Minimal Sets of Turns for Breaking Cycles in Graphs Modeling Networks

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***Abstract*—We propose an algorithm that provides for deadlock-free and livelock-free routing, in particular in wormhole routed networks. The proposed algorithm requires nearly minimal amount of resources. We model a computer communication system by representing the underlying topology by a connected graph. Let $G = (V, E)$ be an undirected connected graph and C be a cycle of edges in G, containing (not necessarily all different) vertices $v_0, v_1, ..., v_L, v_0, v_1$. A turn (triple of nodes) (a, b, c) belongs to C, if a, b, c are consecutive nodes in C. A turn (a, b, c) breaks cycle C, if this turn belongs to C and to the set of prohibited turns $W(G)$. The set $W(G)$ is connectivity preserving if for any $a, b \in V$ there exists a path in G that is not prohibited. In this paper we consider the problem of constructing minimal cycle-breaking connectivity preserving sets of turns for graphs that model communication networks, as a method to prevent deadlocks. We present a new cycle-breaking algorithm called the Simple Cycle-Breaking (SCB) algorithm that is considerably simpler than earlier algorithms. We prove its properties and present lower and upper bounds for minimal cardinalities of cycle-breaking connectivity preserving sets $W(G)$ for graphs of general topology as well as for planar graphs. We present experimental results on the fraction of prohibited turns and on the dilation that illustrate the effectiveness and the efficiency of the SCB algorithm.**

***Index Terms*—deadlock, livelock, turn prohibition, wormhole routing.**

I. INTRODUCTION

DUE to the availability of low cost workstations with network interface adapters that offer high-performance communications using wormhole techniques, clusters of workstations are emerging as preferred computing environments [1], [2], [3], [4]. However, as wormhole routed messages hold network resources while requesting others, as they traverse the network towards the destination, it is prone to deadlocks under heavy network loads [5], [1], [3]. Deadlocks have been shown to occur due to the presence of cycles in the channel dependency graph (CDG) of the original graph representing the network [5]. Given a graph, constructing the CDG for it is at best tedious. However, cycles of nodes in the CDG graph correspond to "cycles of edges", as defined below, in the original network graph. A similar problem causede by cycles, the so called "livelocks",appears in Ethernet type networks.

In this paper, we propose a simple algorithm that will break all such cycles in the graph of the original communication network. This is done by preventing certain pairs of communication links from being used, sequentially, at some nodes in the graph when forwarding messages.

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Turn prohibition has been used to convert networks with cycles to feed-forward networks to apply techniques that were available exclusively for networks free of cycles [6], [7]. In [6] authors used turn prohibition to facilitate the application of network calculus to general network graphs with cycles and demonstrate that the earlier version of the turn prohibition algorithm significantly outperforms other approaches such as the spanning tree and up/down routing algorithms for breaking cycles, for network utilization and delay. Authors conclude that with the introduction of turn prohibition the restriction of network calculus to feed-forward routing networks may not represent a significant limitation. Authors in [7] invented a new approach, referred to as turnnet algorithm, which is used to convert a network graph with a set of prohibited turns into one without turn prohibitions. The main idea for this is to permit any routing algorithm to use only the information required, i.e., the nodes, the links, and the link metrics. Our focus in this paper is the construction of minimal sets of prohibited turns for any undirected connected graph.

Let us consider an undirected connected graph $G(V, E)$, with $N = |V|$ vertices (nodes), denoted by $a, b, \ldots$, and $M = |E|$ edges, denoted by (a, b), etc, to represent a communication network. A **turn** in G is a triplet of nodes (a, b, c) if (a, b) and (b, c) are edges in G and $a \neq b$. In an undirected graph turns (a, b, c) and (b, c, a) are considered to be the same turn. If the degree of node j is d_j, the total number of turns $T(G)$ in G is given by $T(G) = \sum_{j=1}^{N} \binom{d_j}{2}$. A **path** $P = (v_0, v_1, \ldots, v_{L-1}, v_L)$ of length L, $L \geq 1$ from node a to node b in G is a sequence of nodes $v_i \in V$ such that, $v_0 = a$ and $v_L = b$, and every two consecutive nodes are connected by an edge. Subsequences of the form (v_i, v_k, v_i) are not permitted in a path. Nodes and edges in the path are not necessarily all different. A turn (a, b, c) **belongs** to path $P = (v_0, v_1, \ldots, v_L)$ if $(a, b, c) = (v_i, v_{i+1}, v_{i+2})$, $i = 0, \ldots, L-2$. A set of turns $W(G)$ is called the **set of prohibited turns**, if any path that includes turns from $W(G)$ is **prohibited**. This set is called **connectivity preserving**, if for any $a, b \in V$ there exists a path in G that is not prohibited. Path $P = (v_0, v_1, v_2, \ldots, v_k, v_0, v_1)$ in G is called a **cycle**. If no proper subset of nodes of cycle P forms a cycle, we call P a **simple cycle**. Set $W(G)$ of prohibited turns in G is called **cycle-breaking** if every cycle in G includes at least one turn from $W(G)$. The minimum cardinality of connectivity preserving set $W(G)$ for a given graph G is denoted by $Z(G)$ and the fraction of prohibited turns is denoted by $z(G) = Z(G)/T(G)$. Since prohibition of turns imposes routing constraints, by preventing certain communication paths from being used during the routing of messages in the network, it must be done in a way that minimizes the fraction of link pairs (i.e. turns) that are prevented from being used. The motivation for seeking the minimal fraction of prohibited turns is originally due to Glass and Ni [8]. They have found that reduction in the number of prohibited turns results in a decrease of average path length and the average message delivery time, thereby increasing the throughput. After Glass and Ni showed it for regular topologies such as meshes and tori, this conclusion was confirmed by other authors [9], [10], [11] for irregular topologies as well. Experimental data [12], [13] show that there is a considerable gain of approximately 7-8% in the maximum sustainable throughput in the network, for each percentage point reduction in the fraction of prohibited turns. Similar to spanning tree approaches, prohibiting a carefully selected set of the turns in the network, provides deadlock freedom. However, unlike the spanning tree based approaches, the cycle-breaking approach allows all communication links in the network to be used. The only restriction is that some pairs of communication links, namely, those that form the prohibited turns, are prevented from being used sequentially.

