REALIZATION OF A SYSTEM OF LOGICAL FUNCTIONS BY MEANS OF AN EXPANSION IN ORTHOGONAL SERIES

M. G. Karpovskii and É. S. Moskalev

UDC 62-50

A method is proposed for the realization of systems of logical functions consisting in the formation of a certain piecewise-constant function from the given system of functions, and the realization of this new function by means of an expansion in an orthogonal series in Haar's basis. The problem of optimal choice of the piecewise-constant function for a given system of Boolean functions is solved with respect to minimal equipment.

1. Let there be given a system of Boolean functions of m arguments

$$y_0 = \varphi_0(x_0, x_1, \ldots, x_{m-1}), \; y_{k-1} = \varphi_{k-1}(x_0, x_1, \ldots, x_{m-1}). \tag{1}$$

We put $x = \sum_{i=0}^{m-1} x_i 2^i$; $y = \sum_{i=0}^{k-1} y_i 2^i$. Then the system (1) can be rewritten in the form $y = \varphi(x)$. (The argument x of the function $\varphi(x)$ will be denoted by $\bar{x} = (x_0, x_1, \ldots, x_{m-1})$.)

Let us consider the following method of realizing the system (1).

We complete the function $y = \varphi(x)$ in the interval [0, 1) to a piecewise-constant function $\Phi\left(\frac{x}{2^m}\right)$ in the following way:

$$\Phi\left(\frac{x}{2^m}\right) = \varphi(i), \quad i \leqslant x < i+1 \quad (i = 0, 1, \ldots, 2^m - 1). \tag{2}$$

The function $\Phi\left(\frac{x}{2^m}\right)$ is bounded, has a finite number of discontinuities of the first kind, i.e., belongs to the space L^2 [1]. We represent the function $\Phi\left(\frac{x}{2^m}\right)$ in the form of the orthogonal series

$$\Phi\left(\frac{x}{2^m}\right) = \sum_{i=0}^{\infty} c_i \psi_i\left(\frac{x}{2^m}\right), \tag{3}$$

where $\{\psi_i\}$ is a complete orthogonal system of basis functions.

By formula (3), a device which realizes $\left(\Phi \frac{x}{2^m}\right)$ must contain (Fig. 1): a) a memory block for the storage of coefficients, b) a device generating the values of the basis functions, c) multiplication and addition circuits.

2. We take as the basis Haar's system of functions, defined in the interval [0, 1) in the following manner:

$$H_0^{(0)}\left(\frac{x}{2^m}\right) \equiv 1;$$

Leningrad. Translated from Avtomatika i Telemekhanika, No. 12, pp. 119-129, December, 1967. Original article submitted February 4, 1967.

... at the remaining points ...

$$(i = 0, 1, \ldots, j = 1, 2, \ldots, 2^i).$$

The forms of the first Haar functions are given in Fig. 2.

It follows directly from (2) and (4) that: a) for the function $\Phi\left(\frac{x}{2^m}\right)$, $c_i^{(j)} = 0$ for $i \geq m$, $j = 1, 2, \ldots, 2^i$, i.e., the number of terms of the series (3) in the expansion of $\Phi\left(\frac{x}{2^m}\right)$ in a system of Haar functions does not exceed 2^m; b) since Haar functions take on only the values 0.1, −1, the calculation of $\Phi\left(\frac{x}{2^m}\right)$ by formula (3) reduces to the algebraic summation of certain coefficients $c_i^{(j)}$, i.e., no multiplication unit is needed in the realization; c) the device generating the values of the basis functions $H_i^{(j)}\left(\frac{x}{2^m}\right)$ from the binary code of the argument $(x_0, x_1, \ldots, x_{m-1})$ is realized simply by means of a shift register. The above considerations show the usefulness of the Haar functions as the basis system.

The complexity of the device [2] realizing $\Phi\left(\frac{x}{2^m}\right)$ from the block diagram of Fig. 1 is defined basically by the complexity of the memory unit, i.e., by the number of nonzero coefficients, when the Haar system is used as the basis. (This is connected with the fact that the memory block complexity depends exponentially on m, while the complexity of the shift register and adder depend linearly on m and on k). Then, to reduce the memory volume the argument x is transformed by a special block σ in such manner as to reduce the number of nonzero coefficients of the series (3).

