

REALIZATION OF PARTIALLY-DEFINED BOOLEAN FUNCTIONS BY EXPANSIONS IN ORTHOGONAL SERIES

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A method is proposed for the synthesis of circuits realizing partially-defined Boolean functions and functions of p -valued logic (p a prime number), by means of expansion in orthogonal series. The problems of finding the optimal completion of the given function and finding the optimal linear transformation of arguments, for which the number of nonzero coefficients of the corresponding series is minimized, are solved.

1. The problem of realizing partially-defined Boolean functions is one of the most complicated problems arising in the synthesis of switching devices.

In the present work methods are used for the synthesis of partially-defined functions, using methods based on the expansion of these functions in orthogonal series. Such methods were considered for the realization of completely-defined functions in [1, 2].

Methods based on expansion in orthogonal series can sometimes be more effective than classical methods. The basic advantages of methods based on expansion in orthogonal series are connected with the circumstances that

- in the use of these methods the structure of the device is independent of the functions realized and is given in advance; minimization of complexity reduces to minimization of the number of coefficients in the series;
- the problem of minimization, both in the case of realization of one function and in the case of the joint realization of a system can be solved by analytical methods, practically completely excluding enumeration;
- the methods of constructing and minimizing devices realizing Boolean functions can easily be generalized to the case of functions of p -valued logic (the only essential difference in this case is that the basic functions now longer take on two real values, but certain p complex values).

Aside from this, it must be noted that the analysis of spectral and autocorrelation characteristics, obtained by expansion in orthogonal series of systems of Boolean functions allows a number of problems, arising in the synthesis of circuits by classical means [2], to be very simply solved.

2. Let there be given a partially-defined Boolean function of m arguments. Let us fix the methods of completing its definition and the numbering of the arguments x_0, x_1, \dots, x_{m-1} . We put $x = \sum_{j=1}^{m-1} x_j 2^{m-1-j}$ and repre-

sent the given function $y = \varphi(x_0, x_1, \dots, x_{m-1})$ in the form $y = \varphi(x/2^m)$. The function $y = \varphi(x/2^m)$ is given at binary-rational points $0, 2^{-m}, 2 \cdot 2^{-m}, \dots, (2^m - 1)2^{-m}$ of the segment $[0, 1)$. We complete $\varphi(x/2^m)$ on $[0, 1)$ to piecewise-constant in the following way:

$$\Phi(x/2^m) = \varphi(s/2^m), \quad s \leq x < x + 1 \quad (s = 0, 1, \dots, 2^m - 1). \quad (1)$$

Then $\Phi(x/2^m)$ belongs to the space L^2 [3] and may be represented in the form

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where $\{\psi_j(x/2^m)\}$ is a complete system of orthogonal functions.

Representation (2) gives the possibility of realizing $\Phi(x/2^m)$ by means of a) a generator of basis functions $\{\psi_j(x/2^m)\}$; b) a memory block for storage of the coefficients C_j ; c) a multiplication and a summation device. In [1] it was pointed out that it is expedient to take a system of Haar functions as the basis function system for the realization of $\Phi(x/2^m)$.

The nonnormalized complete orthogonal Haar system $\{H_l^{(j)}(x/2^m)\}$ is defined on $[0, 1]$ in the following way:
 $H_0^{(0)}(x/2^m) \equiv 1$,

$$H_l^{(j)}\left(\frac{x}{2^m}\right) = \begin{cases} 1 & \text{for } \frac{x}{2^m} \in \left[\frac{2j-2}{2^{l+1}}, \frac{2j-1}{2^{l+1}}\right), \\ -1 & \text{for } \frac{x}{2^m} \in \left[\frac{2j-1}{2^{l+1}}, \frac{2j}{2^{l+1}}\right), \\ 0 & \text{at other points of the segment } [0,1) \\ & (l = 0, 1, \dots, m-1; j = 1, 2, \dots, 2^l). \end{cases} \quad (3)$$

Using the Haar functions as the basis system, the number of nonzero coefficients of series (2) does not exceed 2^m for any $\Phi(x/2^m)$, and at the same time

$$\Phi(x/2^m) = C_0^{(0)} H_0^{(0)}(x/2^m) + \sum_{l=0}^{m-1} \sum_{j=1}^{2^l} C_l^{(j)} H_l^{(j)}(x/2^m). \quad (4)$$

Since $H_l^{(j)}(x/2^m) \in \{-1, 0, 1\}$ for any l, j, x , there is no need for the multiplier in the realization of $\Phi(x/2^m)$ by formula (4), and the generator $H_l^{(j)}(x/2^m)$ is realized very simply. From (3) it follows that for any fixed $x/2^m \in [0, 1)$ there exist exactly $m+1$ basis functions such that $H_l^{(j)}(x/2^m) \neq 0$. Thus, if the terms of series (4) are summed by means of an accumulator, then for any x the calculation of $\Phi(x/2^m)$ requires not more than $m+1$ cycles.