Let G be a connected graph with minimum degree δ. Consider a set of R cycles in G such that no more than r cycles are covered by the same turn. Then [12], the number of prohibited turns $Z(G)$ and fraction of prohibited turns $z(G)$ satisfy the following inequalities:

$$Z(G) \geq M - N + 1, \tag{1}$$

$$z(G) \geq \frac{R}{rT(G)}, \tag{2}$$

and

$$Z(G) \geq M - N + \binom{\delta - 1}{2} + 1, \delta > 2. \tag{3}$$

Bound (3) is tight. For example, in the Petersen graph (Fig. 1) with $M = 15$, $N = 10$, and $\delta = 3$, the number

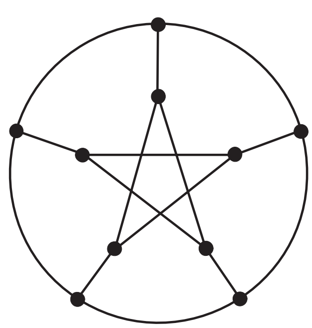

Fig. 1. The Petersen graph with minimum degree $\delta = 3$ and $Z(G) = 7$

of prohibited turns, $Z(G) = 7$. As another example, in a two-dimensional $p \times p$ mesh with $N = p^2$ and $M = 2p(p-1)$, the total number of turns is $T(G) = 6(p-2)^2 + 12(p-2) + 4$. Since $\delta < 3$, we use (1) to obtain the fraction of prohibited turns to be $z(G) = \frac{p^2-2p+1}{6p^2-12p+4}$, which is the exact result, as shown in [14].

In the rest of the paper we propose a new cycle breaking algorithm (the SCB algorithm) and prove its properties in Section II, followed by an investigation of planar graphs in Section III. In Section IV we introduce the notion of distance dilation caused by SCB in the network. In Section V we present our experimental results and offer conclusions in Section VI.

II. A General Algorithm for Construction of Minimal Cycle-Breaking Sets of Turns

In this section we present an algorithm, called the Simple Cycle-Breaking (SCB) algorithm, that is much simpler than those in [15], [12], [10]. Earlier cycle-breaking algorithms in these publications were complicated and involved as many as ten steps per each recursive call, whereas the SCB algorithm has only three steps and is easy to understand. The complexities of having to deal with cut nodes have totally been eliminated in the SCB algorithm, and, as will be shown, it has the same time complexity as the algorithms suggested previously.

Lemma 1: In any connected graph G , there exists a connected subgraph H which consists of non-cut nodes only of the original graph G and is connected to the rest of G via at most one cut node $c \in G\backslash H$ only (i.e., if $a \in H$, $b \in G\backslash H$, then $c \in P(a,b)$, where $P(a,b)$ is any path from node a to node b).

Proof: If G has no cut nodes, then $H = G$. Suppose G has cut nodes. Let S_i be the set of connected components of G obtained by deleting cut node $c_i (i = 1, 2, \ldots)$ from G. Consider the union $\bigcup_i S_i$. Let $H \in \bigcup_i S_i$ be the connected component with the smallest number of nodes. This component does not include any cut nodes from the original graph (otherwise it would not be the smallest component). Thus, if H is obtained by deleting cut node c from graph G, then H is a connected subgraph which is connected to $G\backslash H$ via one cut node c only. ■

Lemma 1 will be used below to prove properties of a new algorithm for obtaining a minimal cycle-breaking set of turns.

Given a connected graph $G(V,E)$, the SCB algorithm creates two sets: the set $W(G)$ of prohibited turns and the set $A(G)$ of permitted turns. It also labels all nodes by natural numbers starting with 1, in the order they are selected by the algorithm. In the beginning, $W(G) = \emptyset$, $A(G) = \emptyset$, and all nodes are unlabeled. If $|V| = N$, the algorithm consists of $N-1$ stages (recursive calls). Each stage consists of 3 steps described below.

1) If $|V| = 2$, label the nodes by the smallest unused natural numbers, select and delete the node with label $\ell = N-1$ and return sets $W(G)$ and $A(G)$. Otherwise, go to Step 2
2) Select a non-cut node a of the minimum degree d, such that
$$2\binom{d}{2} \leq \sum_{i=1}^{d}(d_i - 1), \tag{4}$$
where d_i are the degrees of the neighbors of a (nodes adjacent to a). Prohibit all turns of the form (b,a,c) and include them in $W(G)$. Permit all turns of the form (a,b,c) and include them in $A(G)$. Label a by the smallest unused natural number $\ell(a)$.

3) Delete node a to obtain a graph $G' = G \setminus a$ and go to Step 1 for G'.

Note that at the stage of the algorithm when node a is selected, all other undeleted nodes are unlabeled. In fact, they will be labeled later. As a result, turn (b, a, c) is prohibited *iff* $\ell(a) < \ell(b)$ and $\ell(a) < \ell(c)$. Also, it will be shown below that in any connected graph there exists a node that satisfies conditions for being selected at Step 2 of the algorithm.

Theorem 1: The SCB algorithm has the following four properties.

Property 1. Any cycle in G contains at least one turn from $W(G)$.

Property 2. SCB preserves connectivity; for any two nodes $a, b \in V$, there exists a path between a and b that does not include turns from $W(G)$.

