convolution over G of the functions a and f. Spectrum of the convolution of two functions is equal to product of the spectra of the functions multiplied by the order of the group [6, p. 360] ("convolution theorem"). Hence the required function a must be such that

$$S_a(\omega) \cdot S_f(\omega) = \begin{cases} d, \\ g & \text{if } \omega = 0 \quad (\text{where } S_a(\omega) \text{ spectrum} \\ 0, & \text{if } \omega \neq 0 \qquad\qquad a(q)). \end{cases} \tag{8}$$

Let

$$d = \frac{g}{g_a} S_f(0) = \frac{\sum_{x \in G} f(x)}{g_a} \tag{9}$$

and

$$S_a(\omega) = \begin{cases} 1/g_a, & \text{if } \omega \in G_a, \\ 0, & \text{if } \omega \notin G_a. \end{cases} \tag{10}$$

Then equality (8) (and hence also (6)) will follow from (9), (10), and by (3), (4), (10), for every $q \in G$,

$$a(q) = \sum_{\omega \in G_a} S_a(\omega) \chi_\omega(q) = 1/g_a \sum_{\omega \in G_a} \chi_q(\omega). \tag{11}$$

The function $\chi_q(\omega)$, $\omega \in G_a$, is a character of the subgroup G_a. Hence, using (3) and (11), we obtain

$$\sum_{\omega\in G_a}\chi_a(\omega)\in\{0,g_a\};\qquad a(0)=1,\ a(q)\in\{0,1\}. \quad (12)$$

Thus if $a(q)$ is defined by (11) and d is defined by (9), then a, f, and d satisfy (6) and $a(q)\in\{0,1\}$. We now prove (7). Consider a homomorphism of the group of characters of G onto the group of characters of G_a. The kernel $G_a{}^{\perp}$ of this homomorphism is defined by the condition:

$$q\in G_a{}^{\perp} \text{ iff } \chi_q(\omega)=1 \text{ for all } \omega\in G_a$$

or

$$q\in G_a{}^{\perp} \text{ iff } a(q)=1.$$

Since $G_a{}^{\perp}$ is a subgroup of G (this subgroup is isomorphic to quotient group G/G_a), and the order of $G_a{}^{\perp}$ is g/g_a, it follows that

$$\sum_{q\in G}a(q)=g/g_a.$$

C. The proof of Theorem 1 generates a simple method for constructing a checking equation (2) or (6) for the given function f. This method reduces to the following operations:

1) Compute the generalized Fourier spectrum S_f of f by (3), (5).

2) Construct a subgroup G_a of G such that if $\omega\in G_a$ and $\omega\neq 0$, then $S_f(\omega)=0$.

3) Construct $a\colon G\to\{0,1\}$ by (11), and d by (9).

The spectrum S_f may be calculated by using the highly effective algorithm of the fast Fourier transform on the group G [7].

We now illustrate the error detecting technique described above for binary adders and multipliers.

Example 1: Consider error detection in an n-bit binary adder using Theorem 1. Let

$$X=\sum_{p=0}^{n-1}x_p2^{n-1-p}\qquad Y=\sum_{p=0}^{n-1}y_p2^{n-1-p},$$

$$(x_p,y_p\in\{0,1\})$$

and $f(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1})=X+Y$. Then $(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1})\in G_2{}^{2n}$ where $G_2{}^{2n}$ is the group of binary $2n$-vectors with respect to componentwise addition mod 2.

According to (3), the characters of $G_2{}^{2n}$ are the Walsh functions:

$$W_\omega(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1})=(-1)^{\sum_{p=0}^{n-1}(\omega_px_p+\omega_{n+p}y_p)},$$
$$(\omega_p\in\{0,1\};\ r=0,\cdots,2n-1). \quad (13)$$

Letting $\omega=(\underbrace{0,\cdots,0}_{r},1,0,\cdots,0)$ $(r=0,\cdots,2n-1)$, we put

$$R_{r+1}=W_\omega(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1}) \quad (14)$$

(these functions are known as the Rademacher functions [2], [5]).

Then by (13), (14),

$$x_p=0.5(1-R_{p+1})\qquad y_p=0.5(1-R_{n+p+1}),$$
$$p=0,\cdots,n-1 \quad (15)$$

and

$$X+Y=2^n-1-\sum_{p=0}^{n-1}(R_{p+1}+R_{n+p+1})2^{n-2-p}. \quad (16)$$

Consequently, in view of (4), (14),

$$S_f(\omega)=S_{X+Y}(\omega)=$$
$$\begin{cases}2^n-1, & \text{if } \omega=(\underbrace{0,\cdots,0}_{2n});\\ -2^{n-2-p}, & \text{if } \omega=(0,\cdots,0,1,\underbrace{0,\cdots,0}_{2n-p-1}),\\ & \omega=(0,\cdots,0,1,\underbrace{0,\cdots,0}_{n-p-1});\ (p=\le -1);\\ 0, & \text{otherwise}\end{cases} \quad (17)$$

In accordance with (17), we set

$$G_a=\left\{\omega\,\middle|\,\sum_{r=0}^{2n-1}\omega_r=2k\qquad (k=0,1,\cdots,n)\right\}.$$

Then

$$g_a=2^{2n-1},\ G_a{}^{\perp}=\{(\underbrace{0,\cdots,0}_{2n}),(\underbrace{1,\cdots,1}_{2n})\}.$$

Since

$$\sum_{X,Y}f(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1})=2^{2n}(2^n-1)$$

it follows from (9) that $d=2(2^n-1)$, and we have the following checking equation for binary adders:

$$f(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1})+f(x_0\oplus 1,\cdots,x_{n-1}\oplus 1,y_0\oplus 1,\cdots,y_{n-1}\oplus 1)=2(2^n-1)(\text{mod } 2). \quad (18)$$

(Henceforth, the symbol $\oplus$ with "(mod 2)" to the right of the equation stands for addition mod 2.)

Example 2: We now consider error detection in a binary multiplier. Let

$$X=\sum_{p=0}^{n-1}x_p2^{n-1-p}\qquad Y=\sum_{p=0}^{n-1}y_p2^{n-1-p},$$
$$(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1})\in G_2{}^{2n},$$
$$f(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1})=X\cdot Y.$$

Then, in view of (14), (15),

$$X\cdot Y=0.25(2^n-1)^2-0.5(2^n-1)\cdot\sum_{p=0}^{n-1}(R_{p+1}+R_{n+p+1})2^{n-2-p}+\sum_{p_1,p_2=0}^{n-1}R_{p_1+1}\cdot R_{n+p_2+1}\cdot 2^{2n-4-p_1-p_2}. \quad (19)$$

Now it follows from (13), (14) that

$$R_{p_1+1}\cdot R_{n+p_2+1}=W_\omega(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1})$$

where

$$\omega = (\underbrace{0,\cdots,0}_{p_1},1,\underbrace{0,\cdots,0}_{n+p_2-p_1},1,0,\cdots,0)$$

and so, in view of (4), (19), we have

$$S_f(\omega) = S_{X\cdot Y}(\omega) = \begin{cases} 0.25\,(2^n-1)^2, & \text{if } \omega = (\underbrace{0,\cdots,0}_{2n}); \\ 0.5\,(2^n-1)\,2^{n-2-p}, & \text{if } \omega = (\underbrace{0,\cdots,0}_{p},1,0,\cdots,0), \\ & \omega = (\underbrace{0,\cdots,0}_{n+p},1,0\cdots,0); \\ 2^{2n-4-p_1-p_2}, & \text{if } \omega = (\underbrace{0,\cdots,0}_{p_1},1,\underbrace{0,\cdots,0}_{n+p_2-p_1},1,0,\cdots,0). \\ 0, & \text{otherwise, } (p,p_1,p_2 = \le n-1). \end{cases} \tag{20}$$

