ETECTION OF SIGNALS IN COMMUNICATION CHANNELS BY FOURIER TRANSFORMS OVER FINITE GROUPS

.G. Karpovsky
ollege of Engineering
oston University

E.A. Trachtenberg
Department of Electrical
and Computer Engineering
Drexel University

BSTRACT

Detection of signals corrupted by noise is considered. A family of suboptimal group ilters is constructed using two opposing performance criteria of computational effect-veness and of approximating (in Hilbert-Schmidt norm) the optimal Wiener filter. The eneral equations of such a group filter are given and the numerical solution of the roblem of choosing the best suboptimal group filter is discussed for small values of n, or a variety of groups and for the first order Markov and random sinus processes cor-upted by white noise.

. INTRODUCTION

Fast Transforms are playing an increasingly important role in applied engineering practice. For example, the matrices of the DFT (Discrete Fourier Transform) are composed of eigenvectors of linear time-invariant systems. That fact together with the existence of Fast Fourier Transform Algorithms provides a powerful means for spectral analysis and synthesis of such systems. [1,2]. As another example we mention the WHT (Walsh-Hadamard Transform) whose representing matrix is composed of eigenvectors of linear dyadic-invariant systems and it provides a powerful means for spectral analysis, synthesis and optimization of such systems, which find widespread applications in Computer Engineering [3]. These two are examples of use of group transforms (the cyclic group in the case of DFT and the dyadic group on the case of WHT) in the theory of group-invariant systems and signals. Elements of a general theory of such systems and signals are presented in [4,5,10].

In [2,4,5,6,7,8,18,19] such group-invariant systems have been used to approximate classical discrete time-invariant systems. The advantages of a group system approximant are due to convenience of implementing a group operation as compared with usual arithmetic of addition/subtraction and because group transforms possess fast algorithms for multiplication of their representing matrices by vectors. The latter is due to the Kronecker product representation of these matrices by some other matrices of smaller dimension. It has been shown that the computational performance of group transforms is of the same order of magnitude $n\sum_{i=1}^{m} n_i$ computer operations to multiply a matrix of a group transform by an n-vector, if the underlying group G is of order $n = \prod_{i=1}^{m} n_i$, where n_i is the order of a normal subgroup G_i in the direct product representation of the group $G = \prod_{i=1}^{m} G_i$; and the transforms which perform "best" computationally are the DFT and WHT for which the matrix multiplication mentioned above requires $n \log_2 n$ computer operations only.

As to accuracy of approximation, which is another criterion of performance of a group system, it depends upon the system which is approximated and upon the chosen find the group among all the groups of a given order n which maximizes the accuracy of approximation of a given system. That is conjectured to be an extremely difficult problem [5] and several approaches to its numerical solution for small values of n and or special classes of systems have been undertaken [2,5,6,7,8,12,13,14,18,19].

In that paper we consider the classical problem of Wiener filtering of random sinus nd 1st order Markov process in the presence of white noise. Using the two opposing erformance criteria of computational effectiveness and statistical performance (which s identified as the accuracy of approximation in the Hilbert-Schmidt norm), a variety f suboptimal group filters is investigated for small n (up to n=64). Some facts and efinitions from harmonic analysis over finite groups are presented in the next secion. General equations of a group filter are given in Section III, where the problem of hoosing the best group is formulated. The results of its numerical solution for $n \leq 64$ nd for a variety of Abelian and Non-Abelian groups are given and analyzed in Section IV. t is shown that the cyclic and the dyadic are often not the best groups to use and the se of non-commutative groups may be advantageous to that of a commutative because for he same speed we have a better approximation.

I. FOURIER ANALYSIS OVER FINITE GROUPS

Let G be an arbitrary finite group with n elements and K any field of characterstic char K. In the space $L_{G,K} = \{f: G \to K\}$ the elements of the non-equivalent bsolutely irreducible representations of G over the field K will be used as an orthoonal basis.

Recall (see [9]), that representation R_ω of degree d_ω in a linear space V over (dim V = d_ω) is defined as a homomorphism R_ω: $G \to GL(d_\omega, K)$, where $GL(d_\omega,K)$ is the roup of all invertible ($d_\omega \times d_\omega$)-matrices over K. The value of representation R_ω at he point $t \in G$ will be denoted by $R_\omega(t)$.

Two representations R_{ω_1}, R_{ω_2} of the same degree $d_{\omega_1} = d_{\omega_2}$ are said to be equivaent if there exists an invertible ($d_\omega \times d_\omega$)-matrix Q over K such that

$$Q^{-1} R_{\omega_1}(t) \; Q = R_{\omega_2}(t) \text{ for every } t \in G.$$

A representation R_ω in a linear space V over K is said to be irreducible if V has o proper R_ω-invariant subspaces, and is absolutely irreducible, if it remains irreucible in any extension of K. It is assumed that

i) char K = 0, or char K $\nmid$ n – char K does not divide the order n of G.

ii) K is such that if R_ω is an irreducible representation of G in a linear space V ver K, then R_ω is absolutely irreducible, i.e. K is the so-called splitting field or G[9].

Conditions for K to be a splitting field for a given group G and construction methods for absolutely irreducible representations of G in K, are considered in algeraic literature for a great variety of groups G and fields K (see [9]). We note that $K = \mathbb{C}$ ($\mathbb{C}$ is the field of complex numbers) is the splitting field for every G. In that ase the complete orthogonal basis in $L_{G,\mathbb{C}}$ consists of elements of the non-equivalent

(GL(V) is the group of automorphisms of a linear space V of dim V = d_ω over **C**). And R_ω is represented by the unitary ($d_\omega \times d_\omega$)-matrix $R_\omega(t)$, $t\in G$. The field **C** was used in [5], where elements of the general theory of systems over groups were presented and finite Galois fields $GF(q^s)$ were used in [10] in problems of error detection and error correction in computation channels and in error correcting codes.

