8. Noncooperative Games

8.1 Basic Concepts

- **How I learned basic game theory in 1986.**
- **Game theory =** ***theory of strategic interaction***
- **Each agent chooses a strategy in order to achieve his goals with the knowledge that other agents may react to his strategic choice.**

DEFINITION 8.1. ***A*** **strategy** ***is a plan of action that specifies what a player will do in every circumstance.***

Matching Pennies (a zero-sum game)

- **Eva and Esther each places a penny on the table simultaneously.**
- **They each decide in advance whether to put down heads (H) or tails (T). [This is a choice, NOT a random event as when a coin is flipped.]**
- **If both are heads (H) or both are tails (T), then Eva pays Esther \$1.**
- **If one is heads and the other tails, then Esther pays Eva \$1.**

		ESTHER	
		H	**T**
EVA	**H**	-1, *1*	1, -1
	T	1, -1	-1, *1*

		Esther	
		H	**T**
Eva	**H**	*1* -1	*−1* 1
	T	*−1* 1	*1* -1

- **Eva and Esther are the players.**
- **H and T are strategies, and $\{H, T\}$ is each player's strategy space.**
- **1 and -1 are payoffs.**
- **Each box corresponds to a strategy profile.**
- **For example, the top-right box represents the strategy profile $\langle H, T \rangle$**

- **Deterministic strategies (without random elements) are called pure strategies.**
  - **In this example, H and T are both pure strategies.**
  - **The strategy space is the set of pure strategies.**
  - **Strategies that employ more than one pure strategy with positive probability are called mixed strategies.**
  - **For now, I will talk about only pure strategies**

Coordination Game

- **Again, both players put pennies on the table simultaneously, each choosing either *H* or *T*.**
- **But here, the payoffs are different from those of the previous example.**
- **If Vanesa and Miguel can coordinate and put down the same thing, they both win, otherwise they both get nothing.**
- **This is not a zero-sum game.**

Vanesa \ Miguel	H	T
H	2, *1*	0, *0*
T	0, *0*	1, *2*

- **Can we use game theory to predict the outcomes of strategic interaction?**
- **What strategies should we expect Vanesa and Miguel to adopt in their coordination game?**
- **Unfortunately, game theory has a number of different "solution concepts" that sometimes predict different outcomes.**
- **The most commonly used solution concept is the Nash Equilibrium, named after John Nash. Sometimes we call it simply an equilibrium.**

> DEFINITION 8.2. ***A* Nash equilibrium*, is a strategy profile in which each player has chosen the strategy that is the best response to the strategies of the other players.***

- **Equivalently, an equilibrium is a strategy profile in which no player would want to deviate from his chosen strategy.**

		Miguel	
		H	**T**
VANESA	**H**	2, *1*	0, *0*
	T	0, *0*	1, *2*

- $\langle H, H \rangle$ **is an equilibrium, because** H **is Vanesa's best response to Miguel's** H **and Miguel's** H **is his best response to Vanesa's** H**.**
- **Also,** $\langle T, T \rangle$ **is an equilibrium for the same reason.**
- **There is an equilibrium composed of mixed stragies as well, but we will not discuss that yet.**

		MIGUEL	
		H	**T**
VANESA	**H**	2, *1*	0, *0*
	T	0, *0*	1, *2*

- $\langle H, T \rangle$ **is NOT an equilibrium because given Miguel's** T**, Vanesa would want to deviate to** T **herself.**
- **Of course, Vanesa prefers the equilibrium** $\langle H, H \rangle$ **and Miguel prefers** $\langle T, T \rangle$**.**
- **But the Nash concept does not choose between them.**

- **Note: in this game, each player moves without knowing what the other player will do.**
  - **The concept of Nash equilibrium is retrospective.**
  - **In equilibrium, each player is happy he has chosen the strategy that he has chosen.**
  - **If there are regrets, then the strategy profile is not an equilibrium.**
- **If, as an economist, you rely on Nash equilibria, then you must believe that people will choose strategies that later turn out to be correct, even though they couldn't have known that at the time.**

