Remarks on History of Abstract Harmonic Analysis

Radomir S. Stanković, Jaakko T. Astola[1], *Mark G. Karpovsky*[2]
Dept. of Computer Science, Faculty of Electronics, 18 000 Niš, Serbia
[1]Tampere International Center for Signal Processing
Tampere University of Technology, Tampere, Finland
[2]Dept. of Electrical and Computer Engineering, Boston University
8 Saint Marry's Street, Boston Ma 02215, USA

ABSTRACT

This paper review history of work in the development of abstract harmonic analysis, which is a mathematical discipline attempting to extend the classical Fourier analysis to various groups.

1 INTRODUCTION

Fourier analysis is aimed at representation of a complex signal as the superposition (mostly linear combination) of simple signals reflecting the structure of the group G on which the initial signal has been defined.

Theory of Fourier analysis is complex and sometimes complicated, however, many algorithms derived from it, as for instance, spectral estimation, convolution analysis, filtering, etc., are relatively simple and their concrete applications up to some extent are possible without a deeper study of mathematical foundations of the Fourier analysis. However, derivation of yet another sequence of numbers is not the goal of the transition into the spectral domain, but rather achieving a better insight into the essence of the problem considered. The complete understanding and correct interpretation of the derived results can hardly be achieved without a certain knowledge of mathematical foundations of the Fourier analysis.

2 TRIGONOMETRIC SERIES

Theory of trigonometric series can be dated back at the beginning of 18th century. Mathematicians of that time have been using trigonometric series, in particular for their various astronomical calculations.

In 1729, Euler formulated and have been starting solving the problem of interpolation as the problem of determination of the function values in an arbitrary point x if known its values for $x = n$, where n is an integer.

In 1747, Euler applied the method he disclosed to a function ϕ derived from analysis of movement of planets, and represented ϕ in the form of a trigonometric series. This method derived in 1729, Euler had published in 1753 [11]. This article actually contains what is now called the Fourier series, and Euler provided also formulaes to determine the coefficients in the series by the integral of the function considered. Therefore, it may be stated justifiably that the trigonometric series of a function has been presented for the first time in 1750 to 1751.

Figure 1: Euler.

Figure 2: d'Alambert.

In 1754, d'Alambert [9] has considered the problem of series representation of the reciprocal value of the mutual distance of two planets as a function of their position as a series in cosinusoidal functions whose argument is the integer multiple of the value of the angle between the rays from the centers of planets. In this article, d'Alambert also provided formulae for determination of the coefficients of this series in terms of finite integrals.

In [11], [12], Euler derived trigonometric series of some functions in a way completely different from that he previously used. Similar results have been derived at about the same time by Lagrange [25] and Daniel Bernoulli [3].

It is interesting to notice that neither Euler nor La-

Figure 3: Lagrange.

Figure 4: Daniel Bernoulli.

grange commented the interesting feature that it was a non-periodic function among the functions considered. From a letter by Lagrange to d'Alambert dated on August 15, 1768 [26], it can be concluded that they realized that latter in the context of some other problems.

In 1757, Clairaut derived a cosinus series representation of a function derived in a study of the movement of the Sun.

Figure 5: Clairaut.

In 1777, in solving some astronomical problems, Euler determined coefficients in the series representation of a trigonometric function by a method equal to that used nowadays [13],

In the research work discussed above and related publications there are examples of trigonometric series of various classes of functions, however, the essential question about the possibility of representation of an arbitrary function by a trigonometric series, remained unsolved until the works by J. B. Fourier, whose main ideas are contained in his book [14].

3 FOURIER AND HIS WORK

3.1 Biography and Work

Jean Baptiste Joseph Fourier was born in Auxerre in north-central France on March 21, 1768, and left without parents when he was nine old. Thanks to the recommendations of few family friends to the Bishop of Auxerre, Fourier joined a military school run by Benedictines of Saint-Maur. As a student he demonstrated his skills and an extraordinary gift for mathematics. However, due to middle class origins, Fourier, whose father was a tailor, has not be allowed to persuade towards the carrier of an artillery officer, although applied with a strong recommendation by Legendre, and had to accept the position of a lecturer of mathematics in a military school in France.

Figure 6: Fourier.

Figure 7: Legendre.

Due to his public activity during the French Revolution, when served as a publicist, recruiting agent, and a member of the Citizens Committee of Surveillance, Fourier was arrested in 1789 for defending the victims from the terror of revolutionaries, and had problems to escape from the guillotine and fury of former co-allies. In 1794, Fourier was selected among 500 candidates for new teachers at the Normal School just established in Paris. However, when this School failed shortly after that, Fourier already prove himself as an outstanding scholar and in 1795, was awarded by a professorship at the prestigious École Polytechnique in Paris starting as a superintendent of lectures on fortification, and then as a lecturer on analysis. At that time, Lagrange and Monge have been also teaching at the same school.

Figure 8: Monge.

In 1798, both Monge and Fourier joined a group of scholars in the military campaign of Emperor Napoleon Bonaparte to Egypt, where Fourier has been appointed governor of the southern Egypt. After the defeat by Britishers in 1801, Fourier returned to France at the position of prefect of Départment of Isére and lived in Grenoble. Besides his administrative duties, Fourier become appointed secretary of the *Institut d' Egypts*, and in 1809 completed a major work on ancient Egypt, *Préface historique*. In the same year, Napoleon awarder Fourier with the title of a Baron.