In the present work we use the field **C** of complex numbers. Let $\hat{G} = \{R_\omega\}$ denote the set of all non-equivalent unitary representations of G, indexed so that $\{R_\omega\}$ is of degree d_ω. $\hat{G}$ is the dual object for G, its cardinality equals the number of conjugate classes of G, and we have

$$\sum_{R_\omega \in \hat{G}} d_\omega^2 = n, \tag{1}$$

where n is the cardinality of G.

All the irreducible unitary representations are listed for the symmetric group of permutations $G = S_3 = \{0,(132),(12),(13),(23)\}$ in Table 1 and for the quaternion group Q_2 (with generators a and b, $b^2 = a^2$, $bab^{-1} = a^{-1}$ and with j as its identity) in Table 2:

$t\in S_3$ \ $R_\omega \in \hat{S}_3$	0	1	2
0	1	1	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
(132)	1	1	$\frac{1}{2}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$
(123)	1	1	$\frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}$
(12)	1	-1	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
(13)	1	-1	$\frac{1}{2}\begin{pmatrix}-1 & -\sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix}$
(23)	1	-1	$\frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$

Table 1

$t\in Q_2$ \ $R_\omega \in \hat{Q}_2$	0	1	2	3	4
j	1	1	1	1	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
a	1	1	-1	-1	$\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$
a^2	1	1	1	1	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$
a^3	1	1	-1	-1	$\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$
b	1	-1	1	-1	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$
ab	1	-1	-1	1	$\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$
a^2b	1	-1	1	-1	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$
a^3b	1	-1	-1	1	$\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$

Table 2, $i = \sqrt{-1}$

In the important case of Abelian groups, G may be represented as a direct product of its cyclic subgroups

$$G = H_1 \times \ldots \times H_m,\ t \in G,\ t = (t_1, \ldots, t_m),\ t_\ell \in \{0, 1, \ldots n_\ell - 1\},$$

n_ℓ is a power of a prime number, the group operation is component wise addition mod n_ℓ $\ell = 1,2,\ldots,m$. In this case $d_\omega = 1$ for all $R_\omega \in \hat{G} = \hat{H}_1 \times \ldots \times \hat{H}_m$, $\hat{G}$ is the multiplicative group of characters which is isomorphic to G and $\hat{H}$ is isomorphic to H .e. $R_\omega = (R_{\omega_1}, \ldots, R_{\omega_m})$, $R_{\omega_\ell} \in \{0, 1 \ldots n_\ell - 1\}$ and we have

$$R_\omega(t) = \prod_{\ell=1}^{r} \exp\,(2\pi i R_{\omega_\ell} t_\ell / n_\ell), \tag{2}$$

$$R_{\omega_\ell}, t_\ell \in \{0, 1, \ldots, n_\ell - 1\}, \quad i = \sqrt{-1}$$

nd, if $n_1 = n_2 = \ldots = n_m$, then $R_\omega(\cdot)$ are known as Chrestenson functions and for $n_1 = n_2 = \ldots = n_m = 2$ as Walsh function [3, 5,11].

Let $f: G \to C$. Using the orthogonality relations [9] for the n functions $\{R_\omega^{(s,t)}\}$, $s,t = 1,2,\ldots,d_\omega$, $R_\omega \in \hat{G}$, we can define the Fourier transform F_G: $f \to \hat{f}$ and the inverse ourier transform F_G^{-1}: $\hat{f} \to f$ as follows:

$$\hat{f}(\omega) \triangleq d_\omega / n \sum_{t \in G} f(t)\, R_\omega(t^{-1}), \tag{3}$$

$$f(t) = \sum_{R_\omega \in \hat{G}} \mathrm{trace}\,(\hat{f}(\omega)\, R_\omega(t)), \tag{4}$$

here t^{-1} is the inverse of t in G.

omputation of Fourier F_G and inverse Fourier F_G^{-1} transforms can be done using fast lgorithms and it is based on the following representation of elements of $\hat{G}$ by Kronecker roduct of matrices over K. Let G be a group, isomorphic to a direct product of some roups H_ℓ, $\ell = 1,2,\ldots,m$, $G = \prod_{\ell=1}^{m} H_\ell$. In that case (see [9]):

$$R_\omega(t) = \prod_{\ell=1}^{m} R_{\omega_\ell}(t_\ell), \tag{5}$$

here $R_{\omega_\ell} \in \hat{H}_\ell$, $t_\ell \in H_\ell$.

or the case K = C it was proved in [11,15] that the computation of f or $\hat{f}$ requires $\sum_{\ell=1} n_\ell$ multiplications and additions and n memory locations. These results were eneralized in [10] for the case of an arbitrary field K such that char K = 0 or ar $K \nmid n$ and K is a splitting field for G. For the Fourier transform over the group defined by (3, (4) the usual properties of linearity, group translation, group convolution theorem, Plancherel, Poisson, Wiener-Chinchine Theorems, etc., are valid see e.g. [3]).

III. SUBOPTIMAL GROUP FILTERS.

Let $u = (u(0), u(1), \ldots, u(n-1))^T$ and $e = (e(0), e(1), \ldots, e(n-1))^T$ be the zero-mean vectors of the uncorrelated signal and noise with covariance matrices B_{uu} and B_{ee} respectively, where T stands for the transpose of a row vector.