- **But if players are in an equilibrium historically, it makes sense to think that they will stay in that equilibrium if they play the same game again. Here, the equilibrium serves as a focal point.**
- **There are some very special cases in which a group of completely rational players (members of the species *homo economicus*) would be able to figure out in advance how play in a Nash equilibrium.**
- **The are other equilibrium concepts (e.g. subgame-perfect equilibrium), for which, in a broad class of cases, it is more natural to conclude that a group of rational agents will play the equilibrium.**

8.2 Games with Sequential Moves

- **Suppose we have a game in which Vanesa puts down her penny first.**
- **Miguel sees her move, and knows whether it is heads or tails.**
- **Then Miguel puts down his penny.**

		Miguel	
		H	**T**
Vanesa	**H**	2, *1*	0, *0*
	T	0, *0*	1, *2*

- **What would happen in this game?**
  - **The answer is clear.**
  - **Vanesa will choose *H* and force Miguel to choose *H* also.**
  - **Is this a Nash equilibrium? The only Nash equilibrium?**
  - **We must model the strategies properly.**

- From what strategies does Vanesa choose?
  - H or T as before.
- What about Miguel? What are his strategy choices?
  - H and T are not strategies for Miguel!
  - A strategy must tell you what to do in every situation that you know about.
  - Miguel knows what Vanesa has done.
  - So his strategy must reflect his knowledge of what she has done.

- Miguel's possible strategy choices are the following (with my own nicknames):
  - Always H. If I see H, I will choose H;
    if I see T, I will choose H.
  - Copy. If I see H, I will choose H;
    if I see T, I will choose T.
  - Opposite. If I see H, I will choose T;
    if I see T, I will choose H.
  - Always T. If I see H, I will choose T;
    if I see T, I will choose T
- The set of these strategies is Miguel's strategy space.

- This normal-form game can be represented as follows:

MIGUEL

VANESA	Always H	Copy	Opposite	Always T
H	1 2	1 2	0 0	0 0
T	0 0	2 1	0 0	2 1

- But what are the Nash equilibria?

MIGUEL

VANESA	Always H	Copy	Opposite	Always T
H	* 1 2	* 1 2	0 0	0 0
T	0 0	2 1	0 0	* 2 1

- There are 3 Nash equilibria
  - $\langle H,$ Always H$\rangle$
  - $\langle H,$ Copy$\rangle$
  - $\langle T,$ Always T$\rangle$.

		Miguel			
		Always H	Copy	Opposite	Always T
Vanesa	H	* *1* 2	* *1* 2	*0* 0	*0* 0
	T	*0* 0	*2* 1	*0* 0	* *2* 1

- **If Miguel chooses Always T, couldn't he force Vanesa to choose T?**
- **Perhaps not: the strategy Always T is not time consistent.**
  - **It represents an "idle threat."**
  - **Vanesa may think that if she chooses H, Miguel would deviate from his chosen strategy and choose H also.**

- **Nash equilibrium does not capture the idea of time inconsistency.**
- **We need a different concept: subgame-perfect equilibrium (which is a special case of Nash equilibrium).**
- **And this requires a different game structure: the extensive-form game.**

8.3 Noncooperative Games: Normal and Extensive Forms

- Game theory is not an important tool of economists when dealing with competitive markets.
  - Many small agents
  - Each agent is a price-taker.
  - Each agent uses prices for decision making, but no other external information.
  - Each agent has a very small influence on prices, an influence which is ignored in the model.
  - Each agent's strategy has no significant effect on other agents.
  - There cannot be strategic interaction except in an aggregate sense.

- Game theory is important in models with a small number of agents.
- Game theory provides solution concepts for economic (and other) models.
  - Answers the question: What will agents do?
  - But the answers are not unique.
    - One solution concept may offer many possible solutions—or no solutions.
    - Different game-theoretic solution concepts may give different sets of solutions.
    - Economists do not always agree on which concepts are best and which solution is most likely to correspond to real world behavior.
  - Experimental evidence suggests that agents frequently deviate from the various solution concepts.