The coefficients of $C_l^{(j)}$ are calculated by the formula

$$C_l^{(j)} = \frac{1}{2^{m-l}} \sum_{x=0}^{2^m-l} \Phi\left(\frac{x}{2^m}\right) H_l^{(j)}\left(\frac{x}{2^m}\right) \quad (l=0, 1, \dots, m-1; j=1, 2, \dots, 2^l). \quad (5)$$

Here, if L_l is the number of coefficients $C_l^{(j)}$ ($j=1, 2, \dots, 2^l$), then $0 \leq L_l \leq 2^l$. In particular, from (3) and (5) we have

$$C_{m-1}^{(j)} = \frac{1}{2} \left(\Phi\left(\frac{2j-2}{2^m}\right) - \Phi\left(\frac{2j-1}{2^m}\right) \right) \quad (j=1, 2, \dots, 2^{m-1}) \quad (6)$$

and $0 \leq L_{m-1} \leq 2^{m-1}$.

According to (4) the complexity of realizing $\Phi(x/2^m)$ is basically determined by the complexity of the memory block, which it is natural to assume to be a monotonically increasing function of $\sum_{l=0}^{m-1} L_l$. (This is connected

with the exponential dependence of the memory block complexity on m for almost all functions, while the complexities of the other blocks depend linearly on m).

The methods of completing the function and of numbering the arguments substantially influence $\Phi(x/2^m)$ and, consequently, the number of coefficients of the corresponding series. In this connection it is useful to consider

minimized.

3. We assume that the method σ of numbering the arguments is fixed, and we shall find the optimal completion of the given function. Since $0 \leq L_l \leq 2^l$, we shall seek the completion in the following way. We first find the class K_{m-1} of completions for which minimum L_{m-1} is attained, then in K_{m-1} we determine the subclass $K_{m-2} \subseteq K_{m-1}$ of completions, in which minimum L_{m-2} is attained over all completions in K_{m-1} , etc. In this an absolute minimum is reached for L_{m-1} and certain relative minima for $L_{m-2}, L_{m-3}, \dots, L_1$.

The class K_{m-1} is constructed in the following way. If φ_σ is the initial partially-defined function for a given method σ of numbering its arguments, then K_{m-1} is composed of all functions obtained by the completions of φ_σ for which, if $R_\sigma^{(m-1)} \in K_{m-1}$, then the number of vectors $(x_0, x_1, \dots, x_{m-2})$, such that

$$R_\sigma^{(m-1)}(x_0, x_1, \dots, x_{m-2}, 0) = R_\sigma^{(m-1)}(x_0, x_1, \dots, x_{m-2}, 1), \quad (7)$$

is maximal over all completions of φ_σ .

The class $K_{m-2} \subseteq K_{m-1}$ is formed by all functions $R_\sigma^{(m-2)}$ of the class K_{m-1} for which the number of vectors $(x_0, x_1, \dots, x_{m-3})$ is such that

$$\begin{aligned} & R_\sigma^{(m-2)}(x_0, x_1, \dots, x_{m-3}, 0, 0) + R_\sigma^{(m-2)}(x_0, x_1, \dots, x_{m-3}, 0, 1) \\ &= R_\sigma^{(m-2)}(x_0, x_1, \dots, x_{m-3}, 1, 0) + R_\sigma^{(m-2)}(x_0, x_1, \dots, x_{m-3}, 1, 1), \end{aligned} \quad (8)$$

is maximal over all completions of the class K_{m-1} . The classes $K_{m-3} \supseteq K_{m-4} \supseteq \dots \supseteq K_0$ are constructed analogously. Any function φ_σ in K_0 is found for prescribed φ_σ relatively simply and does not require search. Thus, from a given partially-defined function and chosen σ one of the completely-defined functions $R_\sigma^{(0)}$ is constructed such that it is completed analogously to (1) to a piecewise-constant $\Phi_\sigma(x/2^m)$, and then represented in the form of series (2) in the system $\{H_l^{(j)}(x/2^m)\}$, then, as is evident from (3), (5), and (6), for L_{m-1} an absolute minimum is attained, and for $L_{m-2}, L_{m-3}, \dots, L_1$ certain relative minima.