In accordance with (20), we set

$$G_a = \Big\{\omega \Big| \sum_{p=0}^{n-1} \omega_p = 2K_1, \quad \sum_{p=0}^{n-1} \omega_{n+p} = 2K_2 (k_1,k_2 = 0,\cdots,n)\Big\}.$$

Then

$$g_a = 2^{2n-2},\ G_a^{\perp} = \{\underbrace{(0,\cdots,0)}_{2n}, \underbrace{(0,\cdots,0,1,\cdots,1)}_{n}, \underbrace{(1,\cdots,1,0,\cdots,0)}_{n}, \underbrace{(1,\cdots,1)}_{2n}\}.$$

Since in this case

$$\sum_{X,Y} f(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1}) = 2^{2n-2}(2^n-1)^2$$

it follows from (9) that $d = (2^n-1)^2$ and we have the following checking equation for binary multipliers:

$$\begin{aligned} &f(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1}) \\ &\quad + f(x_0,\cdots,x_{n-1},y_0 \oplus 1,\cdots,y_{n-1} \oplus 1) \\ &\quad + f(x_0 \oplus 1,\cdots,x_{n-1} \oplus 1,y_0,\cdots,y_{n-1}) \\ &\quad + f(x_0 \oplus 1,\cdots,x_{n-1} \oplus 1,y_0 \oplus 1,\cdots y_{n-1} \oplus 1) \\ &\quad = (2^n-1)^2 (\text{mod } 2). \end{aligned} \tag{21}$$

To end this section, we consider the special case of Theorem 1 for $d = 0$.

Corollary 1: For any $f: G \to C$ such that $\Sigma_{x\in G} f(x) = 0$, there exists $a: G \to \{0,1\} (a \neq 0)$ with the property: for every $x \in G$,

$$\sum_{q\in G} a(q) f(x^* q^{-1}) = 0$$

and

$$\sum_{q\in G} a(q) = \frac{g}{g_a}$$

where g_a is the order of an arbitrary subgroup G_a of G such that if $\omega \in G_a$ then $S_f(\omega) = 0$.

Proof: Follows directly from (6), (7), (9).

III. Error Detection by Linear Nonhomogeneous Equations Over a Group

A. Suppose that besides the device or program for calculation of the function $f: G \to C$ we have another (independent) device or program for calculation of some function $\Phi: G \to C$, such that the calculation of $\Phi(x)$ for all $x \in G$ is "simple" in the sense that the probability of errors in $\Phi(x)$ is small. (An example is $\Phi(x) = x_r (r = 0,1,\cdots,m-1)$, where x_r is the rth component of the vector x.)

We shall use $\Phi(x)$ to detect errors in the calculation of $f(x)$.

Let there exist $q_1,\cdots,q_{L(f)},S_1,\cdots,S_{L(\Phi)} \in G$ and $d \in C$ such that, for given f and Φ, and for every $x \in G$

$$\begin{aligned} f(x) + f(x^* q_1^{-1}) + \cdots + f(x^* q_{L(f)}^{-1}) &= \Phi(x) \\ &+ \Phi(x^* S_1^{-1}) + \cdots + \Phi(x^* S_{L(\Phi)}^{-1}) + d. \end{aligned} \tag{22}$$

Equation (22) is the general form of a linear nonhomogeneous equation with coefficients 0 or 1 over the group G; it generates another method of error detection, illustrated (for the case $L(\Phi) < L(f)$) in Fig. 2. In this case detection of an error for given x requires $\max(L(f),L(\Phi))$ elementary additions (errors in the calculation of $\Phi(x)$ may also be detected by the block diagram of Fig. 2 if there are no errors in the calculation of $f(x)$). The use of the block diagram of Fig. 2 instead of that of Fig. 1 may result in a significant decrease in $L(f)$, so that nonhomogeneous checking equations may be more effective for error detection (see Example 3).

B. Using the notation of (22), we denote

$$a(q) = \begin{cases} 1, & \text{if } q \in \{0,q_1,\cdots,q_{L(f)}\}; \\ 0, & \text{otherwise;} \end{cases} \tag{23}$$

$$b(q) = \begin{cases} 1, & \text{if } q \in \{0,S_1,\cdots,S_{L(\Phi)}\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then by (12) and (23)

$$\sum_{q\in G} a(q) f(x^* q^{-1}) = \sum_{q\in G} b(q) \Phi(x^* q^{-1}) + d. \tag{24}$$

Set

$$\Omega_f = \{\omega | S_f(\omega) = 0\},\ \Omega_\Phi = \{\omega | S_\Phi(\omega) = 0\},$$

$$\Omega_{f,\Phi}(\alpha) = \left\{\omega \left| \frac{S_\Phi(\omega)}{S_f(\omega)} = \alpha \right.\right\}.$$

Theorem 2: For any two functions $f,\Phi: G \to C$, there exist $a: G \to \{0,1\}$, $b: G \to \{0,1\}$ $(a \neq 0, b \neq 0)$, and $d \in C$ such that a,b,f,Φ,d satisfy (24), and if for some α the sets $\Omega_f \cup \Omega_{f,\Phi}(\alpha) \cup \{0\}$ and $\Omega_\Phi \cup \Omega_{f,\Phi}(\alpha) \cup \{0\}$ contain subgroups $G_a(\alpha)$ and $G_b(\alpha)$ of G, of orders $g_a(\alpha)$ and $g_b(\alpha) = \alpha g_a(\alpha)$, respectively, and $G_a(\alpha) \cap \Omega_{f,\Phi}(\alpha) = G_b(\alpha) \cap \Omega_{f,\Phi}(\alpha)$, then