- **A game is a formal mathematical construction.**
- **Uses a special terminology.**
- **There are several types of games, each with a different construction.**
- **Normal-form games have simple structures, but they are poor descriptions of the agents' choices and actions.**
- **Extensive-form games use a tree structure with nodes and branches.**
  - **Nodes describe agents with defined histories in the game.**
  - **Different branches of the tree describe the results of different choice of the players.**

DEFINITION 8.3. ***A summary of game-theory terminology:***

- ***the definitions below will be repeated as we present examples***
- ***more definitions will be added***

1. **Player:** ***an agent who makes choices in order achieve certain objectives.***
2. **Nature:** ***an agent who makes choices in a mechanical or random way without any objectives in mind.***
   - ***Nature is not counted as a player in the game, but nature may be labelled Player 0.***
   - ***Game theory with only one player and nature is called*** **decision theory.**

3. **Action or Move:** *a choice that a player can make.*

4. **Play:** *a complete sequence of actions taken by all players and nature up to the end of the game.*

5. **Node:** *a member of the sequence of actions. At each node, one of the players or nature must choose an action.*

6. **History:** *a sequence of moves by all players and nature up to a given stage of the game. Each node has one history and vice versa.*

7. **Stage:** *sometimes it is convenient to decompose a game into a sequence of time periods. Each time period is then called a "stage."*

8. **Games of perfect information:** *Games with a finite number of possible nodes such that at every stage, the player knows the history of the game.*

9. **Games of imperfect information:** *Games in which a player at some stages does not have exact knowledge of the history.*

10. **Information Set:** *Given a player's information, the set of possible histories at a given stage of the game.*

    - *In games of perfect information, every information set contains only one history (because the player always know the history).*
    - *In games of imperfect information, some information sets may contain more than one history.*
        - *These are the histories that given his information, a player thinks may have occurred.*
        - *In a given play of the game, each player faces a sequence of information sets.*

11. **Rule:** *describes which actions a player can take under various circumstances.*

12. **Strategy:** *a function that maps each information set that a player might face to an action permitted by the rules.*

    - *A strategy must specify what a player will do in every* **possible** *circumstance.*

13. **Strategy space:** *the set of all strategies that a player may choose under the rules.*

    - *If you know each players strategy space, then you know the rules.*

14. **Strategy profile:** *a vector of strategies, one for each player.*

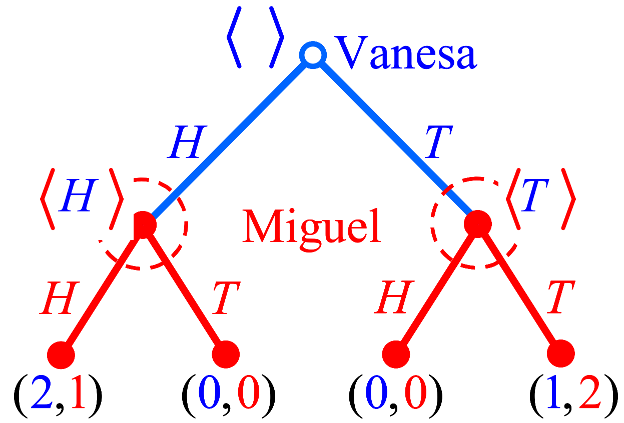
15. **Payoff:** *the utility or value that a player receives at the end of the game.*

16. **Payoff profile:** *a vector that describes the payoffs that go to each player.*