Property 3. The set $W(G)$ of prohibited turns generated by SCB algorithm is *minimal* (*irreducible*).

Property 4. For any graph G, $W(G) \leq T(G)/3$, where $T(G)$ is the total number of turns in G.

Proof of Property 1: Consider the node a with the minimum label $\ell(a)$ in any cycle C in G. Then in the turn (b, a, c) $(b, a, c \in C)$, $\ell(a) < \ell(b)$ and $\ell(a) < \ell(c)$. Thus, turn (b, a, c) is prohibited and cycle C is broken. ■

Proof of Property 2: The proof is by induction. Consider the first selected node a, $\ell(a) = 1$. Since a is a non-cut node, after all turns of the form (b, a, c) are prohibited and node a is deleted, there still exists a path from any node x to any node y, where $x, y \in G \setminus a$. Also, since all turns of the form (a, b, c) are permitted, there exists a path from a to any node $x \in G$. Now assume that connectivity is preserved after the first n stages of the algorithm, so that the next selected node a has label $\ell(a) = n + 1$. Node a is a non-cut node in the graph that remains after deletion of the first n selected nodes. Therefore, after prohibition of all turns (b, a, c) there still exits a path between any two unlabeled nodes x and y. Consider now paths from a labeled node u, $\ell(u) \leq \ell(a)$ to another previously labeled node v, $\ell(v) < \ell(a)$, or to an unlabeled node y. If such a path P does not include a turn of the form (b, a, c), where b and c are unlabeled, it remains permitted. Now suppose P includes such a turn (Fig. 2). Then, let x be the first unlabeled node in the path from u to v or from u to y, and z be the last unlabeled node in the path from u to v. Now we can replace the part of P from x to y, or from x to z, respectively, by a path that does not include a (such a path exists, since a is a non-cut node) and obtain a path P^*. Let x' be the node already labeled that immediately precedes x in P and in P^*, and z' be the labeled node that immediately follows z in P and P^* (in the case when such a node exists). Since all turns (x', x, w) and (w, z, z') are permitted, path P^* does not contain prohibited turns, and connectivity is preserved at the $(n + 1)$th stage of the algorithm. Thus, Property 2 is proved by induction. ■

Proof of Property 3: Consider a prohibited turn (b, a, c). Since connectivity is preserved and a is a non-cut node, there exists a permitted path (b, P, c) from b to c that does not include a. Adding edges (a, b) and (c, a) to this path, we obtain a cycle $C = (a, b, P, c, a, b)$. Since turns of the form (a, b, x)

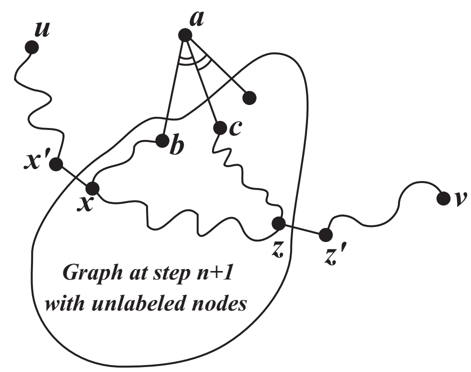

Fig. 2. Figure depicting the state of the graph at step n+1 of the SCB algorithm. Path $P = (u, \ldots, x', x, \ldots, b, a, c, \ldots, z, z', \ldots, v)$ is prohibited due to the prohibited turn at node a. Path $P* = (u, \ldots, x', x, \ldots, z, z', \ldots, v)$ is permitted since it does not involve any prohibited turns.

and (a, c, y) are permitted, the only prohibited turn in C is (b, a, c). By removing this turn from $W(G)$, we would create a cycle in G and violate the cycle-breaking Property 1. Thus, set $W(G)$ is minimal. ■

Proof of Property 4: At the stage of the algorithm when node a is selected (recursive call $\ell(a)$), all turns (b, a, c) become prohibited, and all turns (a, b, c) become permitted, where $\ell(a) < \ell(b)$ and $\ell(a) < \ell(c)$. The number of prohibited turns is $\binom{d}{2}$ where d is the degree of node a; the number of permitted turns is $\sum_{i=1}^{d}(d_i - 1)$, where d_i, $(i = 1, \ldots, d)$ are degrees of all neighbors of a. Let us prove that in any graph there exists a node that satisfies inequality (4). Using Lemma 1, consider a subgraph that consists of non-cut nodes and at most one cut node, connecting this subgraph to the remaining part of the graph. Select a non-cut node a of the minimum degree d among all non-cut nodes in this subgraph. If a is not adjacent to the cut node, then inequality (4) is obviously satisfied. Suppose now that all nodes with minimum degree d are adjacent to the cut node with degree $d' < d$. Then the selected node a has at most $d' - 1$ neighbors of degree d, while at least $(d - 1) - (d' - 1) = d - d'$ of its neighbors have degrees at least $d+1$. Thus the number $\zeta(a)$ of permitted turns at this stage of the algorithm is

$$\zeta(a) \geq (d'-1)(d-1)+(d-d')d+(d'-1) = d(d-1) = 2\binom{d}{2}.$$

Hence, the inequality (4) is satisfied in all cases, which means that the number of permitted turns is larger than the number of prohibited turns by at least a factor of two. Since this is true for each stage of the algorithm, it follows that $W(G) \leq T(G)/3$. ■

The prohibition rule for the SCB algorithm can be expressed in different terms. Let us call edge (a, b) **positive**, if $\ell(a) < \ell(b)$, and negative otherwise. Then the path P is prohibited *iff* it includes a pair of consecutive edges such that **the first edge is negative** and **the second one is positive**. Then the connectivity means that SCB algorithm labels nodes in such a way that for any two nodes there exists a path between them in which all positive edges (if any) precede all negative ones (if any).