In Fig. 1 a unitary transform is utilized which is represented by a matrix U (see [7,2]):

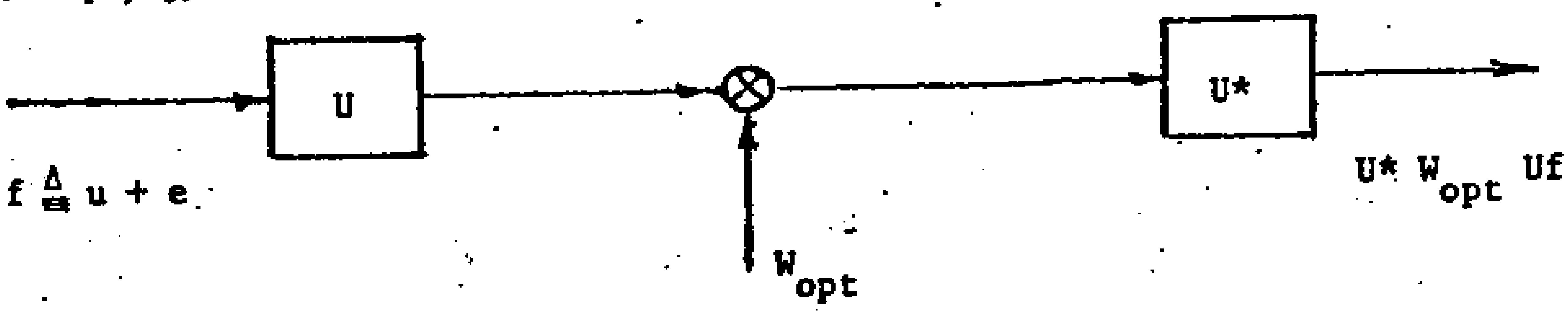

Fig. 1

W_{opt} is the (nxn) matrix of optimal Wiener filter that is the solution of the following problem:

$$\min_W \{1/n\, E(||Wf - u||)\} = 1/n\, E(||W_{opt} f - u||), \tag{6}$$

where $E(\cdot)$ represents the expected value operator and for an (nxn) matrix A, the norm $||A|| = \operatorname{trace} AA^*$, A^* is the transposed and complex conjugate matrix for A. It is known (see e.g. [2,7]) that W_{opt} can be computed from:

$$W_{opt}(B_{uu} + B_{ee}) = B_{uu}, \tag{7}$$

and assuming invertibility of $B_{uu} + B_{ee} \triangleq B_{ff}$, the minimal dispersion squared in (6) is determined by:

$$D(\varepsilon_{opt}) \triangleq 1/n\, E(||W_{opt} f - u||) = 1/n \operatorname{trace}(B_{uu} - B_{uu} B_{ff}^{-1} B_{uu}) \tag{8}$$

The filtering in Fig. 1 using a transform U is performed in the following three steps which determine the overall amount of computer operations:

(i) Uf

(ii) $(UW_{opt}U^*)(Uf)$

(iii) $U^*((UW_{opt}U^*)(Uf))$.

In the case of direct Wiener filtering (when U is and identity matrix U=I) only n^2 computer operations are required at step (ii) for matrix multiplication. The direct Wiener filtering is therefore the fastest. If U is the KLT (Karhunen-Loeve Transform which diagnolizes W_{opt}, that is it diagonalizes both B_{uu} and B_{ee} and is generally represented by a non-structured arbitrary unitary matrix) then we need n^2 operations for matrix multiplication at step (i); n operations to multiply a diagonal matrix $UW_{opt}U^*$, by the vector Uf at step (ii) and again n^2 for matrix multiplication at step (iii). That amounts to $2n^2 + n$ computer operations.

As to the statistical performance, the transform U in Fig. 1 does not affect $D(\varepsilon_{opt})$ in (8) and (6). However, that mathematically optimal dispersion is not achievable in reality (because e.g. of roundoff) and the idea of a suboptimal filter is to attribute that acceptable degradation in performance to improving the computational abilities of the filtering scheme. That is, the transform U must possess a fast algorithm for matrix multiplication (like DFT or WHT) so that steps (i), (iii) can be performed fast. Then the suboptimal filtering results in reducing the amount of operations at step (ii) to the order of n at the expense of nullifying all of the off-diagonal entries in $UW_{opt}U^*$ (in case of Abelian group based transforms) or by structuring $UW_{opt}U^*$ to a canonical block diagonal form, uniquely determined by the group G (in case of non-commutative group, see [5]). In that paper, group transforms will be compared with KLT based filters (in case of filtering the 1st order Markov process and random sinus corrupted by white noise) for which the results are known [2,7,12,13]. The KLT is known to be statistically optimal, that is, its matrix consist of eigenvectors of W_{opt} defined by (7). Other known transforms considered here are the DCT (Discrete Cosine Transform) which is asymptotically equivalent to the KLT for the first order Markov process [16,17] and the DFT (Discrete Fourier Transform) which is represented by the matrix of characters of the cyclic group $G = C_n$ of integers 0,1,...n-1 with addition mod n as the group operation. We shall consider two non-Abelian groups, namely S_3 (the symmetric group of third order), and Q_2 (the quaternion group of the order 8). Their duals $\hat{S}_3$, $\hat{Q}_2$, $\hat{C}_n$ are described in Section 2. We shall use the direct products $S_3 \times C_n$, $C_n \times S_3$, $Q_2 \times C_n$, $C_n \times Q_2$, $S_3 \times Q_2$, $Q_2 \times S_3$, $S_3 \times S_3$, $Q_2 \times Q_2$. The corresponding duals are computed by (5) using the Kronecker product property of group representations. The number $n \sum_{\ell=1}^{m} n_\ell$ of operations which is needed to compute the Fourier transform (3), (4) (see Section 2) is the upper bound on the computational complexity. The real amount of operations depends upon the number of 0's in all the elements of the dual $\hat{G}$ for a given group G. For Example, for $G = S_3$ there are four 0's among the elements of $R_\omega = 2$(see Table 1). Therefore, to compute (3) or (4) for $S_3 \times C_2$ we need not $12(6 + 2) = 96$ computer operations but only $12 \cdot 8 - 2 \cdot 4 = 88$ operations. Analogously, there are sixteen 0's among the elements of $R_\omega = 4 \in \hat{Q}_2$ (see Table 2). Hence, e.g., to compute (3) or (4) for $Q_2 \times Q_2$ we need $64 \cdot 16 - 8 \cdot 16 - 8 \cdot 16 = 3 \cdot 2^8 = 768$ computer operations.

