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The Zhang-Watari Transform: A Discrete, Real-Valued, Generalized Haar Transform*

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The relationship between the 2-Dimensional multiple-valued (complex-valued) Haar transform and the 2-Dimensional real valued Zhang-Watari transform of patterns is studied and a method is disclosed to compute the Haar- (more properly, Watari)- Spectrum of a pattern by using only real arithmetic. It is shown that to extend the straight forward 1-D results to the 2-D case, a special permutation operation has to be introduced. This result is closely related to that known for 2-D Chrestenson and Zhang-Hartley transforms, except that a different choice of pattern partition and permutation is required.

Keywords: Spectral techniques; orthogonal transforms; real-valued transforms

1. INTRODUCTION

Zhang Gongli [1] disclosed at the International Symposium on Multiple-Valued Logic in Winnipeg, Canada, two new discrete, real-valued orthogonal transforms closely related to the Chrestenson [2] and the Watari [3] transforms, in a similar way as the Hartley [4] is

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related to the Fourier transform [5]. Real-valued transforms are appealing, since they have a lower computational complexity as compared with complex-valued transforms. The first of these new real-valued transforms was later called Zhang-Hartley [6] and some of its properties have been studied [7,8]. The present paper analyzes properties of the second of the above mentioned transforms. The Watari transform, as a canonic extension of the Haar transform [9] to the multiple-valued case seems to have inherited the rather restricted attention paid to the Haar transform. Except for their completeness, orthogonality, and some applications to pattern recognition and logic design [10], as well as some recent applications to the study of genetic algorithms [11] and picture processing [12] the *missing* properties of the Haar and Watari functions are best known: they are *not* closed under product, and therefore they do *not* have a convolution-product theorem, their Kernels do *not* build a group, they *do* have fast algorithms [13,14], but they have *no* Kronecker recursive structure and they exhibit *no* symmetry between indices and arguments [15,16]. The aim of this paper is to disclose some *positive* properties of the multiple-valued real Haar transform, which we call hereafter "Zhang-Watari" transform. The main result concerns the relationship between the 2-Dimensional Watari and Zhang-Watari transforms of patterns by introducing a special permutation. The result is similar to that disclosed in [8] for the Chrestenson and Zhang-Hartley transforms, however, a quite different kind of partition and permutation is needed, as will be shown in a later section.

In section 2 the required notation and basic definitions will be stated. The main result is disclosed in section 3. The fourth section is devoted to the analysis of further properties.

2. NOTATION AND BASIC CONCEPTS

2.1. Main Definitions

DEFINITION 1 Let $G_p=\{0,1,\dots,p-1\}$. A multiple-valued function is a mapping $(G_p)^m \to G_p$. The elements $x \in (G_p)^m$ will be interpreted as m-tuples $(x_{m-1},\dots,x_1,x_0)$ with $x_i \in G_p$, as well as natural numbers from $\{0,1,\dots,p^m-1\}$ according to $x=\Sigma_i x_i\, p^i$. Since there is a bijection between both interpretations, they will be used freely without danger of confusion.

DEFINITION 2 Let $\oplus$ denote the modulo p addition. It follows that $(G_p, \oplus)$ is a cyclic group of order p. By allowing the same symbol to express a componentwise modulo p addition, the extension group $((G_p)^m, \oplus)$ is obtained. This group is abelian. The additive inverse of $x \in (G_p)^m$ will be written x'. It follows that $x \oplus x' = 0$.

DEFINITION 3 The primitive p-th roots of unity are defined in the following way:

$$u_p = e^{j2\pi/p} = \cos(2\pi/p) + j\sin(2\pi/p);$$

where $j := \sqrt{-1}$

The powers of the primitive roots build a multiplicative cyclic group of order p. Let this group be denoted by U_p. It follows:

$$U_p = \{u_p^0 = 1, u_p^1 = u_p, u_p^2, \ldots, u_p^{p-1}\}$$

Finally let the symbol "*" denote complex conjugation. It is simple to prove that

$$(u_p^x)^* = u_p^{x'}$$

holds.