Table 1 presents an example of an initial function φ_σ , for which the classes K_3, K_2, K_1, K_0 and the function $R_\sigma^{(0)}$ are constructed.

We shall now find a numbering of the arguments for which the piecewise-constant function $\Phi_\sigma(x/2^m)$, constructed above, would have minimal $\sum_{l=0}^{m-1} L_l$. Here for arbitrary σ the above-described method of completing the

function φ_σ to piecewise constant $\Phi_\sigma(x/2^m)$ is conserved. Let us fix a certain numbering of the arguments $x = (x_0, x_1, \dots, x_{m-1})$. Then any other numbering can be obtained by premultiplying the vector x by a matrix containing a single one in each row and each column, and zeros otherwise. (Here and below all matrix operations are understood in the field of residues modulo 2.) Thus, the problem of finding an optimal numbering can be considered as the problem of finding the corresponding matrix of ones and zeros.

We shall solve a general problem, in which a single constraint is imposed on the class Ξ of matrices of zeros and ones under consideration, consisting in the condition that the matrices of class Ξ are not degenerate. Premultiplication of x by $\sigma \in \Xi$ requires a special block at the input to the device; it can be shown that this block is always realized in not more than $(2m^2/\log_2 m) - 2m$ adders modulo 2. The block diagram of the entire device consists of the series connection of two blocks, realizing the functions σ and φ_σ .

The optimal matrix $\sigma_{opt} \in \Xi$ will be sought by analogy to the optimal completion. We first find the class Ξ_{m-1} of matrices for which L_{m-1} is minimal for the corresponding $\Phi_\sigma(x/2^m)$. Then we find in Ξ_{m-1} the subclass $\Xi_{m-2} \subseteq \Xi_{m-1}$, for which L_{m-2} is minimal, etc.

4. Let us consider the problem of constructing Ξ_{m-1} . We first assume that the matrix $\sigma \in \Xi$ is fixed.

x_0	x_1	x_2	x_3					
0	0	0	0	1	1	1	1	1
0	0	0	1	0	0	0	0	0
0	0	1	0	*	0	0	0	0
0	0	1	1	0	0	0	0	0
0	0	0	0	*	a_1	0	0	0
0	0	0	1	*	a_1	0	0	0
0	0	1	0	0	0	0	0	0
0	0	1	1	1	1	1	1	1
1	1	0	0	*	a_2	a_6	a_8	0
1	1	0	1	*	a_2	a_6	a_8	0
1	1	1	0	*	a_3	a_6	a_8	0
1	1	1	1	*	a_3	a_6	a_8	0
1	1	1	0	*	a_4	a_7	a_8	0
1	1	0	1	*	a_4	a_7	a_8	0
1	1	0	0	*	a_5	a_7	a_8	0
1	1	1	0	*	a_5	a_7	a_8	0
1	1	1	1	*	a_5	a_7	a_8	0

$a_1, a_2, a_3, a_4, a_5 \in \{0, 1\}; a_6 \in \{a_2, a_3\}; a_7 \in \{a_4, a_5\}; a_8 \in \{a_6, a_7\}$

We form for $\varphi_\sigma(x_0, x_1, \dots, x_{m-1})$ the system of completely defined characteristic functions $\varphi_{t,\sigma}(x)$ ($x = (x_0, x_1, \dots, x_{m-1}), t \in \{0, 1\}^m$):

$$\varphi_{t,\sigma}(x) = \begin{cases} 1 & \text{for } \varphi_\sigma(x_0, x_1, \dots, x_{m-1}) = t, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

We construct further the cross correlation function $B_\sigma(\tau)$ of the function $\varphi_{t,\sigma}(x)$:

$$B_\sigma(\tau) = \sum_{x \in \{0,1\}^m} \varphi_{0,\sigma}(x) \varphi_{1,\sigma}(x \oplus \tau) \pmod{2}. \quad (10)$$

(Here and below the symbol \oplus denotes the componentwise sum of vectors x and τ the modulo shown at the right of the expression; $\{0, 1\}^m$ is the set of all binary vectors of length m ; $x, \tau \in \{0, 1\}^m$.)