In general, the fraction of prohibited turns yielded by the SCB algorithm is considerably smaller than the upper bound of $1/3$. The only class of graphs where the fraction is exactly $1/3$ is the complete graphs K_n with $|V| = n$ and $|E| = n(n-1)/2$. Indeed, the closer is a graph to a complete one, the larger is the fraction of prohibited turns, as shown by the following theorem which provides a better upper bound on the ratio $|W(G)|/T(G)$.

Theorem 2: Let $G = (V, E)$ be a connected graph with N nodes and M edges. The fraction of prohibited turns $z_{SCB}(G)$ yielded by the SCB algorithm satisfies the upper bound:

$$\begin{aligned} z_{SCB} &= \frac{|W(G)|}{T(G)} \\ &\leq \frac{1}{3} - \frac{2N-3-\sqrt{8\beta+1}}{3[2N+(\beta-1)(\sqrt{8\beta+1}+3)]}, \quad (5) \end{aligned}$$

where $\beta = M - N + 1$.

Proof: When a node is selected in the course of the SCB algorithm, all its edges are deleted. Thus, if d_ℓ is the degree of node with label ℓ at the stage when it is selected, then

$$\sum_{\ell=1}^{N} d_\ell = M. \quad (6)$$

The total number of prohibited turns is

$$|W(G)| = \sum_{\ell=1}^{N} \frac{d_\ell(d_\ell-1)}{2}. \quad (7)$$

Note that, for the SCB algorithm,

$$d_{\ell+1} \geq d_\ell - 1. \quad (8)$$

(Otherwise, the nodes would be selected in the opposite order). Obviously, $d_N = 0$ and $d_{N-1} = 1$. The quadratic sum (7) under the constraint (6) achieves maximum if the values of d_ℓ are maximally unequal, so that some of them are as large as possible. Looking at the sequence (d_ℓ) in the backward direction, from $\ell = N$ to $\ell = 1$, one can see that, because of (8), the sequence can increase only by 1 from one term to the another: $d_N = 0, d_{N-1} = 1, d_{N-2} \leq 2, \ldots, d_{N-i} \leq i$. Hence, there exists a subsequence (d_{ℓ_j}) such that $d_{\ell_j} = j$, where j takes on all integer values from 0 to a certain k. The value of $|W(G)|$ achieves its maximum, if k is the largest integer that satisfies two conditions. On one hand,

$$M \geq \sum_{j=0}^{k} j = \frac{k(k+1)}{2}. \quad (9)$$

On the other hand, since the graph remains connected through the course of the algorithm, the number of remaining edges should be no smaller than the number of remaining nodes:

$$M - \frac{k(k+1)}{2} \geq N - (k+1). \quad (10)$$

The number of prohibited turns in the nodes of the subsequence (d_{ℓ_j}) is

$$\sum_{j=1}^{k} \frac{j(j-1)}{2} = \frac{(k+1)k(k-1)}{6}. \quad (11)$$

The upper bound on $|W(G)|$ is obtained for the value of k (not necessarily an integer) that turns (10) into equality, i.e. for the root of the equation:

$$M - N = \frac{(k-2)(k+1)}{2}. \quad (12)$$

Hence,

$$k = \frac{1+\sqrt{8(M-N+1)+1}}{2} = \frac{1+\sqrt{8\beta+1}}{2}. \quad (13)$$

Then, by (11),

$$\begin{aligned} |W(G)| &\leq \frac{(M-N+1)\left(\sqrt{8(M-N+1)+1}+3\right)}{6} \\ &\leq \frac{\beta(\sqrt{8\beta+1}+3)}{6}. \quad (14) \end{aligned}$$

Now let us estimate the total number of turns. According to the proof of Property 4 in Theorem 1, if the degree of the selected nodes is k, then there exist at least k other nodes with the sum of degrees at least k^2. The total number of turns at these $k+1$ nodes is minimal, if all degrees are equal: $d = k$. Since the graph is connected, the remaining $N-k-1$ nodes add at least $N-k-1$ turns. Thus the total number of turns $T(G)$ obeys inequality

$$T(G) \geq \frac{(k+1)k(k-1)}{2} + N - k + 1, \quad (15)$$

where k is given by (13).

It follows that the fraction of prohibited turns $z(G)$ is upperbounded by

$$\begin{aligned} z(G) &\leq \frac{|W(G)|}{T(G)} \\ &\leq \frac{1}{3}\left[1 - \frac{2N-3-\sqrt{8\beta+1}}{2N+(\beta-1)(\sqrt{8\beta+1}+3)}\right]. \quad (16) \end{aligned}$$

■

Bound (16) is tight. It is achieved, in particular, for a tree $(M = N-1)$, for a ring $(M = N)$, and for a complete graph $K_N \left(M = \frac{N(N-1)}{2}\right)$.

Note that bound (16) converges to $1/3$ *iff* the cyclomatic number $\beta = M - N + 1 = \omega(N^{2/3})$. It will be shown below (see Section 3) that for some classes of graphs, the SCB algorithm guarantees that the fraction of prohibited turns is substantially smaller than that given by bound (16).

The complexity of the SCB algorithm is at most $O\left(N^2\Delta\right)$, where Δ is the maximum node degree in $G(V, E)$.

III. Planar Graphs

Planar graphs defined as those which can be embedded in a plane without any crossing edges form an important class of graphs. A large number of physical problems such as transportation highways (without underpasses), telecommunication networks, and physical circuit or component layout problems are modeled by planar graphs. For example, for proper operation, all physical layout problems in a printed circuit board and VLSI designs involve constructing conductive (metallic) signal pathways that must be prevented from crossing each other, these problems naturally map into planar graphs. In VLSI chips, either the entire chip or large sections of the chip are modeled by planar graphs [16]. In [16], [17] authors introduced turn prohibition in Network-On-Chips (NOC) architectures in which multiple processing elements are networked on one VLSI chip, in which the layout is planar. In this section we present constructive upper bounds on minimal fraction of turns, $z(G)$ to be prohibited to break all cycles in any planar graph G.

