9. Competitive Equilibria and Welfare

- In this section, we define the idea of equilibrium in competitive markets.
- We will also define concepts of welfare.
- We will ask in what ways competitive markets affect the welfare of the people in those markets.

9.1 An Introduction to Markets

DEFINITION 9.1. **Economics** *is the* **scientific study** *of* **human behavior** *associated with the* **production** *and* **distribution** *of the* **"necessities and conveniences of life."**

- The quoted phrase is from Adam Smith.
- Marx stressed that economics was about human behavior, not about things.

- **By scientific study, we mean a way of learning about the world through the use of the scientific method.**
  - **Whether or not economics as taught and studied in most of the world uses the scientific method is a question much debated by economists and philosophers of science.**
  - **If economics is a science, it is clearly a very difficult science.**
    - **An enormous number of variables.**
    - **Limited possibilities for experimentation.**
    - **Ideological pressure.**
    - **Difficulty in rejecting bad hypothesis.**
  - **Sometimes economics seems more like low-level mathematics than like a science.**
  - **Economists prove theorems.**

- **Production and distribution are the two principal classes of economic activities.**
- **Economic systems are codes of human behavior developed in society for organizing production and distribution activities.**
  - **Some of these codes are legal, enforceable by the coercive powers of the state.**
  - **Other codes are enforced by private economic incentives**
- **Economic systems help a society determine**
  - **what goods (and services) will be produced**
  - **how they will be produced**
  - **who gets various goods, and how much**
  - **who gives up the services used as inputs in production.**

- Some economic systems make use of voluntary exchange to distribute goods and services.
- Voluntary exchange is welfare improving for those who exchange.
  - Proof: it is voluntary.
  - If some person's welfare would not improve, he will not make the exchange.
  - Exceptions?
- Voluntary exchange may have negative effects on third parties.
  - Externalities related to the commodities that are exchanged.
  - Externalities of the economic and social system required for voluntary exchange.
  - The system shapes our lives!

- A market is a focal point for the voluntary exchange of specified commodities.
- People who want exchange the specified commodities meet at the market.
- Markets lower search costs.
- Barter is a system for direct exchange.
  - "Double coincidence of wants"
  - High search costs.
  - If markets are used, barter with n commodities requires markets for n^2 types of exchanges.

- **Selling** and **buying** is a system for exchange that uses a medium of exchange (money).
  - A two step process.
  - First you exchange the commodities you want to give up for money (selling)
  - Then you exchange money for the commodities that you want to acquire (buying)
  - You have to search twice rather than once (with barter), but search costs are much lower.
  - If markets are used, selling and buying n commodities requires markets for only n types of exchanges.

9.2 Models of Market Economies

DEFINITION 9.2. *The* total initial endowment *of an economy is a vector of commodities ω which is assumed to be available to the economy at the beginning of the period under examination.*

DEFINITION 9.3. *A* household *is a group of economic agents who make their economic decisions as a unit.* [Give me a break!]

DEFINITION 9.4. *An* allocation *is an assignment of commodities to households. An allocation is denoted by a vector of the form $\langle x_1, ..., x_I \rangle$, where*

- *I is the number of households,*
- *and each component x_i is a vector of commodities assigned to household i.*

DEFINITION 9.5. ***An initial allocation specifies how the initial endowment is assigned to households. The initial allocation can be denoted by a vector of the form $\langle \omega_1, ..., \omega_I \rangle$***

- ***where ω_i is a vector of commodities belonging to household i at the beginning of the period,***
- ***and where $\sum_i \omega_i = \omega$.***

The vector ω_i may be called the initial endowment of household i.

DEFINITION 9.6. ***A pure exchange economy is an economy with an initial endowment ω_i defined for each household and in which voluntary exchange may occur among households.***

- ***Voluntary exchange is the only economic activity modeled in an exchange economy.***
- ***Voluntary exchange leads to a final allocation $\langle x_1, ..., x_I \rangle$ among the households.***

DEFINITION 9.7. ***An allocation $\langle x_1, ..., x_I \rangle$ in a pure exchange economy is feasible if $\sum_i x_i \leq \omega$.***

- **Note that the final allocation in a pure exchange economy is always feasible.**

- **This is because exchange does not change the total amounts of the various commodities.**

- **Let $u_i(\cdot)$ denote the utility function of household i.**

- **Then $\langle u_1(x_1), ..., u_I(x_I) \rangle$ is the utility profile induced by an allocation $\langle x_1, ..., x_I \rangle$.**

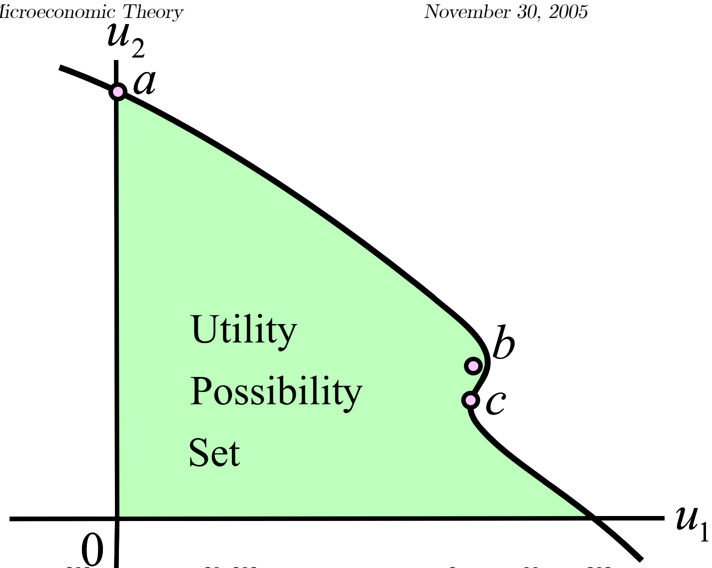
DEFINITION 9.8. ***An allocation $\langle x_1, ..., x_I \rangle$ is said to Pareto dominate an allocation $\langle x'_1, ..., x'_I \rangle$ if***

$$\langle u_1(x_1), ..., u_I(x_I) \rangle > \langle u_1(x'_1), ..., u_I(x'_I) \rangle,$$

that is, if $\langle x_1, ..., x_I \rangle$ leaves at least one household better off than $\langle x'_1, ..., x'_I \rangle$ does, and leaves no household worse off. In this case, we say that $\langle x'_1, ..., x'_I \rangle$ is Pareto dominated by $\langle x_1, ..., x_I \rangle$.

DEFINITION 9.9. *A feasible allocation is said to be* Pareto optimal (PO) *if it is not Pareto dominated by any other feasible allocation.*

- **If an allocation is not PO, then every household's utility can be increased.**
- **Non-PO allocations are inefficient, in the sense that utility is being wasted.**
- **PO allocations are better than the allocations they dominate, but they may be less desirable than other non-PO allocations.**



- **The above utility-possibility set contains all utility combinations induced by feasible allocations.**
- **c is on the utility frontier but it does not represent a PO allocation, because c is dominated by b.**
- **b is not on the frontier, so it cannot be PO.**
- **a is PO, but it may not be as desirable as either b or c, because one person has everything.**

DEFINITION 9.10. *In a pure exchange economy, the demand $x_i(\cdot)$ of household i with endowment ω_i is a correspondence that satisfies*

- $x_i(p) = \text{argmax}_{x_i}\{u_i(x_i) \mid px_i \leq p\omega_i\}$ *(utility maximization), and*
- $px_i(p) = p\omega_i$ *(Walras Law)*

for all price vectors $p \in \mathbb{R}_+$.