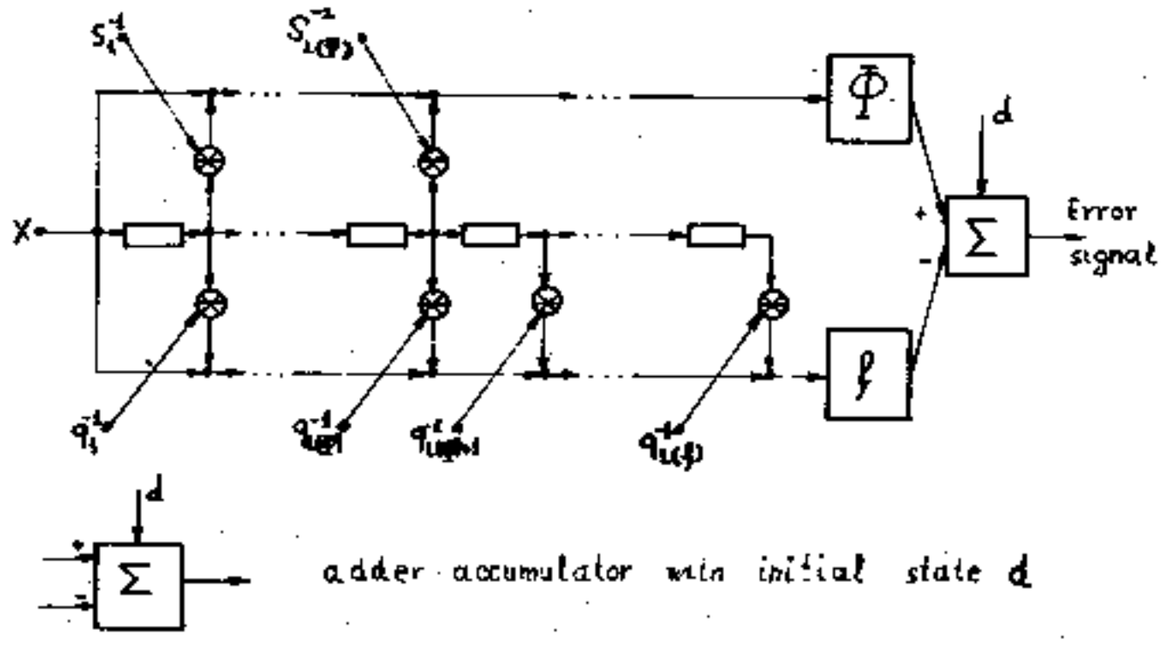


Fig. 2. Error detection by a linear nonhomogeneous equation.

$$\sum_{q\in G} a(q) + \sum_{q\in G} b(q) = \frac{g}{g_a(\alpha)}\left(1+\frac{1}{\alpha}\right). \tag{25}$$

Proof: By (24) and the convolution theorem, we have

$$S_a(\omega)S_f(\omega) = \begin{cases} S_b(0)\cdot S_\Phi(0) + \dfrac{d}{g}, & \text{if } \omega = 0; \\ S_b(\omega)\cdot S_\Phi(\omega), & \text{if } \omega \neq 0. \end{cases} \tag{26}$$

(S_a, S_b, S_f, S_Φ denote the spectra of a, b, f, Φ.)

Assume that for some α all the conditions of Theorem 2 are satisfied. Let

$$S_a(\omega) = \begin{cases} \dfrac{1}{g_a(\alpha)}, & \text{if } \omega \in G_a(\alpha); \\ 0, & \text{if } \omega \notin G_a(\alpha); \end{cases} \tag{27}$$

$$S_b(\omega) = \begin{cases} \dfrac{1}{g_b(\alpha)}, & \text{if } \omega \in G_b(\alpha); \\ 0, & \text{if } \omega \notin G_b'(a); \end{cases}$$

and

$$d = \frac{1}{g_a(\alpha)}\sum_{x\in G} f(x) - \frac{1}{g_b(\alpha)}\sum_{x\in G}\Phi(x). \tag{28}$$

We now prove that (26) (and hence also (24)) will follow from (27) and (28). We consider all possible cases.

1) Let $\omega = 0$; then by (27), (28) ●

$$S_a(0)\cdot S_f(0) = \frac{1}{g_a(\alpha)}\cdot\frac{1}{g}\sum_{x\in G} f(x)$$

$$= \frac{1}{g}\left(\frac{1}{g_b(\alpha)}\sum_{x\in G}\Phi(x) + d\right) = S_b(0)\cdot S_\Phi(0) + g^{-1}d.$$

2) Let $\omega \neq 0$ and $\omega \in G_a(\alpha) \cap G_b(\alpha)$.

a) If $\omega \in \Omega_f$, then $\omega \in \Omega_\Phi$, since for every α, $\Omega_f \cap \Omega_{f,\Phi}(\alpha) \subseteq \Omega_\Phi \cap \Omega_{f,\Phi}(\alpha) = \emptyset$ (the empty set), $S_f(\omega) = S_\Phi(\omega) = 0$ and $S_a(\omega)\cdot S_f(\omega) = S_b(\omega)\cdot S_\Phi(\omega) = 0$.

b) If $\omega \in \Omega_{f,\Phi}(\alpha)$, then $S_\Phi(\omega)/S_f(\omega) = \alpha$ and, since $g_b(\alpha) = \alpha\cdot g_a(\alpha)$, $S_a(\omega)\cdot S_f(\omega) = (1/g_a(\alpha))S_f(\omega) = (1/g_a(\alpha))(S_\Phi(\omega)/\alpha) = (1/g_b(\alpha))S_\Phi(\omega) = S_b(\omega)\cdot S_\Phi(\omega)$.

3) Let $\omega \in G_a(\alpha)$ and $\omega \notin G_b(\alpha)$. Since $G_a(\alpha)\cap\Omega_{f,\Phi}(\alpha) = G_b(\alpha)\cap\Omega_{f,\Phi}(\alpha)$, it follows that for every α $\omega \notin \Omega_{f,\Phi}(\alpha)$, so that $\omega \in \Omega_f$ and $S_f(\omega) = 0$; but if $\omega \notin G_b(\alpha)$, then $S_b(\omega) = 0$, and thus

$$S_a(\omega)\cdot S_f(\omega) = S_b(\omega)\cdot S_\Phi(\omega) = 0.$$

(The case $\omega \notin G_a(\alpha)$ and $\omega \in G_b(\alpha)$ may be treated similarly.)

4) Let $\omega \notin G_a(\alpha)$, $\omega \notin G_b(\alpha)$. Then $S_a(\omega) = S_b(\omega) = 0$ and $S_a(\omega)\cdot S_f(\omega) = S_b(\omega)\cdot S_\Phi(\omega) = 0$. Thus (26) follows from (27) and (28). Furthermore, if we consider the subgroups $G_a{}^\perp(\alpha)$ and $G_b{}^\perp(\alpha)$ as in the proof of Theorem 1 for $G_a{}^\perp$, we see that $a(q), b(q) \in \{0,1\}$ for every $q \in G$, and

$$L(f) + 1 = \sum_{q\in G} a(q) = \frac{g}{g_a(\alpha)},$$

$$L(\Phi) + 1 = \sum_{q\in G} b(q) = \frac{g}{g_b(\alpha)} \tag{29}$$

whence, using the fact that $g_1(\alpha) = \alpha\cdot g_a(\alpha)$, we have (25).

Example 3: Let f be the function defined on the dyadic group $G_2{}^3$ by Table I.

We take $\Phi(x) = x_2$ (the spectra S_f, S_Φ are shown in Table I). Then by Table I

$$\Omega_{f,\Phi}(\alpha) = \begin{cases} \{(0,0,0),(0,0,1)\}, & \text{if } \alpha = 2; \\ \emptyset, & \text{if } \alpha \neq 2; \end{cases}$$

$$\Omega_f = \{(0,1,0),(0,1,1)\};$$

$$\Omega_\Phi = \{(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\};$$

$$G_a(2) = \Omega_f \cup \Omega_{f,\Phi}(2);$$

$$G_b(2) = \Omega_\Phi \cup \Omega_{f,\Phi}(2) = G;$$

$$G_a(2)\cap\Omega_{f,\Phi}(2) = G_b(2)\cap\Omega_{f,\Phi}(2) = \Omega_{f,\Phi}(2);$$

$g_a(2) = 4$; $g_b(2) = 8$. The conditions of Theorem 2 are satisfied for $\alpha = 2$. The functions S_a, S_b, a, b are shown in Table I. Since it follows from (28) that $d = 0$, we have from Table