To obtain the suboptimal group filter which is the best approximation to a given W_{opt} defined by (7) we denote (see [5]) the following set of all impulse response matrices of group systems over a given group G:

$$\mathrm{Cir}(G) = \{H | H = (H^{(t,\zeta)}), H^{(t,\zeta)} = h(\zeta^{-1} o t),\ h: G \to \mathbb{C}\} \quad (9)$$

here $\circ$ denotes the group operation.

t can be shown then that the best group filter approximation to the optimal Wiener ilter is the unique solution of the following minimization problem:

$$\min_{\epsilon \mathrm{Cir}(G)} \{1/n\ E(||H(u+e) - u||) = 1/n\ E(||H_{opt}(u+e) - u||) \triangleq D^G(\epsilon_{h_{opt}}) \quad (10)$$

hat is the action of a group filter is described by group convolution:

$$y(t) \triangleq (h \circledast u)(t) = \sum_{\zeta \epsilon G} h(\zeta^{-1} o\ t) u(\zeta). \quad (11)$$

here $\circledast$ stands for group convolution of h, u: $G \to \mathbb{C}$, y: $G \to \mathbb{C}$ is the output, the input ignal vector $u = (u(0),\ldots,u(n-1))^T$ is treated as a centralized random function defined n the group G i.e. u: $G \to \mathbb{C}$. The problem (10) was considered in [2,4,7] for dyadic nd cyclic groups. It will be shown that by using other Abelian and non-commutative roups, the approximation error may be reduced (see also [5]).

We denote for the uncorrelated signal and noise respectively:

$$B_{\hat{u}\hat{u}}(\omega) \triangleq E(\hat{u}(\omega)\ \hat{u}(\omega)^{*}) = E(d^2_{\omega}/n^2 \sum_{\zeta \epsilon G} u(\zeta)\ R_{\omega}(\zeta^{-1}) \sum_{t \epsilon G} \overline{u(t)}\ R^{*}_{\omega}(t^{-1})$$

$$= d^2_{\omega}/n^2 \sum_{\zeta, t \epsilon G} B_{uu}^{(\zeta,t)}\ R_{\omega}(\zeta^{-1} o t) \quad , \quad R_{\omega} \epsilon \hat{G}\ ; \quad (12)$$

here $B_{uu} = (B_{uu}^{(\zeta,t)})$, $B_{uu}^{(\zeta,t)} \triangleq E^{(\zeta,t)}(uu^{*})$.

$$B_{\hat{e}\hat{e}}(\omega) = d^2_{\omega}/n^2 \sum_{\zeta, t \epsilon G} B_{ee}^{(\zeta,t)}\ R_{\omega}(\zeta^{-1} o t) \quad , \quad R_{\omega} \epsilon \hat{G}\ ; \quad (13)$$

here $B_{ee} = (B_{ee}^{(\zeta,t)})$, $B_{ee}^{(\zeta,t)} \triangleq E^{(\zeta,t)}(ee^{*})$.

It can be shown (see [6]) that the minimal dispersion in (10) is being achieved for $H_{opt} \epsilon$ Cir(G) defined as follows:

$$\hat{h}_{opt}(\omega)((B_{\hat{u}\hat{u}}(\omega) + B_{\hat{e}\hat{e}}(\omega)) = d_{\omega}/n\ B_{\hat{u}\hat{u}}(\omega) \ , \ R_{\omega} \epsilon \hat{G} \quad . \quad (14)$$

The dispersion squared achieved by utilizing the optimal group filter is computed by:

$$D^G(\epsilon_{h_{opt}}) = \sum_{R_{\omega} \epsilon \hat{G}} 1/d_{\omega}\ \mathrm{trace}\ (B_{\hat{u}\hat{u}}(\omega) - B_{\hat{u}\hat{u}}(\omega)\ (B_{\hat{u}\hat{u}}(\omega) + B_{\hat{e}\hat{e}}(\omega))^{-1} B_{\hat{u}\hat{u}}(\omega)) \ . \quad (15)$$

Given the group G of order n, all the computations in optimal group filters are being done using the corresponding algorithms of Fast Fourier Transforms [10,11,15]. The results of comparing computational effectiveness of different transforms in the problem of suboptimal filtering of Fig. 1 are given in Table 3 (see [2,14,16] for the computational requirements of DFT and DCT in that problem).

n \ Transform	KLT	DCT	$\hat{C}_n$(DFT)	$\widehat{C_{n/8} \times Q_2}$	$\widehat{Q_2 \times Q_2}$
8	136	92	56	104	
16	528	252	144	272	
32	2080	652	352	736	
64	8256	1612	832	2048	1600

Table 3: Number of operations required for various suboptimal filters.

Statistical performance of different transforms will be compared in the next section. It follows that the cyclic group (DFT) is the best computationally and the dyadic group has the same computational complexity.

The KLT is assumed to be represented by an arbitrary (nxn)-unitray matrix and it requires $2n^2 + n$ computer operations to perform the optimal filtering in Fig. 1.

The dispersion $D^G(\varepsilon_{h_{opt}})$ in (15) depends upon the choice of the group G. That poses a difficult problem of selecting the optimal group G' of the given order for which the dispersion $D^{G'}(\varepsilon_{h_{opt}})$ is maximal. In the next section we are going to use the groups S_3, Q_2, C_n and their direct products in order to investigate the statistical performance of the corresponding group filters for small values of n.

IV. NUMERICAL RESULTS.

We consider various group filters in this section in the problem of filtering the 1st order Markov process with the covariance matrix

$$B_{uu} = (\rho^{|s-\ell|}), 0 < \rho < 1, \quad s,\ell = 0,1,\ldots,n-1; \tag{16}$$

and the random sinus process $x(t) = a \sin(\lambda t + \alpha)$ with the phase α distributed uniformly on the segment $[0,2\pi]$ and with the covariance matrix:

$$B_{uu} = (a^2/2 \ \cos(\lambda(s-\ell))), \ s,\ell = 0,1,\ldots,n-1; \tag{17}$$

where $a^2/2 = 1$.

These signals are assumed to be corrupted by the white noise with identity as its covariance matrix. The signal and noise are assumed to be uncorrelated.

The KLT is computed for the 1st order Marcov process in [7]. In the case of the andom sinus, the matrix of eigenvectors of the corresponding W_{opt} in (7) (the KLT) was omputed for each n.

