METHODS FOR ANALYZING THE CORRECTING POWER OF AUTOMATA

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Methods are described for analyzing the correcting power $\eta(f)$ of abstract automata and automata with arbitrary structural alphabets. Methods of evaluating $\eta(f)$ are considered for various ways of specifying automata and for various classes of errors.

We are given the automaton $M = (X, Y, Q, \delta, \lambda)$, where $\delta: X\times Q \to Q$ is the transition function; $\lambda: X\times Q \to Y$ is the output function; X, Y, Q are the input, output, and state alphabets, respectively; let f be the "input-output" mapping realized by this automaton. We shall always assume that M is the minimal automaton and that Y = Q. By an error in M we shall mean an ordered pair (u, u'). The automaton corrects the set of errors R if and only if for any (u, u') $(u, u' \in R)$,

$$f(u) = f(u'). \tag{1}$$

By the correcting power f on the set R we shall mean the number $\eta_R(f)$ of errors corrected. Here we consider evaluation of $\eta_R(f)$ for a specified automaton. We let $|X| = n_x$ represent the power of X, $|Q| = n_q$ the power of Q, and $J_m(X)$ the set of input words whose length does not exceed m. We shall assume that the empty word also belongs to $J_m(X)$, while the mapping $f = f_M$ is defined for M in the following manner:

$$f_M = \lambda(\delta(x_m, \delta(x_{m-1}, \ldots, \delta(x_1, q_0))\ldots)). \tag{2}$$

Here $x = x_1x_2\ldots x_m \in J_m(X)$ $(x_i \in X)$, and q_0 is the initial state of M.

Let $\{J_m^{(i)}(X)\}$ $(i = 1,2,\ldots,N_m)$ be the factor-set of $J_m(X)$ with respect to Myhill equivalence [1] and let $|J_m^{(i)}(X)|$ be the power of the class $J_m^{(i)}(X)$.

Theorem 1. For any m > 0,

$$\eta_{J_m^2(X)}(f_M) = \sum_{i=1}^{N_m} |J_m^{(i)}|^2.$$

The proof of Theorem 1 consists in the following: a necessary and sufficient condition for correction of the error (x, x') $(x, x' \in J_m(X))$ in M is that the words x and x' be Myhill equivalent (with respect to the mapping f_M).

Theorem 1 is not always convenient for calculations of $\eta_{J_m^2(X)}(f_M)$, so that we shall give upper and lower bounds for it.

Theorem 2. Let N be the power of the factor-semigroup of the free semigroup J(X) with respect to the semigroup of words over the alphabet X that are Myhill-equivalent to the empty word. Then for any m,

$$\frac{n_x^{2m+2} - 2n_x^{m+1} + 1}{N(n_x^2 - 2n_x + 1)} \leqslant \eta_{J_m^2(X)}(f_M) \leqslant \frac{n_x^{2m+2} - 2n_x^{m+1} + 1}{n_x^2 - 2n_x + 1}.$$

Proof. Since $\sum_{i=1}^{N_m} |J_m^{(i)}| = |J_m(X)|$ for any m, then

$$N_m\left(\frac{|J_m(X)|}{\lambda_m}\right)^2 \leqslant \sum_{i=1}^{N_m} |J_m^{(i)}(X)|^2 \leqslant |J_m(X)|^2. \tag{3}$$

Moreover, $J_m(X)=J_{m-1}(X)\cup X^m$, so that $|J_m(X)|=|J_{m-1}(X)|+n_x^m$ and, consequently,

$$J_m(X)=\frac{n_x^{m+1}-1}{n_x-1}. \tag{4}$$

From the definition of N and N_m, for any m,

$$N \geqslant N_m. \tag{5}$$

Theorem 2 follows from (3)-(5) and Theorem 1.

We note that for the power N of the factor-semigroup $J_m(X)$ with respect to the sub-semigroup of words Myhill-equivalent to the empty word, we have the following estimates [1]:

$$n_q \leqslant N \leqslant n_q^{n_q}. \tag{6}$$

Let us now look at another method of determining $\eta_R(f)$. Assume the automaton M is given. We let $\tilde{J}_m(X)$ be the set of input words of length m and assume that $f = f_M$ is defined as follows:

$$f_M(x)=\lambda(\delta(x_m, \delta(x_{m-1}, \ldots, \delta(x_1, q_0))\ldots)), \tag{7}$$

where $x=x_1x_2\ldots x_m\in\tilde{J}_m(X);\ x_j\in X,\ q_0\in Q.$

For the mapping f_M we construct the system of characteristic functions $\{f_i\}$ (i = = 0,1,...,n_q - 1):

$$f_i(x)=\begin{cases}1 & \text{for } f_M(x)=q_i;\\ 0 & \text{in all other cases}\end{cases} \tag{8}$$