Fourier have been keeping this position of the prefect also in 1814, when Napoleon returned from his exile at Elba. Fourier went to Lyon to inform Burbons that the city of Grenoble will surrender to Napoleon and his supporters. The answer was that he would be responsible for the safety of the city. When Fourier returned, the Grenoble already capitulate, and his loyalty to Napolen has been re-established after some remarks by the Emperor to his strongly supported associate scientist, mathematician and Egyptologist. After the end of Hundred Days, during the Restoration, Fourier run into troubles for his political past. In this situation, Fourier was helped by at that time prefect of Paris, his former student and friend, who supported his appointment as the Director of the Bureau of Statistics.

Besides much engaged with various respectable administrative duties, Fourier has been preforming mathematical research in the theory of equations, and mathematical physics.

3.2 Scientific Work

Already in the age of sixteen, Fourier has found a new proof of the rule formulated by Descartes about the number of positive and negative roots in a polynomial. In the age of twenty-one he delivered his first memoir before the Academy of Sciences on the resolution of numerical equations of all degrees.

Fourier worked on the book *Analyse des équations déterminées*, published in 1831 by his friend Louis Marie Marie Navier, where he anticipated linear programming [18].

However, fundamental contribution by Fourier is in mathematical physics. He studied the flow of heat between two regions of different temperature. This question was very important in producing guns, therefore, a very important problem also for military authorities, and the same problem was discussed already by Sir Isaac Newton, who provided an estimation of the temporal rate of cooling in terms of the difference between the temperature of an object and his environment. However, Newton did not solve the spatial rate of change, since it depends on several factors as, for instance, the geometric shape of the object, heat conductivity of the object, and initial distribution of the temperature on its boundary. This problem requires application of the analytic tool of the continuum and solution of partial differential equations. Fourier provided a solution of the problem considered by showing that the initial distribution of the temperature mast be expressed as a sum of infinitely many sine and cosine terms, which is now called the trigonometric or Fourier series.

Figure 9: Descartes.

3.3 Main Contribution

The main contribution of Fourier to mathematical physics can be summarized as follows.

For a reasonably behaved function on $(-\pi, \pi)$, the Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where the coefficients are determined as

$$c_n = \frac{1}{2\pi} = \int_{-\pi}^{\pi} e^{-inx} dx.$$

The term reasonably behaved means $f(x)$ is piecewise smooth on $-\pi, \pi$, i.e., it is continuous and differentiable except possibly at a finite number of finite jump discontinuities. In these points, the function takes values $f(x) = f(x^-) + f(x^+)/2$, that is, as the mean between the values $f(x)$ just to the left and right of the point x.

Fourier viewed non-periodic functions as a limiting case of periodic functions with the period approaching to infinity. In this case, the Fourier series is replaced by the Fourier integral that represents a continuous distribution of sine waves over all frequencies.

$$S_f(w) = \int_{\infty}^{\infty} f(x)e^{-2\pi iwx}dx,$$

and

$$f(x) = \int_{-\infty}^{\infty} S_f(w)e^{2\pi iwx}dw.$$

3.4 Presentation of Work and Publishing

In 1807, Fourier completed his work "Propagation of Heat in Solid Bodies" and presented it before the French Academy of Science on December 12. The claim that any function defined in a finite closed interval can be represented in the form of a series of sine and cosine functions with suitably assigned coefficients, has not been completely accepted. The sceptical Referees were Laplace, Lagrange, Monge, and Lacroix [1]. However, to encourage the author to continue and improve his research results, the Academy assigned as a subject for the award scheduled in 1812 the subject *The Mathematical Theory of the laws of the propagation of heat and the comparison of the results of this theory with exact experiment.* Fourier submitted a revised paper in 1811. The group of judges, among which the previous Referees, awarded the memoir by Fourier consisting of two parts, however, criticized the lack of mathematical rigor and rejected the paper for publication in the *Memoirs of the Academy.*

3.5 Recognition and Rewarding

It should be noticed that his political past, revolutionary activity and strong relationships with Napoleon, did not prevent Fourier to be recognized for his scientific research and achievements after the Restoration of Monarchy in France. Fourier has been nominated for the member of the French Academy of Science in 1816, and Louis XVIII refused his consent that time. In 1817, Fourier has been elected to the Academy od Sciences, and became the permanent Secretary of the Division of Mathematical Sciences in 1822, when he published his awarded memoir *Theorié analytique de la chaleur*, (Analytical Theory of Heat), which is widely considered as his major contribution to the mathematical physics. In 1826, Fourier become a member of French Academy, and in 1826 followed Laplace at the position of the President of the Council for Improving the Polytechnic School. In 1828, Fourier was appointed as a member of a committee of French government for encouragement of literature.

Fourier attitude to the research and mathematics can be expressed by his often used saying *Profound study of nature is the most fertile source of mathematical discoveries.* However, exactly that was a source of many criticisms of Fourier mathematical work by some mathematicians at that time as, for instance, Lagrange, Poisson, and Biot.

[1]In some publications, it is noticed that the committee consisted of Laplace, Lagrange, and Legendre

Figure 10: Biot.

4 Further Development of Fourier Analysis

The results by Fourier were expressed by Dirichlet and Riemman with stronger precision and formalism. The work by Dirichlet has been published in the *Crelle's Journal* in 1828.

Riemann, who was a student of Dirichlet write in the introduction to his habilitation thesis on Fourier series that Dirichlet wrote "the first profound paper about this subject" [24].

Poisson provided foundations for the work by Dirichlet and Riemman, as can be found in the *Journal of the École Polytechnique* from 1813 to 1823 and in *Memoirs de l' Académie* for 1823. He also studied the Fourier integral.

Poisson summation formula is a relation between the sum of a function f over all integers and a corresponding sum of the Fourier transform s_f. If the normalization of the Fourier transform is correctly adjusted, it can be written as $\sum F(n) = \sum S_f(n)$. Some conditions must be applied to ensure convergence.

