Deadlock Prevention in Multiprocessor Systems with Wormhole Routing

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Abstract

In this chapter the problem of constructing minimal cycle-breaking connectivity preserving sets of turns for graphs that model regular or near regular multiprocessor systems, as a method to prevent deadlocks is investigated. Cycle-breaking provides for deadlock-free wormhole routing defined by turns prohibited at some nodes. The lower and upper bounds for minimal cardinalities of cycle-breaking connectivity preserving sets for several classes of graphs such as homogeneous meshes, p-ary n-cubes, cube-connected cycles, hexagonal and honeycomb meshes and tori, Hamiltonian graphs and others are obtained and presented along with some preliminary experimental results.

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1. Introduction

In previous chapter, we analyzed communication networks that are irregular, where nodes have arbitrary number of adjacent neighbor nodes. Because of the way they evolve in an ad hoc manner, networks of workstations (NOWS) are as a rule irregular. This chapter is devoted to procedures that guarantee deadlock-free wormhole routing in multiprocessor systems with regular or almost regular interconnection topologies. The approach is based on minimizing the number of turns that are prohibited and therefore are not available for routing. The regularities in the structure of networks make it possible to derive simple and efficient solutions for the turn prohibition problem. Thus, the general algorithms developed previously for arbitrary topologies, e.g., (Dally & Seitz, 1987; Boppana & Chalasani, 1993; Chalasani & Boppana, 1995; Duato, Yalamancili, & Ni, 1997; Ni & McKinley, 1993), (Dally & Seitz, 1986), (Duato, 1993), (Dally & Aoki, 1997), (Zakrevski, Jaiswal, Levitin, & Karpovsky, 1999; Zakrevski, Jaiswal, & Karpovsky, 1999; Zakrevski, Mustafa, & Karpovsky, 2000; Lysne, Skeie, Reinemo, & Theiss, 2006; Schroeder, Birrel, Burrows, Murray, Needham, Rodeheer, et al., 1990; Sancho & Robles, 2000), (Skeie, Lysne, & Theiss, 2002; Sancho et al., 2004; Pellegrini et al., 2004, 2006), (Mustafa et al., 2005; Levitin et al., 2006; Levitin et al., May, 2009, 2010) are not used here. Instead, optimal or asymptotically optimal solutions of the turn prohibition problem for general classes of special topologies, prevalent in multiprocessor systems are presented. These solutions are obtained by application of simple rules, run-time complexities of which do not exceed $O(N)$ (i.e., linear in the number of nodes N), and, in many cases, is $O(1)$ (i.e., constant). The memory requirements for computing the solutions do not exceed $O(\log N)$. The proposed turn prohibition rules can be easily implemented for execution in a distributed way.

It should be pointed out that turn prohibition algorithms are, in fact, pre-routing procedures; they do no prescribe any specific routing policy, but just restrict the set of turns permitted for use in routing tables. Therefore, they are compatible with any routing algorithm, in particular, with the fully adaptive minimal routing (of course, paths that include prohibited turns are excluded from consideration during the construction of the routing tables).

A few particular regular topologies have been considered in several papers (Glass & Ni, 1994; Horst, 1996; Decayeux & Seme, 2005; Nocetti, Stojmenovic, & Zhang, 2002; Parhami & Kwai, 2001; Stojmenovic, 1997; Dolter, Ramanathan, & Shin, 1991), (Dally & Seitz, 1986). This chapter presents methods applicable to a number of classes of popular regular graphs, such as homogeneous meshes, p-ary n-cubes, cube connected cycles, hexagonal and honeycomb meshes and tori and Hamiltonian graphs.

The dimension-ordered routing (DOR) (Min et al., 2004) has been popular for meshes. However, as shown in Section 3, the use of DOR algorithm results in prohibition of much larger fraction of turns in the network than the approach developed in the present chapter. For multi-dimensional meshes, the fraction of turns prohibited by DOR tends to 1/2. Methods developed in this chapter guarantee that the fraction of prohibited turns never exceeds 1/4.

Section 2 introduces and studies embedded graphs and homogeneous meshes. A number of well known regular topologies are analyzed in Section 3. Section 4 discusses the dilation of the average distances as a result of turn prohibitions. Conclusions are presented in Section 5.

Certain notations, definitions, lower bounds, and other basic graph theoretic concepts used in this chapter are presented in Section 2 of the previous chapter

2. Embedded Graphs and Homogeneous Meshes

Consider a graph $G=(V,E)$ which is embedded in the n-dimensional real space R^n, so that each node $\mathbf{x}$ is a point in R^n.

Definition 1 *An embedded graph G is a* ***homogeneous mesh****, if each node $\mathbf{x}$ has a degree $d=2t$, and if $\mathbf{x}\in V$, then its neighbors are nodes $\mathbf{x}\pm\mathbf{a}_i, i=1,2,\ldots,t$, where $\mathbf{a}_i$ are vectors in R^n and elements of a set $D=\{\pm\mathbf{a}_i, i=1,\ldots,t\}$.*

Several important topologies, such as multi-dimensional meshes and tori, can be embedded into n-dimensional real spaces and can be considered as homogeneous meshes.

We call $\mathbf{a}\in D$ **positive**, $\mathbf{a}>0$, if the first non-zero component of $\mathbf{a}$ is positive, otherwise $\mathbf{a}$ is negative, $\mathbf{a}<0$. For example, in a two dimensional space, $(0,1)>0$, and $(-1,1)<0$.

Consider the following turn prohibition rule for homogeneous meshes. Turn $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)=(\mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2)$ is prohibited *iff* $\mathbf{x}_2-\mathbf{x}_1<0$ and $\mathbf{x}_3-\mathbf{x}_2<0$. Let $W(M_D)$ be a set of prohibited turns for a homogeneous mesh M_D.

Theorem 1 *As described, the turn prohibition rule has the following properties.*

1. For any mesh M_D and any $\mathbf{x},\mathbf{y}\in V$ there exists a path from $\mathbf{x}$ to $\mathbf{y}$ not containing any turns from $W(M_D)$.

2. For any cycle in M_D there exists a turn which belongs to the cycle and also belongs to $(W(M_D)$, the set of prohibited turns.

3. The set of prohibited turns is minimum

4. The minimum fraction of prohibited turns for a homogeneous mesh M_D with size of D equal to d is

$$z(G) = \frac{1}{4}\left(1 - \frac{1}{d-1}\right). \tag{4}$$

Proof.