Then from the definition of the correcting power $\eta_R(f_M)$ of (8) it follows that

$$\eta_R(f_M)=\sum_{i=0}^{n_q-1}\ \sum_{(x,x')\in\tilde{J}_m^2(X)} f_i(x)\cdot f_i(x'). \tag{9}$$

Let us consider methods of determining $\eta_R(f)$ when the automaton M is specified by a transition table. To be specific, as before we shall assume that Q = Y. By an error in the input signal (state) for M we shall mean the ordered pair $\gamma_x = (x_i, x_j)$, where $x_i, x_j\in X$ $(\gamma_q=(q_i, q_j);\ q_i, q_j\in Q)$. The automaton M corrects the set of errors $\Gamma_x\subseteq X^2$ $(\Gamma_q\subseteq Q^2)$ if and only if for any $(x_i, x_j)\in\Gamma_x$ $((q_i, q_j)\in\Gamma_q)$

$$\delta(x_i, q)=\delta(x_j, q);\quad (\delta(x, q_i)=\delta(x, q_j)).$$

For the given automaton M we construct the system of characteristic functions $\{f_i\}$ in the following manner (i = 0,1,...,n_q - 1):

$$f_i(x, q)=\begin{cases}1 & \text{for } \delta(x, q)=q_i;\\ 0 & \text{in all other cases}\end{cases} \tag{10}$$

In accordance with (9), the correcting power of M with respect to errors in the input signals (states) alone is determined by the relationship

$$\eta_{\Gamma_x}(\delta)=\sum_{i=0}^{n_q-1}\ \sum_{\gamma_x\in\Gamma_x} f_i(x, q)\cdot f_i(x', q). \tag{11}$$

$$\left(\eta_{\Gamma_q}(\delta)=\sum_{i=0}^{n_q-1}\ \sum_{\gamma_q\in\Gamma_q} f_i(x, q)\cdot f_i(x, q')\right). \tag{12}$$

$$\eta_{\Gamma_q \times \Gamma_x}(\delta) = \sum_{i=0}^{n_q-1} \sum_{\gamma_x, \gamma_q} f_i(x, q) \cdot f_i(x', q'). \qquad (13)$$

Let us look at one more method of determining $\eta_R(\delta)$ for the case in which the error set R consists of all possible errors in the input signals (states), while the states (input signals) are error-free. In this case,

$$\Gamma_x = X^2 \times Q; \ (\Gamma_q = Q^2 \times X).$$

We let $H_x(i,j)$ represent the number of states q_i in the j-th row and let $H_q(i,j)$ represent the number of states q_j in the i-th column of the transition table.

Theorem 3.

$$\eta_{\Gamma_x}(\delta) = \sum_{i=0}^{n_q-1} \sum_{q=0}^{n_q-1} H_q(i,j) \cdot (H_q(i,j) - 1). \qquad (14)$$

$$\left(\eta_{\Gamma_q}(\delta) = \sum_{i=0}^{n_q-1} \sum_{x=0}^{n_x-1} H_x(i,j) \cdot (H_x(i,j) - 1) \right). \qquad (15)$$

The proof of the theorem follows from the fact that for a fixed state q_i the error (x_m, x_k) is corrected if and only if the elements $\delta_{m,i}$ and $\delta_{n,i}$ of the (δ_{ji}) transition table coincide.

We assume that f_M for automaton M is determined by (7), and that in this case the set of input words $\tilde{J}_m(X)$ forms a commutative group G_m in terms of whose group operation it is possible to describe the classes of errors on the set $\tilde{J}_m(X)$. We let * represent the group operation for G_m and assume that the class R of errors considered is defined as follows:

$$R = \{(x, x') \mid x, x' \in G_m;\ x * (x') \in \Gamma_x\}. \ (\Gamma_x \subseteq G_m^2). \qquad (16)$$

As we shall show later, most naturally defined classes of errors (for example, errors of a specified multiplicity) satisfy this condition. Then from (3) we have:

Theorem 4.

$$\eta_{G_m^2}(f_M) = \sum_{\gamma \in \Gamma_x} \sum_{i=0}^{n_q-1} B_{*,i}(\gamma), \qquad (17)$$

where

$$B_{*,i}(\gamma) = \sum_{x \in G_m} f_i(x) f_i(x * \gamma^{-1}). \qquad (18)$$

We shall refer to the function $B_{*,i}(\gamma)$ as the autocorrelation function of the characteristic function $f_i(x)$ on the group G_m.

The discrete functional transformation that matches the initial function f_i with its autocorrelation function $B_{*,i}(\gamma)$ is a functional transformation of the convolution type [3] on group G_m. Thus, to simplify the construction of $B_{*,i}(\gamma)$ in certain cases we find it useful to use the relationship between the initial function f_i and the autocorrelation function $B_{*,i}(\gamma)$ in terms of the double spectral transformation of f_i. This resembles the relationship between a discrete lattice function and its ordinary autocorrelation function in terms of the double discrete Laplace transform [4].