- For now, we assume that for all i, $x_i(p)$ is a function.

DEFINITION 9.11. *The excess demand of household i is given by $x_i(p) - \omega_i$.*

DEFINITION 9.12. *The excess supply of household i is given by $\omega_i - x_i(p)$, the negative of excess demand.*

DEFINITION 9.13. *A set of prices p is market clearing if the aggregate excess demand for each commodity ℓ is zero when p_ℓ is positive and nonpositive when $p_\ell = 0$, i.e. if*

$$\begin{aligned} \sum_i [x_{i\ell}(p) - \omega_{i\ell}] &= 0 \text{ if } p_\ell > 0 \\ \sum_i [x_{i\ell}(p) - \omega_{i\ell}] &\leq 0 \text{ if } p_\ell = 0 \end{aligned}$$

Note that positive aggregate excess supply at a zero price is consistent with market-clearing prices.

DEFINITION 9.14. *An allocation $\langle x_1, ..., x_I \rangle$ is said to be supported by prices p if for all i, $x_i = x_i(p)$.*

DEFINITION 9.15. *Given an initial endowment $\langle \omega_1, ..., \omega_I \rangle$, a* **competitive equilibrium** *of a pure exchange economy is a set of prices p^* and a nonnegative allocation $\langle x_1^*, ..., x_I^* \rangle$ such that*

- *$\langle x_1^*, ..., x_I^* \rangle$ is supported by p^* and*
- *p^* is market clearing.*

- **Later we will show:**

PROPOSITION 9.1. **(First Fundamental Theorem of Welfare Economics)** *Every competitive equilibrium yields a Pareto-optimal allocation.*

PROPOSITION 9.2. **(Second Fundamental Theorem of Welfare Economics)** *Under assumptions to be described later, every Pareto-optimal allocation can be implemented as a competitive equilibrium for some set of initial endowments.*

9.3 Economies with Production and Exchange

- **Let firms be denoted by the subscript j.**
- **Given the production set Y, supply (and derived demand when negative) $y_j(\cdot)$ of firm j is given by by the vector that solves the profit maximization problem:**

$$y_j(p) = \underset{y_j}{\operatorname{argmax}}\{py_j \mid y_j \in Y\}.$$

- **Let θ_{ij} represent household i's share of the profits of firm j, such that for for all i, $\sum_i \theta_{ij} = 1$.**

- **In an economy with production and exchange, for all price vectors p, the demand $x_i(\cdot)$ of household i satisfies:**
  - **utility maximization**

$$x_i(p) = \underset{x_i}{\operatorname{argmax}}\{u_i(x_i) \mid px_i \leq p\omega_i + \sum_j \theta_{ij} py_j\}$$

  - **Walras' Law:**

$$px_i(p) = p\omega_i + \sum_j \theta_{ij} py_j$$

DEFINITION 9.16. *An* `allocation` *in an economy with production and exchange is an assignment of commodities to households and of production activities to firms, denoted by a vector of the form* $\langle x_1, ..., x_I, y_1, ..., y_J \rangle$, *where*

- I *is the number of households, and* J *is the number of firms;*
- *each* x_i *is a vector of commodities assigned to household* i;
- *each* y_j *is the production activity of firm* j.

DEFINITION 9.17. *An allocation $\langle x_1, \ldots, x_I, y_1, \ldots, y_J \rangle$ is* **feasible** *if*

$$\sum_i x_i = \omega + \sum_j y_j$$

and $y_j \in Y$ for all j.

DEFINITION 9.18. *A feasible allocation $\langle x_1, \ldots, x_I, y_1, \ldots, y_J \rangle$ is said to be* **supported** *by prices p if for all i and j, $x_i \in x_i(p)$ and $y_j \in y_j(p)$.*

DEFINITION 9.19. *A set of prices p is* **market clearing** *if there is a feasible allocation $\langle x_1, \ldots, x_I, y_1, \ldots, y_J \rangle$ supported by prices p such that*

$$\sum_i x_{i\ell} = \sum_j y_{j\ell} + \sum_i \omega_{i\ell} \text{ when } p_\ell > 0$$

$$\sum_i x_{i\ell} \leq \sum_j y_{j\ell} + \sum_i \omega_{i\ell} \text{ when } p_\ell = 0$$

Note that positive aggregate excess supply at a zero price is consistent with market-clearing prices.

PROPOSITION 9.3. *(Walras' Law for markets.) Suppose that Walras' law holds for demand. Then, if $p \gg 0$, and if all but one market clears, then the last market must clear also.*

PROOF. Suppose, without loss of generality, that the market clears for all goods $\ell < L$.

- Walras's law requires that for each i,

$$\sum_{\ell<L} p_\ell x_{i\ell} + p_L x_{iL}$$
$$= \sum_{\ell<L} p_\ell(\omega_{i\ell} + \sum_j \theta_{ij} y_{j\ell}) + p_L(\omega_{iL} + \sum_j \theta_{ij} y_{jL})$$

- so that

$$\sum_{\ell<L} p_\ell(x_{i\ell} - \omega_{i\ell} - \sum_j \theta_{ij} y_{j\ell})$$
$$= -p_L(x_{iL} - \omega_{iL} - \sum_j \theta_{ij} y_{jL})$$

- **It follows that:**

$$\sum_{\ell<L} p_\ell \left(\sum_i (x_{i\ell} - \omega_{i\ell}) - \sum_j y_{j\ell} \right)$$
$$= -p_L \left(\sum_i (x_{iL} - \omega_{iL}) - \sum_j y_{jL} \right)$$

- **Recall that $p \gg 0$.**
- **By the assumption that markets clear for $\ell < L$, we know that all the terms in brackets on the left must be 0**
- **Since $p_L > 0$, the term in brackets on the right must also be 0.**
- **Thus the market for L clears as well.**

■

DEFINITION 9.20. ***Given an initial endowment $\langle \omega_1, ..., \omega_I \rangle$, a [general] competitive equilibrium of an economy with production and exchange is a set of prices p^* and a feasible allocation $\langle x_1^*, ..., x_I^*, y_1^*, ..., y_J^* \rangle$ such that***

- ***$\langle x_1^*, ..., x_I^*, y_1^*, ..., y_J^* \rangle$ is supported by p^* and***
- ***p^* is market clearing.***