Theorem 1. The class Ξ_{m-1} is composed of the nondegenerate matrices σ_{m-1} , satisfying the condition

$$\sigma_{m-1} \tau_{\text{opt}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad (11)$$

where

$$B_\sigma(\tau_{\text{opt}}) = \min_{\tau \neq (00 \dots 0)} B_\sigma(\tau).$$

It follows from Theorem 1 that the problem of finding the class Ξ_{m-1} reduces to finding the minimum of the cross correlation function $B_\sigma(\tau)$, where the initial matrix σ may be arbitrarily fixed.

Let us present still another way to calculate the cross correlation function $B_\sigma(\tau)$. Rather than the system of Haar functions we take as the basis the system of Walsh functions [3]. The piecewise-constant function obtained by the completion of the sequence of coefficients of the expansion $\Phi_\sigma(x/2^m)$ in a Walsh basis is denoted by $W(\Phi_\sigma)$.

Theorem 2. For any $\sigma \in \Xi$

$$B_\sigma(\tau) = 2^{2m} W(W(\Phi_{0,\sigma}) W(\Phi_{1,\sigma})), \quad (12)$$

where $\Phi_{0,\sigma}(x/2^m); \Phi_{1,\sigma}(x/2^m)$ are piecewise-constant, obtained by completion of $\varphi_{0,\sigma}(x), \varphi_{1,\sigma}(x)$.

TABLE 2

x_0	x_1	x_2	x_3	φ_{E_4}	φ_{1, E_4}	φ_{0, E_4}	$16W(\Phi_0, E_4)$	$16W(\Phi_1, E_4)$	$B_{E_4}(\tau)$	$\varphi_{E_4}^*$
0	0	0	0	1	1	0	2	3	0	1
0	0	0	1	0	0	1	2	3	0	1
0	0	1	0	*	0	0	0	1	2	*
0	0	1	1	0	0	1	0	1	0	0
0	1	0	0	*	0	0	0	1	1	0
0	1	0	1	*	0	0	0	-1	1	0
0	1	1	0	0	0	1	0	-1	0	0
0	1	1	1	1	1	0	2	1	0	0
1	0	0	0	*	0	0	2	1	2	*
1	0	0	1	*	0	0	0	-1	0	1
1	0	1	0	*	0	0	0	-1	0	*
1	0	1	1	*	0	0	2	-3	0	*
1	1	0	0	*	0	0	2	-3	0	*
1	1	0	1	*	0	0	2	-1	0	*
1	1	1	0	*	0	0	2	-1	0	*
1	1	1	1	*	0	0	0	1	0	*
1	1	1	1	*	0	0	0	1	0	*

Theorem 2 gives a method of calculating the cross correlation function $B_{\sigma}(\tau)$ in terms of the double Walsh transform.

Example. We find for the function φ_{σ} in the previous example (Table 1) (for $\sigma = E_4$, where E_4 is the 4×4 unit matrix) the class $\Xi_{m-1} = \Xi_3$. The functions φ_{E_4} , φ_{0, E_4} , φ_{1, E_4} , $W(\Phi_0, E_4)$, $W(\Phi_1, E_4)$, B_{E_4} are given in Table 2,

where $\tau_{opt} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. The class Ξ_3 is defined by the condition $\sigma_3 \in \Xi_3$. If and only if $\sigma_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. We take

$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. The function φ_{σ_3} is given in Table 2; for it $L_3 = 0$, while for the initial function $\varphi_{E_4} L_3 = 2$.

5. Let us present a recurrent method of constructing the classes $\Xi_{m-2} \supseteq \Xi_{m-3} \supseteq \dots \supseteq E_1$, where the matrices of class Ξ_{m-s} minimize L_{m-s} . Simultaneously with Ξ_{m-s} we shall construct recurrently the system of auxiliary partially-defined functions $\varphi^{(s)}(x/2^{m-s+1})$.