We let X represent the spectral transformation over G_m that matches a function specified on G_m to the sequence of coefficients in its expansion (Fourier coefficients) for the characters of G_m [2]. We let X^{-1} and $\bar{X}$, respectively, represent the functional transformations that are the inverse and the complex conjugate of X. We note that if the original of the transformation is defined on G_m, then we can also assume the image of these transformations to be a function defined on G_m. Since the characters are complex-valued functions, the image $X(f)$ of the mapping f is also a complex-valued function.

Theorem 5 [3].

$$B_{*,l}(\gamma)=|G_m|^2 X^{-1}(X(f_l)\cdot\overline{X(f_l)})(\gamma). \quad (19)$$

Theorem 6. Assume we are given two automata M and M_α, realizing, respectively, the mappings f_M and f_{M_α} over the same group G_m; here $\alpha\in G_m$ and for any $x\in G_m$,

$$f_{M_\alpha}(x)=f_M(x*\alpha), \quad (20)$$

and then

$$\eta_{G_m^l}(f_M)=\eta_{G_m^l}(f_{M_\alpha}). \quad (21)$$

Proof. It follows from (18) and (20) that $B_{*,1}(\gamma)=\sum_{x\in G_m} f_l(\alpha,x)\, f_l(\alpha,x,\gamma^{-1})$, $(\gamma\in G^{m^l})$; hence, when we allow for (17), we see that (21) is valid.

Theorem 6 demonstrates the invariance of the correcting power of the automaton under a shift over G_m.

Let us assume that M is specified by the transition function $\delta(x,q)$; its input signals (states) form the group G with respect to the operation $*(\boxtimes)$, and the class of errors defined as

$$\Gamma_x=\{(x,x')\,|\,x,x'\in X;\ \ x*(x')^{-1}\in X\}, \quad (22)$$

$$\Gamma_q=\{(q,q')\,|\,q,q'\in Q;\ \ q\boxtimes(q')^{-1}\in Q\}. \quad (23)$$

Then it follows from (11), (12) that the correcting power of automaton M with respect to errors in the input signals (states) alone is determined by the expression

$$\eta_{\Gamma_x}(\delta)=\sum_{i=0}^{n_q-1}\sum_{\gamma_x\in\Gamma_x} B_{*,i}(\gamma_x), \quad (24)$$

where

$$B_{*,i}(\gamma_x)=\sum_{x,q} f_i(x,q)\cdot f_i(x*\gamma_x^{-1}q), \quad (25)$$

$$\eta_{\Gamma_q}(\delta)=\sum_{i=0}^{n_q-1}\sum_{\gamma_q\in\Gamma_q} B_{\boxtimes,i}(\gamma_q). \quad (26)$$

Here

$$B_{\boxtimes,i}(\gamma_q)=\sum_{x,q} f_i(x,q)\cdot f_i(x,q\boxtimes\gamma_q^{-1}). \quad (27)$$

When there are simultaneous errors in the input signals and the states,

$$\eta_{\Gamma_q\times\Gamma_x}(\delta)=\sum_{i=0}^{n_q-1}\sum_{\gamma_x,\gamma_q} B_{*,\boxtimes,i}(\gamma_x,\gamma_q), \quad (28)$$

where

$$B_{*,\boxtimes,i}(\gamma_x,\gamma_q)=\sum_{x,q} f_i(x,q)\cdot f_i(x*\gamma_x^{-1},q\boxtimes\gamma_q^{-1}). \quad (29)$$

Turning to evaluation of the correcting power of automata with structural input and internal alphabets, we assume that the input signals and the states of automaton M are coded by p-ary vectors of corresponding length.

Let us assume that the mapping realized by M is defined by formula (7); we first con- ... of determining the correcting power with respect to l-tuple errors (both arithmetic and nonarithmetic) in input words of length m for an automaton with k in[put] channels.

For any $x \in \tilde{I}_m(X)$, where $x = (x_1, x_2, \ldots, x_m)$ $(x_i \in X)$, we construct the $k \times m$ matrix $\tilde{x}$ [s]
the i-th column of $\tilde{x}$ forms a code set for the letter x_i.