1. Consider a path $P = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k)$ from node $\mathbf{x}_0$ to $\mathbf{x}_k$, where $\mathbf{x}_{i+1} = \mathbf{x}_i + \mathbf{b}_i$; $i = 0, \ldots, k-1$, $\mathbf{b}_i \in D$. The corresponding sequence of edges is $S = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{b}_{k-1})$. Note that path P is prohibited *iff* there exists a pair of consecutive edges $(\mathbf{b}_{i-1}, \mathbf{b}_i)$ in S such that $\mathbf{b}_{i-1} > 0$ and $\mathbf{b}_i < 0$. It follows from Definition 1 that if S forms a path from $\mathbf{x}_0$ to $\mathbf{x}_k$, then any permutation of $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{k-1}$ also corresponds to a path from $\mathbf{x_0}$ to $\mathbf{x_k}$, since the mesh is homogeneous and $\mathbf{x}_k = \mathbf{x}_0 + \sum_{i=0}^{k-1} \mathbf{b}_i$. Then there exists a permutation $S' = (\mathbf{b}'_0, \mathbf{b}'_1, \ldots, \mathbf{b}'_{k-1})$ of S in which all negative vectors (if any) precede all positive ones (if any). The corresponding path $P' = (\mathbf{x}_0, \mathbf{x}'_1 = \mathbf{x}_0 + \mathbf{b}'_0, \ldots, \mathbf{x}_k = \mathbf{x}'_{k-1} + \mathbf{b}'_{k-1})$ has no prohibited turns and thus, nodes $\mathbf{x}_0$ and $\mathbf{x}_k$ are connected.

2. Consider a cycle $C = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_0, \mathbf{x}_1)$ and the corresponding cycle of edges $S = (\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{b}_0)$, where $\mathbf{b}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$, $i = 0, 1, \ldots, k-1$; $\mathbf{b}_k = \mathbf{x}_0 - \mathbf{x}_k$. Note that $\sum_{i=0}^{k} \mathbf{b}_i = 0$. Therefore, among vectors $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_k$ must be both positive and negative ones. Since sequence S starts and ends with the same vector (either positive or

negative), it must include at least one pair $\mathbf{b}_{i-1}, \mathbf{b}_i$, where $\mathbf{b}_{i-1}$ is positive and $\mathbf{b}_i$ is negative. Thus, the corresponding cycle is prohibited.

3. Let us consider cycles of length four, $C = (\mathbf{x}_0, \mathbf{x}_1, \ \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_0, \mathbf{x}_1)$, where $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{b}_0$, $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{b}_1$, $\mathbf{x}_3 = \mathbf{x}_2 - \mathbf{b}_0 = \mathbf{x}_3 + \mathbf{b}_1$. All sets of turns corresponding to different choices of nodes $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$ are disjoint. Hence, in order to break all cycles, it is necessary to prohibit at least one turn in each of such cycles. Indeed, according to our prohibition rule, in the sequence of edges $(\mathbf{b}_0, \mathbf{b}_1, -\mathbf{b}_0, -\mathbf{b}_1, \mathbf{b}_0)$ exactly one turn is prohibited (e.g., if $\mathbf{b}_0, \mathbf{b}_1 > 0$, then turn $(\mathbf{b}_1, \mathbf{b}_0)$ is prohibited). Thus, the set of prohibited turns is the smallest possible.

4. Obviously, in the set $D = \{\pm \mathbf{a}_i, i = 1, \dots, t\}$ exactly $t = \frac{d}{2}$ vectors are positive, and the other half are negative. Therefore

$$z(G) = \frac{\binom{d/2}{2}}{\binom{d}{2}} = \frac{1}{4}\left(1 - \frac{1}{d-1}\right). \qquad +$$

Remarkably, the result 4 does not depend on the choice of the coordinate system and on the particular topology of the mesh. For example, Figure 1 shows two different topologies which have the same node degree d and, thus, the same $z(G)$.

Figure 1. Different topologies with the same degree $d = 6$ have the same fraction $z(G)$.

It is interesting to compare (4) with the fraction of prohibited turns when one uses the popular DOR algorithm [28]. For the case of an n-dimensional mesh the fraction of prohibited turns given by (4) is $\frac{n-1}{2(2n-1)}$. The DOR algorithm prohibits a portion of the turns equal to $\frac{n-1}{2n-1}$, i.e., twice as large as our approach.

A more general situation can be described as follows. Consider an embedded graph $G=(V,E)$ that consists of m different types of nodes, $V=\bigcup_{k=1}^{m} V_k$ such that all nodes of type k have the same degree d_k, and if $\mathbf{x} \in V_k$, then its neighbors are $\mathbf{x}+\mathbf{a}_{ki}$, $i=1,2,\ldots,d_k$. Let $d_k=d_k^{(+)}+d_k^{(-)}$, where $d_k^{(+)}$ and $d_k^{(-)}$ are the numbers of positive and negative vectors, respectively, in the set $A_k=\{\mathbf{a}_{ki}\}$. We call such embedded graphs *multicomponent meshes*.

Suppose we prohibit all turns $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$, such that $\mathbf{x}_1-\mathbf{x}_2<0$ and $\mathbf{x}_3-\mathbf{x}_2<0$, or, alternatively, such that $\mathbf{x}_1-\mathbf{x}_2>0$ and $\mathbf{x}_3-\mathbf{x}_2>0$. Let us call such turns "negative" or, respectively, "positive". Assuming that the connectivity is preserved and following the same reasoning, as in the proof of Theorem 1, we obtain Corollary 1.

Corollary 1 *Prohibition of all negative or of all positive turns in graph G described above breaks all the cycles in G. The fraction of prohibited turns $z(G)$ obeys an upper bound*

$$z(G) \le \frac{\min\left\{\sum_{k=1}^{m}\rho_k \binom{d_k^{(-)}}{2}, \sum_{k=1}^{m}\rho_k \binom{d_k^{(+)}}{2}\right\}}{\sum_{k=1}^{m}\rho_k \binom{d_k}{2}}, \tag{5}$$

where ρ_k is the density of nodes of type k.

Here, as usual (Honkala, Karpovsky, & Levitin, 2006) the density ρ_k of a subset V_k of nodes in an infinite embedded graph $G(V, E)$ is defined as follows. Consider a ball $B(R)$ of radius R in R^n. Then

$$\rho_k = \limsup_{R\to\infty} \frac{|V_k \bigcap B(R)|}{|V \bigcap B(R)|}.$$

Note that if $\mathbf{y} = \mathbf{x} + \mathbf{a}$, where $\mathbf{a} > 0$, then $\mathbf{x} = \mathbf{y} + \mathbf{b}$, where $\mathbf{b} = -\mathbf{a} < 0$. Therefore, $\sum_{k=1}^{m} \rho_k d_k^{(-)}$. However, for some structures prohibition of positive vs. negative turns can give rather different results, as shown by Example 1.

Example 1 *The embedded graph in Figure 2 has three different types of nodes with degrees 2, 3, and 5, each with a density of* $\rho = 1/3$. *As shown in the enlarged view, all positive turns prohibited at the node of degree 5, and all negative turns prohibited at nodes of degree 2 and degree 3. Prohibition of negative and positive turns yields different fractions of prohibited turns equal to* $3/7$ *and* $1/7$, *respectively.*

Figure 2. A multicomponent mesh with three different types of nodes of degrees 2, 3, and 5. In the enlarged view we show all positive turns prohibited at the node of degree 5, and all negative turns prohibited at nodes of degree 2 and degree 3.