The class Ξ_{m-1} was constructed in point 4. We put

$$\varphi^{(1)}(x/2^m) = \varphi_{\sigma_{m-1}}(x_0, x_1, \dots, x_{m-1}) \left(x = \sum_{j=0}^{m-1} x_j 2^{m-1-j}; \sigma_{m-1} \in \Xi_{m-1} \right). \tag{13}$$

Let now the class Ξ_{m-s} and the function $\varphi^{(s)}(x/2^{m-s+1})$ already be constructed and let us construct $\Xi_{m-(s+1)}$ and $\varphi^{(s+1)}(x/2^{m-s})$. For this we first construct the function

$$\tilde{\varphi}^{(s)}\left(\frac{x}{2^{m-s}}\right) = \varphi^{(s)}\left(\frac{2x}{2^{m-s+1}}\right) + \varphi^{(s)}\left(\frac{2x+1}{2^{m-s+1}}\right) \tag{14}$$

(in (14) we have assumed $a + * = 2a$, $* + * = *$, where $*$ is an indeterminate value). We form for $\tilde{\varphi}^{(s)}(x/2^{m-s})$ the system of characteristic functions $\varphi_t^{(s)}(x)$ ($t = 0, 1, \dots, 2^s$; $x \in \{0, 1\}^{m-s}$) by analogy to (9) and the overall cross correlation function

$$B^{(s)}(\tau) = \sum_{t < q} \sum_{x \in \{0, 1\}^{m-s}} \varphi_t^{(s)}(x) \varphi_q^{(s)}(x \oplus \tau) \pmod{2} \tag{15}$$

$(x, \tau \in \{0, 1\}^{m-s}).$

ing the condition

$$\sigma_{m-(s+1)} = \begin{pmatrix} \sigma^{(s)} & 0 \\ 0 & E_s \end{pmatrix} \sigma_{m-s}, \quad (10)$$

where $\sigma^{(s)} \tau_{\text{opt}}^{(s)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ and $B^{(s)}(\tau_{\text{opt}}^{(s)}) = \min_{\tau \neq (00 \dots 0)} B^{(s)}(\tau)$, $\sigma_{m-s} \in \Sigma_{m-s}$, $\sigma^{(0)} = \sigma_{m-1}$, $\tau_{\text{opt}}^{(0)} = \tau_{\text{opt}}$, $\sigma_m = E_m$.

E_s is the $s \times s$ unit matrix.

Theorem 3 is a generalization of Theorem 1 and gives a method recurrent in s for calculating $\sigma_{m-(s+1)}$. The function $\varphi^{(s+1)}(x/2^{m-s})$ is constructed from $\tilde{\varphi}^{(s)}(x/2^{m-s})$ and $\sigma^{(s)}$ found by Theorem 3 in the following way:

$$\begin{aligned} \varphi^{(s+1)}(\sigma^{(s)}x) &= \tilde{\varphi}^{(s)}\left(\frac{x}{2^{m-s}}\right), \quad x = (x_0, x_1, \dots, x_{m-1}), \\ \varphi^{(s+1)}\left(\frac{x}{2^{m-s}}\right) &= \varphi^{(s+1)}(x) \quad \left(x = \sum_{j=0}^{m-1} x_j 2^{m-1-j}\right). \end{aligned} \quad (17)$$

Theorem 3 and (17) gives a method recurrent in s for finding the optimal linear transform $\sigma_{\text{opt}} = \sigma_1$.

For the previous example (Tables 1, 2) Table 3 presents

$$\varphi^{(1)}\left(\frac{x}{16}\right) = \varphi_{\sigma_1}(x_0, x_1, x_2, x_3), \quad \tilde{\varphi}^{(1)}\frac{x}{8}; \quad \tilde{\varphi}_0^{(1)}(x), \tilde{\varphi}_2^{(1)}(x), \quad (\tilde{\varphi}_1^{(1)}(x) = \tilde{\varphi}_3^{(1)}(x) \equiv 0) \text{ и } B^{(1)}(\tau).$$

From Table 3 it follows that $\tau_{\text{opt}}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. By Theorem 3 we have

$$\sigma^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Further, Table 3 presents $\varphi^{(2)}(x/8)$ constructed by formula (17) and for it $\tilde{\varphi}^{(2)}(x/4)$ and $B^{(2)}(\tau)$; $\tau_{\text{opt}}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\sigma^{(2)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$$\sigma_1 = \sigma_{\text{opt}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

One adder modulo 2 is required to realize the block σ_{opt} . Table 4 presents the initial partially-defined function φ_{E_4} , the function $\varphi_{\sigma_{\text{opt}}}$, their optimal completions $R_{E_4}^{(0)}$, $R_{\sigma_{\text{opt}}}^{(0)}$ (cf. point 3) and their expansion coefficients in the basis $\{H_i(i)(x/16)\}$ in the following order: $C_0^{(0)}$, $C_0^{(1)}$, $C_1^{(1)}$, $C_1^{(2)}$, $C_2^{(1)}$, \dots , $C_3^{(8)}$. From Table 4 it follows that if for the initial function $L = 6$, then after the optimal transformation of arguments $L = 2$.