The DCT is asymptotically equivalent to the KLT of the Markov process (see [17]).

The statistical performances of suboptimal group filters for the 1st order arkov process corrupted by white noise, are compared in Table 4 (see also Figures 2 nd 3). Those for the random sinus corrupted by white noise, are given in Table 5 see also figures 4 and 5).

ρ \ n	6	8	12	16	18	24	30	32	36	40	42	48	54	56	60	64
.9	S_3	Q_2	$C_2 x S_3$	C_{16}	C_{18}	C_{24}	C_{30}	C_{32}	C_{36}	C_{40}	C_{42}	C_{48}	C_{54}	C_{56}	C_{60}	C_{64}
.99	S_3	Q_2	$C_2 x S_3$	$C_2 x Q_2$	$C_3 x S_3$	$C_3 x Q_2$	$C_5 x S_3$	$C_4 x Q_2$	$S_3 x S_3$	$Q_2 x C_5$	$S_3 x C_7$	$S_3 x Q_2$	$S_3 x C_9$	$Q_2 x C_7$	$S_3 x C_{10}$	$Q_2 x Q_2$

Table 4: Group with the optimal statistical performance for the 1st order Markov process.

λ \ n	6	8	12	16	18	24	30	32	36	40	42	48	54	56	60	64
.01	C_6	C_8	C_{12}	C_{16}	$S_3 x C_3$	$S_3 x C_4$	$S_3 x C_5$	$Q_2 x C_4$	$S_3 x C_6$	$Q_2 x C_5$	$S_3 x C_7$	$S_3 x C_8$	$S_3 x C_9$	$Q_2 x C_7$	$S_3 x C_{10}$	$Q_2 x C_3$
	S_3	Q_2	$S_3 x C_2$	$Q_2 x C_2$												$Q_2 x Q_2$
			$C_2 x S_3$	$C_2 x Q_2$												
.05	C_6	Q_2	$C_2 x S_3$	$C_2 x Q_2$	$S_3 x C_3$	$S_3 x C_4$	$S_3 x C_5$	$Q_2 x C_4$	$S_3 x S_3$	$Q_2 x C_5$	$S_3 x C_7$	$S_3 x Q_2$	$S_3 x C_9$	$Q_2 x C_7$	$S_3 x C_{10}$	$Q_2 x Q_2$
	S_3															
.1	S_3	Q_2	$C_2 x S_3$	$C_2 x Q_2$	$S_3 x C_3$	$S_3 x C_4$	$S_3 x C_5$	$C_4 x Q_2$	$C_6 x S_3$	$C_5 x Q_2$	$C_7 x S_3$	C_{48}	C_{54}	C_{56}	C_{60}	C_{64}

Table 5: Group with the optimal statistical performance for the random sinus.

In the case of 1st order Markov process (see Table 4), the use of various non-Abelian groups, as ρ increases, results in improved statistical performance as compared with DFT. That is compensated for by the increased number of computations. For example, (see figures 2,3) for ρ=0.9, n=64, the replacement of C_{64} by $Q_2 \times Q_2$ results in 10.82% improvement in statistical performance. The price for that however (see Table 3) is nearly 100% loss of speed: 1600 computer operations instead of 832 to perform the filtering. Similarly, in the case of random sinus, as λ decreases, the use of various non-Abelian groups results in improved statistical performance as compared with DFT. For example, (see figures 4,5) for n=64 and λ=0.01 (or λ=0.05) the statistical gain is 17,59% (or 20.95% for) $Q_2 \times Q_2$ as compared with C_{64}. That is however compensated for by the increase in speed of nearly 100% in DFT as compared with the group $Q_2 \times Q_2$ (see Table 3).

We note that $D^G(\epsilon_{h_{opt}})$ increases as ρ decreases (in case of the 1st order Markov process) and as λ increases (in case of the random sinus). See, for example, Table 6 for the 1st order Markov for $G = S_3 x C_2$.

ρ	0.5	0.6	0.7	0.8	0.9	0.92	0.94	0.96	0.99
$D^{S_3 \times C_2}(\epsilon_{h_{opt}})$	0.4545	0.4294	0.3944	0.3437	0.2628	0.2398	0.2128	0.1800	0.1111

Table 6

That happens because B_{uu} approaches the identity matrix as ρ decreases in case of 1st order Markov process. In the corresponding case of random sinus, B_{uu} approaches the all 1's matrix as λ decreases i.e. correlation between u-components increases and the dispersion decreases. In other words, as λ increases, B_{uu} approaches the identity matrix and the dispersion increases.

The order of groups G_1, G_2 in their direct product affects the dispersion without affecting the computational effectiveness. That gives the designer more freedom in choosing corresponding transforms and a great variety of fast group transforms can always be used to choose the best from using (14) and (15). Group filters might also find their use in practical situations in which we do not know computationally good approximating transforms for the KLT for a given stochastic process.