DEFINITION 4 The discrete Watari functions for $x \in (G_p)^m$ are defined as follows [3]:

$$w_{rp^i+k}(x) = W_{i,p}^{k,r}(x)$$

$$= \begin{cases} \sqrt{p^i}\, u_p^{lr} & \text{if } x \in \{(pk+l)p^{m-i-1}, \ldots, \\ & (pk+l+1)p^{m-i-1} - 1\} \\ & l = 0, \ldots, p-1 \\ 0 & \text{otherwise} \end{cases}$$

where $i = 0, 1, \ldots, m-1$; $k = 0, 1, \ldots, p^i - 1$; $r = 0, 1, \ldots, p-1$. However if $r = 0$ then k is allowed to take only the value 0. In this way the index 0 of $w_0(x)$ is uniquely defined. Even though the exponent i (in $rp^i + k$) could have in this case an arbitrary value, it is also defined as 0. As will be seen below, i represents the degree of the function. It makes sense to assign a zero degree to the constant function $w_0(x) = 1$.

DEFINITION 5 The Watari Kernel is a p^m by p^m matrix, whose row-entries are taken from the Watari functions after increasing degree (i), permutation type of non-zero elements (r) and order (k).

$$\mathbf{W}_p(m) := [W_{i,p}^{k,r}(x)]_{i,k,r};$$

$$x = 0, \dots, p^m - 1.$$

The Watari Kernel is a regular matrix. Its inverse is given by:

$$\mathbf{W}_p(m)^{-1} = p^{-m}\mathbf{W}_p^T(m)^*$$

since the Watari functions are orthogonal.
The complex conjugate of the Kernel is simply

$$\mathbf{W}_p^*(m) := [W_{i,p}^{k,r'}(x)]_{i,k,r};$$

$$x = 0, \dots, p^m - 1.$$

DEFINITION 6 The discrete Zhang-Watari functions [1] are given by:

$$ZW_{i,p}^{k,r}(x) = zw_{rp^i+k}(x):$$

$$= \begin{cases} \sqrt{p^i}\,(\cos(\frac{2\pi}{p}lr) + \sin(\frac{2\pi}{p}lr)); \\ \qquad x \in \{(pk+l)p^{m-i-1}, \dots, \\ \qquad (pk+l+1)p^{m-i-1} - 1\} \\ \qquad l = 0, \dots, p-1 \\ 0 \qquad \text{otherwise} \end{cases}$$

where $i = 0, 1, \dots, m-1$; $k = 0, 1, \dots, p^i - 1$; $r = 0, 1, \dots, p-1$. (See Definition 4 for the special case $i = k = r = 0$.)

It becomes apparent that the Zhang-Watari functions are actually the sum of the real and imaginary parts of the corresponding generalized Haar (i.e. Watari) functions. As such they are *real valued*. Finally, it has been shown, that the Zhang-Watari functions are complete and orthogonal [1, 16, 17].

DEFINITION 7 The p^m by p^m matrix

$$\mathbf{ZW}_p(m) := [ZW_{i,p}^{k,r}(x)]_{i,k,r}$$

represents the Zhang-Watari Kernel. It becomes apparent that the Kernel is regular and orthogonal, but non-symmetric.

$$\mathbf{ZW}_p(m)\,\mathbf{ZW}_p^T(m) = p^m I(m),$$

where $I(m)$ is the p^m by p^m identity matrix;

$$\mathbf{ZW}_p^{-1}(m) = \frac{1}{p^m}\,\mathbf{ZW}_p^T(m).$$

Moreover, $\mathbf{ZW}_p(m) = \mathrm{Re}\,[\mathbf{W}_p(m)] + \mathrm{Im}\,[\mathbf{W}_p(m)]$.

In what follows, whenever p is known the Kernel may simply be expressed as $\mathbf{ZW}(m)$. Similarly for $\mathbf{W}(m)$.

DEFINITION 8 A pattern is considered to be representable as an array of pixels. Pixels are atoms of a picture and exhibit a single color. The size of a pixel is determined by the desired geometric and chromatic resolution. By defining an injective mapping from the set of colors into the non-negative integers (reals) it is possible to give an equivalent representation to patterns as a matrix with integer (real) entries. Operations among patterns will be expressed as operations among the corresponding matrices. A matrix M of dimension p^m by p^n will be written as $M_{(m,n)}$.