An important characteristic of a planar graph is the number of edges in the shortest cycle known as its **girth**.

Lemma 2: The average degree in a planar graph with n nodes and girth g obeys inequality

$$\overline{d} \le \frac{2g}{g-2} - \frac{4g}{N(g-2)}. \tag{17}$$

Proof: Let G be a planar graph with F faces and girth $g(G) = g$. Since each edge belongs to either one or two faces, it follows that $2M \ge \sum_{j=1}^{F} g_j \ge Fg$ where g_j is the number of edges of face j. Hence

$$F \le \frac{2M}{g}. \tag{18}$$

Substituting (18) into the Euler equation $F = M - N + 2$, we obtain

$$M \le \frac{g(N-2)}{g-2}. \tag{19}$$

Thus the average node degree is

$$\overline{d} = \frac{2M}{N} \le \frac{2g}{g-2} - \frac{4g}{N(g-2)}.$$

■

It is seen that the upper bound on $\overline{d}$ given by (17) decreases monotonically with girth g.

Theorem 3: If G is a planar graph without triangles, then

$$z(G) \le \frac{1}{4}. \tag{20}$$

Proof: With no triangles in a planar graph, the girth of the graph is $g(G) = g \ge 4$. Then the average degree $\overline{d}$ in (17) becomes $\overline{d} \le 4 - 8/N$, which means that a planar graph without triangles contains at least two nodes of degree less than four. Note that any subgraph of G is also a planar graph with girth at least four and an average degree $\overline{d} \le 4 - 8/n$, where n is the number of nodes in the subgraph. By Lemma 1, there exists a subgraph H of G that consists of non-cut nodes only and connected to the rest of the graph by at most one cut node. Consider a subgraph of G formed by subgraph H and this cut node. It follows that this graph contains a non-cut node of degree at most 3.

At every step of the execution of the SCB algorithm (see Section II), a minimum degree node is selected according to the rule (4). Let $A_i (i = 1, 2, 3)$ be the number of nodes of degree i that were selected during the execution of the algorithm. Since the last node left is a node of degree zero and all edges are deleted in the course of the algorithm, we have

$$A_1 + A_2 + A_3 = N - 1, \tag{21}$$

$$A_1 + 2A_2 + 3A_3 = M. \tag{22}$$

Hence,

$$A_2 + 2A_3 = M - N + 1 = F - 1. \tag{23}$$

Note that the number of prohibited turns in the SCB procedure is given by

$$Z = A_2 + 3A_3. \tag{24}$$

Hence, an upper bound for Z would correspond to a maximal A_3 and a minimal A_2. Obviously, the deletion of a degree 2 node decreases the number of faces by 1, and the deletion of a degree 3 node decreases this number by 2. The algorithm terminates when the number of faces is reduced to 1. It is easy to show, using (17), (18), and the Euler equation, that for girth $g \ge 4$,

$$\overline{d} \le \frac{4(F-2)}{F(g-2)+4} + 2 \le \frac{4(F-2)}{2(F+2)} + 2 = 4 - \frac{8}{F+2}.$$

Hence, for any graph with girth $g \ge 4$ ($F \le 5$), $\overline{d} = 3 - \frac{1}{7} < 3$. Thus any such graph has non-cut nodes of degree 2 or 1. Therefore, $A_3 \le (F-4)/2$ and $A_2 \ge 3$ (provided $F \ge 4$). Then, using (23) and (24), we obtain:

$$Z = \frac{3}{2}(A_2 + 2A_3) - \frac{A_2}{2} = \frac{3}{2}(M - N + 1) - \frac{A_2}{2}.$$

Finally, since $A_2 \ge 3$, it follows that

$$Z(G) \le Z \le \frac{3}{2}(M - N). \tag{25}$$

Here, $Z(G)$ is the minimum number of prohibited turns for G. To estimate the total number of turns $T(G)$, we note that there are two cases; first, when the average degree $\overline{d} = \frac{2M}{N}$ is $3 \le \overline{d} \le 4 - \frac{8}{N}$, and second, when $\overline{d} < 3$. In the first case, $T(G)$ is minimal if nodes are of degree 3 and degree 4 only and in the second case if nodes are of degree 2 and degree 3 only. Assuming first that nodes are of degree 3 and 4 only, we determine that $N_3 = 4N - 2M$ and $N_4 = 2M - 3N$, where N_3 and N_4 designate the number of nodes of degree 3 and degree 4, respectively, in the graph. It follows that $T(G) = 6(M - N)$, and the fraction of prohibited turns $z(G) = \frac{Z(G)}{T(G)}$ is

$$z(G) \le \frac{1}{4}. \tag{26}$$

For the case when $2 \le \overline{d} < 3$, $T(G)$ is minimal if there are N_2 nodes of degree 2 and N_3 nodes of degree 3 only. Then $N_2 = 3N - 2M$, and $N_3 = 2(M - N)$, and $T(G) = 4M - 3N = 6(M - N) + (3N - 2M) \ge 6(M - N)$ (since $\overline{d} = \frac{2M}{N} < 3$). Hence in both cases the upper bound for the fraction of prohibited turns is

$$z(G) \le \frac{1}{4}. \tag{27}$$

■

Theorem 4: If G is a planar graph with a girth $g \ge 6$, then

$$z(G) \le \frac{2}{g+6} - \frac{(g-2)(g-6)}{(g+6)\left[g(N-8)+6N\right]}, \tag{28}$$