In 1806, Poisson has been appointed for a Full Professor at the École Polytechnique in succession to Fourier who went to Grenoble. Poisson become a baron in 1821 for remaining faithful to Burbons during the Hundred Days.

Cauchy has shown that in a work by Poisson on the convergence of Fourier series was non-rigorous. However, Dirichlet wrote that a proof provided by Cauchy "does not include certain functions for which the convergence is incontestable".

Figure 11: Poisson.

Figure 12: Dirichlet.

Figure 13: Riemman.

Notice that many useful properties of Fourier series follows from the orthogonality and homomorphism of the basis functions. Similar properties are true for a various classes of functions, as for example, Bessel functions and orthogonal polynomials. However, some of the properties are missed when the homomorphism property is not satisfied. In many cases, such classes of functions are derived as solutions of some differential equations. A more general classes are generated as solutions of the Sturm-Liouvile problems.

In 1799, Parseval published a formula for the sum of squares of the coefficients of a trigonometric series in terms of integrals, which is now called the Parseval theorem. This theorem can be viewed as the particular case of the Plancharel theorem.

The introduction of the Lebesgue integral in his PhD thesis in 1902 and the book in 1904, provided foundations for formulation of the Riesz-Fischer theorem in 1907, showing that any square-summable sequence $\{c_n\}$, for $n \in \{-\infty, \infty\}$ is the sequence of Fourier coefficients of an L^2 function on the interval $(-\pi, \pi)$, thus, Fourier coefficients are an isometric linear mapping between two L^2 spaces.

In 1910, Plancharel proved a result which is called the Plancharel formula, which shows that the Fourier transform is an isometric mapping of L^2 into L^2.

5 GROUP REPRESENTATIONS AND HARMONIC ANALYSIS

Harmonic analysis cannot be separated from theory of group representations, which are used as a basis replac-

Figure 14: Lebesgue.

Figure 15: Plancherel.

ing the role of exponential functions in classical Fourier analysis.

In other words, harmonic analysis is an extension of the classical Fourier analysis derived by replacing the real line R by an arbitrary group G. In this respect, it should be distinguished

1. Abelian,
2. Non-Abelian groups.

The Fourier analysis on an Abelian group G is defined in terms of the corresponding *group characters*.

However, multiplicative characters are not sufficient to extend the Fourier analysis to non-Abelian groups. In this case, *group representations* are required, which can be viewed as a generalization of multiplicative characters by increasing the dimension of them. Notice that for Abelian groups all the representations are single-dimensional and reduce to group characters.

Definition 1 *(Group representations)*
For a group G, a representation is a homomorphism of G into the group of linear transformations on V [2]*, where V is a complex vector space.*

For finite groups, the generalizations are possible by allowing to replace the complex field C by any field the

[2] This group is usually called the *general linear group* $GL(V)$. It is often assumed that the representations are matrix-valued, i.e., the group $GL(V)$ is isomorphic to the general linear group $GL(r_w, C)$ of non-singular $(r_w \times r_w)$ complex matrices. The number r_w is called the degree of the representation $\mathbf{R}_w$ and the r_w^2 functions $r_{i,j}$ on G are the matrix entries of $\mathbf{R}_w$ in the given basis.

characteristic of which is relatively prime to the order of the group G that is considered. The *modular theory* due largely to R. Brauer, removes this restriction to the characteristic of the field.

The group G and the vector space V are often topologized and the group action is normally assumed to be continuous. [3] When G topologized, for discussion of abstract harmonic analysis, the following *topological groups* should be distinguished

1. Compact,

2. Locally compact,

3. Non-compact.

Abstract harmonic analysis is a branch of harmonic analysis that extends the definition of the Fourier transforms for functions defined on various groups, and the above mentioned classes of groups will be discussed for both Abelian and non-Abelian groups.

5.1 Group Characters

In 1882, Heinrich Weler introduced multiplicative characters for an arbitrary finite Abelian group G.

The definition of a group character was discussed in the late 1870s by Dedekind [2]. He defined a character on a finite Abelian group G to be a homomorphism from G to the multiplicative group of nonzero complex numbers, and orthogonality relations have beeb previously discovered. Dedekind also defined what he called the *group determinant* and noticed that it can be factored nicely, when the group is Abelian. Dedekind conjectured that this factorization can be extended to non-Abelian groups. In that respect, in 1896, Dedekind communicated with Frobenius, who had published in the same year a paper on group characters and presented these results to the Berlin Academy on July 16, 1896. It should be noticed that in this paper Frobenius did not relate the group characters to the group representations. However, this research has been continued based on a paper by Dedekind from 1885, further supported by the communication with Dedekind started on April 12, 1896 by letter of Dedekind to Frobenius. In this communication, many interesting results can be found. For instance, in a letter of Frobenius to Dedekind on April 26, 1896, Frobenius presented the *irreducible characters* for the *alternating groups* A_4, A_5, the *symmetric groups* S_4, S_5, and the group $PSL(2,7)$ of order 168.

Due to this work, in 1897, Frobenius introduced the notion of group characters. After studying the work of Molien [30], [31], and reformulation some of these results in terms of matrices, Frobenius has shown that the group characters defined by him in 1897 are traces of *irreducible representations.* In a letter to Dedekind on

[3]Notice that a multiplicative character $\chi_w(x)$ is a representation on the single.dimensional space C of complex numbers, and the action by an element $g \in G$ is the multiplication by $\chi_w(g)$.

Figure 16: Dedekind.

Figure 17: Frobenius.

February 1924, Frobenius said that Molien investigated non-commutative multiplication and obtained general results from which the properties of group determinants can be derived as special cases.