Example 2 *The embedded graph called the "Brick Mesh" is shown in Figure 3. There are five types of nodes in this mesh; type 1 nodes are of degree 4, and type 2, type 3, type 4, and type 5 nodes are of degree 3, as shown in the insert. Considering the building block of this mesh, shown as the darker rectangular region in the figure, one* *determines that the density of each of the degree 3 node types is 1/6 and the density of the degree 4 node type is 1/3. If we consider the prohibition of the negative turns as shown in the enlarged view in Figure 3 , we determine that the fraction of prohibited turns is* $z \leq 1/6$.

Figure 3. Multicomponent "Brick Mesh" in which degree 3 and degree 4 nodes have densities of 1/6 and of 1/3 respectively. Five different node types are identified in the enlarged view by the numbers adjacent to the nodes.

Another interesting topology is the honeycomb mesh (see Section 3, Figure 7 (B)).

In general, the bounds in (5) depend on the choice of the coordinate system, in particular, on the order of the coordinates.

Note also that the prohibition rule given above for a multicomponent mesh does not guarantee, in general, the preservation of connectivity. However, it can be shown that for a two-component mesh ($m = 2$) connectivity is always preserved, provided that $d_k^{(+)} > 0$ and $d_k^{(-)} > 0$ for $k = 1,2$. For example, for the honeycomb mesh (Figure 7 (B)), $m = 2$, $d_1^{(+)} = 2$, $d_1^{(-)} = 1$, $d_2^{(+)} = 1$, $d_2^{(-)} = 2$, $\rho_1 = \rho_2 = 1/2$ and $z(G) = 1/6$ (see Section 3).

Homogeneous meshes considered so far in this section are of infinite extent with infinite number of nodes. Now finite D-Meshes $M_D(p_1,\ldots,p_n)$ and finite wraparound D-meshes $M_D^W(p_1,\ldots,p_n)$ will be defined.

Let $D=\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\ldots,\pm\mathbf{a}_t\}$, $\mathbf{a}_i\in\mathrm{R}^n$, $i=1,2,\ldots,t$ and $d=2t$ be the degree of every node. Then $n\le t$, (otherwise the mesh is embedded in a space of a smaller dimensionality), and there are n linearly independent vectors in D. Henceforth we will assume that there exists a basis $B=\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}$, $B\subseteq D$ such that any point in the mesh can be represented as a linear combination of vectors from B with integer coefficients. Denote $\mathbf{C}=\mathbf{A}^{-1}$ where $\mathbf{A}$ is the matrix with columns $\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_n$. Then any node $\mathbf{x}$ in the mesh can be represented in basis B as $\tilde{\mathbf{x}}=\mathbf{Cx}=(\tilde{x}^{(1)},\tilde{x}^{(2)},\ldots,\tilde{x}^{(n)})$, where all $\tilde{x}^{(i)}$ are integers, $i=1,2,\ldots,n$.

Let $p_1,p_2,\ldots,p_n$ be positive integers, $p_i\ge 2$, $i=1,2,\ldots,n$.

Definition 2 *A graph $G(V,E)$ is a finite D-mesh $M_D(p_1,p_2,\ldots,p_n)$ if $V=\{\mathbf{x}\mid \tilde{x}^{(i)}\in\{0,1,\ldots,p_i-1\}, i=1,\ldots,n\}$. Then $(\mathbf{x},\mathbf{y})\in E$ if $\mathbf{C}(\mathbf{x}-\mathbf{y})\in D_{\mathbf{C}}$ or $\mathbf{C}(\mathbf{y}-\mathbf{x})\in D_{\mathbf{C}}$, where $D_{\mathbf{C}}=\{\pm\mathbf{Ca}_i\mid i=1,\ldots,t\}$*

$=\{\pm(1,0,0,\ldots,0),\pm(0,1,0,\ldots,0),\ldots,\pm(0,0,0,\ldots,1),\ \pm\mathbf{Ca}_{n+1},\ldots,\pm\mathbf{Ca}_t\}$.

Example 3 *Let $n=2$ and*

$D=\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3\}=\left\{\pm\left(\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right),\pm\left(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right),\pm(1,0)\right\}$. *Note that $\mathbf{a}_3=\mathbf{a}_1-\mathbf{a}_2$, and*

$$\mathbf{A}=\begin{bmatrix}\tfrac{1}{2} & -\tfrac{1}{2}\\ \tfrac{\sqrt{3}}{2} & \tfrac{\sqrt{3}}{2}\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}1 & \tfrac{\sqrt{3}}{3}\\ -1 & \tfrac{\sqrt{3}}{3}\end{bmatrix},$$

and $D_{\mathbf{C}}=\{\pm(1,0),\pm(0,1),\pm(1,-1)\}$. The finite mesh $M_D(4,3)$ is shown in Figure 4 .

Figure 4. Finite D-Mesh $M_D(4,3)$ with $D=\left\{\pm\left(\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right),\pm\left(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right),\pm(1,0)\right\}$ and $D_{\mathbf{C}}=\{\pm(1,0),\pm(0,1),\pm(1,-1)\}$.

Next finite wraparound meshes $M_D^W(p_1,p_2,\ \ldots,p_n)$ are defined. Let p_i be positive integers larger than 2. We will also assume that for the set $D=\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\ldots,\pm\mathbf{a}_t\}$, $(\mathbf{a}_i\in \mathrm{R}^n, n\le t)$, vectors $\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_n\}$ are linearly independent and each $\mathbf{a}_{n+j}=\sum_{i=1}^n u^{(i)}\mathbf{a}_j\,(j=1,\ldots,t-n)$, where $c^{(i)}$ are integers, such that $|u^{(i)}|\le p_i-1$. Let $\mathbf{U}_1=\left(u_1^{(1)},u_1^{(2)},\ldots,u_1^{(n)}\right)$ and $\mathbf{U}_2=\left(u_2^{(1)},u_2^{(2)},\ldots,u_2^{(n)}\right)$ be vectors with $u_1^{(i)},u_2^{(i)}$ $\in\{0,1,\ldots,p_i-1\}$. Denote $\mathbf{U}_3=\mathbf{U}_1\oplus\mathbf{U}_2$, if $u_3^{(i)}=\ u_1^{(i)}+u_2^{(i)}\bmod p_i$, $i=1,2,\ldots,n$.