TABLE I

$x_3x_2x_1x_0$	x_0 x_1 x_2	$f(x)$	$S_f(w)$	$S_\phi(w)$	$S_a(w)$	$S_b(w)$	$a(q)$	$b(q)$
0	0 0 0	1	0.25	0.5	0.25	0.125	1	1
1	0 0 1	0	-0.25	-0.5	0.25	0.125	0	0
2	0 1 0	1	0	0	0.25	0.125	0	0
3	0 1 1	1	0	0	0.25	0.125	0	0
4	1 0 0	-1	0.5	0	0.	0.125	1	0
5	1 0 1	1	0.5	0	0	0.125	0	0
6	1 1 0	-1	-0.25	0	0	0.125	0	0
7	1 1 1	0	0.25	0	0	0.125	0	0

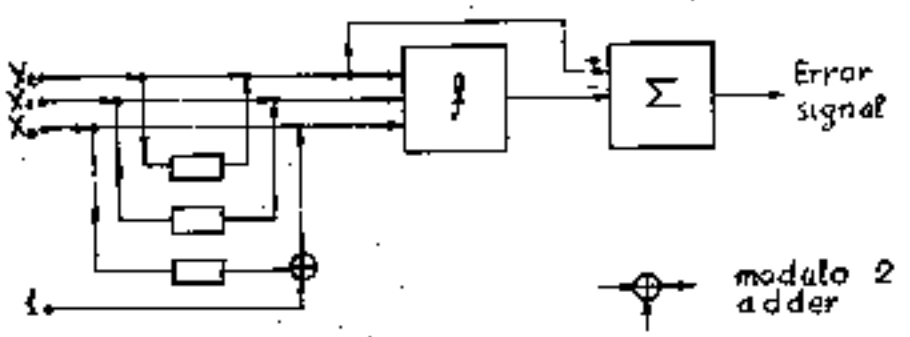
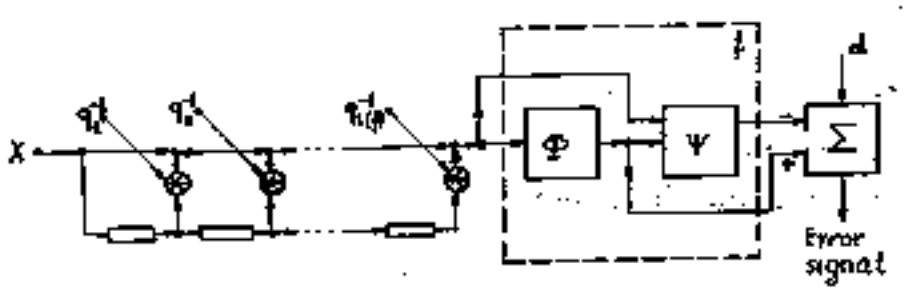


Fig. 3. Error detection for Example 3.



$$f(x) = \Psi(x,\Phi(x))$$

Fig. 4. Error detection by (24) with $a(q) = b(q)$ $(q = 0,1,\cdots,g-1)$.

I the following nonhomogeneous checking equation:

$$f(x_0,x_1,x_2) + f(x_0 \oplus 1,x_1,x_2) = x_2 \pmod 2.$$

(A block diagram of error detection for this example is shown in Fig. 3.) To find an optimal nonhomogeneous checking equation (24) for given f and Φ, i.e., an equation minimizing the complexity $\Sigma_{q\in G}a(q) + \Sigma_{q\in G}b(q)$, one must consider all α such that α or α^{-1} is a divisor of the order g of the group G.

C. We now consider the special case of the checking equation (24) in which $a(q) = b(q)$ for all $q \in G$. The block diagram Fig. 2 is now considerably simplified, and Φ may be chosen as a function realized by some subnetwork implementing f (see Fig. 4).

Corollary 2: For any two functions f,Φ: $G \to C$, there exist a: $G \to \{0,1\}$ $(a \not\equiv 0)$ and $d \in C$ such that a,f,Φ,d satisfy (24) with $a(q) = b(q)$ for all $q \in G$, and

$$\sum_{q\in G} a(q) = \frac{g}{g_a(1)} \tag{30}$$

where $g_a(1)$ is the order of an arbitrary subgroup $G_a(1)$ contained in $(\Omega_f \cap \Omega_\Phi) \cup \Omega_{f,\Phi}(1) \cup \{0\}$.

Proof: Corollary 2 follows from Theorem 2 with $\alpha = 1$ and $G_a(1) = G_b(1)$. In this case $g_a(1) = g_b(1)$ and, by (27), $S_a = S_b$, $a = b$ and (30) follows from (29).

Note that the "system redundancy method" mentioned above is a special case of error detection method generated by Corollary 2. In this case $f(x) = \Phi(x)$ for all $x \in G$, $(\Omega_f \cap \Omega_\Phi) \cup \Omega_{f,\Phi}(1) = G$, $d = 0$, $g_a(1) = g$ and $\Sigma_{q\in G}a(q) = 1$.

D. We now consider a usage of error detection methods generated by Theorems 1, 2 and Corollaries 1, 2 for finite automata. Let $M_f = (X,Q,Y,q_0,\delta,\lambda)$ be the given finite automaton ($X = \{0,1,\cdots,n_x - 1\}$ be the input alphabet; Q the set of internal states; $Y \subset C$ the output alphabet; $q_0 \in Q$ the initial state; δ: $X \times Q \to Q$ the transition function; λ: $Q \to Y$ the output function); and $x = (x_0,x_1,\cdots,x_{m-1}) \in X^m$. Set

$$f(x) = \lambda(\delta(x_{m-1},\cdots,\delta(x_1,\delta(x_0,q_0))\cdots)). \tag{31}$$

The function f now may be considered as a function defined on the group G of the all vectors $(x_0,x_1,\cdots,x_{m-1}) \in X^m$ with the respect to componentwise modulo n_x addition and all the error detection methods of Theorems 1,2 and Corollaries 1,2 may be used for f.

Example 4: Let the automaton M_f be defined by the transition diagram Fig. 5(a) and suppose that a network implementing M_f contains a subnetwork implementing automaton M_Φ defined by the transition diagram Fig. 5(b). Let us try to use M_Φ to detect errors in M_f for $m = 3$.

The functions f and Φ constructed for M_f and M_Φ by (31) and the spectra S_f and S_Φ of f and Φ are shown in Table II. Then $\Omega_f = \emptyset$, $\Omega_\Phi = \emptyset$, $\Omega_{f,\Phi}(1) = \{(0,1,1), (1,0,0),(1,1,1)\}$. Let $G_a(1) = G_b(1) = \{(0,0,0),(0,1,1), (1,0,0),(1,1,1)\}$ ($g_a(1) = g_b(1) = 4$). (The functions S_a, S_b, a and b are also shown in Table II.)

By (28), $d = 1/4\ (\Sigma_{x\in G} f(x) - \Sigma_{x\in G}\Phi(x)) = 2$; thus, by Table II, and Corollary 2 we have the checking equation:

$$f(x_0,x_1,x_2) + f(x_0,x_1 \oplus 1,x_2 \oplus 1) = \Phi(x_0,x_1,x_2) + \Phi(x_0,x_1 \oplus 1,x_2 \oplus 1) + 2 \pmod 2.$$

IV. Tests for Error Detection Networks—Recognition of the "Simplest" Functions

A. It follows from Theorems 1,2 and Corollaries 1,2 that the complexity of the error detection block diagrams (Figs. 1, 2, and 4) depends on the orders $g_a, g_a(\alpha), g_a(1)$ of the selected subgroups $G_a, G_a(\alpha), G_a(1)$. To minimize the complexity, therefore, one should choose subgroups of the maximal possible order. The measures $\Sigma_{q\in G} a(q)$ and $\Sigma_{q\in G} b(q)$ of the complexity are always divisors of the order g of the original group G.