It is interesting to notice that Molien studied the results by Frobenius in group theory and applied them to investigate polynomial invariants of finite groups. In particular, Molien studied how many times a given irreducible representation of a finite group appear in a complete reduction of the representation of the group on the vector space of homogeneous polynomials of degree n over the complex numbers. In 1898, Molien introduced a generating function to compute the number of times the irreducible character occurs.

In 1898, Frobenius introduced the notion of *induced characters* and the *tensor product of characters*, and a theorem called now the *Fobenius Reciprocity Theorem.*

In 1990 and 1991, Frobenius completely determined characters of the symmetric and alternating groups, respectively, published in two separated papers. Further advent in application of group characters, Frobenius provided by studying the structure of groups called nowadays the *Frobenius groups*

The theory of groups characters developed by Frobenius, was nicely presented by Burnside in [4].

6 Group Representations

In their work started in 1904 and 1905 respectively, Burnside and I. Schur [34], Vol. 1, discussed matrix representations, i.e., homomorphism into the group of invertible matrices of given dimensions. In their ap-

Figure 18: Burnside.

Figure 19: Schur.

proach, group representations were complex-valued column vectors and the linear transformations are viewed as matrices.

Burnside is often credited as a founder of the theory of finite groups and his work complemented and sometimes compete with the work by Frobenius.

Schur was a student of Frobenius and made a considerable contribution by his own work or in collaboration with Frobenius.

In 1925, Schur returned to the group representation theory due to the development of theoretical physics exploiting it. Then, he provide a complete description of the rational representations of the general linear group.

Figure 20: Emmy Noether.

However, Emmy Noether replaced the matrices by linear transformations of a vector space, and therefore, his definition of the group representations is equal to that used nowadays. This approach to the definition of group representations has been reconfirmed when considered groups where infinite-dimensional representations are necessarily required, as for instance, the *Lie groups*.

Figure 21: Lie.

Burnside pointed out that in the case of finite groups every finite-dimensional representation is equivalent to a representation by unitary matrices and the complete reducibility follows from the unitarity. Burnside also pointed out that if Q is a mapping between irreducible representations in two spaces V_1 and V_2, then $Q = 0$ or Q is invertible. Schur had shown that if $V_1 = V_2$, then Q is a scalar. Schur also proved the orthogonality of inequivalent irreducible unitary representations of finite groups.

Frobenius introduced the notion of induced representations as a way to define a representation R of a group G from a representation R_i of a subgroup G_i of G.

7 Finite Groups

The harmonic analysis on finite groups is performed in terms of irreducible unitary representations, or their characters, for non-Abelian and Abelian groups, respectively. This approach has been developed first for the symmetric and alternating groups in the work by Frobenius and Young, who introduced the *Young diagrams* for manipulating with irreducible representations.

7.1 Finite Abelian Groups

Notice that when G is an Abelian groups, the set of group characters $\chi_w(x)$ form a multiplicative group Γ isomorphic to G. Therefore, a function $f(x)$ on a finite Abelian group G of order $|G| = g$ can be represented as

$$f(x) = g^{-1} \sum_{w \in \Gamma} S_f(w) \chi_w(x),$$

where $\Gamma = \{\chi_w(x)\}$, $x \in G$ is the set of characters of G, and

$$S_f(w) = \sum_{x \in G} f(x) \chi_w(x)^{-1}.$$

7.2 Finite Non-Abelian groups

In the case of finite non-Abelian groups the Fourier transform is defined in terms of finite-dimensional irreducible unitary representations $\mathbf{R}_w(x)$, $x \in G$, as

$$f(x) = \sum_{w=0}^{K-1} Tr(\mathbf{S}_f(w) \mathbf{R}_w(x)),$$

where K is the number of equivalency classes of unitary irreducible representations which form the dual object Γ for G, and $Tx(\mathbf{Q}$ denotes the trace of a square matrix $\mathbf{Q}$, i.e., the sum of elements of the min diagonal of $\mathbf{Q}$.

The Fourier coefficients are $(r_w \times r_w)$ matrices, where r_w is the dimension of the representation $\mathbf{R}_w$,

$$\mathbf{S}_f(w) = r_w g^{-1} \sum_{u=0}^{g-1} f(u)\mathbf{R}_w(u^{-1}),$$

where g is the order of G.

Finite groups are compact groups, and definition of Fourier transform is a simplified version of the Fourier transform for arbitrary compact groups.

8 Compact non-Abelian groups

Extensions of Fourier analysis to compact non-Abelian groups are due to the Peter-Weyl theorem formulated by H. Weyl and his student and associate F. Peter, first for the case of non-Abelian Lee groups [32]. The main contribution consists in the observation that not the finiteness of a group ensures existence of main properties of the Fourier representations, but existence of an averaging procedure over the group [19]. In other words, it is required the existence of an invariant integral that assigns a finite volume to the group. In this case, the *Haar integral* plays and an important role.

Figure 22: Weyl.

Figure 23: Haar.

In the case of non-Abelian groups it is necessary to distinguish the left and right invariance. For instance, an integral on a topological group G is the right invariant if

$$\int_G f(xa)dx = \int_G f(x)dx,$$

for all $a \in G$.

It is proved by Haar in 1933 that a right invariant integral exists for locally compact groups. This integral is now called the Haar integral. Notice that local compactness is implied by the existence of a right invariant integral as shown by Andre Weil in his book [35].

Figure 24: Weil.

The main idea by Peter and Weyl, which provides possibility to extend the abstract harmonic analysis, has been to use an infinite dimensional representation and its decomposition by means of spectral theory for bounded operators on Hilbert space [32].