Definition 3 *A graph $G(V,E)$ is a wraparound D-Mesh $M_D^W(p_1,p_2,\ldots,p_n)$ if $V=\{\mathbf{x}\,|\,\tilde{x}^{(i)}\in\{0,1,\ldots,p_i-1\},i=1,2,\ldots,n\}$ and the edge $(\mathbf{x},\mathbf{y})\in E$ if there exists a vector $\mathbf{h}$ such that $\tilde{\mathbf{x}}\oplus\tilde{\mathbf{h}}=\tilde{\mathbf{y}}$, and $\tilde{h}=\tilde{b}$ for some $b\in D$. (Here, $\tilde{\mathbf{x}}=\mathbf{Cx}$, $\tilde{\mathbf{h}}=\mathbf{Ch}$, $\tilde{\mathbf{y}}=\mathbf{Cy}$, and $\tilde{\mathbf{b}}=\mathbf{Cb}$.)*

Example 4 *Let $n=2$ and $D=\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3\}=\left\{\pm\left(\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right),\pm\left(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right),\pm(1,0)\right\}$. As in Example 3, select $\mathbf{a}_1=(1/2,\sqrt{3}/2)$ and $\mathbf{a}_2=(-1/2,\sqrt{3}/2)$. Then $\mathbf{C}=\begin{bmatrix}1 & \tfrac{\sqrt{3}}{3}\\ -1 & \tfrac{\sqrt{3}}{3}\end{bmatrix}$,*

$\mathbf{a}_3=\mathbf{a}_1-\mathbf{a}_2$, and $\mathbf{Ca}_3=\tilde{\mathbf{a}}_3=(1,-1)$. With this neighborhood definition, the wraparound mesh $M_D^W(5,4)$ is shown in Figure 5 . This wraparound mesh has five wraparound cycles $(\tilde{\mathbf{x}},\tilde{\mathbf{x}}\oplus(0,1),\tilde{\mathbf{x}}\oplus 2\cdot(0,1),\tilde{\mathbf{x}}\oplus 3\cdot(0,1),\tilde{\mathbf{x}})$ of length 4, where $\oplus$ stands for addition of vectors such that first components are added modulo 5 and the second components are added modulo 4, four wraparound cycles $(\tilde{\mathbf{x}}, \tilde{\mathbf{x}} \oplus (1,0), \tilde{\mathbf{x}} \oplus 2\cdot(1,0), \tilde{\mathbf{x}} \oplus 3\cdot(1,0), \tilde{\mathbf{x}} \oplus 4\cdot(1,0), \tilde{\mathbf{x}})$ of length 5, and one wraparound cycle $(\tilde{\mathbf{x}}, \tilde{\mathbf{x}} \oplus (-1,1), \tilde{\mathbf{x}} \oplus 2\cdot(-1,1), \tilde{\mathbf{x}} \oplus 3\cdot(-1,1), \ldots, \tilde{\mathbf{x}} \oplus 19\cdot(-1,1), \tilde{\mathbf{x}})$ of length 20. In the figure a path from node $\tilde{\mathbf{x}} = (3,1)$ to node $\tilde{\mathbf{y}} = (1,2)$, $P = \big((3,1),(2,1),(1,1),(1,2)\big)$ is shown using thick lines. Note that all turns along this path are permitted.

Figure 5. Wraparound D-Mesh $M_D^W(5,4)$ with $D =$ $\{\pm\left(-1/2, \sqrt{3}/2\right), \pm(1,0)\}$, $D_{\mathbf{C}} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ and a permitted path from node $(3,1)$ to $(1,2)$.

To construct sets of prohibited turns for $M_D(p_1, P_2, \ldots, p_n)$ or $M_D^W(p_{1,2}, \ldots, p_n)$ we will introduce a total ordering of nodes in these meshes.

Definition 4 *If* $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in V$ *where* V *is the set of nodes in* $M_D(p_1, p_2, \ldots, p_n)$ *or* $M_D^W(p_1, p_2, \ldots, p_n)$, *we will say that* $\tilde{\mathbf{x}} > \tilde{\mathbf{y}}$ *if* $\tilde{\mathbf{x}}^{(i)} > \tilde{\mathbf{y}}^{(i)}$ *where* i *is the smallest integer such that* $\tilde{\mathbf{x}}^{(i)} \neq \tilde{\mathbf{y}}^{(i)}$ $(\tilde{\mathbf{x}} = \mathbf{Cx},\ \ \tilde{\mathbf{y}} = \mathbf{Cy})$.

Theorem 2 *For a finite mesh* $M_D(p_1, p_2, \ldots, p_n)$ *or a wraparound mesh* $M_D^W(p_1, \ldots, p_n)$, *let the set of prohibited turns* $F = \{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}) \mid \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}} \in V$ *and* $\tilde{\mathbf{y}} > \tilde{\mathbf{x}}$, $\tilde{\mathbf{y}} > \tilde{\mathbf{z}}\}$. *Then*

1. For any $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in V$ there exists a path from $\tilde{\mathbf{x}}$ to $\tilde{\mathbf{y}}$ containing no turns from F.
2. For any cycle there exists a turn in the cycle that belongs to F.

3. The set F is asymptotically optimal if $p_i \to \infty$ $(i = 1, \ldots, n)$, and the minimum fraction z of prohibited turns for $M_D(p_1, p_2, \ldots, p_n)$ or a $M_D^W(p_1, p_2, \ldots, p_n)$ with $|D| = d$ is, asymptotically,

$$\lim_{\substack{p_i \to \infty \\ i=1,\ldots,n}} z = \frac{1}{4}\left(1 - \frac{1}{d-1}\right).$$

Proof.

1. First we will prove that if $\tilde{\mathbf{x}} = \mathbf{Cx} = (\tilde{\mathbf{x}}^{(1)}, \tilde{\mathbf{x}}^{(2)}, \ldots, \tilde{\mathbf{x}}^{(n)})$ and $\tilde{\mathbf{y}} = \mathbf{Cy}$ $= \left(\tilde{\mathbf{y}}^{(1)}, \tilde{\mathbf{y}}^{(2)}, \ldots, \tilde{\mathbf{y}}^{(n)}\right)$, there exists a path from $\tilde{\mathbf{x}}$ to $\tilde{\mathbf{y}}$ in $M_D(p_1, p_2, \ldots, p_n)$ or in $M_D^W(p_1, p_2, \ldots, p_n)$ containing no turns from F. Let $S_+(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \{i | \tilde{\mathbf{x}} \geq \tilde{\mathbf{y}}\}$ and $S_-(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \{i | \tilde{\mathbf{x}} < \tilde{\mathbf{y}}\}$. Consider now a node $\tilde{\mathbf{z}}$ such that $\tilde{\mathbf{z}}^{(i)} = \min(\tilde{\mathbf{x}}^{(i)}, \tilde{\mathbf{y}}^{(i)})$. Obviously, there exists a path from $\tilde{\mathbf{x}}$ to $\tilde{\mathbf{z}}$, such that any next node in the path is smaller than the previous one. Similarly, there exists a path from $\tilde{\mathbf{z}}$ to $\tilde{\mathbf{y}}$ such that any next node is larger than the previous one. Now take the concatenation of these two paths. The turn at node $\tilde{\mathbf{z}}$ is permitted, since $\tilde{\mathbf{z}}$ is smaller than the two neighboring nodes in the path. Thus, there exists a permitted path from $\tilde{\mathbf{x}}$ to $\tilde{\mathbf{y}}$.