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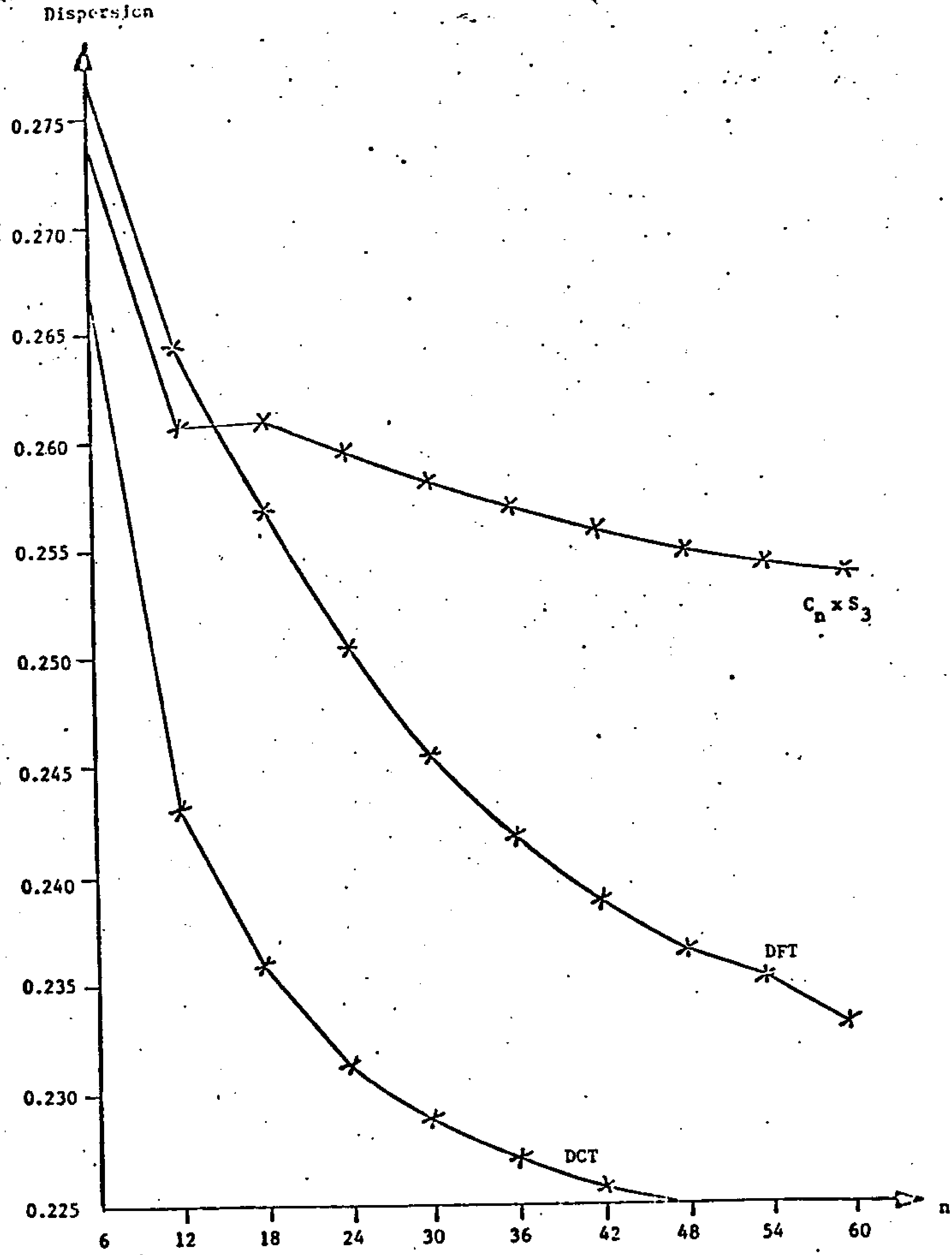
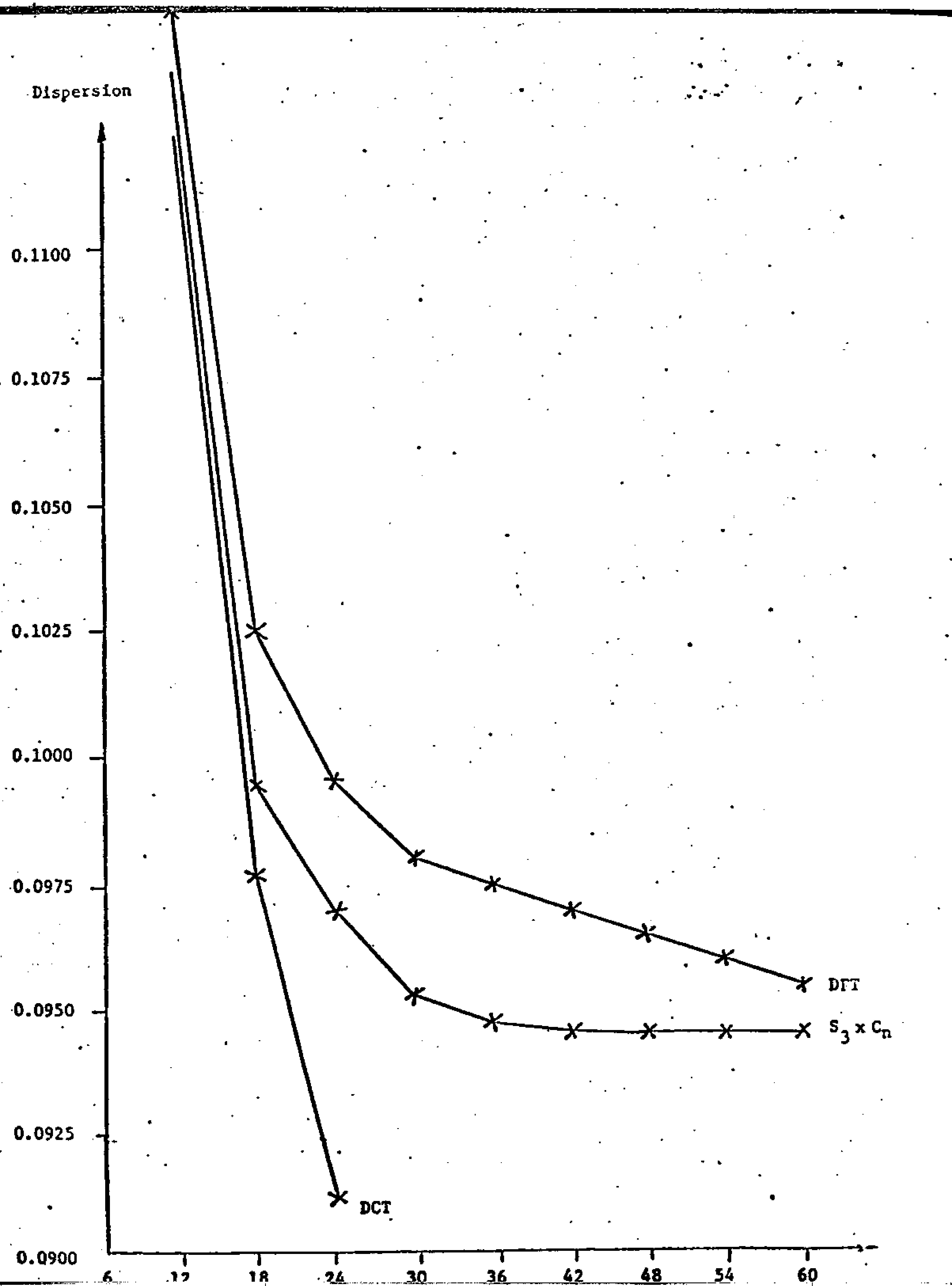


Fig. 2 : Dispersion for the 1st order Markov process, $\rho = 0.9$.