By a nonarithmetic l-tuple error, we mean an error (x, x') for which

$$\|\tilde{x} \ominus \tilde{x}'\| = l, \quad x, x' \in X^m, \tag{30}$$

where the symbol $\ominus$ (mod p) indicates the component-by-component difference of the matri[ces] modulo-p, while $\|\tilde{a}\|$ is the number of nonzero components of the matrix $\tilde{a}$. We let $\eta_{l,m}{}^{(p)}($ represent the number of corrected l-tuple errors determined by (30), and let $A_{k,m}$ re[pre]sent the class of all p-ary $k \times m$ matrices. Since the set of automaton input words f[orms] a commutative group with operation $\ominus$ (mod p), while the error class satisfies condition it follows from (17) that

$$\eta_{l,m}{}^{(p)}(f_M) = \sum_{\|\tilde{\gamma}\| = l} \sum_{i=0}^{n_s - 1} B_{\ominus_{il}}(\tilde{\gamma}), \quad \tilde{\gamma} \in A_{k,m}, \tag{31}$$

where

$$B_{\ominus_{il}}(\tilde{\gamma}) = \sum_{\tilde{x} \in A_{k,m}} f_i(\tilde{x}) \cdot f_i(\tilde{x} \ominus \tilde{\gamma}) \pmod{p}. \tag{32}$$

Let us write the class of arithmetic l-tuple errors. This class is divided into subclasses: a) parallel l-tuple arithmetic errors; b) serial l-tuple arithmetic error[s].

A description of the input-word errors in terms of parallel (serial) arithmetic [er]rors is desirable when the input signals are delivered to the given automaton by a si[ngle] arithmetic device in parallel code (from arithmetic devices in serial codes).

Let us determine these subclasses formally. For the word $x \in X^m$ we construct the k matrix $\tilde{x}$. From $\tilde{x}$ we form the two vectors x^{I} and x^{II}. The length of x^{I} is m, while it components are numbers whose p-ary expansions form the columns of x. The length of x^{I}, k, while its components are numbers whose p-ary expansions form the rows of $\tilde{x}$. The ve[ctor] x^{I} is used to determine parallel arithmetic errors, and x^{II} to determine serial errors. As the weight of x^{I} and x^{II} ($\|x^{I}\|$ and $\|x^{II}\|$, respectively) we use the sum of the arithmetic we[ights] of the vectors x^{I} and x^{II}. By an l-tuple parallel arithmetic error we mean an error (x, x') $(x, x' \in X^m)$ for which

$$\|x^{I} \ominus (x')^{I}\| = l \pmod{p^k}. \tag{33}$$

Similarly, an l-tuple serial arithmetic error (x, x') $(x, x' \in X^m)$ is defined by the relationship

$$\|x^{II} \ominus (x')^{II}\| = l \pmod{p^m}. \tag{34}$$

To evaluate the correcting power of an automaton for arithmetic errors in input w[ords] we employ (29), where the operation $\ominus$ (mod p) is replaced by $\ominus$ (mod p^k) or $\ominus$ (mod p^m), depending o[n] the error subclass.

Let us consider correction of errors in internal automaton states. As errors in internal states we can consider both nonarithmetic and arithmetic errors of multiplicity determined in analogy with errors in the input words (see (30), (33), (34)).

It is not difficult to see that (26) is valid for analysis of the correcting power with respect to errors in the state vector. Depending on the class of errors, it is nec[]essary to make use of the appropriate group operation. Relationship (28) is valid for [er]rors both in input signals and in states, where the group operations employed: $\ominus$ (mod p), $\ominus$ (mod p^k), $\ominus$ (mod p^m) can differ in pairs for differences in the nature of errors in structur[al] input signals and states.

Clearly, for automata with structural p-ary alphabets and errors of multiplicity l, all the relationships (16)-(29) hold. In particular, for nonarithmetic errors, relationship (19) of Theorem 5 reduces to the following form: $B_{\Theta_l}(\gamma) = p^{2k}(X^{(p)})^{-1}(X^{(p)}(f_i)\overline{X^{(p)}(f_i)})(\gamma) \pmod{p}$, where $X^{(p)}$ is the Christenson transform [4].

When p = 2, for nonarithmetic errors $B_{\Theta_l}(\gamma) = 2^{2k}W(W^2(f_i))(\gamma) \pmod{2}$, where W is the Walsh transform [3,4].

When Theorem 5 is employed, it is possible to calculate $B_{*,i}(\gamma)$ (and, consequently, $\eta_R(f)$) with the aid of the "fast Fourier transform" [4], which simplifies computer realization of the algorithm for evaluating $\eta_R(f)$.

The results given are easily generalized to nonminimal automata, and also for the case in which $n_y = |Y| \neq |Q|$. In the first case, if $x = \{x_1, x_2, \ldots, x_{n_x}\}$ is the partitioning of the automaton state set into equivalence classes, we let

$$f_i(x, q) = \begin{cases} 1 & \text{for } \delta(x, q) = q_i, \quad i = 0, 1, \ldots, n_x - 1; \\ 0 & \text{in all other cases}. \end{cases}$$

When $|Y| \neq |Q|$, however, it is necessary to treat M as a Moore automaton, which can always be done [5].

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