DEFINITION 9 The spectrum -(or more precisely the two-sided Watari spectrum)- of a pattern $A_{(m,n)}$ is given by:

$$S_A = \frac{1}{p^{m+n}}\,\mathbf{W}_p(m)\,A\,\mathbf{W}_p^*(n)^T$$

$$= \frac{1}{p^m}\,\mathbf{W}_p(m)\,A\,\mathbf{W}_p^{-1}(n).$$

and the recovery of a pattern from its spectrum may be obtained as follows:

$$A = \mathbf{W}_p^*(m)^T S_A \mathbf{W}_p(n)$$

DEFINITION 10 The two-sided Zhang-Watari spectrum of a pattern $A_{(m,n)}$ is given by:

$$RS_A = \frac{1}{p^{m+n}} \mathbf{ZW}_p(m)\, A\, \mathbf{ZW}_p(n)^T$$

$$= \frac{1}{p^m} \mathbf{ZW}_p(m)\, A\, \mathbf{ZW}_p^{-1}(n).$$

and the recovery of a pattern from its spectrum may be obtained as follows:

$$A = \mathbf{ZW}_p(m)^T RS_A\, \mathbf{ZW}_p(n)$$

DEFINITION 11 Let $P(m) = P_p(m) := [\rho_{\mu,v}]$, $0 \leqslant \mu, v \leqslant p^m - 1$ with $\rho_{0,0} = 1$ and for all $r \in \{1, \ldots, p-1\}$, $i \in \{0, \ldots, m-1\}$ and $k \in \{0, \ldots, p^i - 1\}$

$$\rho_{\mu,v} = \begin{cases} 1 & \text{if } \mu = rp^i + k \text{ and} \\ & \quad v = (p-r)p^i + k \\ 0 & \text{otherwise} \end{cases}$$

It becomes apparent, that for every (r, i, k) there exists a unique pair (μ, v), hence P is a permutation matrix. Moreover P can be defined recursively as shown below:

$$P(0) := 1$$

and for all $k \geqslant 1$:

$$P(k) := \begin{bmatrix} (1\ 0\ \cdots\ 0) \otimes P(k-1) \\ (0\ \cdots\ 0\ 1) \otimes I(k-1) \\ (0\ \cdots\ 0\ 1\ 0) \otimes I(k-1) \\ \vdots \\ (0\ 1\ 0\ \cdots\ 0) \otimes I(k-1) \end{bmatrix}$$

DEFINITION 12

$$A^{\#}_{(m,n)} := P_p(m)\, A_{(m,n)} P_p(n)$$

2.2. Auxiliary Lemmata

To alleviate the reading of the paper the formal proof of every lemma is given in the Appendix.

LEMMA 1

$$P_p(m)\, \mathbf{W}_p(m) = \mathbf{W}_p^*(m)$$

COROLLARY 1.1

$$P_p(m) = p^{-m}\, \mathbf{W}_p(m) \mathbf{W}_p^T(m)$$

COROLLARY 1.2

$$P_p(m)\, P_p(m) = I(m)$$

COROLLARY 1.3

$$P_p(m) = P_p^T(m)$$

LEMMA 2

$$P_p(m)\, \mathrm{Re}\mathbf{W}_p(m) = \mathrm{Re}\mathbf{W}_p(m)$$

$$P_p(m)\, \mathrm{Im}\mathbf{W}_p(m) = -\mathrm{Im}\mathbf{W}_p(m)$$

LEMMA 3 *For every $A_{(m,n)}$ there is a unique decomposition into α and β invariants as shown below:*

$$A_{(m,n)} = \alpha(A_{(m,n)}) + \beta(A_{(m,n)})$$

with

$$\alpha(A_{(m,n)}) = (A_{(m,n)} + A^{\#}_{(m,n)})/2 \text{ and}$$

$$\beta(A_{(m,n)}) = (A_{(m,n)} - A^{\#}_{(m,n)})/2$$

3. RELATIONSHIP BETWEEN THE 2D-WATARI AND THE 2D-ZHANG-WATARI SPECTRUM OF A PATTERN

In the 1-dimensional case there exists a simple relationship between the Watari and the Zhang-Watari spectrum [16], which allows computing the Watari spectrum without using complex arithmetic:

$$S_A = \alpha(RS_A) + j\beta(RS_A), \quad m \geqslant 0, \quad n = 0$$

In what follows it will be shown, that it is possible to establish a direct relationship between the 2D-Watari and 2D-Zhang-Watari spectra, in a way similar to the case of the Chrestenson and Zhang-Hartley spectra [8], except that instead of the even and odd decomposition of patterns, the α and β invariants defined earlier as well as a different permutation matrix are needed.