For every block diagram of Figs. 1, 2, and 4 we can organize the fly test detection by applying some signals x to the input of the block diagram. Any such signal will be called a fly test for the corresponding block diagram. We now consider a problem of finding the minimal set of these tests.

It follows from the results of the previous sections that, for a given checking equation, the set $G_a{}^\perp$ defined by $q \in G_a{}^\perp$ iff $a(q) = 1$ is a subgroup of G. Hence, for every $x \in G$, the set $\{x, x*q_1{}^{-1}, \cdots, x*q_{L(f)}{}^{-1}\}$ generated by the test x is the coset of $G_a{}^\perp$ in G. Thus any system of distinct coset representatives of $G_a{}^\perp$ in G (and, in particular, the subgroup G_a) is a minimal set of tests.

For instance, for the block diagram of Fig. 3 (Example 3) $G_a{}^\perp = \{(0,0,0),(1,0,0)\}$ and the minimal set of tests may be chosen as $\{(0,0,0),(0,0,1),(0,1,1),(1,1,0)\}$.

The minimal number of tests is $g/\Sigma_{q\in G} a(q)$, and for a given f the product of the complexity $\Sigma_{q\in G} a(q)$ of the block diagram and the minimal number of tests is always equal to the order g of the group G (irrespective of the choice of the subgroup G_a).

B. We now consider the class of "simplest" functions f which satisfy the general equation (24) when $\Phi(x) = 0$ for all $x \in G$, $d = 0$, the number of nonzero terms on the left of (24) is two and f: $G \to R$ (where R is the set of real numbers). Then for all $x \in G$

$$f(x) + a(q) f(x*q^{-1}) = 0 \tag{32}$$

but in this case we shall assume that $a(q) \in \{-1,+1\}$. The case $a(q) = -1$ is convenient if $f(x) \geqslant 0$ (or $f(x) \leqslant 0$) for all $x \in$. (All functions of two and many-valued logic, for example, satisfy this condition.)

Our problem is, given f, to find (if possible) $q \in G$ ($q \neq 0$), and $a(q) \in \{-1,+1\}$, such that $f, q, a(q)$ satisfy (32).

In contrast to the previous treatment, the method proposed for solution of this problem will involve not spectra but correlation functions on the group G.

We first construct the following system of characteristic functions f_t: $G \to \{-1,0,1\}$ ($t > 0$):

$$f_t(x) = \begin{cases} (-1)^{\operatorname{sign} f(x)+1}, & \text{if } |f(x)| = t; \\ 0, & \text{if } |f(x)| \neq t; \end{cases}$$

$$\operatorname{sign} f(x) = \begin{cases} 1, f(x) \geqslant 0; \\ 0, f(x) < 0; \end{cases} \tag{33}$$

the corresponding system of autocorrelation functions B_{f_t} on G:

$$B_{f_t}(\tau) = \sum_{x\in G} f_t(x) f_t(x*\tau^{-1}), \tag{34}$$

and total autocorrelation function B_Σ:

$$B_\Sigma(\tau) = \sum_{t>0} B_{f_t}(\tau). \tag{35}$$

(The properties and applications of these autocorrelation functions to the analysis and synthesis of digital devices were studied in [2], [4], [5].)

Theorem 3: A function f: $G \to R$ satisfies (32) for given $q \in G$ ($q \neq 0$), $a(q) \in \{-1,+1\}$ iff

$$(-1)^{0.5(a(q)+1)} B_\Sigma(q) = B_\Sigma(0). \tag{36}$$

Proof: Since for all $x \in G$, $f_t(x) \in \{-1,0,+1\}$, we have for every $q, \tau \in G$, $a(q) \in \{-1,+1\}$

$$(-1)^{0.5(a(q)+1)} f_t(x) f_t(x*\tau^{-1}) \leqslant f_t{}^2(x). \tag{37}$$

It follows now from (34), (35), and (37), in a view of $\Sigma_{t>0}\Sigma_x f_t{}^2(x) = B_\Sigma(0)$ that (36) is satisfied iff

$$f_t{}^2(x) = (-1)^{0.5(a(q)+1)} f_t(x) f_t(x*q^{-1}) \tag{38}$$

but in a view of (33) and $(-1)^{0.5(a(q)+1)} = -a(q)$, the last condition is satisfied iff the condition (32) holds.

Thus the "simplest" functions, for which there exist $q \in G (q \neq 0)$ and $q(q) \in \{-1,+1\}$ satisfy (32), may be recognized by using autocorrelation functions on the group G.

C. Note that for given f there may exist several q and $a(q)$ satisfying (32); all of them may be found simultaneously by calculating the autocorrelation functions B_Σ.

The set of all q satisfying (32) for a given f when $a(q) = -1$ is a subgroup of G ("inertia group" for f), so that the number of q satisfying (36) for $a(q) = -1$ is always a divisor of the order g of G (this is also true for $a(g) = +1$), and this fact may be used to detect errors in the calculation of the autocorrelation function B_Σ.

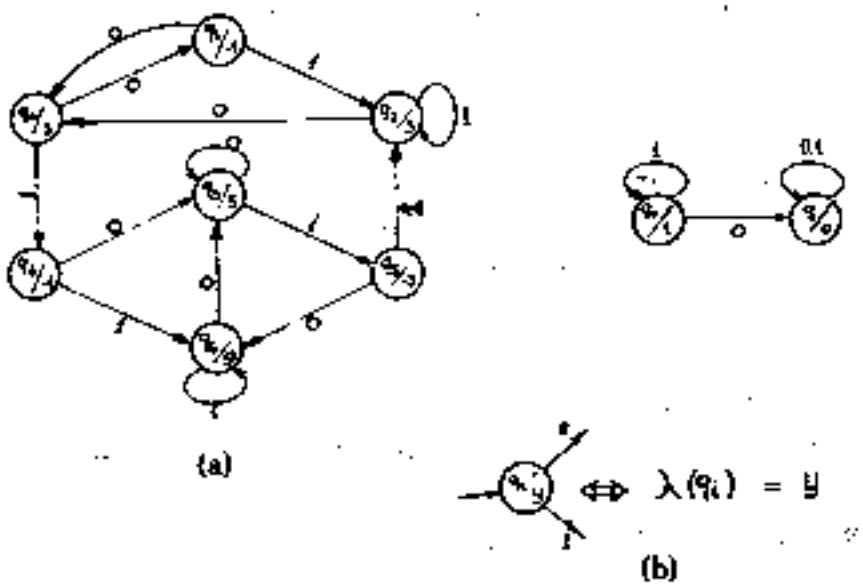


Fig. 5. (a) Transition diagram for automaton M_f from Example 4. (b) Automaton M_* from Example 4.