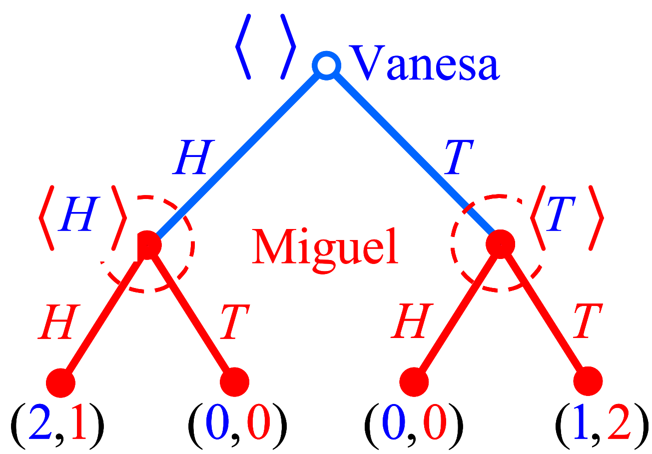
17. **Payoff function:** *a function that maps each strategy profile to a payoff profile.*

8.4 Extensive-Form Games

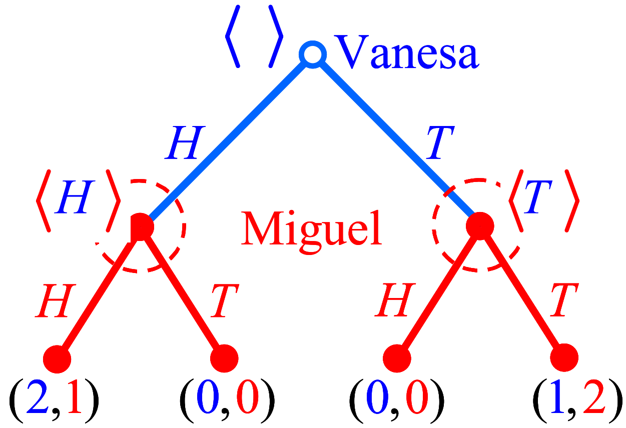
- Here is the coordination game in extensive form.



- The **initial node** (small empty circle) belongs to Vanesa
- Vanesa moves first.
- The branches represent her choice of actions, H or T.



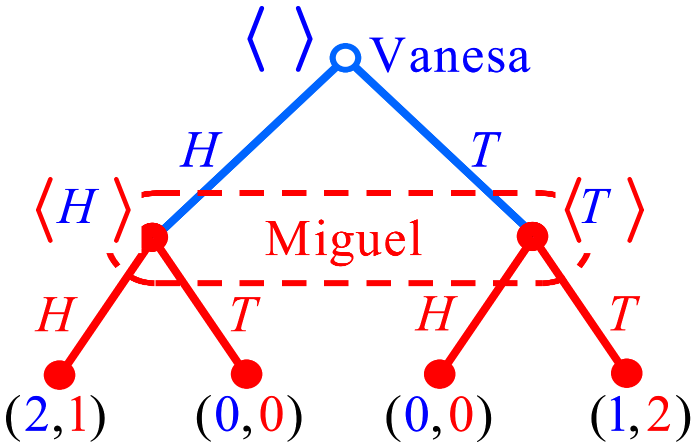
- After Vanesa chooses her action, it is Miguel's turn to move.
- Miguel's turn takes place at one of two possible nodes (small solid circles).
  - Each node corresponds to one history of the game up to that point.
  - Miguel's turn may be played at the node $\langle H \rangle$, if Vanesa chose action H.
  - Or it may be played at $\langle T \rangle$, if Vanesa chose T.



- **Each node belongs to an information set, which represents the player's knowledge about the history of the game.**
  - **A player knows which information set his node belongs to,**
  - **but he doesn't know which of the nodes in the information set is his node.**
  - **In this case, each of Miguel's nodes has it own information set (the dashed circles)...**
  - **because Miguel knows what action Vanesa chose.**