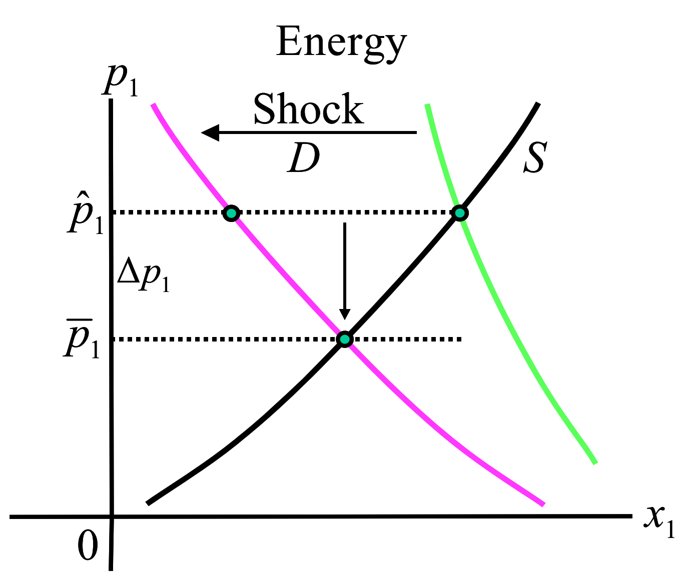
9.4 Partial- vs. General-Equilibrium Analysis

- General equilibrium: prices are such that
  quantity supplied =quantity demanded
  in markets for all goods

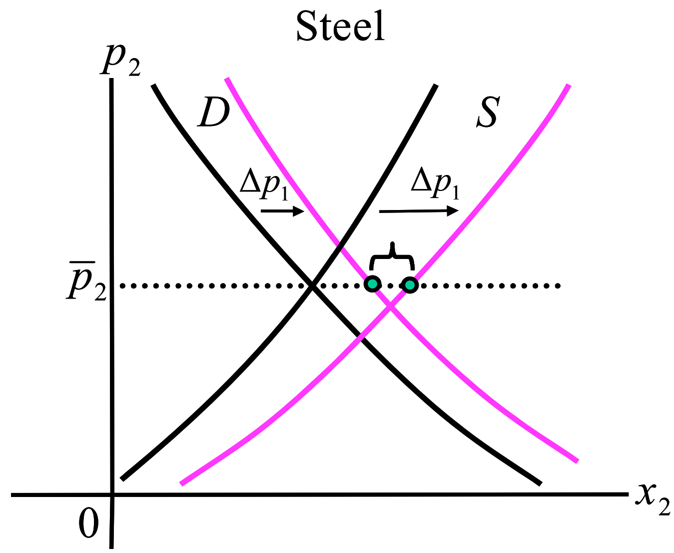
- Partial equilibrium: the prices are such that
  quantity supplied =quantity demanded
  in market for one good, with unknown situation in other markets.

- Given
  - an aggregate demand function $x(p_1, p_2..., p_L)$
  - and an aggregate supply function $y(p_1, p_2..., p_L)$
  - whose values are vectors of commodities
- and given any commodity ℓ (say $\ell = 1$)
- we can fix the other prices at $\bar{p}_2, ..., \bar{p}_L$ and analyze
  - $x_1(p_1) \equiv x_1(p_1, \bar{p}_2, ..., \bar{p}_L)$ and
  - $y_1(p_1) \equiv y_1(p_1, \bar{p}_2, ..., \bar{p}_L)$

- Suppose the economy is in (general) equilibrium.
- Then there is a supply or demand shock in one market.
  - A sudden exogenous change to the quantity supplied or demanded.
- Partial equilibrium analysis: How does the price have to change in the affected market to bring it back into equilibrium?



- **But the change of one price, usually affects the quantities supplied and demanded of many goods.**
- **If only one price is adjusted, then other markets will go out of equilibrium.**



- General equilibrium analysis: **How do all prices have to change to bring all markets back into equilibrium?**

Example 9.1. *Suppose two commodities have demands*

$$x_1 = \frac{p_2 I}{p_1^2} \text{ and } x_2 = \frac{p_1 I}{p_2^2}$$

where $I = 1$, and inelastic supplies $y_1 = y_2 = 1$.

Compare partial and general equilibrium analysis if a shock changes y_1 from 1 to 2.

Solution:

- **Before the shock:**

$$x_1 = p_2/p_1^2 = y_1 \equiv 1 \text{ and } x_2 = p_1/p_2^2 = y_2 \equiv 1$$

so we have

$$p_2 = p_1^2 \quad p_1 = p_2^2$$

$$p_1 = p_1^4$$
$$p_1 = p_2 = 1.$$

- **Now y_1 changes from 1 to 2.**

- In partial equilibrium analysis, we preserve $p_2 = 1$. We have

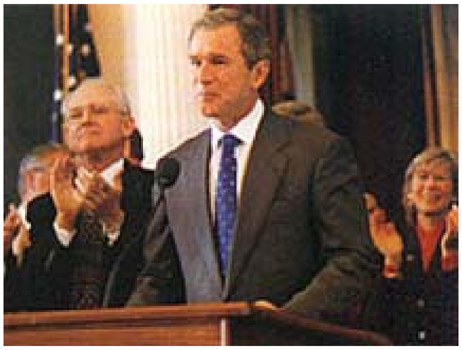
$$\begin{aligned} x_1 &= y_1 \equiv 2 \\ 1/p_1^2 &= 2 \\ p_1 &= 1/\sqrt{2} = 0.71. \end{aligned}$$

- In general equilibrium analysis, we would find the new equilibrium in both markets.

$$x_1 = p_2/p_1^2 = y_1 \equiv 2 \text{ and } x_2 = p_1/p_2^2 = y_2 \equiv 1$$
$$p_2 = 2p_1^2 \qquad\qquad p_1 = p_2^2$$

$$\begin{aligned} p_1 &= 4p_1^4 \\ p_1 &= 1/\sqrt[3]{4} = 0.63 \\ p_2 &= 2 \cdot 0.63^2 = 0.80. \end{aligned}$$

- Which is better:
  - Partial equilibrium analysis? or
  - General equilibrium analysis?
- Which picture of George Bush is better?



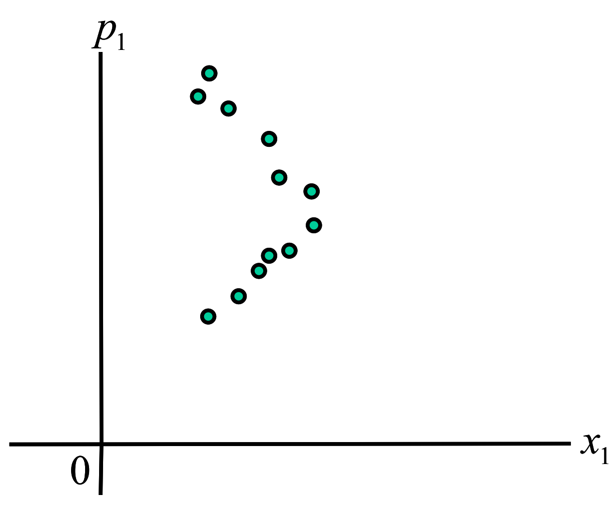
or...

...