6. Let us consider the question of the realization of partially-defined functions in p -valued logic.

First of all we construct a system of functions analogous to the Haar system (3), which it would be useful to utilize as the basis for the realization of functions in p -valued logic. We construct the functions $K_{r,j}(x/p^m)$ in the following way:

x_0	x_1	x_2	x_3	$\varphi_{0,0} - \varphi^{(1)}$	$\varphi^{(1)}\left(\frac{x}{8}\right)$	$\varphi_0^{(1)}(x)$	$\varphi_2^{(1)}(x)$	$B^{(1)}(x)$	$\varphi^{(2)}\left(\frac{x}{8}\right)$	$\varphi_0^{(2)}\left(\frac{x}{4}\right)$	$B^{(2)}(x)$
0	0	0	0	1	2	0	1	0	2	4	0
0	0	0	1	*	0	1	0	2	*	0	2
0	0	1	0	0	0	1	0	2	*	0	2
0	0	1	1	0	2	1	1	0	*	4	0
0	1	0	0	*	*	0	0	0	*	*	*
0	1	0	1	*	*	0	0	0	*	*	*
0	1	1	0	*	*	0	0	0	2	*	*
0	1	1	1	1	*	0	0	0	*	*	*
1	0	0	0	*	*	*	*	*	*	*	*
1	0	0	1	*	*	*	*	*	*	*	*
1	0	1	0	*	*	*	*	*	*	*	*
1	0	1	1	*	*	*	*	*	*	*	*
1	1	0	0	*	*	*	*	*	*	*	*
1	1	0	1	*	*	*	*	*	*	*	*
1	1	1	0	*	*	*	*	*	*	*	*
1	1	1	1	*	*	*	*	*	*	*	*
1	1	1	1	*	*	*	*	*	*	*	*

TABLE 4

x_0	x_1	x_2	x_3	φ_{E_4}	$R_{E_4}^{(0)}$	$c_1^{(0)}$	φ_{opt}	$R_{opt}^{(0)}$	$c_1^{(0)}$
0	0	0	0	1	1	1/8	1	1	1/2
0	0	0	1	0	0	1/8	*	1	1/2
0	0	1	0	*	0	0	*	1	0
0	0	1	1	0	0	0	*	1	0
0	1	0	0	*	0	1/4	*	1	0
0	1	0	1	*	0	-1/4	1	1	0
0	1	1	0	0	0	0	*	1	0
0	1	1	1	1	1	0	*	1	0
1	0	0	0	*	0	1/2	0	0	0
1	0	0	1	*	0	0	*	0	0
1	0	1	0	*	0	0	*	0	0
1	0	1	1	*	0	-1/2	*	0	0
1	1	0	0	*	0	0	*	0	0
1	1	0	1	*	0	0	*	0	0
1	1	1	0	*	0	0	*	0	0
1	1	1	1	*	0	0	*	0	0
1	1	1	1	*	0	0	*	0	0

$$K_{0,0}\left(\frac{x}{p^m}\right) \equiv 1, \tag{18}$$

$$K_{r,l}\left(\frac{x}{p^m}\right) = \exp\left(\frac{2\pi}{p} l r x_l\right) \quad (r=1,2,\dots,p-1; l=0,1,\dots,m-1),$$

where

$$x = \sum_{l=0}^{m-1} x_l p^{m-1-l}, \quad i = \sqrt{-1}.$$

The system $\{K_{r,l}(x/p^m)\}$ contains $(p-1)m+1$ orthogonal functions, each of which (except for $K_{0,0}$) takes on p distinct complex values. The functions $\{K_{r,l}(x/p^m)\}$ do not compose a complete system (the system $\{K_{r,l}(x/p^m)\}$ is a subset of the Krestenson function, [2], and for $p=2$ the system of Rademacher function [2]).