Let there be given a system of Boolean functions (Table 1). (In Table 1 the coefficients $c_i^{(j)}$ are expanded in the order $c_0^{(0)}$, $c_0^{(1)}$, $c_1^{(1)}$, $c_1^{(2)}$, $c_2^{(1)}$, ...) From Table 1 it follows that the number of nonzero coefficients which must be stored in the memory block is equal to 14.

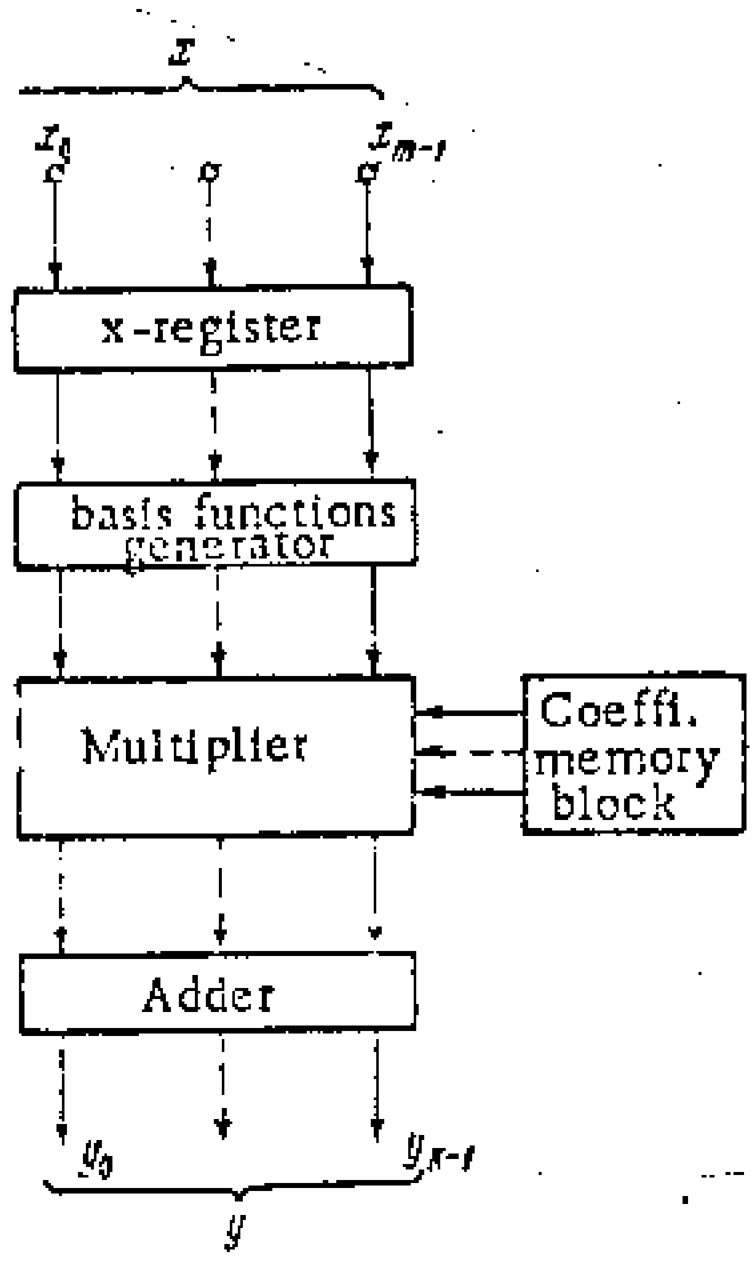

Fig. 1. Block diagram of (m, k)-pole.