- **Answer: it depends on what you want to say.**
- **If the commodity with the demand or supply shock:**
  - **represents a small share of total expenditures**
  - **does not have strong substitutes or complements**
- **Then a change in its price is** `not` **likely to**
  - **have a large income or wealth effect**
  - **have large substitution effects**
  - **shift other demand and supply curves very much.**

- **Models or empirical work in a general-equilibrium framework:**
  - **use a large amount of data**
  - **include many complex relationships (many equations)**
  - **difficult to analyze**
  - **prone to error**
  - **not elegant**
  - **not transparent**
  - **difficult to interpret**
- **Therefore, in such cases, partial equilibrium may be preferable to general equilibrium.**

- **Suppose you see time-series data like this:**



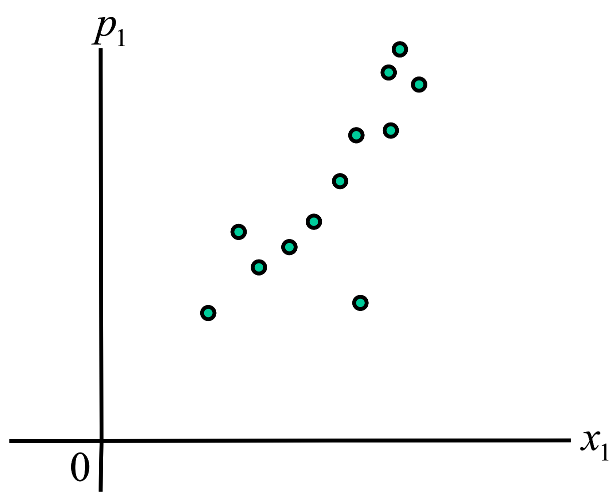
- **What is happening?**
- **Prices may be rigid, possibly set by regulators.**

PROPOSITION 9.4. **(Law of the short side.)** ***If voluntary exchange occurs at disequilibrium prices, then the quantity transacted will be the minimum of quantity supplied and the quantity demanded.***

PROOF. **Exchange is voluntary.** ■

- **The points at the lower prices may be on the supply curve.**
  - **Exchange is supply-constrained**
- **with the points at higher prices on the demand curve**
  - **Exchange is demand-constrained.**

- **Supply and demand are hypothetical constructs, which are not directly observable.**
- **Suppose you see time-series data like this:**



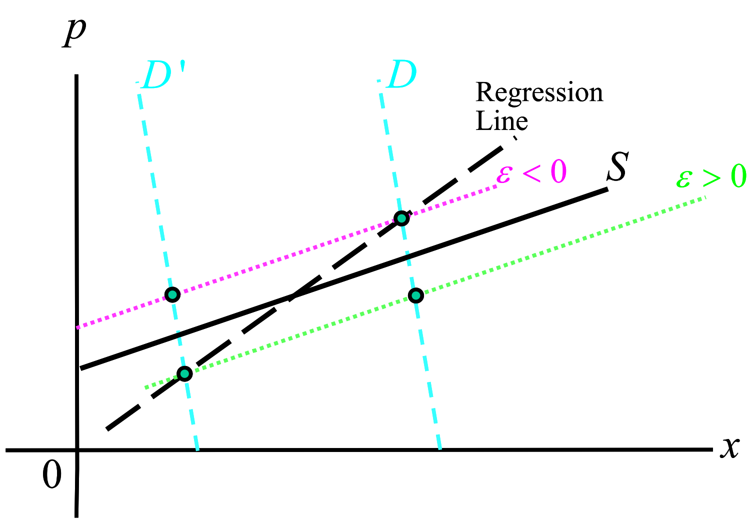
- **What is happening?**
- **Possibly: demand is shifting a lot and supply is more stationary**

- Suppose you have following model of supply:

$$x_s = a + \beta p + \gamma z + \varepsilon$$

- z represents exogenous variables that affect supply (supply-shift variables).
- The data are determined by the intersection of supply and demand.

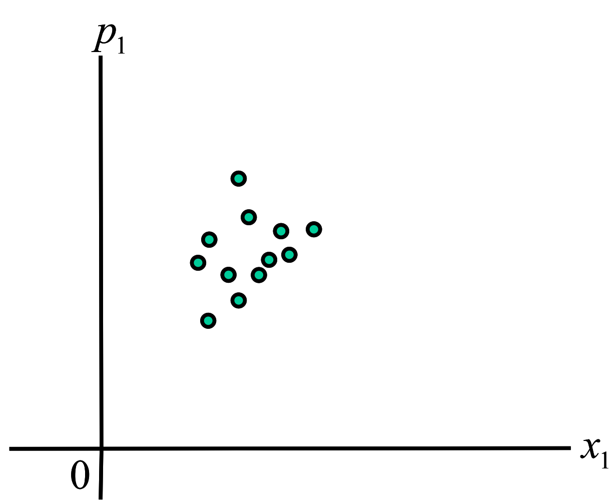
- The points below represent the data you might see.
- Quantities and prices are endogenous.



- Remember: x_s is the dependent variable! The graph doesn't follow mathematical convention.

- **Prices are (negatively) correlated with the error term.**
- **When there is a positive supply shock, the equilibrium price is likely to be low.**
- **When there is a negative supply shock, the equilibrium price is likely to be high.**
- **This causes the regression coefficient β to be biased downwards [towards the price axis].**
- **There are well-known econometric techniques for correcting this situation.**
- **Even in partial-equilibrium models, empirical work must be done with care.**

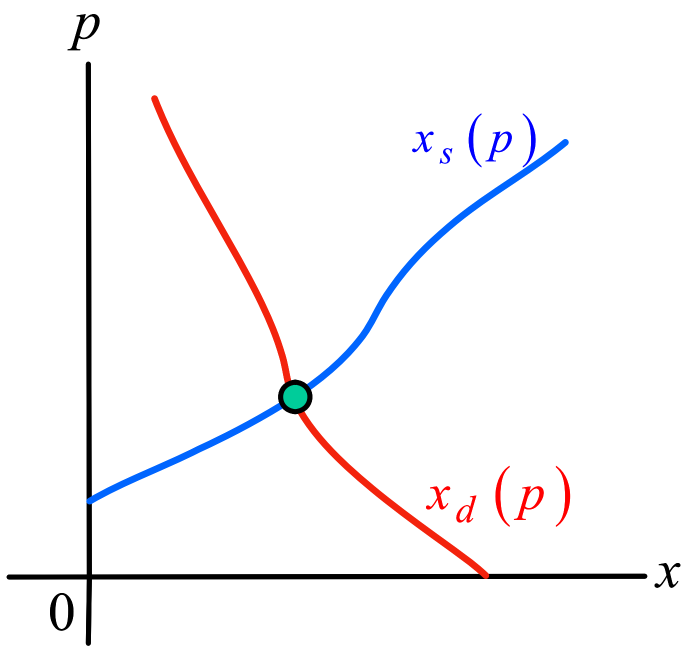
- **Suppose you see time-series data like this.**
- **These may represent a series of equilibria in the given market.**
- **Appropriate data and econometric techniques are needed to identify supply and demand.**