LEMMA 4 *For every pattern $A_{(m,n)}$ holds the following:*

$$RS_A = \operatorname{Re} S_A + \operatorname{Im} S_A P(n)$$

COROLLARY 4.1

$$RS_A = \operatorname{Re} S_A - P(m) \operatorname{Im} S_A$$

LEMMA 5 *The 2D complex-valued Watari spectrum of a pattern $A_{(m,n)}$ can be obtained from the 2D real-valued Zhang-Watari spectrum in the following way:*

$$S_A = \alpha(RS_A) + j\beta(RS_A)P(n)$$

COROLLARY 5.1

$$S_A = \alpha(RS_A) - jP(m)\beta(RS_A)$$

$$S_A = [(RS_A + P(m)(RS_A)P(n)) - j(P(m)(RS_A) - (RS_A)P(n))]/2$$

Lemma 5 and its Corollary 5.1 show that the α invariant and a properly permuted β invariant of the Zhang-Watari spectrum are equivalent to the real and imaginary parts of the Watari spectrum respectively. It becomes apparent, that if $n = 0$ the well known results of the 1D case are obtained [16], since $P(0) = 1$.

COROLLARY 5.2 *If for a given pattern $A_{(m,n)}$*

$$\beta(RS_A) = \beta(RS_A) \cdot P(n)$$

holds, then $S_A = \alpha(RS_A) + j\beta(RS_A)$ as in the 1D-case.

LEMMA 6

$$\beta(RS_A) = \beta(RS_A) \cdot P(n)$$

$$\textit{iff}\ \operatorname{Re}\mathbf{W}(m) \cdot A_{(m,n)} \cdot \operatorname{Im}\mathbf{W}(n)^t = 0$$

4. PROPERTIES OF THE 2D WATARI AND ZHANG-WATARI SPECTRA OF PATTERNS

For a given p let $A_{(m,n)}$, $B_{(m,n)}$, $C_{(n,t)}$, $J_{(m,m)}$ and $K_{(m,m)}$ be real-valued patterns. Moreover let $J_{(m,m)} = (K_{(m,m)})^{-1}$ and $u, v \in \mathbb{R}$

P1: Linearity

$$S_{(uA+vB)} = uS_A + vS_B$$

$$RS_{(uA+vB)} = uRS_A + vRS_B$$

This property follows directly from the definitions of the respective spectra.

P2: Preservation of the Product

$$S_{AC} = p^n S_A S_C$$

$$RS_{AC} = p^n RS_A RS_C$$

P3: Preservation of Inversion

$$S_J = p^{-2m}(S_K)^{-1}$$

$$RS_J = p^{-2m}(RS_K)^{-1}$$

P4: Roots

If $m=n=1$ the Watari Kernel reduces to the Chrestenson Kernel, and the Zhang-Watari Kernel to the Zhang-Hartley one. It becomes apparent, that the same property holds for the 2D spectra of patterns of size p by p. (See Definition 4. If $m=1$ then $i=0$ and consequently $k=0$. It follows that $p^i = p^{m-i-1} = 1$ and $x \in \{0, \ldots, p-1\}$. All entries are powers of u_p, which characterizes the Chrestenson Kernel.)

P5: About Examples

The smallest representative example is of dimension p^2 by p^2 (Cf. P4). The simplest matrix $\mathbf{W}_p(2)$ is obtained for $p=4$, since this matrix has only real and imaginary elements (and no elements with both real and imaginary non-zero parts).