Dispersion
0.1100
0.1075
0.1050
0.1025
0.1000
0.0975
0.0950
0.0925
0.0900
DFT
S3 x Cn
DCT
6
12
18
24
30
36
42
48
54
60

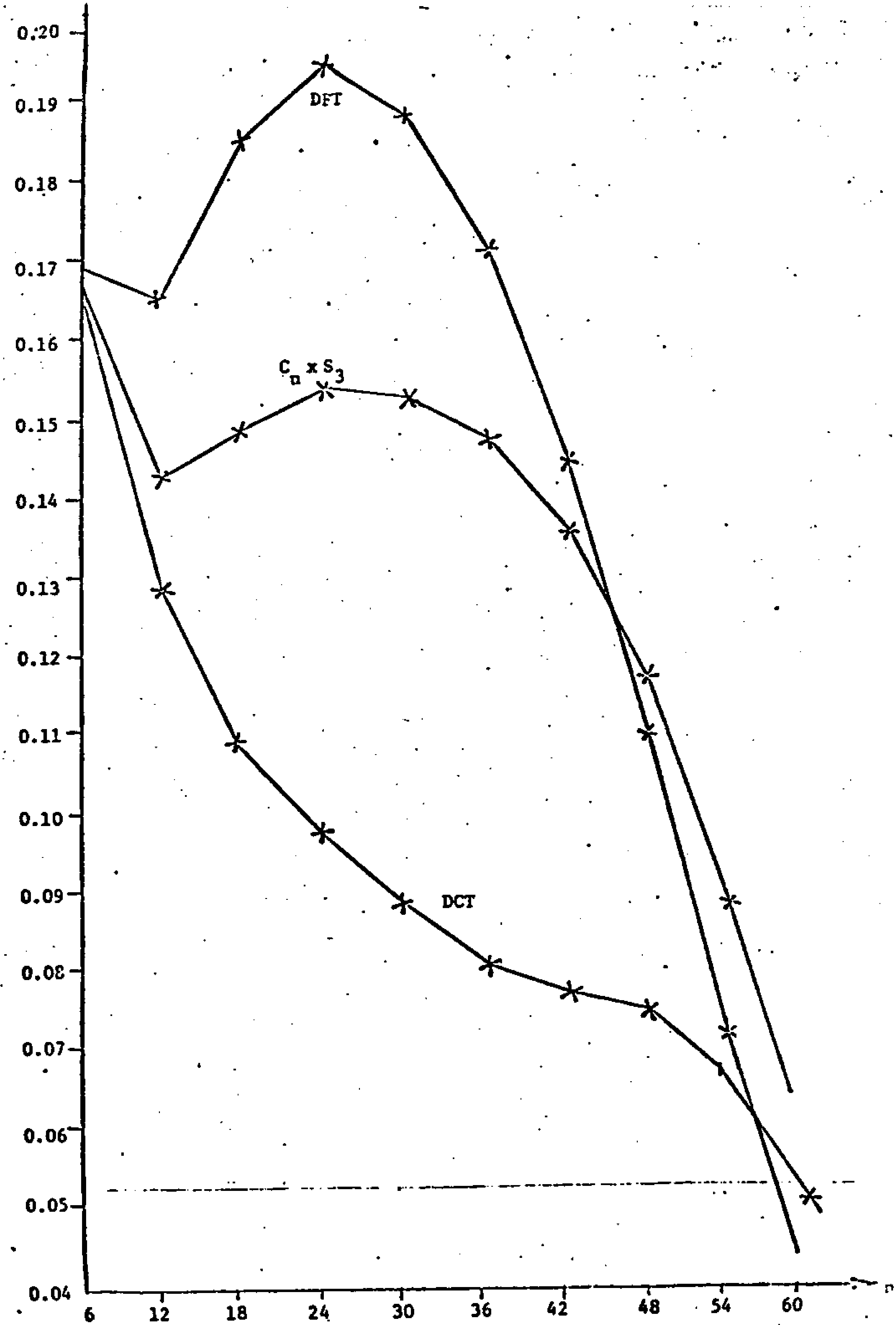


Fig. 4: Dispersion for the random sinus, $\lambda = 0.1$.