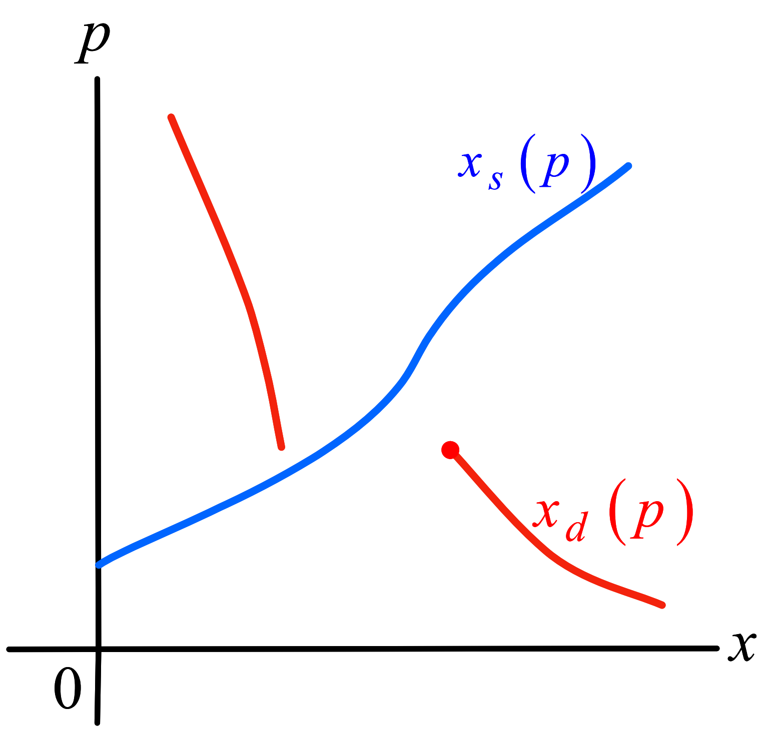
9.5 The Existence of General Equilibrium

[I'll do a simplified version this semester. We present a more general version next semester in EC 703.]

- **Where is the equilibrium in this graph?**



- **Where is the equilibrium here?**

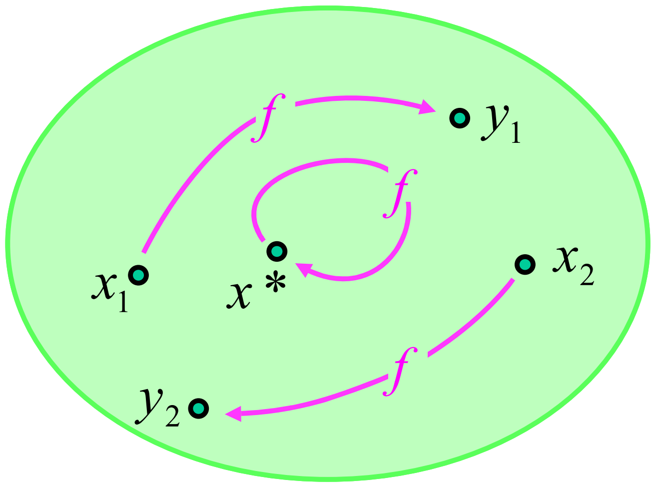


- **The proof of existence of equilibrium is a proof that curves (or higher dimensional manifolds) cross!**

- **The proof of that curves cross is equivalent to the proofs of fixed-point theorems.**

- **Here we will talk about the Brouwer Theorem; Mookherjee will do more general theorems.**

PROPOSITION 9.5. **(Brouwer Fixed-Point Theorem)** ***Let X be a compact convex set, and suppose $f : X \to X$ is a continuous function. Then there is a point $x^* \in X$ such that $f(x^*) = x^*$.***

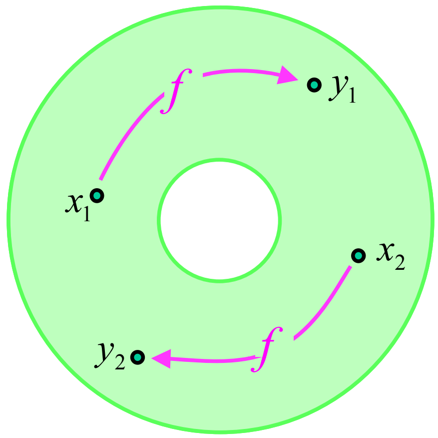


- **Interested students may find the proof in the mathematical notes on the course website, but you will not be responsible for it in EC 701.**

- **Think of f as a function that transforms X within its own boundaries. The function may rotate X, shrink it, deform it, etc., but the theorem says that there is at least one point that f will not move.**

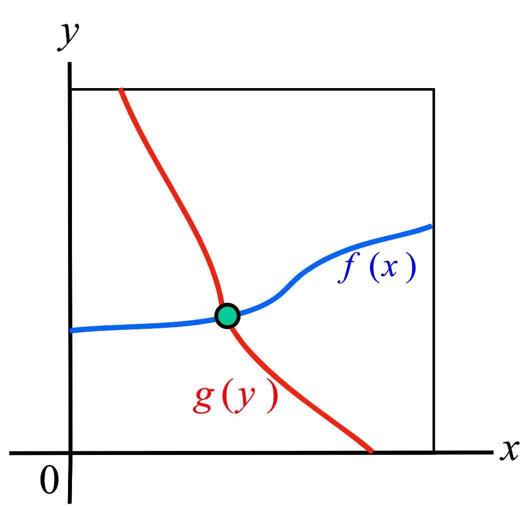
  - **For example, when the earth rotates, all the points on the axis of rotation remain fixed.**

- **Note that the theorem fails for the rotation of an annulus: no point remains fixed.**



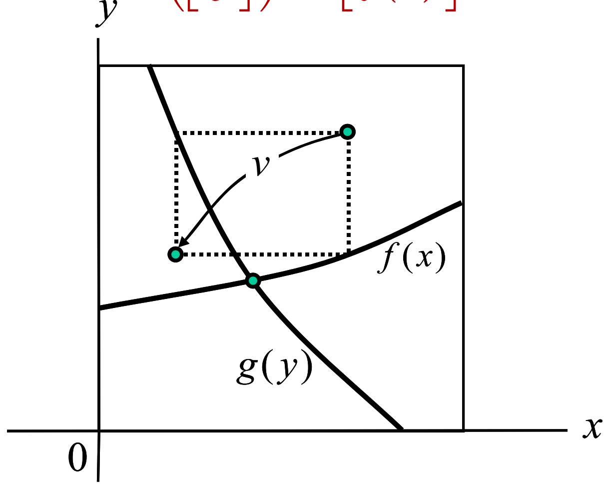
- **The fixed-point theorem has an important corollary.**

PROPOSITION 9.6. ***Consider the unit square, and suppose $y = f(x)$ and $x = g(y)$ are both continuous functions, so that the graph of f connects the vertical sides and the graph of g connects the horizontal sides. Then the two curves must intersect.***



PROOF. Define the function $v : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$v\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} g(y) \\ f(x) \end{bmatrix}.$$



- Because f and g are continuous, v must be continuous.
- v has a fixed point $\begin{bmatrix} x^* \\ y^* \end{bmatrix}$.
- The curves must intersect there:
  - $y^* = f(x^*)$, $x^* = g(y^*)$

∎

- This corollary applies to surfaces in higher dimensional spaces as well.
- But the dimensions of the surfaces (manifolds) must add up to the dimension of the space in which the graphs are drawn.
  - Consider two roads in three-dimensional space: they will not intersect if one passes over the other one.