In short, for compact non-Abelian groups the Peter-Weyl theorem explains determination of harmonics as representatives of each equivalence class of representations. From each equivalence class of representations, a representation is selected as a harmonic to define an analogue to the classical Fourier transform.

More precisely, the Peter-Weyl theorem for compact groups shows that the Fourier series of a function f on G is

$$f(x) = \sum_{R_w \in \Gamma} r_w \sum_{i,j=0}^{r_w-1} S_f^{(i,j)}(w) R_w^{(i,j)}(x),$$

where Γ, the dual object of G, is a collection of all equivalence classes of irreducible unitary representations $\mathbf{R}_w$ of G.

The Fourier coefficients are determined as

$$S_f^{(i,j)}(w) = \langle f, R_w^{(i,j)} \rangle = \int_G f(x)(R_w^{(i,j)})^{-1}(x)dx.$$

This series applies to functions which are square-integrable in that the norm

$$||f|| = \left\{ \int_G |f(x)|^2 dx \right\}^{\frac{1}{2}},$$

is finite, and the Fourier series for $f(x)$ is equal to f in the mean-square sense of

$$||f||^2 = \sum_{R_w \in \Gamma} r_w \sum_{i,j=0}^{r_w-1} |S_f^{(i,j)}|^2.$$

This formula is called the Plancharel formula for G.

In the case of compact Abelian groups, by the Schur lemma, the irreducible representations are single-dimensional and, thus, the dual object Γ is the dual group of all continuous homomorphisms χ_w of G into the unit circle. Then, the Fourier series for f on G is

$$f(x) = \sum_{\chi \in \Gamma} S_f(w)\chi_w(x),$$

and the Fourier coefficients are numbers

$$S_f(w) = \int_G f(x)\chi_w^{-1}(x)dx.$$

From 1923 to 1938 Weyl developed the theory of compact groups in terms of matrix representations. In the case of compact Lie groups, he proposed a fundamental character formula.

There are compact groups that are not Lie groups, however, the representation theory and, therefore, harmonic analysis on such groups are highly incomplete.

9 Locally Compact Abelian Groups

To discuss the harmonic analysis on locally compact groups, recall that the real line R is a locally compact Abelian group. The Fourier integral S_f of a function f on the real line R, defined for all real numbers w by

$$S_f(w) = \frac{1}{2\pi}\int_{\infty}^{\infty} f(x)e^{-iwx}dx,$$

is an example of the Fourier transform on locally compact Abelian groups. The constant $1/2\pi$ can be viewed as the normalization of the Haar integral on R.

Notice that for f integrable over the real line, i.e., $f \in L^1$, the spectrum S_f is well defined. However, the integrability of f does not imply the integrability of S_f, with integrability understood in the Lebesgue sense. Therefore, generalized methods of summability are required.

If $f \in L^2$, i.e., f is both integrable and square integrable, then S_f is also square-integrable and f is equal to the Fourier integral in the means-square sense, i.e., the Plancherel formula is valid

$$\int_{-\infty}^{\infty} |S_f(w)|^2 dw = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

The classical Fourier analysis on R has been extended to an arbitrary locally compact Abelian group G due to the *Pontryagin duality*, also called *Pontryagin-van Kampen duality* which can be briefly summarized as follows.

For a locally compact Abelian group, the set of unitary multiplicative characters under the pointwise multiplication expresses the structure of a locally compact Abelian group $\hat{G}$. This group, when topologized with the topology of uniform convergence of compact sets is

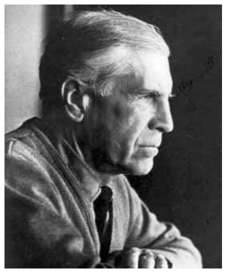

Figure 25: Pontryagin.

Figure 26: van Kampen.

the dual group for G. The group $\hat{G}$ has also a dual group, called dual dual $\hat{\hat{G}}$ [21], since there is a canonical continuous homomorphism of G into $\hat{\hat{G}}$, i.e., if $x \in G$, then the corresponding member of $\hat{\hat{G}}$ evaluated on a character $\chi_w \in \hat{G}$ has the value of $\chi_w(x)$.

The Pontryagin duality states that this homomorphism $G \to \hat{\hat{G}}$ is a *homeomorphism*, i.e., a topological isomorphism, of G onto $\hat{\hat{G}}$.

This result has been exploited by Andre Weil [35] to define the Fourier transform pair for functions f on G as

$$\begin{aligned} S_f(w) &= \int_G f(x)\overline{\chi_w(x)}dx, \\ f(x) &= \int_{\hat{G}} S_f(w)\chi_w(x)dw, \end{aligned}$$

where dx and dw are suitably normalized Haar integrals on G and $\hat{G}$, respectively.

Thus, the inversion formula is valid for integrable continuous function f whose Fourier transforms are integrable.

The foundations for the theory of locally compact Abelian groups and their duality has been established Lev Semenovich Pontryagin in 1934. In his approach it was exploited the structure theory and assumed that the group is second countable and either compact or discrete. This was imposed to cover the general locally compact Abelian groups by E.R. van Kampen in 1935 and André Weil in 1953.

It has been shown by Rudin that the duality theorem and harmonic analysis can be establish without referring to the structure theory [33].

10 Non-compact non-Abelian Groups

For compact groups, either Abelian or non-Abelian, the Fourier transform has been defined in terms of finite-dimensional irreducible unitary representations. In the case of compact Abelian groups, the representations are single-dimensional. For locally compact Abelian groups, the representations are again single-dimensional. However, for locally compact non-Abelian groups, the irreducible infinite-dimensional representations are required.

Notice that, in general, a non-compact group G may have representations that are not unitarizable in a Hilbert space.