A numerical example to illustrate the main result would require 13 matrices of dimension 16 by 16. (4 of them to compute and express the Watari spectrum of a pattern and two additional to separate the real and imaginary parts of this Spectrum. Another 4 would be required to compute and display the Zhang-Watari spectrum; one more to compute the #-operation on this spectrum and finally 2 matrices to represent the α

and β invariants!). The decision of giving in this paper informations on the meaning and implications of the Lemmata instead of an example was taken, since a numerical example would only express in form of (possibly difficult to remember) explicit matrices, what the main Lemma expresses by using symbolic matrices.

5. FUTURE WORK

The next task to be undertaken is the evaluation of these transforms with respect to real world applications. Preliminary results [16] show that the Watari power spectrum of patterns with high regularity may be used for error detection and location. This suggests further work in, for instance, error detection and location in layouts of VLSI memories as well as in testing multiprocessor systems consisting of an array of processors. Furthermore a generalization of the work of S. Khuri [11], who used the Walsh and Haar transforms to investigate the performance of binary coded GAs, could be considered to study the GA-hardness of problems, when the coding of individuals is chosen to be non-binary.

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APPENDIX

LEMMA 1

$$P_p(m)\,\mathbf{W}_p(m) = \mathbf{W}_p^*(m)$$

Proof Recall that $\forall\, x \in (G_p)^m$

$$w_{rp^i+k}(x) = W_{i,p}^{k,r}(x)$$

$$= \begin{cases} \sqrt{p^i}\, u_p^{lr} & \text{if } x \in \{(pk+l)p^{m-i-1}, \ldots, \\ & (pk+l+1)p^{m-i-1} - 1\} \\ & l = 0, \ldots, p-1 \\ 0 & \text{otherwise} \end{cases}$$

Since $r = 1, \ldots, p-1$,

$$\left(\sqrt{p^i}\, u_p^{lr}\right)^* = \sqrt{p^i}\, (u_p^{lr})^*$$

$$= \sqrt{p^i}\, u_p^{lr'} = \sqrt{p^i}\, u_p^{l(p-r)}$$

This means that the complex conjugate of a Watari function is another Watari function of the same degree and order. The result of the operation may be summarized with the following expression:

$$(w_{rp^i+k}(x))^* = (W_{i,p}^{k,r}(x))^*$$

$$= W_{i,p}^{k,(p-r)}(x) = w_{(p-r)p^i+k}(x)$$

An analysis of Definition 11 shows that pre-multiplying a Watari Kernel with $P(m)$ exchanges the rows in positions $rp^i + k$ and $(p-r)p^i + k$. As seen above, this is the same result obtained by complex conjugation. ***

LEMMA 2

$$P_p(m)\,\mathrm{Re}\,\mathbf{W}_p(m) = \mathrm{Re}\,\mathbf{W}_p(m)$$

$$P_p(m)\,\mathrm{Im}\,\mathbf{W}_p(m) = -\,\mathrm{Im}\,\mathbf{W}_p(m)$$

Proof From Lemma 1, pre-multiplying a Watari Kernel with $P(m)$ is equivalent to taking the complex conjugate of the Kernel. This means that the real part of the Kernel will be preserved and the imaginary part of the kernel will be scaled by (-1). ***

LEMMA 4 *For every pattern $A_{(m,n)}$ holds the following:*

$$RS_A = \mathrm{Re}\,S_A + \mathrm{Im}\,S_A P(n)$$

Proof To simplify the syntax of the proof, the following notation will be introduced:

$$\mathfrak{R} = \mathrm{Re}\mathbf{W}(q);$$

$$\mathfrak{I} = \mathrm{Im}\mathbf{W}(q);$$

$$A = A_{(m,n)};$$

$$d = p^{-(m+n)}$$

where $q = m$ or $q = n$ depending on whether the Kernel is at the left or the right hand side of A respectively.