- Proving the existence of general equilibrium may be conceptualized as proving that multidimensional supply and demand curves intersect.
- But because of many technicalities, the proofs that I have seen all apply the standard statement of a fixed-point theorem.
- Monotone preferences lead to "infinite" demand for goods with zero prices, so that excess demand is usually not defined on price vectors that contain zeros.
- Therefore, proofs of the existence of general equilibria require complicated mathematical machinery, such as the Kakutani fixed point theorem, which is valid for set-valued functions.
- The following proposition, not proved here, requires only the Brouwer fixed-point theorem.

PROPOSITION 9.7. **(Existence of general equilibrium: simplified version)** *Suppose that in a pure-exchange economy the aggregate (market) excess demand function* $z(p)$

- *is defined for all* $p \in \mathbb{R}^L_+$, $p \neq 0$.
- *is continuous,*
- *homogeneous of degree zero, and*
- *satisfies Walras' Law.*

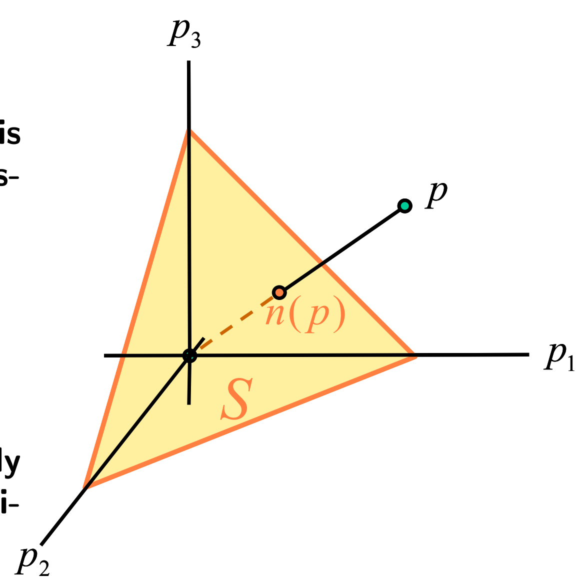
Then, there is a price vector p^* *such that* $z(p^*) \leq 0$, *with* $z_\ell(p^*) = 0$ *for* $p^*_\ell > 0$.

- **Because demand and excess demand are homogeneous of degree zero, we can normalize price vectors in a variety of ways.**
  - **We can establish a `numeraire` by dividing all prices by the price of a given commodity (say p_1), in which case all price vectors will have the form $(1, p_2, \ldots, p_L)$**
  - **Or we can divide all price vectors by the sum of the prices.**
    - **let** $S = \{p \mid p \in \mathbb{R}^L_+, \sum_\ell p_\ell = 1\}$
    - S **is called the unit simplex of** $\mathbb{R}^L$

- **Define a normalization function:**

$$n(p) \equiv \frac{1}{\sum_\ell p_\ell} p$$

  - **If** $\hat{p} = n(p)$, **and** $z(p)$ **is a homogeneous-degree-0 excess-demand function**
    - **then** $z(\hat{p}) = z(p)$
    - **and** $\sum_\ell \hat{p}_\ell = 1$, **so that** $\hat{p} = n(p) \in S$.



- **Therefore, we need consider only prices in S to find a general equilibrium.**

- **To prove there is a general equilibrium, we will**
  - **start with prices $p \in S$,**
  - **raise prices with positive excess demand, yielding p'**
  - **but do nothing to prices with excess supply.**
  - **Then we renormalize to $n(p') \in S$.**
  - **We will have:**
    - $n(p') > p$ **for goods with high excess demand**
    - $n(p') < p$ **for goods with high excess supply.**

- **This is how we think prices adjust over time in the economy,**
- **but time does not appear in the general equilibrium model, which is static.**
- **We define a function $f : S \to S$, such that $f(p) = n(p')$.**
- **This function must have a fixed point, p^*.**
- **Excess demand does not change prices at p^*.**
- **Therefore excess demand must be zero at p^*.**
- **Conclusion: p^* is a set of general-equilibrium prices.**

Formal Proof of Existence.

- Let $S = \{p \mid p \in \mathbb{R}^L_+, \sum_\ell p_\ell = 1\}$
  - S is called the unit simplex of $\mathbb{R}^L$
- Define a normalization function:

$$n(p) \equiv \frac{1}{\sum_\ell p_\ell} p$$

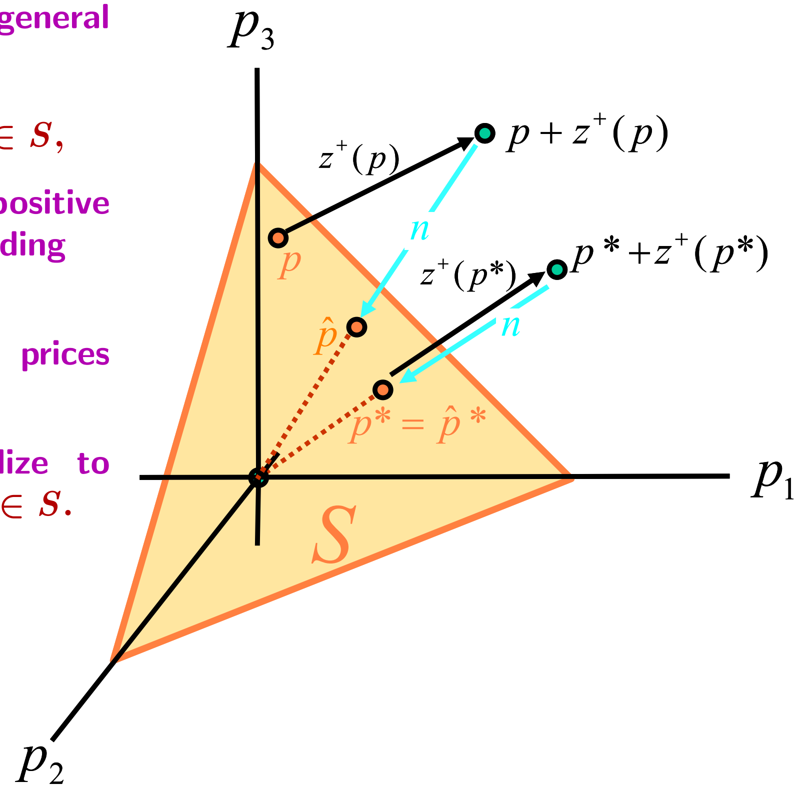
- then, by hdz, $z(n(p)) = z(p)$,
- and since $\sum_\ell n_\ell(p) = 1$, $n(p) \in S$.
- Therefore, we need consider only prices in S to find a general equilibrium.

- Define $z^+(p)$ by

$$z^+_\ell(p) = \begin{cases} z_\ell(p) & \text{for } z_\ell(p) > 0 \\ 0 & \text{for } z_\ell(p) \leq 0 \end{cases}.$$

- or $z^+(p) = \max\{z(p), 0\}$.
- $z^+(p)$ represents only the positive components of excess demand.