It may be said that for non-Abelian groups which are not compact, there is no a general theory that would preserve at least some of the properties of the classical Fourier transform, as for example, the Plancharel theorem. However, many particular cases are considered, for example, $SL(n, F)$, in which case the representations of infinite dimensions are used.

In a series of publications, Harish-Chandra discussed extensions of harmonic analysis to noncompact real semi-simple Lie groups, providing also the Plancherel theorem in 1952 [20]. This work was preceded by the research done by Gelfand and Raikov in 1943, pointing out that in principle, there should exists a sufficient number of irreducible representations to perform harmonic analysis on locally compact groups [17].

Figure 27: Gelfand.

For instance, G. W. Mackey used the notion of induced representations to deal with measure-theoretic foundations for infinite-dimensional representation theory [28], [29].

The work by A.A. Kirilov, started in his doctoral thesis in 1962, [22] provided a basis for the work by L. Auslander and C.C. Moore [1], and B. Konstant for extension of harmonic analysis to some solvable groups.

Some brief reviews of these results can be found in [19] and [23].

Table 1 summarizes definitions of Fourier representations on various groups.

Table 1: Groups and transforms.

Group	Transform
Circle	$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$,
$R/2\pi Z$	$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$
Real line	$f(x) = \int_{-\infty}^{\infty} S_f(w)e^{2\pi iwx}dw$
R	$S_f(w) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iwx}dx$
Finite	$f(x) = \frac{1}{\lvert G\rvert}\sum_w S_f(w)w(x)$
Abelian	$S_f(w) = \sum_{y\in G} f(y)\overline{w(y)}$
Finite	$f(x) = \sum_{w=0}^{K-1} Tr(\mathbf{S}_f(w)\mathbf{R}_w(x))$
non-Abelian	$\mathbf{S}_f(w) = r_w g^{-1}\sum_{u=0}^{g-1} f(u)\mathbf{R}_w(u^{-1})$
Compact	$f(x) = \sum_{\chi\in\Gamma} S_f(w)\chi_w(x)$
Abelian	$S_f(w) = \int_G f(x)\chi_w^{-1}(x)dx$
Compact	$f(x) = \sum_{R_w\in\Gamma} r_w \sum_{i,j=0}^{r_w-1} S_f^{(i,j)}(w)R_w^{(i,j)}(x)$
non-Abelian	$S_f^{(i,j)}(w) = \langle f, R_w^{(i,j)}\rangle = \int_G f(x)(R_w^{(i,j)})^{-1}(x)dx$
Locally	$f(x) = \int_{\hat{G}} S_f(w)\overline{\chi_w(x)}dw$
compact	$S_f(w) = \int_G f(x)\overline{\chi_w(x)}dx$
Abelian	

References

[1] Auslander, L., Moore, C.C., "Unitary representations of solvable Lie groups", *Mem. Amer. Math. Soc.*, 62, 1966.

[2] Beauchy, J.A., *Rings and Modules*, Class Notes.

[3] Bernoulli, D., "Reflexions et eclaricissement sur les nouvelles vibrations des corde", *Memories de l'Academie Royale des Sciences et Belles Letters*, Berlin, 1753.

[4] Burnside, W., *The Theory of Group of Finite Orders*, 1897, sewcond edition 1911, Dover reissue 1955.

[5] Butzer, P.L., "Dirichlet and his role in the founding of mathematical physics", *Arch. Internat. Hist. Sci.*, 37, 118, 1987, 49-82.

[6] Butzer, P.L., Jansen, M., Zilles, H., "Zum bevorstehenden 125. Todestag des Mathematikers Johann Peter Gustav Lejeune Dirichlet (1805-1859)", *Mitbegrnder der mathematischen Physik im deutschsprachigen Raum, Sudhoffs Arch.*, 68, 1, 1984, 1-20.

[7] Butzer, P.L., Jansen, M., Zilles, H., "Johann Peter Gustav Lejeune Dirichlet (1805-1859): Genealogie

und Werdegang", *Dürerner Geschichtsblätter*, 71, 1982, 31-56.

[8] Clairaut, A.-C., *Hist. de l'Acad. des Sci.*, Paris, 1754, 545, ff, publ. 1759.

[9] d'Alambert, J. le R. "Researchers sur diferentes points importants du systeme du monde", 1754, Vol. 2, p. 66.

[10] Dirichlet, Johann Peter Gustav Lejeune, *Vorlesungen uber Zahlentheorie*, Braunschweig, 1863, (Lectures in Numberr Theory).

[11] Euler, L., *Nova Acta Acad. Sci. Petrop.*, 5, 1754-1755, 164-204, publ. 1760 = Opera (1), 14, 435-542-84.

[12] Euler, L., Opera (1), 15, 435-497.

[13] Euler, L., *Nova Acta Acad. Sci. Petrop.*, 11, 1793, 114-132, publ 1798 = Opera (1), 16, Part 1, 333-355.

[14] Fourier, J., *Theorie analitique de la chaleur*, Euvres 1, see also the paper from 1811 published in *Mem. de l'Acad. des Sci.*, Paris, (2), 4, 1819-1820, 135-155.

[15] Frobenius, G., "Über Gruppencharahtere", *Sitzubgsber. Preuss. Akad. Wiss.*, Berlin, 1896, 985-1021. (Gesammelte Abhandlungen, Springer-Verlag, 1968, 1-37.)

[16] Frobenius, G., "Über die Primfactoren der Gruppendeterminante", *Sitzngsber. Preuss Akad. Wiss. Berlin*, 1896, 1343-1382. (Gesammelte Abhandlungen, Springer-Verlag, 1968, 38-77.)