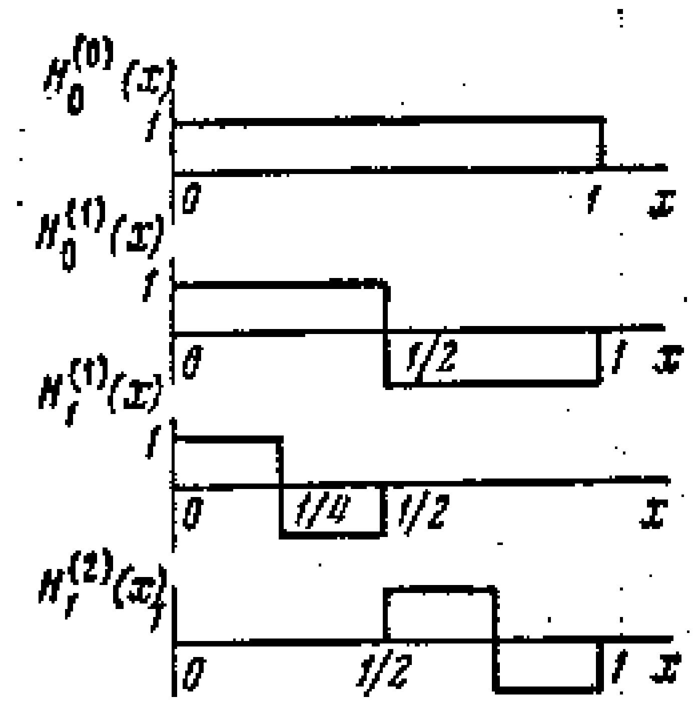

Fig. 2. Graphs of the Haar functions.

[illegible]	[illegible]	[illegible]	[illegible]	0011	01	1	0
0100	01	1	—16	0100	11	3	—8
0101	10	2	—4	0101	11	3	—8
0110	01	1	4	0110	10	2	0
0111	11	3	—8	0111	10	2	0
1000	11	3	—16	1000	00	0	0
1001	01	1	0	1001	11	3	0
1010	01	1	—8	1010	01	1	0
1011	10	2	—16	1011	10	2	0
1100	00	0	16	1100	10	2	—24
1101	00	0	—8	1101	01	1	—8
1110	10	2	0	1110	00	0	8
1111	00	0	16	1111	11	3	—24

We realize the following linear [3] transformation of the argument:

$$x_0' = x_1 \oplus x_2; \qquad x_2' = x_2 \oplus x_3;$$
$$x_1' = x_0 \oplus x_1 \oplus x_2 \oplus x_3; \qquad x_3' = x_0 \oplus x_1 \oplus x_2.$$

In Table 1 (at the right) are represented the Boolean functions y_0 and y_1 of the arguments $x_0', x_1', \ldots, x'_{m-1}$, the corresponding function $\varphi\left(\frac{x'}{16}\right)$ and the sequence of coefficients of the Haar expansion of this function. From Table 1 it follows that after transformation of the arguments the number of nonzero coefficients has been reduced to eight.

From the comparison of the Boolean functions y_0 and y_1 of the arguments $(x_0, x_1, \ldots, x_{m-1})$ and $(x_0', x_1', \ldots, x'_{m-1})$ it follows that this transformation of the arguments is also useful for the realization of the Boolean functions y_0 and y_1 by the ordinary methods of Boolean algebra.

3. Further, we solve with certain constraints the problem of finding the transformation σ_{min} for which the number of nonzero coefficients of the series (3) $(L(\sigma))$ reaches a minimum, i.e.,

$$L(\sigma_{min}) = \min_{\sigma \in \Xi} L(\sigma). \tag{5}$$

We consider for the class Ξ the set of m-dimensional linear Boolean functional [3].

Any function $\sigma \in \Xi$ can be represented by a matrix of order $m \times m$, $\sigma = \|\sigma_{ij}\|$, where $\sigma_{ij} \in \{0, 1\}$, such that $\sigma(x)$ satisfies the condition

$$\sigma(\bar{x}) = \sigma\bar{x}.$$

(Here and everywhere below, operations on matrices are understood in the field of residues modulo 2. Only nondegenerate transformations of the arguments, for which $|\sigma| = 1$, are considered.)

We define the function

$$\varphi_\sigma\left(\frac{x}{2^m}\right) = \varphi(\sigma^{-1}\bar{x}). \tag{6}$$

Then φ_σ is a comb function, defined at the points $0, \frac{1}{2^m}, \ldots, \frac{2^m-1}{2^m}$ of the interval $[0, 1)$. We complete φ_σ at the remaining points of the interval $[0, 1)$ to piecewise-constant $\Phi_\sigma\left(\frac{x}{2^m}\right)$:

$$\Phi_\sigma\left(\frac{x}{2^m}\right)=c_0^{(0)}(\sigma)H_0^{(0)}\left(\frac{x}{2^m}\right)+\sum_{i=0}\sum_{j=1}c_i^{(j)}(\sigma)H_i^{(j)}\left(\frac{x}{2^m}\right). \tag{8}$$