Table 5 gives for the case $p=3, m=2$ the values of $K_{r,l}(x/9)$. Here

$$a = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \bar{a} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

$K_{r,l} \left(\frac{x}{9} \right)$	x								
	0	1	2	3	4	5	6	7	8
$K_{0,0}$	1	1	1	1	1	1	1	1	1
$K_{1,0}$	1	1	1	\bar{a}	\bar{a}	\bar{a}	\bar{a}	\bar{a}	\bar{a}
$K_{2,0}$	1	1	1	\bar{a}	\bar{a}	\bar{a}	\bar{a}	\bar{a}	\bar{a}
$K_{1,1}$	1	\bar{a}	\bar{a}	1	\bar{a}	\bar{a}	1	\bar{a}	\bar{a}
$K_{2,1}$	1	\bar{a}	\bar{a}	1	\bar{a}	\bar{a}	1	\bar{a}	\bar{a}

TABLE 6

$M_{r,l}^{(j)} \left(\frac{x}{9} \right)$	x								
	0	1	2	3	4	5	6	7	8
$M_{0,0}^{(0)}$	1	1	1	1	1	1	1	1	1
$M_{1,0}^{(1)}$	1	1	1	\bar{a}	\bar{a}	\bar{a}	\bar{a}	\bar{a}	\bar{a}
$M_{2,0}^{(2)}$	1	1	1	\bar{a}	\bar{a}	\bar{a}	\bar{a}	\bar{a}	\bar{a}
$M_{1,1}^{(1)}$	1	\bar{a}	\bar{a}	0	0	0	0	0	0
$M_{1,1}^{(2)}$	0	0	0	1	\bar{a}	\bar{a}	0	0	0
$M_{1,1}^{(3)}$	0	0	0	0	0	0	1	\bar{a}	\bar{a}
$M_{2,1}^{(1)}$	1	\bar{a}	\bar{a}	0	0	0	0	0	0
$M_{2,1}^{(2)}$	0	0	0	1	\bar{a}	\bar{a}	0	0	0
$M_{2,1}^{(3)}$	0	0	0	0	0	0	1	\bar{a}	\bar{a}

We now construct with the aid of $\{K_{r,l}(x/p^m)\}$ the complete system $\{M_{r,l}^{(j)}(x/p^m)\}$:

$$M_{0,0}^{(0)} \left(\frac{x}{p^m} \right) = K_{0,0} \left(\frac{x}{p^m} \right) \equiv 1,$$

$$M_{r,l}^{(j)} \left(\frac{x}{p^m} \right) = \begin{cases} K_{r,l} \left(\frac{x}{p^m} \right) & \text{for } x \in [(j-1)p^{m-l}, jp^{m-l}), \\ 0 & \text{elsewhere in } [0, 1). \end{cases} \quad (19)$$

Table 6 gives for the case $p = 3, m = 2$ the values of $M_{r,l}^{(j)}(x/9)$.

Let now there be given a completely-defined function in p -valued logic; let us fix the method σ of numbering its arguments and complete it by analogy to (1) to a piecewise-constant $\Phi_\sigma(x/p^m)$.

Theorem 4. The functions $\{M_{r,l}^{(j)}(x/p^m)\}$ compose a complete system of orthogonal functions, where in the series expansion of an arbitrary function $\Phi_\sigma(x/p^m)$ in $\{M_{r,l}^{(j)}(x/p^m)\}$ with coefficients $C_{r,l}^{(j)}$ the equality

$$C_{r,l}^{(j)} = 0 \quad \text{for } l \geq m \quad (r = 1, 2, \dots, p-1; j = 1, 2, \dots, p^l).$$

is satisfied.

It follows from Theorem 4 that in the series expansion of any $\Phi_\sigma(x/p^m)$ in $\{M_{r,l}^{(j)}(x/p^m)\}$ the number of coefficients does not exceed p^m and from (19) for any x^* there exist exactly $m(p-1) + 1$ functions $M_{r,l}^{(j)}(x/p^m)$ such that $M_{r,l}^{(j)}(x^*/p^m) \neq 0$, i.e., the calculation of $\Phi_\sigma(x/p^m)$ by an accumulator requires not more than $m(p-1) + 1$ cycle

This points out the utility of using the system $\{M_{r,l}^{(j)}(x/p^m)\}$ as the basis for the realization of functions of p -valued logic. (We note that the Haar system is a particular case of the system $\{M_{r,l}^{(j)}(x/p^m)\}$ for $p = 2$.)

7. Let us now consider the question of the realization of partially-defined functions of p -valued logic using series expansions in the basis $\{M_{r,l}^{(j)}(x/p^m)\}$.