- To prove there is a general equilibrium, we
  - start with prices $p \in S$,
  - raise prices with positive excess demand, yielding $p + z^+(p)$
  - but do nothing to prices with excess supply.
  - Then we renormalize to $\hat{p} = n(p + z^+(p)) \in S$.



- We will have:
  - $\hat{p} > p$ for goods with high excess demand
  - $\hat{p} < p$ for goods with high excess supply.
- This is how we think prices adjust over time in the economy,
- but time does not appear in the general equilibrium model, which is static.
- We define a function $f : S \to S$, such that $\hat{p} \equiv f(p) \equiv n(p + z^+(p))$.
- S is compact and convex.
- $z(\cdot)$ is continuous $\Longrightarrow z^+(\cdot)$ is continuous, because $z^+(p) = \max\{z(p), 0\}$, and maximization is a continuous function.
- Therefore, f is continuous.
- By Brouwer fixed-point theorem, the function f must have a fixed point, say $p^* = f(p^*)$.

- We have:

$$p^* = n(p^* + z^+(p^*)) = \lambda(p^* + z^+(p^*))$$

  for some $\lambda > 0$.

- Solution for $z^+(p^*)$:

$$z^+(p^*) = \gamma p^*$$

$$\text{where } \gamma = \frac{1-\lambda}{\lambda}$$

- If $\gamma < 0$, then $z^+(p^*)$ has negative values; contradicts definition of z^+.
- If $\gamma > 0$, $z_\ell^+(p^*) > 0$ when $p_\ell^* > 0$
  - $\Longrightarrow z_\ell^+(p^*) = z_\ell(p^*)$ when $p_\ell^* > 0$
  - $\Longrightarrow p^* z(p^*) = p^* z^+(p^*) = \gamma(p^* \cdot p^*)$
  - But $p^* z(p^*) = 0$ by Walras Law
  - and $p^* \cdot p^* > 0$, contradiction!

- Therefore $\gamma = 0$, $z^+(p^*) = 0$.
- $z(p^*) \leq 0$
- But $p^* z(p^*) = 0$,
- so $z_\ell(p^*) = 0$ when $p_\ell^* > 0$
- and $z_\ell(p^*) \leq 0$ when $p_\ell^* = 0$.
- p^* are equilibrium prices.

■

PROPOSITION 9.8. **(The First Fundamental Theorem of Welfare Economics)** ***Suppose a pure-exchange economy has households characterized by local nonsatiation and Walras Law. Then any competitive equilibrium has a Pareto-optimal allocation.***

PROOF. **This proof is easy!**

- **Let $\langle x_1, \ldots, x_I \rangle$ be the allocation associated with the competitive-equilibrium prices p^*.**
- **Then $\langle x_1, \ldots, x_I \rangle \equiv \langle x_1(p^*), \ldots, x_I(p^*) \rangle$, where $x_i(\cdot)$ is the demand function for household i.**
- **And let $\langle \omega_1, \ldots, \omega_I \rangle$ be the initial endowments.**

- **Suppose that $\langle x'_1, \ldots, x'_I \rangle$ Pareto dominates $\langle x_1, \ldots, x_I \rangle$.**
- **Then $x'_i \succ x_i(p^*)$,**
- $\Longrightarrow p^* x'_i > p^* x_i(p^*)$ **[revealed preference]**
- $\Longrightarrow$

$$\sum_i p^* x'_i > \sum_i p^* x_i(p^*) = \sum_i p^* \omega_i.$$

- **Let $x' \equiv \sum_i x'_i, \quad x(p^*) \equiv \sum_i x_i(p^*)$ and $\omega \equiv \sum_i \omega_i$.**
- **Then $p^* x' > p^* \omega$.**

- In scalar notation:

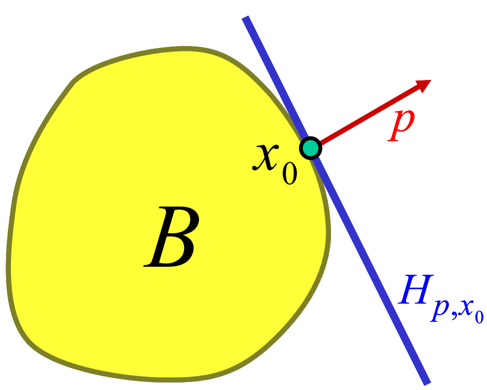
$$\sum_{\ell} p^*_\ell x'_\ell > \sum_{\ell} p^*_\ell \omega_\ell$$

- But this implies that for at least one ℓ with $p_\ell > 0$, we have $p_\ell x'_\ell > p_\ell \omega_\ell$.
- $\implies x'_\ell > \omega_\ell$.
- So $\langle x'_1, ..., x'_I \rangle$ is not feasible.

■

- The proof of the Second Fundamental Theorem of Welfare Economics requires the use of a corollary of the separating hyperplane theorem.

Definition 9.21. *H is a* `supporting hyperplane` *of the set $B \subset \mathbb{R}^n$ if H is a hyperplane of dimension $n-1$ that intersects the boundary of B with the property that B lies entirely on one side of H.*



PROPOSITION 9.9. ***(Supporting Hyperplane Theorem) Suppose $B \subset \mathbb{R}^n$ is a closed convex set, and suppose x_0 is a point on the boundary of B. Then B has a supporting hyperplane H_{p,x_0} that intersects B at x_0. It follows that there is a vector $p \neq 0$, such that for all $b \in B$, $pb \leq px_0$.***

- **This proposition is proved by applying the *separating hyperplane theorem* to each member of a sequence of points converging to x_0 from outside of B (see MC p949).**
- **Note that by taking the negative of the p provided by the proposition, we can find another p such that for all $b \in B$, $pb \geq px_0$.**

PROPOSITION 9.10. **(Second Fundamental Theorem of Welfare Economics)** ***If the utility functions of households are continuous, quasiconcave and monotonic, then any Pareto-optimal allocation $x^* \equiv \langle x_1^*, ..., x_I^* \rangle$ can be implemented as a competitive equilibrium for some vector of initial endowments.***

- **The Second Fundamental Theorem says a social planner can implement whatever Pareto Optimal allocation she likes as a competitive equilibrium. But to accomplish this she must in general...**
  - **reallocate the initial endowments (or make wealth transfers–see below), and**
  - **announce a vector of equilibrium prices.**

- **The proof of this proposition is an application of the supporting hyperplane theorem.**
- **Suppose a social planner wants an individual to demand a particular vector of goods x_0.**
  - **The planner can apply the supporting hyperplane theorem to x_0 and its upper contour set $G(x_0)$ to produce the budget frontier (a hyperplane) and a price vector p such that $px_0 \leq px$ for all $x \in G(x)$.**
  - **The she can set income $y = px_0$.**
  - **Then the individual will demand x_0, because every preferred bundle of goods will violate the budget constraint.**

- **The second fundamental theorem presents the planner with a more difficult task:**
  - **She must find a one price vector p^* that simultaneously causes every household to demand its allocated bundle of goods x_i^*, ...**
  - **even though each household may have a different utility function.**
  - **This is accomplished by aggregating the x_is and all the $G(x_i)$ sets, before applying the supporting hyperplane theorem.**

- The strategy of the proof is as follows:
  - We set the initial allocation $\langle \omega_1, \ldots, \omega_I \rangle$ equal to the designated PO allocation $\langle x_1^*, \ldots, x_I^* \rangle$.
  - By summing the upper contour sets of the x_i^* s, we define a convex set G in commodity space with the following property:
    - if each $X \in G$ were distributed appropriately among the households, . . .
    - the resulting allocations would Pareto dominate $\langle x_1^*, \ldots, x_I^* \rangle$.