Let $L_i(\sigma)$ be the number of nonzero coefficients of the form $c_i^{(j)}(\sigma)$ $(j=1, 2, \ldots, 2^i)$. Then

$$L(\sigma)=\sum_{i=0}^{m-1}L_i(\sigma).$$

Since $0 \le L_i(\sigma) \le 2^i$ $(i=0, 1, \ldots, m-1)$, we shall seek the function $\sigma_{\min}$ in the following manner:

a) we find $\sigma_1 \in \Xi$ such that

$$L_{m-1}(\sigma_1)=\min_{\sigma\in\Xi}L_{m-1}(\sigma); \tag{9}$$

b) we find $\sigma_2 \in \Xi$ such that

$$L_{m-2}(\sigma_2\sigma_1)=\min L_{m-2}(\sigma\sigma_1), \tag{10}$$

where the minimum is taken over all $\sigma \in \Xi$ such that

$$L_{m-1}(\sigma\sigma_1)=L_{m-1}(\sigma_1);$$

c) we find $\sigma_i \in \Xi$ $(i=1, 2, \ldots, m-1)$ such that

$$L_{m-i}(\sigma_i\sigma_{i-1}\ldots\sigma_1)=\min L_{m-i}(\sigma\sigma_{i-1}\ldots\sigma_1), \tag{11}$$

where the minimum is taken over all $\sigma \in \Xi$ such that

$$L_{m-i+s}(\sigma\sigma_{i-1}\ldots\sigma_1)=L_{m-i+s}(\sigma_{i-s}\sigma_{i-s-1}\ldots\sigma_1) \quad (s=1, 2, \ldots, i-1). \tag{12}$$

Finally, we put for $\sigma_{\min}$

$$\sigma_{\min}=\sigma_{m-1}\sigma_{m-2}\ldots\sigma_1. \tag{13}$$

Here it must be kept in mind that the function $\sigma_{\min}$ thereby found does not give an absolute minimum for $L(\sigma)$ in the general case, but gives an absolute minimum for $L_{m-1}(\sigma)$ and certain conditional minima for $L_{m-i}(1)$ $(i=2, 3, \ldots, m-1)$.

4. We consider first the problem of finding σ_1, satisfying condition (9). We note that for $\Phi_\sigma\left(\frac{x}{2^m}\right)$ for any σ $c_{m-1}^{(j)}(\sigma)=0$ if and only if

$$\Phi_\sigma\left(\frac{2j-2}{2^m}\right)=\Phi_\sigma\left(\frac{2j-1}{2^m}\right) \quad (j=1, 2, \ldots, 2^{m-1}). \tag{14}$$

We shall seek the solution to this problem by means of the expansion of $\Phi_\sigma\left(\frac{x}{2^m}\right)$ in a Fourier series in the Rademacher–Walsh system of functions [1]. The Rademacher functions are defined in the following way:

The Walsh functions are defined as all possible products of the Rademacher functions $R_i\left(\frac{x}{2^m}\right)$ $(i = 1, 2, \ldots, m)$ in the following way:*

$$W_0\left(\frac{x}{2^m}\right) = R_0\left(\frac{x}{2^m}\right) \equiv 1, \; W_{p_1p_2\ldots p_l}\left(\frac{x}{2^m}\right) = R_{p_1}\left(\frac{x}{2^m}\right)R_{p_2}\left(\frac{x}{2^m}\right)\ldots R_{p_l}\left(\frac{x}{2^m}\right), \tag{16}$$

where $0 < p_1 < p_2 < \ldots < p_l$.

It can be shown that in the expansion of $\Phi_\sigma\left(\frac{x}{2^m}\right)$ in Walsh functions, as in the expansion in Haar functions, the number of nonzero coefficients $c_{p_1p_2\ldots p_l}(\sigma)$ does not exceed 2^m, and $c_{p_1p_2\ldots p_l}(\sigma) = 0$ for $p_l > m$.

We put

$$c_{p_1p_2\ldots p_l}(\sigma) = c_\alpha(\sigma), \tag{17}$$

where $\alpha = \sum_{i=1}^{l} 2^{p_i - 1}$, and then $\Phi_\sigma\left(\frac{x}{2^m}\right) = \sum_{\alpha=0}^{2^m-1} c_\alpha(\sigma) W_\alpha\left(\frac{x}{2^m}\right)$.