and

$$z(G) \le \frac{2}{g+6}. \tag{29}$$

Proof: Note that, by (17), for girth $g = 6$ the average degree becomes $\overline{d} \le 3 - \frac{6}{N} < 3$. For the case of $g \ge 6$, if there are only N_2 nodes of degree 2 and N_3 nodes of degree 3, we get $N_2 = 3N - 2M$, $N_3 - 2(M - N)$, and the total number of turns, $T(G) = N_2 + 3N_3 = 4M - 3N$. Since $T(G)$ achieves minimum if the degrees take values closest to the given average degree, it follows that $T(G) \geq 4M - 3N$. By the same argument that is given in the first paragraph of the proof of Theorem 3, there will always be non-cut nodes of degree at most 2 available for selection at every step of the algorithm; and therefore $A_3 = 0$. From $A_1 + A_2 = N - 1$ and $A_1 + 2A_2 = M$, we obtain that $Z = A_2 = M - N + 1 = F - 1$ and the upper bound for the fraction of prohibited turns will be

$$z(G) \leq \frac{M - N + 1}{4M - 3N}.$$

Substituting $x = M/N$, we get

$$z(G) \leq \frac{1}{4} - \frac{1/4 - 1/N}{4x - 3}. \quad (30)$$

The right-hand side of (30) is a monotonically increasing function of x. Note that from $\overline{d} = 2M/N = 2x$ we get

$$x \leq \frac{g}{g-2} - \frac{2g}{N(g-2)}.$$

Substituting the maximum value of x into (30) we obtain

$$z(G) \leq \frac{2}{g+6} - \frac{(g-2)(g-6)}{(g+6)[g(N-8)+6N]},$$

and

$$z(G) \leq \frac{2}{g+6}.$$

■

The bound (28) is tight as shown by the following example.

Example. Consider the planar graph of girth $g = 8$ shown in Fig. 4, with $N = 32$, $M = 40$, and $T = 64$. For this graph, $Z = M - N + 1$ and $z = \frac{9}{64}$ is equal to the right-hand part of inequality (28).

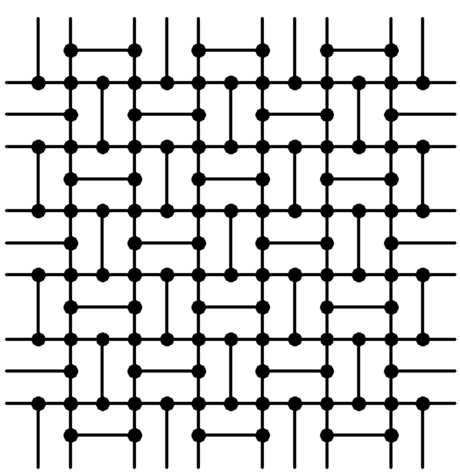

Fig. 3. An infinite planar graph with average degree $\overline{d} = 10/3$

To avoid misunderstanding, let us point out that it is planarity and girth constraints that result in (20) and (28), but not the limits on the average degree alone. It is easy to construct graphs with average degree $\overline{d}$ arbitrarily close to 2, for which $z(G)$ is arbitrarily close to $1/3$.

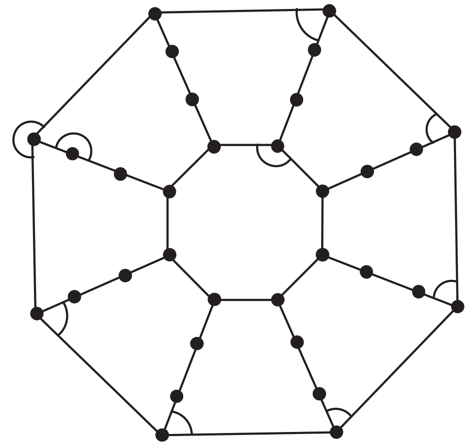

Fig. 4. Planar graph with girth $g = 8$, $N = 32$, $M = 40$, and $T = 64$. Prohibited turns are shown as arcs.

For girth $g(G) = 5$ planar graphs, the average degree in (17) becomes

$$\overline{d} \leq \frac{10}{3} - \frac{20}{3N}. \quad (31)$$

This bound is tight and is achieved for the infinite graph of Fig. 3.

For $N < 20$ ($F < 12$) the average degree $\overline{d} < 3$ and it follows that such graphs would always have a node of degree 2.

Conjecture. If G is a planar graph with girth $g(G) = 5$, then $z(G) \leq 1/5$.

Note that Theorems 3 and 4 do not apply to non-planar graphs. For example, for bipartite graph $K_{4,4}$ with $N = 8$ nodes, we have $z(K_{4,4}) = 14/48 > 1/4$.

The proofs of Theorem 3 and Theorem 4 suggest a somewhat more general result.

Theorem 5: If in the course of the SCB algorithm, all selected nodes are of degree 2 or less, then the solution given by SCB is optimal, and,

$$|W(G)| = M - N + 1. \quad (32)$$

Proof: The result follows immediately from the expressions (23) and (24), and lower bound (1). ■

In particular, the SCB algorithm provides an optimal solution for 2-dimensional rectangular mesh and honeycomb mesh – two popular network topologies.

IV. DISTANCE DILATION

Consider now the notion of dilation in a network topology due to turn prohibitions. Paths that involve prohibited turns are prohibited and are not used for communication. Thus, one side effect of turn prohibitions is that, prohibiting certain paths from being used for message routing, may increase distances between some nodes. The net result of this is that the average distance of the network graph will be increased. To facilitate the investigation of this phenomenon, we introduced the notion of distance dilation which we define as the ratio of the average distance after turn prohibitions to the average distance without any turn prohibitions. When the dilation is 1 it would imply that the turn prohibitions have not caused any lengthening of the average distance. For example, in complete graphs the fraction of prohibited turns achieves the upper bound, but the dilation is 1.