TABLE II

x,w;q	$x_0x_1x_2$	f(x)	φ(x)	$S_f(w)$	$S_\phi(w)$	$S_*(w)$	a(q)
0	0 0 0	-1	0	1.125	0.125	0.25	1
1	0 0 1	-1	0	1.875	-0.125	0	0
2	0 1 0	3	0	-1.125	-0.125	0	0
3	0 1 1	3	0	0.125	0.125	0.25	1
4	1 0 0	5	0	-0.125	-0.125	0.25	0
5	1 0 1	-3	0	-1.875	0.125	0	0
6	1 1 0	5	0	-0.875	0.125	0	0
7	1 1 1	-2	1	-0.125	-0.125	0.25	0

To calculate the autocorrelation functions B_{f_t} one can use (34) and the "eveness relation": $B_{f_t}(\tau) = B_{f_t}(\tau^{-1})$ for all $\tau \in G$. However, when the number of different values of f is small, it is more convenient to use the formula [2]:

$$B_{f_t} = g \cdot F_G^{-1}(S_{f_t}\bar{S}_{f_t}) \tag{39}$$

(where $\bar{S}_{f_t}$ is the complex conjugate of the spectrum S_{f_t} of f_t, F_G^{-1} the inverse Fourier transform on G, $(S_{f_t} \cdot \bar{S}_{f_t})(\omega) = S_{f_t}(\omega)\bar{S}_{f_t}(\omega)$). To calculate the spectrum and inverse Fourier transform F_G^{-1} one can use fast Fourier transform on the group G [7].

D. Theorem 3 may be simplified when f is a switching function. In this case G is the dyadic group $G_2^{2^m}, f(x) \in \{0,1\}$ for all $x \in G, f_1 = f, B_\Sigma = B_{f_1}, a0 = -1, B_\Sigma(0) = B_{f_1}(0) = \Sigma_{x \in G} f(x)$ and condition (36) may be replaced by

$$B_{f_1}(q) = B_f(q) = \sum_{x \in G} f(x). \tag{40}$$

Since characters of the dyadic group are Walsh functions one can calculate B_f by (39) and fast Walsh transform [8]. Tables of spectra and autocorrelation functions for a large number of classes of switching functions may be found in [5].

Example 5: Let $f(x_0,x_1,x_2)$ $(m = 3)$ be the switching function defined by Table III. Autocorrelation function B_f is also shown in Table III. Since $B_f(1,1,0) = \Sigma_{x \in G} f(x) = 6$, formulas (40), (32) imply the following checking equation

$$f(x_0,x_1,x_2) - f(x_0 \oplus 1, x_1 \oplus 1, x_2) = 0 \pmod 2.$$

V. Description of the Class of Errors Detected by Linear Equations Over a Group

A. Let us consider the class of errors detected by the general nonhomogeneous linear equation (24), if $a: G \to \{0,1\}$, $b: G \to \{0,1\}$, and $d \in C$ are chosen as described in the proof of Theorem 2.

We shall assume that the result of an error $e: G \to C$ is to replace the given function f by the function $f + e$.

Set $\Omega_e = \{\omega | S_e(\omega) = 0\}$ (where $S_e(\omega)$ is the spectrum of the error function e).

Theorem 4: An error $e: G \to C$ is not detected by a nonhomogeneous equation (24) iff

$$G_a(\alpha) \subseteq \Omega_e \tag{41}$$

where $\omega \in G_a(\alpha)$ iff $S_a(\omega) \neq 0$.

Proof: The error $e: G \to C$ is not detected by (24) iff, for every $x \in G$,

$$\sum_{\epsilon} a(q)(f(x*q^{-1}) + e(x*q^{-1})) = \sum_{q \in G} b(q)\phi(x*q^{-1}) + d. \tag{42}$$

It follows from (42) by (24) that

$$\sum_{q\in G} a(q)e(x^*q^{-1}) = 0 \text{ for every } x \in G \qquad (43)$$

and from (43) by the convolution theorem, that $S_a(\omega) \cdot S_e(\omega) = 0$ for all $\omega \in G$, whence follows (41).

It follows from Theorem 4 that with increase in the order $g_a(\alpha)$ of the subgroup $G_a(\alpha)$ (and hence with decrease in the complexity $\Sigma_{q\in G}a(q)$ of the checking equation (24); see, for example, (25)). the number of errors detected by (24) decreases. For an equation (6), if the set $\Omega_f \cup \{0\}(\Omega_f = \{\omega | S_f(\omega) = 0\}$ is a subgroup of G, it is convenient to take $G_a = \Omega_f$; then, by Theorem 4, an error $e: G \to C$ is not detected iff the domain of nonzero values of the spectrum S_e of the error is a subset of the domain of nonzero values of the spectrum S_f of function f.

B. The class of the most probable errors depends on the implementation of device or program calculating the given function f. We will consider two important examples to illustrate the good error detecting capability of the proposed methods.

We now illustrate the error detection capability of the above methods for the binary adder of Example 1. Block diagram of an n-bit adder is shown in Fig. 6.

We shall consider four classes of error for the adder of Fig. 6: input errors $e_{inp}(X,Y)$, output errors $e_{out}(X,Y)$, carry errors $e_c(X,Y)$, and shift errors $e_{sh}(X,Y)$.

An l-fold input (output) error, $0 < l \leqslant 2n$ $(0 < l \leqslant n+1)$, is said to occur if l binary components of $(x_0, \cdots, x_{n-1}, y_0, \cdots, y_{n-1})$ (of $(f_0(X,Y), \cdots, f_n(X,Y))$ are replaced by arbitrary binary constants (see Fig. 6).

An l-fold carry error, $0 < l \leqslant n$, occurs if l components of the vector $(C_0, \cdots, C_n)$ (see Fig. 6) are replaced by arbitrary binary constants.

An l-fold shift error is a shift by l positions to the right or left in a vector recorded in any of the three registers $X, Y, X+Y$ in Fig. 6. For a right (left) shift by l positions, the vector $(z_0, \cdots, z_{k-1})$ is replaced by

$$\underbrace{(0,\cdots,0}_{l}, z_0, \cdots, z_{k-1-l})\ (z_l, \cdots, z_{k-1}, \underbrace{0, \cdots, 0)}_{l}.$$

Shift errors are probable when the information is transferred to the X and Y registers and from the $X+Y$ register in serial form.

We denote the relative frequence of l-fold errors which are detected of the above four classes by $\eta_{inp}(n,l)$, $\eta_{out}(n,l)$, $\eta_c(n,l)$, $\eta_{sh}(n,l)$, respectively.