5. Let $\varphi_\sigma\left(\frac{x}{2^m}\right)$ take on n distinct values $y_0, y_1, \ldots, y_{n-1}$ $(1 \le n \le 2^k)$.

We construct a function $\varphi_{k,\sigma}\left(\frac{x}{2^m}\right)$, where $k = 0, 1, \ldots, n-1$, in the following manner:

$$\varphi_{k,\sigma}\left(\frac{x}{2^m}\right) = \begin{cases} 1, \text{ if } & \varphi_\sigma\left(\frac{x}{2^m}\right) = y_k, \\ 0 \text{ otherwise.} \end{cases} \tag{18}$$

We complete $\varphi_{k,\sigma}\left(\frac{x}{2^m}\right)$ to a piecewise-constant function $\Phi_{k,\sigma}\left(\frac{x}{2^m}\right)$ as in Section 1 for $\varphi_\sigma\left(\frac{x}{2^m}\right)$. The sequence of expansion coefficients of the function $\Phi_{k,\sigma}\left(\frac{x}{2^m}\right)$ in the basis $W_\alpha\left(\frac{x}{2^m}\right)$, in order of increasing value of α (cf. (17)), will be denoted by $s_{k,\sigma}(\alpha)$.

Further, we put

$$\sum_{h=0}^{n-1} (s_{h,\sigma}(\alpha))^2 = t_\sigma(\alpha). \tag{19}$$

Theorem 1. For arbitrary $\sigma \in \Xi$

$$\sum_{\alpha=0}^{2^{m-1}-1} t_{\sigma_1}(\alpha) - \sum_{\alpha=2^{m-1}}^{2^m-1} t_{\sigma_1}(\alpha) \geqslant \sum_{\alpha=0}^{2^{m-1}-1} t_\sigma(\alpha) - \sum_{\alpha=2^{m-1}}^{2^m-1} t_\sigma(\alpha). \tag{20}$$

0001	10	2	0010	1001	1	4	0
0010	11	3	0001	1100	2	4	0
0011	11	3	0001	1100	3	4	3
0100	01	1	0100	0011	4	4	1
0101	10	2	0010	0011	5	4	1
0110	01	1	0100	0000	6	0	1
0111	11	3	0001	0000	7	0	2
1000	11	3	0001	0000	8	0	2
1001	01	1	0100	0000	9	0	1
1010	01	1	0100	0011	10	4	3
1011	10	2	0010	0011	11	4	3
1100	00	0	1000	1100	12	4	0
1101	00	0	1000	1100	13	4	3
1110	10	2	0010	0110	14	4	2
1111	00	0	1000	1110	15	8	2

Theorem 1 shows that to find σ_I it is necessary to find σ, so that

$$\Delta(\sigma) = \sum_{\alpha=0}^{2^{m-1}-1} t_\sigma(\alpha) - \sum_{\alpha=2^{m-1}}^{2^m-1} t_\sigma(\alpha)$$

attains a maximum.

We now investigate more closely the functional $\Delta(\sigma)$, defined on the set of matrices σ for which $|\sigma| = 1$, and the means for finding its maximum.

6. We construct the piecewise-constant functions $T_\sigma\left(\frac{x}{2^m}\right)$, defined in the interval $[0, 1)$ in the following manner:

$$T_\sigma\left(\frac{x}{2^m}\right) = t_\sigma(\alpha) \tag{21}$$

for $\alpha \leq x < \alpha + 1$.

We shall denote the value of the i-th coefficient of the expansion $T_\sigma\left(\frac{x}{2^m}\right)$ in the Walsh system by $B_\sigma(i)$.

Corollary. For arbitrary $\sigma \in \Xi$

$$B_{\sigma_1}(1) \geqslant B_\sigma(1). \tag{22}$$

Inequality (22) follows from the identity $B_\sigma(1) = \Delta(\sigma)$.