Definitions 9 and 10 may then be written as shown below:

$$S_A = d[\mathfrak{R} + j\mathfrak{I}]A[\mathfrak{R} - j\mathfrak{I}]^T$$

$$= d[\mathfrak{R} + j\mathfrak{I}]A[\mathfrak{R}^T - j\mathfrak{I}^T]$$

$$= d[(\mathfrak{R}A\mathfrak{R}^T + \mathfrak{I}A\mathfrak{I}^T) + j(\mathfrak{I}A\mathfrak{R}^T - \mathfrak{R}A\mathfrak{I}^T)]$$

$$RS_A = d[\mathfrak{R} + \mathfrak{I}]A[\mathfrak{R} + \mathfrak{I}]^T$$

$$= d[\mathfrak{R} + \mathfrak{I}]A[\mathfrak{R}^T + \mathfrak{I}^T]$$

$$= d[(\mathfrak{R}A\mathfrak{R}^T + \mathfrak{I}A\mathfrak{I}^T) + (\mathfrak{I}A\mathfrak{R}^T + \mathfrak{R}A\mathfrak{I}^T)]$$

From Lemma 2 it is known that

$$P(q)\mathfrak{I} = -\mathfrak{I}$$

$$(P(q)\mathfrak{I})^T = -\mathfrak{I}^T$$

Since $P(m)$ is symmetric, transposition leads to

$$\mathfrak{I}^T P(q) = -\mathfrak{I}^T$$

In a similar way it may be shown that $\mathfrak{R}^T P(q) = \mathfrak{R}^T$. Moreover,

$$\begin{aligned} \operatorname{Im} S_A P(n) &= d(\mathfrak{I} A \mathfrak{R}^T - \mathfrak{R} A \mathfrak{I}^T) P(n) \\ &= d(\mathfrak{I} A \mathfrak{R}^T P(n) - \mathfrak{R} A \mathfrak{I}^T P(n)) \\ &= d(\mathfrak{I} A \mathfrak{R}^T + \mathfrak{R} A \mathfrak{I}^T) \end{aligned}$$

The assertion follows directly. ***

LEMMA 5 *The 2D complex-valued Watari spectrum of a pattern* $A_{(m,n)}$ *can be obtained from the 2D real-valued Zhang-Watari spectrum in the following way:*

$$S_A = \alpha(RS_A) + j\beta(RS_A) P(n)$$

Proof

(i)

$$\begin{aligned} (S_A)^* &= d[\mathbf{W}(m) A (\mathbf{W}^*(n))^T]^* \\ &= d[\mathbf{W}(m) A [P(n)\mathbf{W}(n)]^T]^* \\ &= d[\mathbf{W}^*(m) A^* (\mathbf{W}^*(n))^T P(n)] \\ &= d[\mathbf{W}^*(m) A (\mathbf{W}^*(n))^T P(n)] \\ &\qquad \text{(since } A \text{ is real)} \\ &= d[P(m)\mathbf{W}(m) A (\mathbf{W}^*(n))^T] P(n) \\ &= P(m)[d\mathbf{W}(m) A (\mathbf{W}^*(n))^T] P(n) \\ &= P(m) S_A P(n) = (S_A)^\# \end{aligned}$$

(ii)

$$\begin{aligned} (RS_A)^\# &= P(m) RS_A P(n) \\ &= P(m)[\operatorname{Re} S_A + \operatorname{Im} S_A P(n)] P(n) \\ &\qquad \text{(Lemma 4)} \\ &= (\operatorname{Re} S_A)^\# + P(m) \operatorname{Im} S_A \end{aligned}$$

From (i) follows that $(\operatorname{Re} S_A)^{\#} = \operatorname{Re} S_A$, since $\operatorname{Re} S_A$ is real.

(iii)
$$
\begin{aligned}
P(m)\operatorname{Im} S_A &= dP(m)[\mathfrak{I}A\mathfrak{R}^T - \mathfrak{R}A\mathfrak{I}^T] \\
&= dP(m)[\mathfrak{I}A\mathfrak{R}^T] - P(m)[\mathfrak{R}A\mathfrak{I}^T] \\
&= d[P(m)\mathfrak{I}]A\mathfrak{R}^T - [P(m)\mathfrak{R}]A\mathfrak{I}^T \\
&= d[-\mathfrak{I}A\mathfrak{R}^T - \mathfrak{R}A\mathfrak{I}^T] \\
&= -\operatorname{Im} S_A P(n) \qquad \text{(Lemma 2)}
\end{aligned}
$$