We shall complete a prescribed partially-defined function φ_σ in p -valued logic, with fixed method σ of numbering the arguments, by analogy to functions of binary logic. We shall denote the number of nonzero coefficients of the form $C_{r,l}^{(j)}$ ($r = 1, 2, \dots, p-1$; $j = 1, 2, \dots, p^l$) by L_l . Then $0 \leq L_l \leq p^l(p-1)$. The class K_{m-1} of completions minimizing L_{m-1} is composed of functions obtained by the completion of φ_σ , for which, if $R_\sigma^{(m-1)} \in K_{m-1}$, then the number of vectors $(x_0, x_1, \dots, x_{m-2})$ such that

$$\begin{aligned} R_\sigma^{(m-1)}(x_0, x_1, \dots, x_{m-2}, 0) &= R_\sigma^{(m-1)}(x_0, x_1, \dots, x_{m-2}, 1) \\ &= \dots = R_\sigma(x_0, x_1, \dots, x_{m-2}, p-1), \end{aligned} \quad (2)$$

is maximal over all completions of φ_σ . The class $K_{m-2} \subseteq K_{m-1}$ for which L_{m-2} is minimized for the L_{m-1} attained previously is composed of functions for which, if $R_\sigma^{(m-2)} \in K_{m-2}$, then the number of vectors $(x_0, x_1, \dots, x_{m-3})$ such that

$$\begin{aligned} \sum_{s=0}^{p-1} R_\sigma^{(m-2)}(x_0, x_1, \dots, x_{m-3}, 0, s) &= \sum_{s=0}^{p-1} R_\sigma^{(m-2)}(x_0, x_1, \dots, x_{m-3}, 1, s) \\ &= \dots = \sum_{s=0}^{p-1} R_\sigma^{(m-2)}(x_0, x_1, \dots, x_{m-3}, p-1, s), \end{aligned} \quad (3)$$

is maximal over all completions of the class K_{m-1} . The classes $K_{m-3} \supseteq K_{m-4} \supseteq \dots \supseteq K_0$ by analogy. The function $R_\sigma^{(0)}$ in the class K_0 is taken as the completion of the initial function φ_σ .

To find an optimal method of numbering the arguments we shall solve the more general problem, in which the class $\Xi^{(p)}$ of matrices under consideration is composed of all matrices σ nondegenerate over the field of residues modulo p . The elements of σ are $\{0, 1, \dots, p-1\}$ and, correspondingly, the device realizing premultiplication by σ of the vector $\mathbf{x} = (x_0, x_1, \dots, x_{m-1})$ is realized by adders modulo p .

The class $\Xi_{m-1}^{(p)} \subseteq \Xi^{(p)}$ of matrices σ minimizing L_{m-1} is constructed in the following way. We fix $\sigma \in \Xi^{(p)}$ arbitrarily and from the φ_σ obtained we construct the system of characteristic $\varphi_{t,\sigma}(\mathbf{x})$ ($t = 0, 1, \dots, p-1$) by analogy to (9). Further we construct the function $B_\sigma(\tau)$, which will be the generalization of the cross correlation function (cf. (10)) to the p -valued case

$$\begin{aligned} B_{p,\sigma}(\tau) &= \sum_{t_1 < t_2 < \dots < t_p} \sum_{\mathbf{x} \in (0, 1, \dots, p-1)^m} \varphi_{t_1,\sigma}(\mathbf{x}) \varphi_{t_p,\sigma} \\ &\times (\mathbf{x} \oplus \tau) \dots \varphi_{t_p,\sigma}(\mathbf{x} \oplus \tau \oplus \dots \oplus \tau) \pmod{p}. \end{aligned}$$

Theorem 5. The class $\Xi_{m-1}^{(p)}$ is composed of matrices $\sigma_{m-1}^{(p)}$ satisfying the condition

$$\sigma_{m-1}^{(p)} \tau_{\text{opt}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

where $B_{p,\sigma}(\tau_{\text{opt}}) = \min_{\tau \neq (00 \dots 0)} B_{p,\sigma}(\tau)$.

Theorem 5 is a generalization of Theorem 1 to the case of p-valued logic.

The classes $\Xi_{m-2}^{(p)} \supseteq \Xi_{m-3}^{(p)} \supseteq \dots \supseteq \Xi_1^{(p)}$, minimizing $L_{m-2}, L_{m-3}, \dots, L_1$, can be found by analogy to the classes $\Xi_{m-2} \supseteq \Xi_{m-3} \supseteq \dots \supseteq \Xi_1$ for the case $p = 2$ (cf. point 5).

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