- $X^* \equiv \sum_i x_i^*$ must be on the boundary of G.
  - Otherwise, there would be a vector $\hat{X} \ll X^*$ also in G,
  - and the difference between the two vectors could be used to create a Pareto dominating allocation that also adds up to X^* and is therefore feasible.
  - That would violate the Pareto optimality of $\langle x_1^*, \ldots, x_I^* \rangle$ and be a contradiction.
- We apply the supporting hyperplane theorem to X^* and G.
  - This gives us a set of prices p^* which makes everything in the interior of G more expensive than X^*.
  - We use this to show that every vector x_i' that gives a higher utility than x_i^* costs more, so that x_i^* maximizes utility within the budget $p^*\omega_i$.
- We conclude that $\langle x_1^*, \ldots, x_I^* \rangle$ and p^* is a competitive equilibrium.

PROOF. **Here are the details:**

- We set the initial allocation $\langle \omega_1, ..., \omega_I \rangle$ equal to the designated PO allocation $\langle x_1^*, ..., x_I^* \rangle$.
- For each x_i^*, let $G_i(x_i^*) \equiv \{x_i \mid u_i(x_i) \geq u_i(x_i^*)\}$ denote the upper contour sets of each x_i^*. Because of the quasiconcavity assumption each $G_i(x_i^*)$ is a convex set.
- Define the set $G \equiv \sum_i G_i(x_i^*)$. That is, G is the set of all possible sums of the form $x \equiv x_1 + x_2 + ... + x_I$ with each summand drawn from the corresponding set $G_i(x_i^*)$.
- By construction, every $x \in G$ is a vector of commodities that could be distributed to households in an allocation that Pareto dominates $\langle x_1^*, ..., x_I^* \rangle$ or at least provides equal utilities.
- G is the sum of convex sets, and therefore G itself is convex.

- Let $X^* \equiv \sum_i x_i^*$.
- We show that X^* is on the boundary of G.
  - $x_i^* \in G_i(x_i^*)$ for all i, so $X^* \equiv \sum_i x_i^* \in \sum_i G_i(x_i^*) \equiv G$.
  - If X^*were in the interior of G, then there must exist a point $\hat{X} \ll X^*$ with $\hat{X} \in G$.
  - But then, by the definition of G, there must be an allocation $\langle \hat{x}_1, ..., \hat{x}_I \rangle$ where $\hat{x}_i \in G_i(x_i^*)$ and $\hat{X} \equiv \sum_i \hat{x}_i$.
  - It follows that for all i, $u(\hat{x}_i) \geq u(x_i^*)$.

- Let the vector ε be defined by $\varepsilon = \frac{1}{I}\left(X^* - \hat{X}\right)$, so that $\sum_i (\hat{x}_i + \varepsilon_i) = X^*$, which means that the allocation $\langle \hat{x}_1 + \varepsilon_1, ..., \hat{x}_I + \varepsilon_I \rangle$ is feasible.
- And by monotonicity, $u(\hat{x}_i + \varepsilon_i) > u(\hat{x}_i) \geq u\left(x_i^*\right)$ for all i.
- But this contradicts the Pareto Optimality of $\langle x_1^*, ..., x_I^* \rangle$, and we conclude that X^* is not in the interior of G and must be in the boundary.

- We now apply the supporting hyperplane theorem to X^* and G.
  - The theorem implies that there exists a price vector $p^* \neq 0$ such that for all $X \in G$, we have $p^* X \geq p^* X^*$.
  - Monotonicity implies that given any vector $\delta > 0, \ X^* + \delta \in G$.
  - This means that $p^*(X^* + \delta) \geq p^* X^*$ so that $p^* \delta \geq 0$ for all $\delta > 0$
  - It follows that $p^* > 0$.

- We now show that if $u_i\big(x_i'\big) > u_i\big(x_i^*\big)$, then $p^*x_i' > p^*x_i^*$, a condition that implies that $x_i^* \in \text{argmax}\big\{u_i(x_i) \mid p^*x_i \leq p^*x_i^*\big\}$.
  - Given $u_i\big(x_i'\big) > u_i\big(x_i^*\big)$, we know by continuity that there is a $x'' \ll x'$ with $u_i\big(x_i''\big) > u_i\big(x_i^*\big)$
  - Consider the vector $X'' \equiv x_i'' + \sum_{j \neq i} x_j^*$.
  - Because $x_i'' \in G_i\big(x_i^*\big)$, we know that $X'' \in G$.
  - Therefore $p^*X'' \geq p^*X^*$ so that $p^*(X'' - X^*) = p^*\big(x_i'' - x_i^*\big) \geq 0$.
  - Thus $p^*x_i'' \geq p^*x_i^*$ and $p^*x_i' > p^*x_i^*$.
- We have shown that the allocation $\langle x_1^*, \ldots, x_I^*\rangle$ and the price vector p^* forms a competitive equilibrium, which proves the Second Fundamental Theorem.

■

- The Second Fundamental Theorem as proved above requires the social planner to set the initial endowments to the desired final allocation.
- In a real economy reallocating the initial endowment would be completely impractical. Much of the endowment would take the form of inalienable skills or human capital.
- However, the Second Fundament Theorem could be proved in a form in which the initial endowments are left intact. Instead, lump-sum taxes are used to reset each household's budget set.

- **Suppose $\langle \omega_1, ..., \omega_I \rangle$ is the initial allocation and $\langle x_1^*, ..., x_I^* \rangle$ is the desired PO allocation.**

- **Then we could define a vector of lump-sum taxes (and subsidies when negative) $\langle t_1, ..., t_I \rangle$ as follows:**

  - **Set $t_i = p^*(\omega_i - x_i^*)$.**

  - **Then $p^*\omega_i - t_i = p^* x_i^*$, so that the initial allocation $\langle \omega_1, ..., \omega_I \rangle$ with taxes $\langle t_1, ..., t_I \rangle$ would create the same budget constraints as using $\langle x_1^*, ..., x_I^* \rangle$ for the initial allocations.**

- **This yields the following proposition:**

PROPOSITION 9.11. **(Second Fundamental Theorem of Welfare Economics with lump-sum taxes)** ***Let the initial allocation $\langle \omega_1, ..., \omega_I \rangle$ be given. Then, if the utility functions of households are continuous, quasiconcave and monotonic, any Pareto-optimal allocation $x^* \equiv \langle x_1^*, ..., x_I^* \rangle$ can be implemented as a competitive equilibrium by use of an appropriate vector of lump-sum taxes and subsidies.***