Then $\quad (RS_A)^{\#} = \operatorname{Re} S_A - \operatorname{Im} S_A P(n)$

But $\quad RS_A = \operatorname{Re} S_A + \operatorname{Im} S_A P(n)$

(Lemma 4)

It follows:

$$
\begin{aligned}
\operatorname{Re} S_A &= [RS_A + (RS_A)^{\#}]/2 = \alpha(RS_A) \\
\operatorname{Im} S_A P(n) &= [RS_A - (RS_A)^{\#}]/2 = \beta(RS_A) \\
\operatorname{Im} S_A &= \beta(RS_A)P(n)
\end{aligned}
$$

The assertion follows. ***

LEMMA 6

$$\beta(RS_A) = \beta(RS_A)\cdot P(n)$$

$$\text{iff } \operatorname{Re}\mathbf{W}(m)\cdot A_{(m,n)}\cdot \operatorname{Im}\mathbf{W}(n)^t = 0$$

Proof From the proof of Lemma 4 it is known that

$$RS_A = d[\mathfrak{R}A\mathfrak{R}^T + \mathfrak{I}A\mathfrak{I}^T + \mathfrak{I}A\mathfrak{R}^T + \mathfrak{R}A\mathfrak{I}^T]$$

It follows:

$$P(m)\cdot RS_A\cdot P(n)$$

$$= d\,P(m)\,[\Re A\Re^T + \Im A\Im^T + \Im A\Re^T + \Re A\Im^T]\,P(n)$$

$$= d\,[\Re A\Re^T + \Im A\Im^T - \Im A\Re^T - \Re A\Im^T]$$

From where

$$\beta(RS_A) = \frac{1}{2}(RS_A - P(m)RS_A P(n))$$

$$= d\,[\Im A\Re^T + \Re A\Im^T]$$

and

$$\beta(RS_A)\,P(n) = d\,[\Im A\Re^T + \Re A\Im^T]\cdot P(n)$$

$$= d\,[\Im A\Re^T - \Re A\Im^T]$$

This leads to

$$\beta(RS_A) = \beta(RS_A)\cdot P(n)$$

$$\Leftrightarrow$$

$$\Re A\Im^T = -\Re A\Im^T$$

i.e.,

$$\mathrm{Re}\mathbf{W}(m)\cdot A_{(m,n)}\cdot \mathrm{Im}\mathbf{W}^t(n) = 0$$

Property 2 Preservation of the Product

$$S_{AC} = p^n S_A S_C$$

$$RS_{AC} = p^n RS_A RS_C$$

Proof Only the Watari Spectrum will be considered. For the Zhang-Watari spectrum the proof has the same structure.

$$S_{AC} = p^{-(m+t)}\mathbf{W}(m)AC(\mathbf{W}^*(t))^T$$

$$= p^{-(m+t)}\mathbf{W}(m)A((\mathbf{W}^*(n))^T p^{-n}\mathbf{W}(n))C(\mathbf{W}^*(t))^T$$

$$= p^{-(m+n)}(\mathbf{W}(m)A(\mathbf{W}^*(n))^T)\mathbf{W}(n)C(\mathbf{W}^*(t))^T)p^{-t}$$

$$= S_A p^n p^{-(n+t)}(\mathbf{W}(n)C(\mathbf{W}^*(t))^T)$$

$$= p^n S_A S_C$$

***

Property 3 Preservation of Inversion

Only the Watari Spectrum will be considered. For the Zhang-Watari spectrum the proof has the same structure.

$$S_J = p^{-2m}(S_K)^{-1}$$

$$RS_J = p^{-2m}(RS_K)^{-1}$$

Proof

$$S_J = p^{-2m}\mathbf{W}(m)J(\mathbf{W}^*(m))^T$$

$$= p^{-2m}p^m((\mathbf{W}^*(m))^T)^{-1}K^{-1}p^m(\mathbf{W}(m))^{-1}$$

$$= p^{-2m}p^{2m}(\mathbf{W}(m)K(\mathbf{W}^*(m))^T)^{-1}$$

$$= p^{-2m}(p^{-2m}\mathbf{W}(m)K(\mathbf{W}^*(m))^T)^{-1}$$

$$= p^{-2m}(S_K)^{-1}$$

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