2. In every cycle $(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_{(\ell-1)}, \tilde{\mathbf{x}}_\ell)$ where $\tilde{\mathbf{x}}_{(\ell-1)} = \tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_\ell = \tilde{\mathbf{x}}_2$ there exists $i \in \{1, 2, \ldots, \ell\}$ such that $\tilde{\mathbf{x}}_\mathbf{i} > \tilde{\mathbf{x}}_{(\mathbf{i}-1)}$, $\tilde{\mathbf{x}}_\mathbf{i} > \tilde{\mathbf{x}}_{(\mathbf{i}+1)}$, and turn $(\tilde{\mathbf{x}}_{(\mathbf{i}-1)}, \tilde{\mathbf{x}}_\mathbf{i}, \tilde{\mathbf{x}}_{(\mathbf{i}+1)}) \in F$.

3. We will say that the node $\tilde{\mathbf{x}} \in V$ is internal in $M_D(p_1, p_2, \ldots, p_n)$ or in $M_D^W(p_1, p_2, \ldots, p_n)$ if $0 < \tilde{\mathbf{x}}^{(i)} < p_i - 1$ for all $i = 1, \ldots, n$. If $\mathbf{x}$ is an internal node, then in each pair of its neighbors, $\mathbf{x} \pm \mathbf{a}_i$ $(i = 1, \ldots, t)$ one neighbor is larger than $\mathbf{x}$ and the other is smaller than $\mathbf{x}$. Thus for any internal node $\mathbf{x}$ exactly t neighbors are larger than $\mathbf{x}$,

and exactly t neighbors are smaller than $\mathbf{x}$. Hence, for every internal node $\mathbf{x}$ there are $\binom{t}{2}$ turns $(\mathbf{y},\mathbf{x},\mathbf{z})$ which belong to F. Thus,

$$\lim_{\substack{p_i\to\infty\\ i=1,\dots,n}} z \le \frac{\binom{t}{2}}{\binom{2t}{2}} = \frac{1}{4}\left(1-\frac{1}{d-1}\right).$$

On the other hand, similar to the proof of Theorem 1, for any internal node $\mathbf{x}$ there are $\binom{t}{2}$ cycles in $M_D(p_1,p_2,\dots,p_n)$ or in $M_D^W(p_1,p_2,\dots,p_n)$ which contain 4 nodes each that do not have common turns. In the union of these sets for all internal nodes any two cycles do not have common turns. Since ate most $\binom{d}{2}$ turns are prohibited at any non-internal node, the contribution of the non-internal nodes to Z does not exceed their fraction among all nodes, and, therefore, is infinitesimal when $p_i\to\infty \quad (i=1,\dots,n)$. Thus, it follows that

$$\lim_{\substack{p_i\to\infty\\ i=1,\dots,n}} z \ge \frac{1}{4}\left(1-\frac{1}{d-1}\right). \qquad +$$

The set $F=W(M_D(4,3))$ of prohibited turns for the $M_D(4,3)$ with $D=\{\pm\left(1/2,\sqrt{3}/2\right),\ \pm\left(-1/2,\sqrt{3}/2\right),\pm(1,0)\}$ is shown in Figure 4 .

3. Special Topologies

3.1 Finite Meshes and Tori

Meshes and tori have been the most widely used communication network topologies for multiprocessors (Ni & McKinley, 1993; Parhami, 1998). Most recently, ”TOFU”, a 6-dimensional mesh and torus topologies have been used to provide the extremely high performance and fault tolerant interconnection network, achieving 10 petaflops (Ajima, Sumimoto, & Shimizu, 2009). In this section, square meshes are considered first, with each inner node connected with *2n* nodes, where *n* is the dimension of a mesh. Meshes of this type were investigated in (Glass & Ni, 1994), where only 90-degree turns were taken into account. It was shown, that 1/4 of all such turns has to be prohibited. With the more general turn model, our results are in agreement with authors' conclusion in (Glass & Ni, 1994).

Theorem 3 *For n-dimensional p-ary mesh,* M_p^n

$$z(M_p^n) = \frac{(n-1)(p-1)^2}{2p(p-2)+4(n-1)(p-1)^2}, \tag{6}$$

and for n-dimensional p-ary tori, Z_p^n, with $p > 2$,

$$z(Z_p^n) = \frac{p(n-1)+2}{2p(2n-1)}. \tag{7}$$

Proof. To prove the lower bound for meshes we consider the system of all cycles of length 4. There are $R = \binom{n}{2}(p-1)^2 p^{n-2}$ turn disjoint cycles of this type and the total number of turns in M_p^n is equal to

$$T(M_p^n) = n(p-2)p^{n-1} + 4\binom{n}{2}(p-1)^2 p^{n-2}. \tag{8}$$

The lower bound for $Z(M_p^n)$ follows now by observing that at least as many turns must be prohibited as there are turn disjoint cycles.

The lower bound for tori can be proven in a similar way by considering cycles of length 4 and $np^n - 1$ one-dimensional cycles, containing nodes with fixed $n-1$ coordinates.

To prove the upper bound of Theorem 3 for p-ary meshes, we prohibit all turns $(\mathbf{a},\mathbf{b},\mathbf{c})$, where $l(\mathbf{a}) < l(\mathbf{b})$, $l(\mathbf{b}) > l(\mathbf{c})$ and $l(\mathbf{a}), l(\mathbf{b}), l(\mathbf{c})$ are distances in terms of number of hops from node $(0,0,\ldots,0)$ to $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$. The number of prohibited turns is equal to

$$Z(M_p^n) = \binom{n}{2}(p-1)^2 p^{n-2}. \tag{9}$$

Then (6) follows from (8) and (9).

For p-ary tori, each node $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is labeled by $l(\mathbf{a}) = a_1 + a_2 + \ldots, a_n \bmod p$ and the turn $(\mathbf{a},\mathbf{b},\mathbf{c})$ is prohibited if $l(\mathbf{a}) = l(\mathbf{c}) = l(\mathbf{b}) - 1 \bmod p$.

The total number of turns in Z_p^n is equal to $T(Z_p^n) = \binom{2n}{2} p^n$, and the number of prohibited turns is equal to $Z(Z_p^n) = \binom{n}{2} p^n + np^n - 1$. +

3.2 Hexagonal and Honeycomb Meshes

Next, we consider hexagonal meshes (Parhami, 1998; Decayeux & Seme, 2005; Nocetti et al., 2002) in which each node has up to 6 neighbors and honeycomb meshes (Parhami, 1998; Parhami & Kwai, 2001; Stojmenovic, 1997) where each node has up to 3 neighbors, and their corresponding tori. In a hexagonal mesh of size p denoted by HeM_p, peripheral edges form a regular hexagon where each side has p nodes. A honeycomb mesh of size *p*, denoted by HoM_p, where each side of the mesh has *p* hexagonal cells whose centers also form a regular hexagon. The hexagonal and honeycomb tori are degree six and degree three regular topologies, respectively.