[17] Gelfand, I.M., Raikov, D.A., "Irreducible unitary representations of locally compact groups", *Mat. Sb.*, Vol. 13, No. 55, 1943, 301-316.

[18] Grattan-Guinness, I., "Joseph Fourier's anticipation of linear programming", *Operational Research Quaterly*, (1970-1977), Vol. 21, No. 3, September 1970, 361-364.

[19] Gross, K., "On the evolution on noncomutative harmonic analysis", *Amer. Math. Monthly*, Vol. 85, No. 7, 1975, 525-548.

[20] Harish-Chandra, "The Plancherel formula for complex semisimple Lie groups", *Trans. Amer. Math. Soc.*, Vol. 76, No. 3, 1954, 458-528.

[21] Key, S.-H., *Notes on Abstract Harmonic Analysis*, RIM-GARC Lecture Notes Ser., No. 20, Seoul National University, 1994.

[22] Kirlilov, A.A., *Elements of the theory of Representations*, Springer, 1976.

[23] Knapp, A.W., "Group representations and harmonic analysis from Euler to Langlands", Parts 1 and 2, *Notices of AMS*, Vol. 43, No. 4, April 1996, 410-415, and No. 4, May 1996, 537-549.

[24] Koch, H., "Gustav Peter Lejeune Dirichlet", in *Mathematics in Berlin*, Berlin, 1998, 33-40.

[25] Lagrange, J.L., *Misc. Taur.*, 1, 1759 = Euvres 1, 110.

[26] Lagrange, J.L., Euvres, 13, 116.

[27] Lam, T.Y., *Notices of the AMS*, 1998, 361-372, 465-474.

[28] Mackey, G.W., "Infinite-dimensional group representations@, *Bull. Amer. Math. Soc.*, 69, 1963, 628-686.

[29] Mackey, G.W., *The Theory of Unitary Group Representations*, University of Chicago Press, 1976.

[30] Molien, T., "Über Systeme höoherer complexer Zahlen", 1893.

[31] Molien, T., "Über die Invarianten der linearen Substitutionsgruppen", 1897.

[32] Peter, F., Weyl, H., "Die Vollständigkeitder primitiven Darstellungen einer geschlossen kontinuierlichen Grouppe", *Math. Ann.*, 97, 1927, 737-755.

[33] Rudin, W., *Fourier Analysis on Groups*, Interscience, New York, 1962.

[34] Schur, I., *Gesamelte Abhandungen*, Springer-Verlag, Berlin, 1973.

[35] Weil, A., *L' Intégration dans les Groupes Topologiques et ses Applications*, Hermann, Paris, 1938.

10.1 Schur lemma

Schur Lemma can be stated as follows. Let G be a finite group representation in irreducible G-modules V and W. Any G-module homomorphism $f : V \to W$ is either invertible or the zero map. If G is represented over an algebraically closed field F, on irreducible G-modules V and W, then any G-module homomorphism $f : V \to W$ is a scalar.

If π on V and π' on V' are irreducible representations and $E : V \to V'$ is a linear map such that $\pi^{prime}(g)E = E\pi(g)$ for all $g \in G$, then $E = 0$ or E is invertible. Furthermore, if $V = V'$, then E is a scalar.

11 Addendum 1

Group characters

If $P = C$ the group characters are defined as the homomorphisms of G into the unit circle [33], or equivalently to the multiplicative group T of complex numbers with modulus equal 1.

Definition 2 *Complex-valued function $\chi_w(x)$ on G is the group character of G if $|\chi_w(x)| = 1$ for all $x \in G$:*

$$\chi_w(x \circ y) = \chi_w(x)\chi_w(y), \quad \forall x, y \in G.$$

The character $\chi_0(x) = 1$ for all $x \in G$ always exists and is called the principal character.

The set $\Gamma = \{\chi_w(x)\}$ of necessarily continuous characters expresses the structure of a multiplicative group called the dual group for G, with the group operation denoted by $\odot$ and defined as

$$(\chi_1 \odot \chi_2)(x) = \chi_1(x)\chi_2(x), \quad \forall x \in G,$$

for all $\chi_1, \chi_2 \in \Gamma$.

If G is a discrete group, Γ is compact and vice versa. If G is finite, Γ is also finite and isomorphic to G. Thanks to that duality, the index w and argument x in $\chi_w(x)$ have equivalent roles and it is convenient to express that property through the notation $\chi(w, x), x \in G, w \in \Gamma$.

12 Addendum 2

Group Determinant

The group determinant is the determinant of the group matrix, which for a finite group $G = \{g_0, \ldots, g_{n-1}\}$ is an $(n \times n)$ matrix whose the (i,j)-th element is $x_{g_i g_j^{-1}}$, where $\{x_{g_0}, \ldots, x_{g_{n-1}}\}$ is a set of commuting variables.

In a more general setting, the group matrix is viewed as any matrix obtained form a group matrix by assigning values in a ring to the variables.

The notion of group matrices was know before the work of Frobenius. A circulant matrix

$$\begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix},$$

is a group matrix of the cyclic group C_n of order n.

12.1 Addendum 2

The symmetric group S_n of order n is the group of all permutation of n symbols.

The set of even permutations in S_n forms a subgroup which is called the Alternating group and denoted by A_n.

A permutation group is a subgroup of a symmetric group on a set Ω.

13 Frobenius group

If a group G contains a proper non-trivial subgroup H such that $H \cap g^{-1}Hg = \{1_G\}$ for all $g \in G \setminus H$, then there exists a normal subgroup N such that G is the semi-direct product of N and H. Such groups are the Frobenius groups.