We shall now describe a simpler way to construct the function $B_\sigma(i)$ and shall examine how this function depends on the choice of σ. We denote by $x_{k_1}, x_{k_2}, \ldots, x_{k_{n_k}}$ the values of x satisfying the conditions

$$\varphi_{k,\sigma}\left(\frac{x_{ks}}{2^m}\right) = 1 \quad (s = 1, 2, \ldots, n_k). \tag{23}$$

We denote by $b_{k,\sigma}(i)$ the number of pairs $(\bar{x}_{kp}, \bar{x}_{kq})$ such that $\bar{x}_{kp} \oplus \bar{x}_{kq} = \bar{i}$ (where $\bar{i} = (i_0, i_1, \ldots, i_{m-1})$ and

$$i = \sum_{r=0}^{m-1} i_r 2^r \quad (p, q = 1, 2, \ldots, n_k;\ p \neq q)).$$

3	3	3	10	7	4	4	

Theorem 2. For arbitrary $\sigma \in \Xi$ there hold the inequalities

$$B_\sigma(i) = \begin{cases} \dfrac{1}{2^m} & \text{for} \quad i = 0, \\ \dfrac{1}{2^{2m-1}} \sum\limits_{k=0}^{n-1} b_{k\sigma}(i) \text{ for} & i > 0. \end{cases} \tag{24}$$

Theorem 2 gives the possibility of finding the function $B_\sigma(i)$ directly from the functions $\varphi_{k,\sigma}\left(\frac{x}{2m}\right)$.

Example. Let there be given two Boolean functions y_0, y_1 of four arguments x_0, x_1, x_2, x_3 (Table 1). We put

$$\sigma = E_4 = \begin{Vmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{Vmatrix},$$

then $\varphi_{k,E_4}\left(\frac{x}{16}\right)$ is given in Table 2, with $\varphi_{k,E_4}\left(\frac{x}{16}\right) = 1$ for $\varphi_{E_4}\left(\frac{x}{16}\right) = k$, where $k = 0, 1, 2, 3$. We find, by means of Theorem 2, $B_{E_4}(i)$. The values of $b_{k,E_4}\left(\frac{x}{16}\right)$ and $B_{E_4}(i)$ are given in Table 2.

7. We shall now consider how the function $B_\sigma(i)$ depends on the choice of σ.

Theorem 3. For any $\sigma_r, \sigma_j \in \Xi$ there holds

$$B_{\sigma_j}(\bar{i}) = B_{\sigma_r}(\sigma_r \sigma_j^{-1} \bar{i}) \tag{25}$$

(here and further $B_\sigma(\bar{i}) = B_\sigma(i)$ and $\bar{i} = (i_0, i_1, \ldots, i_{m-1})$);

$$i = \sum_{r=0}^{m-1} i_r 2^r.$$

From Theorem 3 it follows that the choice of σ influences only the order of the residues of the function $B_\sigma(i)$, in particular, the value of $B_\sigma(i)$ does not depend on the choice of σ. This circumstance will be utilized to find σ_1 by means of the algorithm described in point 8.

Corollary 1. For arbitrary $\sigma_i \in \Xi$ and $i > 0$ there exists a $\sigma_j \in \Xi$ such that

[illegible]

In fact, from (25) it follows that for σ_j there can be taken any matrix $\sigma \in \Xi$ such that

$$B_{\sigma_1}(1) \geqslant B_\sigma(i). \tag{28}$$

8. Corollaries 1 and 2 given the possibility of finding σ_1 by the following algorithm:

a) we take an arbitrary matrix $\sigma \in \Xi$ such that $|\sigma| = 1$; in particular, it is possible to put $\sigma = E_m$ (where E_m is the unit $m \times m$ matrix);

b) we construct by Theorem 2 the function $B_\sigma(i)$;

c) we find s such that

$$B_\sigma(s) = \max_{\{i\}} B_\sigma(i); \tag{29}$$

d) we find σ_1 from the condition

$$\sigma^{-1}\bar{s} = \sigma_1^{-1}\begin{Vmatrix}0\\0\\\vdots\\0\\1\end{Vmatrix}, \tag{30}$$

where

$$\bar{s} = (s_0, s_1, \ldots, s_{m-1}) \qquad s = \sum_{i=0}^{m-1} s_i 2^i.$$

Example. Let us find σ_1 for the system of two Boolean functions of four arguments given in Table 1:

a) we put $\sigma = E_4$;

b) we construct $B_{E_4}(i)$ (cf. Table 2);

c) we find s from condition (29): $s = 15$, $\bar{s} = (1, 1, 1, 1)$;

d) we find σ_1 from condition (30) for $\sigma = E_4$:

$$\begin{Vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{Vmatrix} \times \begin{Vmatrix}1\\1\\1\\1\end{Vmatrix} = \sigma_1^{-1} \times \begin{Vmatrix}0\\0\\0\\1\end{Vmatrix}.$$