In a hexagonal mesh HeM_p, there are $N = 3p^2 - 3p + 1$ nodes with labels $0,1,\ldots,(N-1)$ with the center node having the label 0 (Dolter et al., 1991). Adjacent nodes of any given node a are identified to have labels $a \pm 1$, $a \pm (3p-1)$, $a \pm (3p-2)$

where arithmetic operations are $\bmod N$. In the corresponding torus, wrap-around edges are also identified using the same adjacency rules. Labels of adjacent nodes are shown in Figure 6 (A) for the case of a size $p = 3$ torus.

Figure 6. Examples hexagonal torus HeT_3 in (A), and honeycomb torus HoT_3 in (B) for $p = 3$, where wraparound links are identified.

In a honeycomb torus, nodes that are connected by the wrap-around edges are those nodes that are mirror symmetric with respect to the three lines passing through the center and normal to each of three edge orientations (Stojmenovic, 1997). These axes are shown as dashed lines in Figure 6(B).

Theorem 4 *For a hexagonal mesh of size* p, HeM_p, *with* $N - 3p^2 - 3p + 1$ *nodes,*

$$z(HeM_p) = \frac{9p^2 - 21p + 13}{45p^2 - 99p + 51}, \tag{10}$$

and for a hexagonal tori of size p,

$$z(HeT_p) = \frac{9p^2 - 15p + 10}{45p^2 - 45p + 15}. \tag{11}$$

Proof. First, note that total number of turns in a HeM_p is equal to:

$T(HeM_p) = 15(3p^2 - 9p + 7) + 6(6p - 12) + 18 = 45p^2 - 99p + 51$.

To prove the lower bound, we consider the set of all turn disjoint $6(p-1)$ triangles, and $3p^2 - 9p + 7$ hexagons and observe that we must prohibit at least as many turns as there are turn disjoint cycles, e.g., triangles and hexagons.

Upper bound on $Z(HeM_p)$ can be obtained as shown in Figure 7 (A).

For the case of hexagonal tori with $N(HeT_p) = 3p^2 - 3p + 1$ nodes, $M(HeT_p) = 3N(HeT_p)$ edges, and $T(HeT_p) = 15N(HeT_p)$ turns, additional $6(2p-1)$ turns have to be prohibited to prevent all wrap-around cycles. Therefore, $6(2p-1)$ cycles must be added to the system of turn-disjoint cycles due to triangles and hexagons. Again, observe that we must prohibit at least as many turns as there are turn disjoint cycles. To prove the upper bound, we cut the wrap-around cycles in the hexagonal torus and prohibit all $6(2p-1)$ turns at the nodes on the border of the resulting mesh.

Figure 7. Prohibited turns for Hexagonal (A) and Honeycomb (B) meshes.

3.3 Locally Complete Tree-Like Topologies

Locally complete tree-like topologies are hybrid topologies incorporating the properties and attributes of its components (Parhami, 1998). Consider a tree $T' = G'(V', E')$ with $M' = |E'|$ undirected edges $\{v_i, v_j\} \in E'$ and $N' = |V'|$ nodes v_i, $i = 0, \ldots, N'-1$. Assume that each node of the tree is now replaced with a complete graph K_n with $n \geq d_i$ nodes where d_i is the degree of node v_i of the original tree T', to obtain the augmented graph $G(V, E)$ which is **locally complete**. The locally complete graph has $N = |V| = N'n$ nodes and $|E| = N' - 1 + N'\binom{n}{2}$ edges. Let us denote the nodes of K_n that replaces node v_i of the original tree by $v_{i,m}$ $(m = 0, 1, \ldots, n-1)$. Embedding of the complete graph K_n is done in such a way that if the v_i is the parent of nodes v_j and v_k, then in the locally complete graph $v_{j,0}$ is connected to node $v_{i,r}$ and $v_{k,0}$ is connected to node $v_{i,s}$, where $r \neq s$ and $r, s \neq 0$ (Figure 8).

Figure 8. Embedding a complete graph K_4 at tree nodes $v_i = 4$ and $v_i = 5$. Port numbers at nodes $v_i = 4$, $v_i = 5$, and the node numbers of the complete graph K_4 are displayed

Theorem 5 *For a locally complete tree-like graph obtained as described above, the fraction of prohibited turns is given by*

$$z(G) = \frac{1}{3}\frac{N'n(n-2)}{N'n(n-2) + 4(N'-1)}. \tag{12}$$

Proof. Since the minimum degree nodes will always be at the leaf node positions of the original tree, the number of prohibited turns in each embedded K_n is given by $Z(K_n) = \binom{n-1}{2} + Z(K_{n-1})$. Solving this recursion equation we obtain $Z(K_n) = \frac{1}{6}n(n-1)(n-2)$. Hence, for the augmented graph G with N' nodes we have

$$Z(G) = N'Z(K_n) = \frac{1}{6}N'n(n-1)(n-2).$$

In embedding a K_n at a tree node of degree d_i, only d_i nodes of the K_n will be connected directly to the original tree. This means that embedding a K_n graph at an original tree node, will create nodes of at most degree n in the locally complete graph. Also, note that when a K_n is embedded at a tree node with degree d_i, there will be $n\binom{n-1}{2}$ turns contributed by the K_n and $(n-1)d_i$ turns contributed by the d_i edges of the original tree. With these observations the total number of turns is

$T(G) = N'n\binom{n-1}{2} + \sum_{i=1}^{N'}(n-1)d_i = N'n\binom{n-1}{2} + (n-1)\sum_{i=1}^{N'}d_i$ or

$$T(G) = \frac{1}{2}N'n(n-1)(n-2) + 2(n-1)(N'-1).$$

Hence,

$$z(G) = \frac{1}{3}\frac{N'n(n-2)}{N'n(n-2)+4(N'-1)}. \qquad +$$

For example, for $n=3$ and $N' \to \infty$, $z(G) = \frac{3}{3+4} = \frac{1}{7}$, and for $n=4$, and $N' \to \infty$, $z(G) = \frac{1}{3}\frac{8}{12} = \frac{2}{9}$.

3.4 Fractahedrons

Fractahedrons have been used by Tandem Computers (Horst, 1996) as topology of choice. A fractahedron with ℓ levels is a tree-like graph, where each tree node is replaced by one K_4, a complete graph with four nodes. The tree is balanced with ℓ levels where all non-terminal nodes are of degree four. An example of a 2-level fractahedron is shown in Figure 9. The set of prohibited turns for this fractahedron is also shown in the figure as arcs drawn between the two edges of each prohibited turn.