Corollary 3: For an n-bit binary adder:

$$\eta_{inp}(n,l) = \begin{cases} 1 - \binom{n}{s}\binom{2n}{2s}^{-1}2^{-s}, & \text{if } l = 2s; \\ 1, & \text{if } l = 2s-1; \end{cases} \quad (s = 1, \cdots, n) \qquad (44)$$

$$\eta_{out}(n,l) = 1 - \delta_{l,n+1}2^{-n-1} \cdot (\delta_{l,n+1} - \text{Kronecker symbol}) \qquad (45)$$

$$\eta_c(n,l) = \eta_{sh}(n,l) = 1, \quad \text{for all } n,l. \qquad (46)$$

Proof: Any l-fold input error may be expressed as

$$e_{inp}(X,Y) = \sum_{i=1}^{S_1} (\alpha_{p_i} - x_{p_i})2^{n-1-p_i} + \sum_{i=1}^{S_2} (\beta_{r_i} - y_{r_i})2^{n-1-r_i} \qquad (47)$$

where

$$S_1 + S_2 = l;\ 0 \leqslant p_1 < \cdots < p_{s_1} \leqslant n-1;\ 0 \leqslant r_1 < \cdots < r_{s_2} \leqslant n-1;\ \alpha_{p_i}, \beta_{r_i} \in \{0,1\}.$$

Since

$$\sum_{X,Y} (\alpha_{p_i} - x_{p_i}) = (-1)^{\alpha_{p_i}+1}2^{2n-1}, \qquad i = 1, \cdots, S_1$$

$$\sum_{X,Y} (\beta_{r_i} - y_{r_i}) = (-1)^{\beta_{r_i}+1}2^{2n-1}, \qquad i = 1, \cdots, S_2 \qquad (48)$$

it follows that

$$\sum_{X,Y} e_{inp}(X,Y) = 2^{2n-1}\left(\sum_{i=1}^{S_1}(-1)^{\alpha_{p_i}+1}2^{n-1-p_i} + \sum_{i=1}^{S_2}(-1)^{\beta_{r_i}+1}2^{n-1-r_i}\right). \qquad (49)$$

Set $E_{inp} = \{e_{inp} | S_1 = S_2 = S, p_i = r_i, \alpha_{p_i} = 1 - \beta_{r_i} (i = 1, \cdots, s)\}$. Then, if $e_{inp} \notin E_{inp}$, we see from (49) that $\Sigma_{X,Y}e_{inp}(X,Y) \neq 0$ and $0 \notin \Omega_{e_{inp}}$. Since $0 \in G_a(\alpha)$, it follows by Theorem 4 that e_{inp} is detected. If $e_{inp} \in E_{inp}$, then by (49) $\Sigma_{X,Y}e_{inp}(X,Y) = 0$ and from (14), (15), and (47) we see that $S_{e_{inp}}(\omega) = 0$ if $\Sigma_{i=0}^{2n-1}\omega_i \neq 1$. Since (see Example 1)

$$G_a(\alpha) = G_a = \left\{\omega \middle| \sum_{i=0}^{2n-1} \omega_i = 2\kappa (\kappa = 0, \cdots, n)\right\}$$

it follows that in this case $G_a \subset \Omega_{e_{inp}}$, so that by Theorem 4, e_{inp} is not detected.

Hence, in view of the fact that the total number of $2S$-fold errors is equal to $\binom{2n}{2s} \cdot 2^{2s}$, and the number of $2s$-fold errors $e_{inp} \in E_{inp}$ is $\binom{n}{s} \cdot 2^s$, we obtain (44).

Any l-fold output error may be expressed as

$$e_{out}(X,Y) = \sum_{i=1}^{l} (\alpha_{p_i} - f_{p_i}(X,Y))2^{n-p_i}, \cdot (\alpha_{p_i} \in \{0,1\}, 0 \leqslant p_1 < \cdots < p_l \leqslant n). \qquad (50)$$

Since $X + Y = \Sigma_{p=0}^{n} f_p(X,Y)2^{n-p}$ $(f_p(X,Y) \in \{0,1\})$, it is readily seen that for any $p_i \in \{0, \cdots, n\}$ and $\alpha_{p_i} \in \{0,1\}$ $\Sigma_{X,Y}f_{p_i}(X,Y) = 2^{2n-1} - \delta_{p_i,0} \cdot 2^{n-1}$ and

$$\sum_{X,Y} (\alpha_{p_i} - f_{p_i}(X,Y)) = (-1)^{\alpha_{p_i}+1}2^{2n-1} + (-1)^{\alpha_{p_i}} \cdot \delta_{p_i,0} \cdot 2^{n-1}. \qquad (51)$$

Since $0 \leqslant p_1 < \cdots < p_l \leqslant n$, it follows from (50), (51) that

$$\sum_{X,Y} e_{out}(X,Y) = 2^{2n-1}\left(\sum_{i=1}^{l}(-1)^{\alpha_{p_i}+1}2^{n-p_i} + (-1)^{\alpha_{p_i}} \cdot \delta_{p_1,0}\right). \qquad (52)$$

TABLE III

x	x_0 x_1 x_2	F	B_f
0	0 0 0	1	6
1	0 0 1	0	4
2	0 1 0	1	6
3	0 1 1	1	6
4	1 0 0	1	6
5	1 0 1	1	4
6	1 1 0	1	6
7	1 1 1	0	4

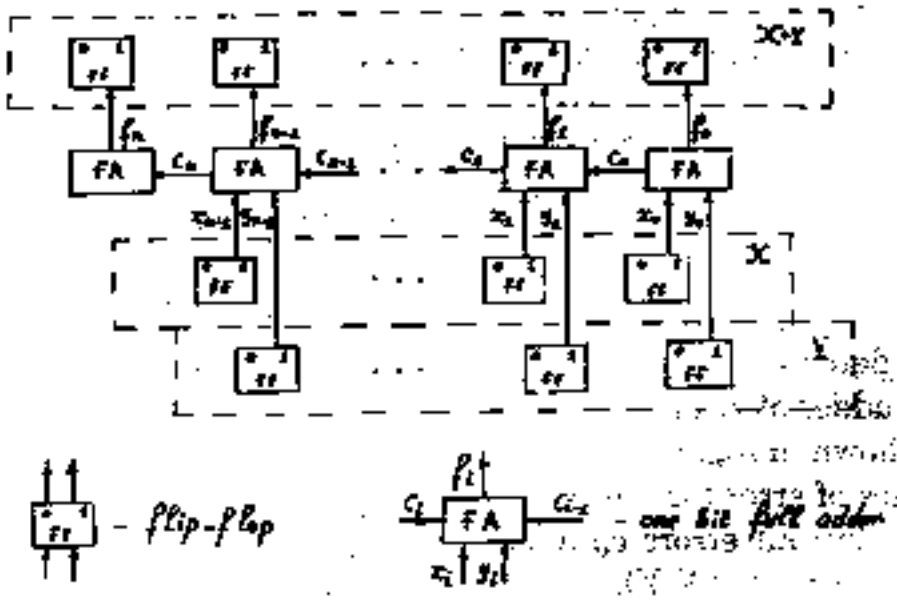


Fig. 6. An n-bit binary adder.

Denote

$$E_{out}(X,Y) = -f_0(X,Y)2^n + \sum_{p=1}^{n} (1 - f_p(X,Y))2^{n-p}. \quad (53)$$

Then, if $e_{out} \neq E_{out}$, it follows from (52) and (53) that $\Sigma_{X,Y} e_{out}(X,Y) \neq 0$, $0 \notin \Omega_{e_{out}}$, and by Theorem 4, e_{out} is detected.

If $e_{out} = E_{out}$, then $\Sigma_{X,Y} e_{out}(X,Y) = 0$, $S_{e_{out}}(0) = 0$, by (53) $S_{e_{out}}(\omega) = -S_{X+Y}(\omega)$ for $\omega \neq 0$, and by Theorem 4, e_{out} is not detected.

Thus the only undetected output error is the $(n+1)$-fold error E_{out}, and this implies (45).