A Frobenius group is a subgroup of $Sym(\Omega)$ which is invertible and in which no element other than 1 fixes two or more points of Ω.

13.1 Second countable space

A second countable space is a topological space satisfying the second axiom of countability.

Axiom of countability is a property of certain mathematical objects that requires the existence of a countable set with certain properties, while without such sets might not exist.

A base B for a topological space X with the topology T is a collection of open sets in T such that every open set in T can be written as a union of elements of B.

13.2 Plancharel theorem

Plancherel theorem states that if X_j and Y_j are DFT spectra of x_k and y_k respectively, then

$$\sum_{r=0}^{n-1} x_k y_k^* = \frac{1}{n} \sum_{j=0}^{n-1} X_j Y_j^*$$

where the star denotes complex conjugation of a value.

Parseval theorem is a special case of the Plancherel theorem and states

$$\sum_{k=0}^{n-1} |x_k|^2 = \frac{1}{n} \sum_{j=0}^{n-1} |X_j|^2.$$

13.3 Manifold

A manifold is a space where near every point the environment is like that in Euclidean space of a given dimension. Since manifolds locally look-like Euclidean space r^n, they are inherently finite-dimensional objects.

A generalization of manifolds is to omit the requirement that a manifold be Hausdorff. It still must be second-countable and locally Euclidean however.

13.4 Dirichlet conditions

Dirichlet formulated conditions that must be met for a function $f(x)$ to have a Fourier transform. These conditions are

1. $f(x)$ must be single.valued,
2. $f(x)$ must have a finite number of extrema in any given interval,
3. $f(x)$ must have a finite number of discontinuities in any given interval,
4. $f(x)$ must be absolutely integrable.

Let f be a piecewise regular real-valued function defined on some interval $[q, r]$, such that f has only a finite number of discontinuities and extrema in $[q, r]$. Then, the Fourier series of this function converges to f when f is continuous and to the arithmetic mean of the left-handed and the right-handed limit of f at a point where it is discontinuous.

13.5 General linear group

$GL(n, F)$ is the general linear group of degree n over a field F (such as R or C), is the group of $(n \times n)$ invertible matrices with entries in F, with the group operation that of ordinary matrix multiplication.

The special linear group $SL(n, F)$ is the subgroup of $GL(n, F)$ consisting of matrices with determinant 1.

The group $GL(n, F)$ and its subgroups are often called linear groups or matrix groups.

A Linear group is an analytic real or complex manifold that is also a group such that the group operations multiplication and inversion are analytical maps. They were introduced by Sophus Lie, a Norwegian mathematician, in 1870 in order to study symmetries of differential equations. The notion arose from some particular classes of partial differential equations similar as the origins of Galois groups are related to the algebraic equations.

Matrix Lee groups are groups of invertible matrices under matrix multiplication. For instance, the group $SO(3)$ of all rotations in three dimensional space is a matrix Lee group. This is a non-Abelian group.

Abstract harmonic analysis is a branch of harmonic analysis considering definition of the Fourier transforms for functions defined on locally compact groups. The theory for locally compact Abelian groups is called the Pontryagin duality.

13.6 Hausdorff space

Suppose that X is a topological space. X is a Hausdorff space, or T_2 space, or separated space, iff given any distinct points x and y, there are a neighbourhood U of x and a neighbourhood V of y that are disjoint. In fancier terms, this condition says that x and y can be separated by neighbourhoods.

Almost all spaces encountered in analysis are Hausdorff, most importantly, the real numbers are a Hausdorff space. More generally, all metric space are Hausdorff. In fact, many spaces of use in analysis, such as topological groups and topological manifolds, have the Hausdorff condition explicitly stated in their definitions.

A set X is called open if any point $x \in X$ is surrounded by elements of X and, thus, it cannot be on the edge of X. For example, the interval $(0, 1)$ consisting of all real numbers with $0 < x < 1$, since if we change x for a little, it will still be a number between 0 and 1. The interval $(0, 1]$ is not open, since if 1 cahnged to the right, it will not stay in $(0, 1]$.

A discrete space is compact iff it is finite.

Every discrete space is first countable, and a discrete space is second countable iff it is countable.

A set is countable if the number of its elements is finite or it has the same number of elements as the natural numbers.

Compact group is a topological group that is also a compact space. It is usually assumed in representation theory that such group is also Hausdorff.

Groups are used to describe symmetries of objects. This is formalized by the notion of a group action, every element of the group acts like a bijective map (or symmetry) on some set. In this case, the group is called a transformation group of the set.

A topological space is compact if every open cover of it has a finite subcover. That is, any collection of open sets has a finite subcollection whose union is still the whole space. In many cases, the term compact space is used for compact Hausdorff spaces.

A cover of a set X is a collection of subsets C of X whose union is X. An open cover is a cover if each of its members are open sets.

Homeomorphism or tolological isomorphism is a special isomorphism between topological spaces with respect to topological porperties.

An isomoprhism is a bijective map of f such that both f and the inverse of it $^{-1}$ are homomorphisms, i.e., mappings that preserve the structure.

Topological property is a property that is invariant under homeomorphisms. Properteis are open and closed sets, interior, closure, neighborhood, limit point, compactness, connectedness, Hausdorff.

13.7 Topological group

A topological group is a group G on which a topology is defined such that the group multiplication $(x, y) \rightarrow xy$ and inversion $x \rightarrow x^{-1}$ are continuous mappings. It follows that each right-translation $x \rightarrow xa$ and the left-translation $x \rightarrow ax$ are homeomorphisms of G and, therefore, the topology of G is completely determined by local behavior of the identity e of G. The group G is locally compact if there exists a compact neighborhood of e. The most important topological groups are Lie groups.