According to this we put

$$\sigma_1^{-1} = \begin{Vmatrix}1&0&1&1\\0&1&1&1\\1&1&0&1\\1&0&0&1\end{Vmatrix}; \qquad \sigma_1 = \begin{Vmatrix}1&1&1&1\\0&0&1&1\\1&0&0&1\\1&1&1&0\end{Vmatrix}.$$

The function $\Phi_{\sigma_1}\left(\frac{x}{16}\right)$ is given in Table 2. (The function $\Phi_{\sigma_1}\left(\frac{x}{16}\right)$ is constructed from the function $\Phi_{E_4}\left(\frac{x}{16}\right)$ (cf. Table 2) by means of relation (6).)

$$\Phi_\sigma\left(\frac{4j-4}{2^m}\right)+\Phi_\sigma\left(\frac{4j-3}{2^m}\right)-\Phi_\sigma\left(\frac{4j-2}{2^m}\right)-\Phi_\sigma\left(\frac{4j-1}{2^m}\right)=0$$
$$(j=1,2,\ldots,2^{m-2}). \qquad (31)$$

We form the function

$$\Phi^{(2)}_{\sigma\sigma_1}\left(\frac{j-1}{2^{m-1}}\right)=\Phi_{\sigma\sigma_1}\left(\frac{2j-2}{2^m}\right)+\Phi_{\sigma\sigma_1}\left(\frac{2j-1}{2^m}\right)(j=1,2,\ldots,2^{m-1}). \qquad (32)$$

From (31) and (32) we have that $c^{(j)}_{m-2}(\sigma\sigma_1)=0$ if and only if

$$\Phi^{(2)}_{\sigma\sigma_1}\left(\frac{2j-2}{2^{m-1}}\right)-\Phi^{(2)}_{\sigma\sigma_1}\left(\frac{2j-1}{2^{m-1}}\right)=0 \quad (j=1,2,\ldots,2^{m-2}). \qquad (33)$$

From the comparison of (33) and (14), and considering (10), it is evident that the problem of finding $\sigma_2\sigma_1$ reduces to the problem of finding σ_1 for the function $\Phi^{(2)}_{\sigma\sigma_1}\left(\frac{x}{2^{m-1}}\right)$, which has been solved for $\Phi_\sigma\left(\frac{x}{2^m}\right)$ in the preceding points. Here the matrix σ_1 for the function $\Phi^{(2)}_{\sigma\sigma_1}\left(\frac{x}{2^{m-1}}\right)$ is of the order $(m-1)\times(m-1)$; we denote this matrix by $\sigma_1^{(2)}$.

Now, in order to satisfy condition (10), it is sufficient to put

$$\sigma_2=\left\|\begin{array}{c:c}\sigma_1^{(2)} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ \hdashline 0\ \ldots\ 0 & 1\end{array}\right\|. \qquad (34)$$

In effect, the following proposition holds.

Theorem 4. For arbitrary $\sigma_i \in \Xi$ such that

$$\sigma_i=\left\|\begin{array}{c:c}\sigma_\nu & 0\\ \hdashline 0 & E_p\end{array}\right\|,$$

where σ_ν is a matrix of order $(m-p)\times(m-p)$, E_p is the unit matrix of order $p\times p$, and for arbitrary $\sigma_r \in \Xi$, we have that

$$L_{m-j}(\sigma_r)=L_{m-j}(\sigma_i\sigma_r) \quad (j=1,2,\ldots,p).$$

Formula (34) follows from Theorem 4 for $p=1$.

5) We now consider the question of finding σ_3. We form the function

$$\Phi^{(3)}_{\sigma\sigma_2\sigma_1}\left(\frac{j-1}{2^{m-2}}\right)=\Phi^{(2)}_{\sigma\sigma_2\sigma_1}\left(\frac{2j-2}{2^{m-1}}\right)+\Phi^{(2)}_{\sigma\sigma_2\sigma_1}\left(\frac{2j-1}{2^{m}}\right)$$
$$(j=1,2,\ldots,2^{m-2}).$$