Now let $e_c(X,Y) = e_c(x_0, \cdots, x_{n-1}, y_0, \cdots, y_{n-1})$ be an l-fold carry error for which $C_{p_i} = \alpha_{p_i}$ (see Fig. 6), where the α_{p_i} are certain binary constants ($i = 1, \cdots, l$; $0 \leq p_1 < \cdots < p_l \leq n-1$). Then we have

$$e_c(\underbrace{0, \cdots, 0}_{2n}) = \sum_{i=1}^{l} \alpha_{p_i} 2^{\alpha p_i + 1}, \quad e_c(\underbrace{1, \cdots, 1}_{2n}) = \sum_{i=1}^{l} (\alpha_{p_i} - 1)2^{\alpha p_i + 1}$$

and

$$e_c(\underbrace{0, \cdots, 0}_{2n}) + e_c(\underbrace{1, \cdots, 1}_{2n}) = \sum_{i=1}^{l} (2\alpha_{p_i} - 1)2^{\alpha p_i + 1} \neq 0 \quad (54)$$

for any $\alpha_{p_i} \in \{0,1\}$ $(i = 1, \cdots, l)$. Now, for an adder (see (18)) we have

$$a(q) = 1 \text{ iff } q = \underbrace{(0, \cdots, 0)}_{2n} \text{ or } q = \underbrace{(1, \cdots, 1)}_{2n}$$

and so it follows from (43) (54) that $\eta_c(n,l) = 1$ for all l.

Similarly, for an arbitrary shift error $e_{sh}(X,Y) = e_{sh}(x_0, \cdots, x_{n-1}, y_0, \cdots, y_{n-1})$ we have

$$e_{sh}(\underbrace{0, \cdots, 0}_{2n}) = 0, e_{sh}(\underbrace{1, \cdots, 1}_{2n}) < 0$$

and $\eta_{sh}(n,l) = 1$ for all $l \in \{1, \cdots, n\}$.

C. Another important example illustrating the good error detecting capability of our methods is the transfer of information from the computer memory. Let the information be transferred from the memory by blocks consisting of $P_0 \cdot P_1$ words ($P_0, P_1 > 1$) (e.g., we have P_0 independent memory devices and the information is transferred from every device by blocks consisting of P_1 words). We consider the every block as a $P_0 \times P_1$ matrix with the elements $f(x_0,x_1)$, where f is the function defined on the group G of the all vectors $x = (x_0,x_1)$ ($x_0 \in \{0, \cdots, P_0 - 1\}$; $x_1 \in \{0, \cdots, P_1 - 1\}$) and G is a direct product of two cyclic groups of the orders P_0 and P_1.

The checking equation (2) for f (construeted by the method of Section II) generates the error detection method for errors arising in the process of storage and transfer of information from the computer memory. The method may be easily implemented by the simple program for checking (2) for some x. (These x may be chosen as a test for block diagram of Fig. 1 for the function f (see Section IV.)

In this case its natural to understand by an l-fold error ($l \in \{1, \cdots, P_0 \cdot P_1\}$) an error e distorting l words in the block, i.e., is the number of x such that $e(x) \neq 0$.

From Theorem 4 if $\Sigma_{x \in G} e(x) \neq 0$ then e is detected by (2). Hence, if each word is the n-bit binary number then we have for the relative frequence $\eta(n,l)$ of detected l-fold errors: $\eta(n,1) = 1$, $\eta(n,2) \geq 1 - (2^n - 1)^{-1}$ and so on. For assymetric errors (i.e., for errors such that $e(x) \geq 0$ (or $e(x) \leq 0$) for all xG) $\eta(n,l) = 1$ for all $l \in \{1, \cdots, P_0 \cdot P_1\}$.

D. In order to look for solutions $a(q)$ of the checking equation (1), we used methods of abstract harmonic analysis on finite commutative groups. When this is done, the complexity of Figs. 1, 2, and 4 depends on the orders of the selected subgroups G_a, $G_a(\alpha)$, $G_a(1)$ of G contained in the sets $\Omega_f \cup \{0\}$, $\Omega_f \cup \Omega_{f,\Phi} \cup \{0\}$, $(\Omega_f \cap \Omega_\Phi) \cup \Omega_{f,\Phi}(1) \cup \{0\}$ (see Theorems 1, 2 and Corollary 2). Generally speaking, therefore, as far as the complexity of the checking block diagrams is conserned, the most suitable functions for this method are functions whose generalized Fourier spectrum contains sufficiently many zeros.

Another limitation on the use of our methods is implied by the fact that they yield only solutions $a(q)$ of (1) for which $\{q \mid a(q) = 1\}$ is a subgroup of the original group G.

The main advantage of the methods proposed for solving (1) lies in their simplicity and convenience from the computational point of view. Thus if the initial functions $f: G \rightarrow C$ are defined analytically, the solution $a(q)$ may often be found analytically too (see Examples 1,2). Note that in the binary case (when G is a dyadic group) the solution $a(q)$ may be determined with the help of the tables in [5], which list the Walsh spectra and autocorrelation functions for a large number of important classes of Boolean functions.

If the function $f: G \rightarrow C$ is specified in tabular form, the solution $a(q)$ may be sought by employing the very effective fast Fourier transform algorithm on G [7], [8].

Another advantage of the methods is their weak dependence on the original group G, so that one obtains a unified set of error detection methods for devices that operate in binary and q-ary ($q > 2$) systems, in systems using residue classes, and so on.

REFERENCES

[1] M. G. Karpovsky and E. S. Moskalev, "Realization of a system of logical functions by means of an expansion in ortholgonal series," *Automat. Remote Contr.*, vol. 28, pp. 1921–1932, 1967 (translated from *Automatika i Telemekhanika*, pp. 119–129, 1962 (Russian)).
[2] ——, "Utilization of autocorrelation characteristics for realization of systems of logical functions," *Automat. Remote Contr.*, vol. 31, pp. 243–250, 1970 (translated from *Automatika i Telemekhanika*, pp. 83–90, 1970 (Russian)).
[3] R. Y. Lechner, "Harmonic analysis of switching functions," in: *Recent Development in Switching Theory*, A. Makhopadhyay, Ed. New York: Academic, 1971.
[4] M. G. Karpovsky and E. S. Moskalev, *Spectral Methods for Analysis and Synthesis of Digital Devices*. Leningrad: Energia, 1973 (Russian).
[5] M. G. Karpovsky, *Finite Orthogonal Series in the Design of Digital Devices*. New York: Wiley, 1976.
[6] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, vol. I. New York: Springer, 1963.
[7] G. Apple and P. Wintz, "Calculation of Fourier transforms on finite Abelian groups," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 233–236, Mar. 1970.
[8] H. C. Andrews and K. L. Caspari, "A generalized technique for spectral analysis," *IEEE Trans. Comput.*, vol. C-19, pp. 16–25, Jan. 1970.

Mark G. Karpovsky was born in Leningrad, USSR, on October 27, 1940. He received the B.S. and M.S. degrees in computer science and the Ph.D. degree in theoretical cybernetics from the Leningrad Electrotechnical Institute, Leningrad, USSR, in 1961, 1963, and 1967, respectively.

Since March 1974 he has been Senior Lecturer at the Department of Mathematics, Tel-Aviv University, Tel-Aviv, Israel. His current research interests are in synthesis fault-tolerant networks, analysis and simulation biological systems, test diagnosis of logical devices, application of abstract harmonic analysis in analysis, and synthesis and optimization of logical networks.

Dr. Karpovsky is a member of the Israel Mathematical Society.