V. Experimental Results

In this section we present the results of our calculations for the fraction of prohibited turns using the SCB algorithm and experiments involving message delivery simulations using Opnet discrete event simulation tools. In all of our calculations and simulations, network topologies were first generated using tools that we developed. All of the topologies were represented by 64-node undirected graphs. For the irregular topologies, we constructed 100 graphs of average degree four. The SCB algorithm was applied to generate minimal turn prohibition sets for each topology and fractions of prohibited turns for each graph were calculated. In Fig. 5, results of these calculations for 100 different graphs are shown as a histogram. It can be seen that no graph had a fraction of prohibited turns larger than 0.19. In Fig. 6, distributions of the average distances before turn prohibitions and after SCB generated turn prohibitions are shown. Given a graph, after turns are prohibited, the average distance may increase. This increase is described in terms of the dilation introduced by turn prohibitions. Results on dilation calculations for irregular graphs are shown in Fig. 7, where we see that the lengthening of the average distance has a mean of approximately 7%.

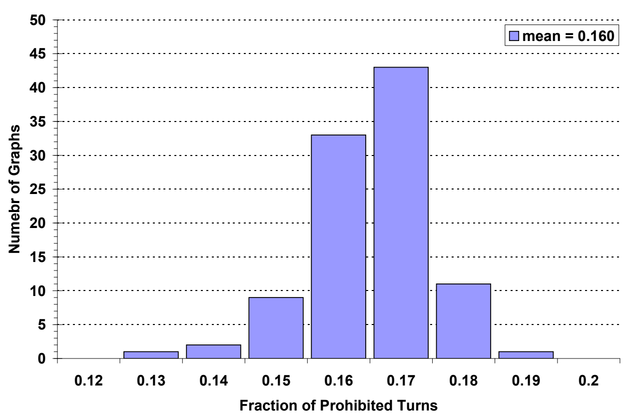

Fig. 5. Histogram for the fractions of prohibited turns in 100 general graphs each with 64 nodes of average degree four

Calculations for planar topologies follow a similar approach. We first generated a number of families of planar graphs, each family with a different girth. In all of our constructions, the faces are all regular and have the same number of edges. For example in girth 3 planar topologies all faces are triangular. After the planar topologies are generated we then applied the SCB algorithm to break all cycles as before. Results of these calculations are shown below. In Fig. 8, the fractions of prohibited turns are shown for families of planar graphs with girths 3 through 8 together with the theoretical upper bounds. In this figure we also show the fraction of prohibited turns of ($z(G) = 4/15$) for icosahedron with 12 nodes.

In Fig. 9, the average distance versus the girth is plotted after the application of the SCB algorithm. Given a graph, as

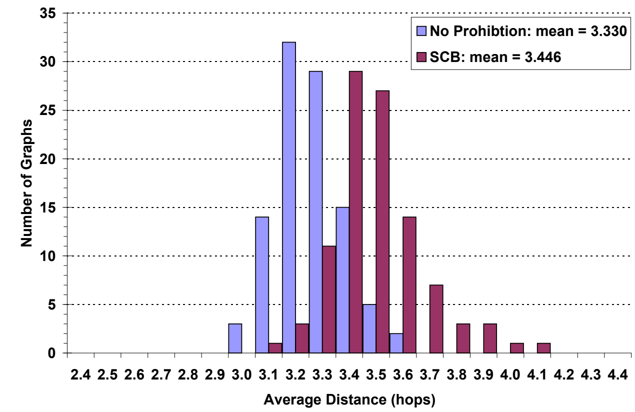

Fig. 6. Distributions of average distance in 100 irregular topologies with 64 nodes before and after SCB generated turn prohibitions

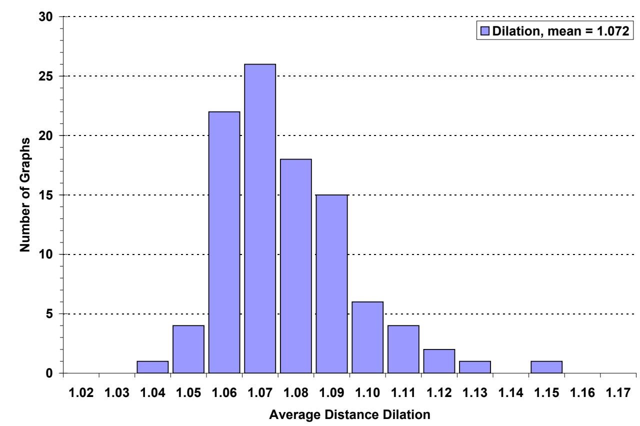

Fig. 7. The average dilation introduced by SCB is 7%

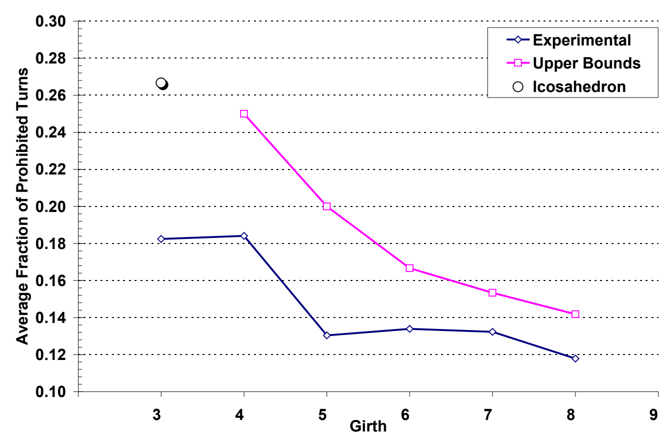

Fig. 8. Fraction of prohibited turns for SCB in 64-node planar graphs as a function of girth

the turns are prohibited, the average distance may increase. This increase is described in terms of the dilation introduced by turn prohibitions.