Theorem 6 *For an ℓ-level fractahedron*

$$z(G) = \frac{2\cdot 3^{\ell} - 1}{3(3^{\ell+1} - 2)}. \tag{13}$$

Proof. For an ℓ-level fractahedron the number of turns which should be prohibited is $Z(G) = 4B$, where $B = 2\cdot 3^{\ell} - 1$ is the number of blocks, or K_4 sub-graphs. To prove the lower bound, we consider the set of all different triangles, and observe that at least as many turns must be prohibited as there are triangles, one in each triangle, to break all cycles. The upper bound for $Z(G)$ can be proven by induction on number ℓ of levels, since only 4 turns within each block should be prohibited.

To calculate $T(G)$, we note that all nodes except those at level ℓ are of degree four. Three nodes at each block at level ℓ are of degree three. Number of nodes at level ℓ that are of degree three is $12\cdot 3^{\ell-1}$, and $T(G) = 12(3^{\ell+1} - 2)$.

Alternatively, since an ℓ-level fractahedron is a tree-like structure with $N' = 2\cdot 3^{\ell} - 1$ tree nodes, each of which is replaced with $n = 4$ node complete graph K_4, substituting these values for N' and n into (11) we obtain (13) directly.

Note that at first level $\ell = 0$, we have only one K_4, and for $\ell = 3$-level fractahedron $z(G) = 53/237$.

Figure 9. Two-level fractahedron with prohibited turns

3.5 Cube Connected Cycles

We will consider now a binary n-cube connected cycles, CCC (Parhami, 1998), where each node of an n-dimensional binary cube is replaced by a cycle of n nodes of degree 3 (see Figure 10 for n = 3). These interconnection networks are popular, since they combine the properties of small node degree and small diameter of the network graph (Harary, 1998). First, we will establish upper and lower bounds with Theorem 7 for a slightly larger class of graphs.

Theorem 7 *If graph G is obtained from d-regular graph H ($d_i = d$ for all i, $d > 2$) with $N(H)$ nodes by replacing each node by the cycle of d nodes, then*

$$\frac{1}{6} + \frac{2}{3dN(H)} \le z(G) \le \frac{1}{6} + \frac{1}{3d}. \qquad (14)$$

Proof. The lower bound can be obtained from $Z(G) \ge M - N + 1$, since for G there are $M(G) = 1.5N(H)d$ edges and $T(G) = 3N(H)d$ turns.

To prove the upper bound, we label all nodes in G as (i, j), where i is the number of the cycle containing the node i in G, and j is the number of a node within each cycle of length d, $i \in \{1,\ldots,N(H)\}$, $j \in \{0,1,\ldots,(d-1)\}$, as shown in Figure 10 . In each cycle, nodes are labeled subsequently. In cycle i we prohibit the turn $((i,d-1),(i,1),(i,2))$. There exist $N(H)$ such turns. Also, for each of $N(H)d/2$ edges between different cycles (edges between cycles in G correspond to edges in H), we prohibit turn (a,b,c), where $a=(i_1,j_1)$, $b=(i_2,j_2)$, $c=(i_2,j_3)$, if $i_1<i_2$ and $j_3=\left(j_2+1\right)\bmod d$. Then it follows that $z(G)\le\frac{|W|}{T(G)}=\frac{1}{6}+\frac{1}{3d}$. +

Figure 10. Labeled binary 3-cube connected cycles

The following theorem is generalization of Theorem 7.

Theorem 8 *If all nodes of 3-regular graph G with N nodes can be covered by k non-intersecting **simple** cycles, then*

$$\frac{1}{6}+\frac{2}{3N}\le z(G)\le\frac{1}{6}+\frac{k}{3N}. \tag{15}$$

Proof. The proof of Theorem 8 is similar to the proof of Theorem 7 (Theorem 8 follows from Theorem 7 for the case of cycles of equal lengths). To illustrate Theorem 8 let us consider the 4-pancake graph (Parhami, 1998). In a 4-pancake graph, nodes have labels that include all 4! = 24 orderings of numbers 1, 2, 3, and 4. For the q-pancake, node $(1,2,\cdots,i-1,i,i+1,\cdots,q)$ is connected to nodes $(i,i-1,\cdots,2,1,i+1,\cdots,q)$ for each i, i.e., $1,2,\cdots,i$ is flipped, like a pancake (Parhami, 1998). In a 4-pancake, nodes that are adjacent to node $(1,2,3,4)$ are $(2,1,3,4)$, $(3,2,1,4)$ and $(4,3,2,1)$ (see Figure 11 (A). For this graph, $N=24, k=4$ and according to Theorem 8 and (3), $z(G)=2/9$.

Another graph, which can be analyzed by Theorem 8, is the Petersen graph

(Harary, 1998), which has the smallest diameter (equal to 2) among all regular graphs of degree 3, shown in Figure 11(B). For this graph N = 10, k = 2 and by Theorem 8 and (3) we obtain z(G) = 7/30.

Figure 11. 4-Pancake graph in (A), and Petersen graph in (B).

3.6 Hamiltonian Graphs With Nodes of Small Degrees

Now we consider graphs with restricted degrees, for which a Hamiltonian path (a path, containing all nodes exactly once (Harary, 1998)) exists. Since Hamiltonian topologies emulate a linear array algorithm efficiently, existence of a Hamiltonian path in a topology is considered as a desirable property (Parhami, 1998). Many regular graphs (e.g. hypercubes and meshes) belong to this class. (Some current multicast techniques in computer networks are path-based (Sivaram, Panda, & Stunkel, 1997; R. V. Boppana, Chalasani, & Raghavendra, 1998) and use the Hamiltonian path to propagate messages from a source node to all destinations.)

Theorem 9 *If graph G with N nodes is Hamiltonian and degree d regular, then*

$$z(G) \le \frac{N(d-2)+4}{2Nd}. \tag{16}$$

Proof. A cycle-breaking set of turns can be constructed by labeling all nodes along the Hamiltonian path and prohibiting all turns, with a middle node having the maximal label. Then all turns can be classified into three groups:

1. Turns between edges, belonging to the path. There are $N-1$ such turns, all of them will be permitted;

2. Turns between an edge from the path and an edge not belonging to the path. There are $M_2 = M - N + 1$ such edges not belonging to the path, and each edge generates one prohibited turn;

3. Turns between edges, not belonging to the path. There are not more than $T_3 \le (N-2)\binom{d-2}{2} + 2\binom{d-1}{2} = N(d-2)(d-3)/2 + 2(d-2)$ such turns. Not more than half of them belong to the constructed set of prohibited turns W (otherwise we can construct W, prohibiting all turns, with a middle node having the *minimal* label).