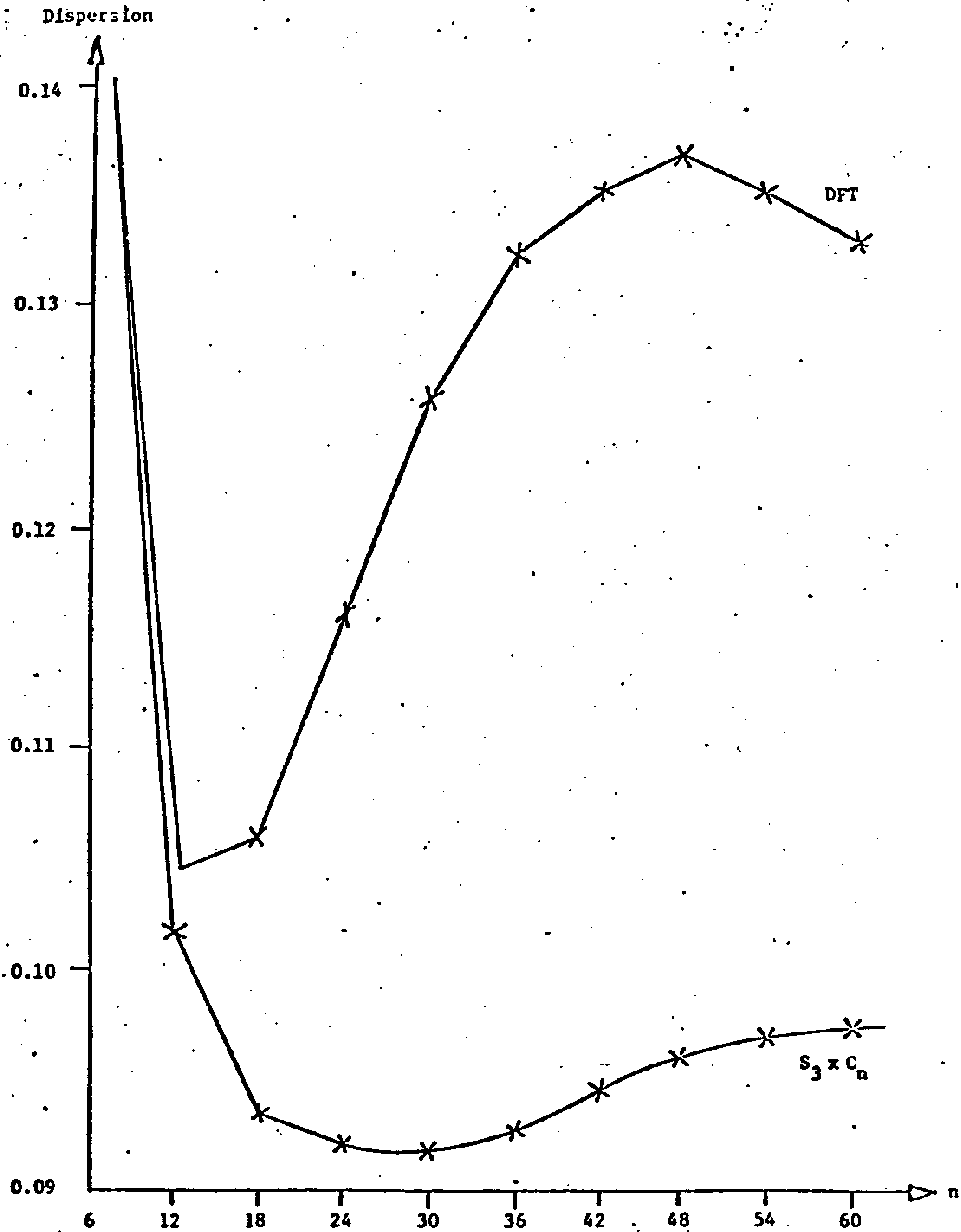


Fig. 5: Dispersion for the random sinus, λ = 0.05.

REFERENCE

1. D.F. Elliott and K.R. Rao, Fast Transforms: Algorithms, Analyses and Applications, Academic Press, 1983.
2. N. Ahmed and K.R. Rao, Orthogonal Transforms for Digital Signal Processing, Springer, 1975.
3. M.G. Karpovsky, Finite Orthogonal Series in the Design of Digital Devices, John Wiley, New York 1976
4. J. Pearl, Optimal Dyadic Models of Time-Invariant Systems, IEEE Trans. on Comp., C-24, 1975, p. 598-602.
5. M.G. Karpovsky, E.A. Trachtenberg, Some Optimization Problems for Convolution Systems over Finite Groups, Information and Control, 34, 3, 1977, p. 227-247.
6. M.G. Karpovsky and E.A. Trachtenberg, Fast Techniques for Signal Detection in Noisy Communication Channels, IEEE Trans. on Acous. Speech and Signal Processing. (Submitted).
7. W.K. Pratt, Generalized Wiener Filtering Computation Technique, IEEE Trans. on Comp., C-21, 1972, p. 636-641.
8. E.A. Trachtenberg, Systems over Finite Groups as Suboptimal Wiener Filters: A Comparative Study, Proc. 1983 International Symposium on Mathematical Theory of Systems and Networks, Beer-Sheva, Isreal 1983.
9. L. Dornhoff, Group Representation Theory, M. Dekker, New York, 1971.
10. M.G. Karpovsky and E.A. Trachtenberg, Fourier Transform over Finite Groups for Error Detection and Error Correction in Computation Channels, Information and Control, 40, 3, 1979, p. 335-358.
11. M.G. Karpovsky, Fast Fourier Transforms on Finite Non-Abelian Groups, IEEE Trans. on Comput. C-26, 1977, p. 1028-1030.
12. J. Pearl. On Coding and Filtering Stationery Signals by Discrete Fourier Trans. form, IEEE Trans. on Infor. Theory, IT-19, 1973, p. 229-232.
13. N. Ahmed, T. Natarajan and K.R. Rao, Discrete Cosine Transform, IEEE Trans. Computers, C-23, 1974, p. 90-93.
14. K.R. Rao, J.G.K. Kuo, M.A. Narasimhan, Slant-Haar Transform, Intern. J. Computer Math., Sec. B, 7, 1979, p. 73-83.
15. P.J. Nicholson, Algebraic Theory of Finite Fourier Transforms, J. of Computer and Systems Science, 5, 1971, p. 524-547.
16. W.M. Chen, C.M. Smith and S.C. Fralick, A Fast Computational Algorithm for the Discrete Cosine Transform, IEEE Trans. on Comm., COM-25, 1977, p. 1004-1009.
17. M.D. Flickner and N. Ahmed, A. Derivation for the Discrete Cosine Transform, Proc. of the IEEE, 70, 9, 1982, p. 1132-1134.
18. E.A. Trachtenberg, Fast Wiener Filtering Computation Technique, Proc. 1979 Intern. Symp. on the Mathematical Theory of Networks and Systems, Vol. 3, Delft, Holland, p. 174-177.

19. E.A. Trachtenberg, Construction of Fast Unitary Transforms Which are Equivalent to Karhunen-Loeye Spectral Representations, Proc. 1980, IEEE Intern. Symp. on EMC, Baltimore, USA, p. 376-379.