Results on dilation calculations are shown in Fig. 10, where we see that as the girth increases the average dilation increases, predicting a better performance for message delivery times for planar topologies with smaller girths. We see that for girths 3 and 4 topologies that we investigated, the dilation were 1.0002

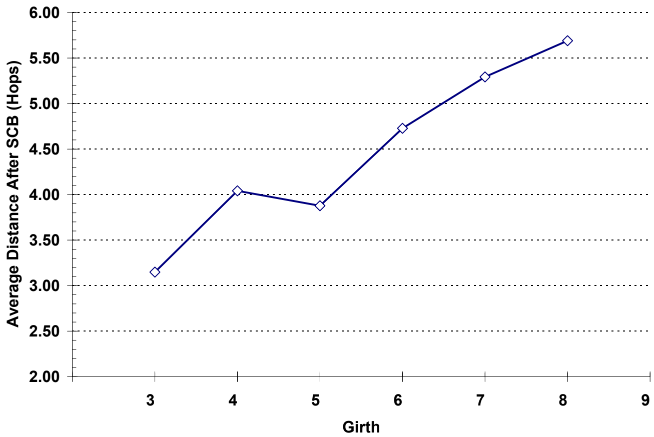

Fig. 9. Average distance in planar topologies with 64 nodes after SCB generated turn prohibitions as a function of girth

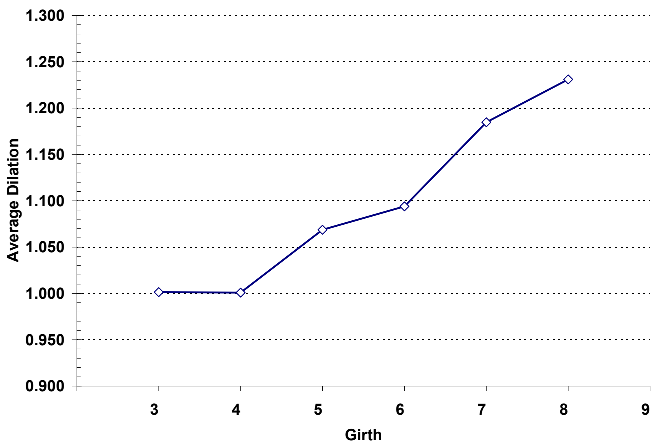

Fig. 10. Average distance dilation in planar topologies with 64 nodes after SCB generated turn prohibitions as a function of girth

and 1.0009 respectively.

In addition to these calculations, we also performed message delivery experiments. In the experiments, we implemented wormhole node models [18], [12], [19] with 16 bidirectional full-duplex ports and a local port. Messages, also known as worms, are generated at a module attached to the local port at the node. All messages in our simulations, 200 flits long, were generated using uniform traffic model with exponential inter-arrival times. As worms are injected into the network via the local channel, the router at the node determines using a routing table, which output port to use to route the message. If the output port is free, it is immediately committed to the incoming message port for the duration of the message, otherwise the message is blocked until the output port is freed up. Routing table at each node is generated using the all-pairs shortest path algorithm with an additional criterion that the selected shortest paths do not include any prohibited turns. This way, both deadlock and live-lock conditions are proactively prevented from occurring during the actual routing of messages. The results on average saturation points obtained in the message delivery experiments with planar topologies are presented in Fig. 11. The results are in agreement with the

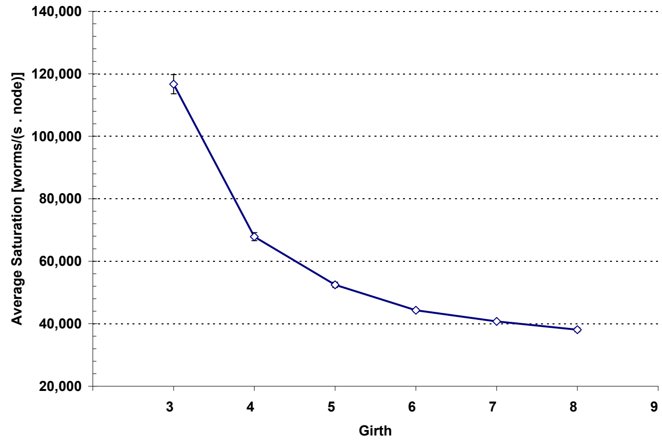

Fig. 11. Saturation points (maximum sustainable message generation rates per second per node) as a function of the girth of planar topologies

anticipated behaviors. In particular, since the average distance increases with girth in planar networks with a given number of nodes, one would expect better performance in networks of smaller girths. Indeed as seen at Fig. **??**, the saturation load increases for smaller girths.

VI. Conclusions

In this paper considers the problem of constructing minimal cycle-breaking sets of turns for graphs that model communication networks. This problem is important for deadlock-free and livelock-free message routing in computer communication networks. We present a new algorithm called the Simple Cycle-Breaking or the SCB algorithm which is considerably simpler than those in [15], [12], [10] and has the same performance and run-time complexity. Earlier cycle-breaking algorithms were complicated, involving as many as ten steps, whereas the SCB algorithm has only three steps and is easy to understand. The complexities of having to deal with cut nodes have totally been eliminated in the SCB algorithm. We also present results on minimal fractions of turns that must be prohibited to break all cycles without loss of connectivity for arbitrary irregular connected graphs and planar graphs, and present simulation results on saturation points for networks of different girths. The proposed algorithm is shown to be very efficient in terms of three basic characteristics: the fraction of prohibited turns, the dilation, and the saturation load.

All turn prohibition approaches considered so far are based on labeling of the vertices of a graph in one way or the other. A turn (a, b, c) is then prohibited either when the label of node b is smaller (or, alternatively, larger) than the labels of nodes a and c. Node labeling, however, may result in prohibiting a larger than necessary number of turns and may effect the efficiency of the algorithms. We may expect that new algorithms which could avoid node labeling would be more efficient.

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