For the number of prohibited turns we then obtain

$$Z(G) \le M_2 + T_3 / 2. \tag{17}$$

The total number of turns is upperbounded by $T(G) = Nd(d-1)/2$. Thus, the fraction of prohibited turns will be upperbounded as

$$z(G) \le \frac{\frac{(d-2)N}{2} + 1 + \frac{N(d-2)(d-3)}{4} + d - 2}{\frac{Nd(d-1)}{2}},$$

$$\le \frac{N(d-2)+4}{2Nd}.$$

We note that only cases $d = 3$, $d = 4$, and $d = 5$ result in new upper bounds. Taking into account the lower bound given by (3), we obtain for $d = 3$

$$z(G) = \frac{N+4}{6N}, \tag{18}$$

and for $d = 4$

$$z(G) \le \frac{N+2}{4N}. \tag{19}$$

For some Hamiltonian graphs we can improve the number of prohibited turns, (17), by dividing T_3 turns between edges, not belonging to the Hamiltonian path, into three groups:

1. Turns (a,b,c), where $a<b$, $b<c$,
2. Turns (a,b,c), where $b<a$, $b<c$,
3. Turns (a,b,c), where $b>a$, $b>c$.

Any one of the two last groups of turns can be selected as a set of prohibited turns, so if there are $F(G)$ turns in the first group, then

$$Z(G) \le M_2 + (T_3 - F(G))/2. \tag{20}$$

The proposed method can be extended to non-Hamiltonian graphs. Let us consider for the case $d=3$ graph G, which has a spanning tree with t leaves (nodes with degree equal to one). Then we label all nodes preserving the tree order and prohibiting turns with maximal labels for middle nodes and have for the number of prohibited turns

$$Z(G) \le M_2 + T_3 = M - N + t,$$

and

$$z(G) \le \frac{N+2t}{6N}. \tag{21}$$

As an example consider the degree-3 regular non-Hamiltonian graph in Figure 12, where a spanning tree with $t=3$ leaves is shown with solid lines. The upper bound for the fraction of prohibited turns for this graph is $z(G) \le \frac{16+2(3)}{6(16)} \le \frac{11}{48}$, and the lower bound (3) is $z(G) \geq 10/48$.

Figure 12. Non-Hamiltonian degree-3 regular topology with $t = 3$. The spanning tree is shown with edges in thick bold lines

4. Distance Dilation

Consider now the notion of dilation in a network topology due to turn prohibitions. Paths that involve prohibited turns are prohibited and are not used for communication. Thus, one side effect of turn prohibitions is that, prohibiting certain paths from being used for message routing, may increase distances between some nodes. The net result of this is that the average distance of the network graph will be increased. To facilitate the investigation of this phenomenon, the notion of distance dilation is introduced.

Definition 5 *The dilation in a graph, is the ratio of the average distance after turn prohibition to the average distance without any turn prohibition.*

When the dilation is 1 it implies that the turn prohibitions have not caused any lengthening of the average distance. For example, for complete graphs the fraction of prohibited turns achieves the upper bound, but the dilation is 1. Similarly for homogeneous and D-meshes, for hexagonal meshes, p-ary n-dimensional meshes, for locally complete tree-like topologies, and for fractahedrons no dilation is introduced by turn prohibitions. In Figure 13 the distance dilations in p-ary n-dimensional tori are shown. For these calculations, we determined the average distance using the shortest distances between all source destination pairs. For the p=3 torus, the largest dilation is less than 5.5%, whereas for tori with $p = 4$, 5, and 6, the largest dilation is less than 3.5%.

Figure 13. Dilation in p-ary n-dimensional tori due to turn prohibition, in $p = 3,\ldots,6$ are displayed

5. Conclusions

In this chapter the problem of constructing minimum cycle-breaking sets of turns for graphs that model communication networks in multiprocessor systems with wormhole routing is considered. This problem is important for deadlock-free and livelock-free message routing in these networks. A series of new algorithms were presented that are used to obtain optimal or close to optimal sets of prohibited turns to prevent deadlock formation during routing. Results on minimum fractions of turns that must be prohibited to break all cycles without loss of connectivity for degree-regular connected graphs are presented in Table 1. The results of calculations for dilations as a result of prohibitions in p-ary n-dimensional meshes and tori are also presented. It is noteworthy that meshes do not suffer from any dilation and the worst case dilation for tori is less than 5.5%.

Table 1. Lower and upper bounds on fractions of prohibited turns, $z(G)$, in minimal cycle breaking sets for several regular and semiregular topologies

Topology	Lower bound on $z(G)$	Upper bound on $z(G)$	Asymptotic Limits for $z(G)$
Homogenous meshes - Theorem 1	$\frac{1}{4}\left(1-\frac{1}{d-1}\right)$		$\frac{1}{4}, d \to \infty$
Complete graph K_n, $n > 2$	$1/3$		$1/3$
n-dimensional p-ary mesh – (6) M_p^n	$\frac{(n-1)(p-1)^2}{2p(p-2)+4(n-1)(p-1)^2}$		$\frac{n-1}{4n-2}$, $p \to \infty$, $\frac{1}{4}, n \to \infty, p \to \infty$
n-dimensional p-ary torus – (7) Z_p^n, $(p > 2)$	$\frac{p(n-1)+2}{2p(2n-1)}$		$\frac{1}{4}, n \to \infty$
Hypercube Z_2^n - Section 3-5	$1/4$		$1/4$, $n \to \infty$
Hexagonal mesh of size p - (10)	$\frac{9p^2-21p+13}{45p^2-99p+51}$		$\frac{1}{5}$, $p \to \infty$
Hexagonal torus of size p - (11)	$\frac{9p^2-15p+10}{45p^2-45p+15}$		$\frac{1}{5}$, $p \to \infty$
Honeycomb mesh of size p - Section 3-2	$\frac{3p^2-3p+1}{18p^2-12p}$		$\frac{1}{6}$, $p \to \infty$
Honeycomb torus of size p - Section 3-2	$\frac{1}{6}+\frac{1}{18p^2}$		$\frac{1}{6}$, $p \to \infty$
Fractahedron with ℓ *levels*- (13)	$\frac{2\cdot 3^{\ell}-1}{3\left(3^{\ell+1}-2\right)}$		$\frac{2}{9}$, $\ell \to \infty$
Cube-connected cycle with $N = 2^n$ *nodes*- (14)	$\frac{1}{6}+\frac{1}{3N}$	$\frac{1}{6}+\frac{1}{3d}$	$\frac{1}{6}$, $n \to \infty$
Hamiltonian graph with max degree $\Delta < 5$ - (16)	-	$\frac{N(\Delta-2)+4}{2N\Delta}$	$\frac{1}{2}-\frac{1}{\Delta}$, $N \to \infty$, (upper bound)
3-regular graph - Section 3-6	$\frac{1}{6}+\frac{2}{3N}$	$\frac{2}{9}+\frac{7}{18N}$	$\frac{2}{9}$, $N \to \infty$, (upper bound)
4-pancake graph - Section 3-5	$2/9$		$2/9$
Petersen graph - Section 3-5	$7/30$		$7/30$
Locally Complete graphs - (12) *obtained by replacing each of* N' *nodes in a tree by a* K_n	$\frac{1}{3}\frac{N'n(n-2)}{N'n(n-2)+4(N'-1)}$		$\frac{1}{3}\frac{n(n-2)}{n(n-2)+4}$, $N' \to \infty$

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