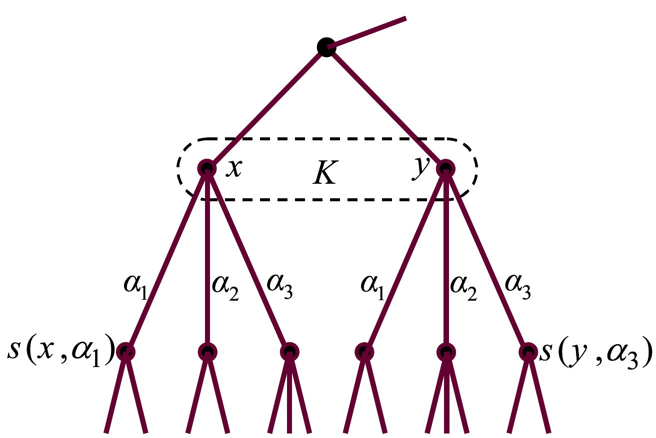
- **At each information set, Miguel can choose either *H* or *T*.**
- **Each of Miguel's choices goes to a terminal node.**
- **At a terminal node, the game ends and payoffs are distributed.**

- In this game, both of Miguel's nodes are in the same information set.



- This means that when Miguel moves, he doesn't know what Vanesa has done.
- When she moves, Vanesa doesn't know what Miguel will do.
- Therefore this game is equivalent to a simultaneous-move game.

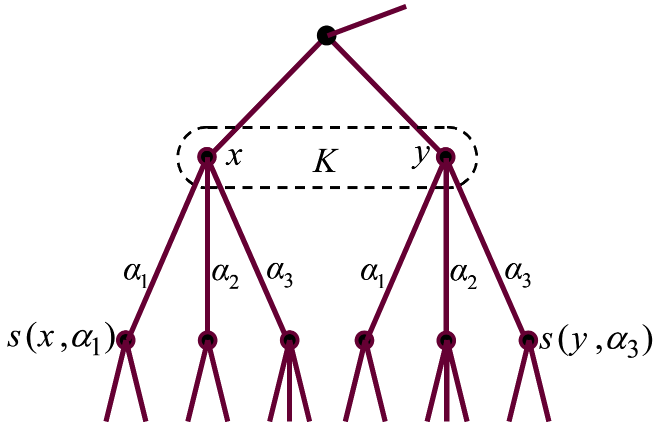
8.5 Formal Definition: Extensive-Form Game



- The game structure is represented by a tree.
- The tree is composed of nodes and branches.
  - Each node represents a state of the game (what has happened so far)

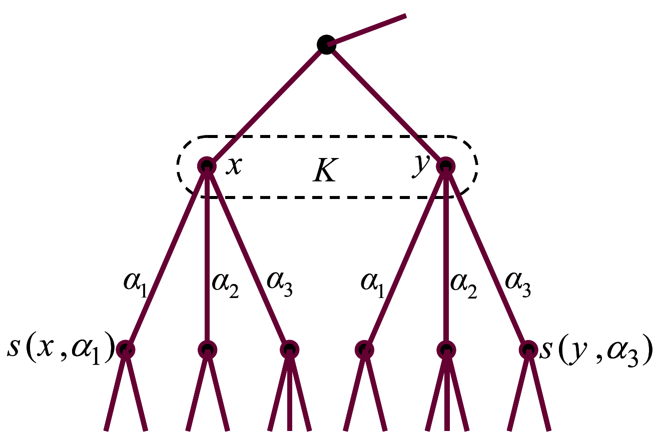
- A set of possible actions $c(x)$ is associated with each node x.
- Each action $\alpha \in c(x)$ defines a branch that begins at x and ends in a **successor** node, $s(x, \alpha)$.
- If z is a successor of x, then x is called the **predecessor** of z, and we write $x = p(z)$.
- The game starts with an **initial** node x_0, which has no predecessors.
- Every other node x has exactly one predecessor, $p(x)$.
- The sequence of nodes and actions beginning with x_0 and leading to x is called the **history** of the game at x.
  - Because $p(x)$ is unique, every node has exactly one history.

- Each node can have any number of successors.
- The number of successors is the same as the number of possible actions associated with the node.
- A node with no successors is called a **terminal** node.
- When a terminal node is reached, the game ends.
- Each terminal node is associated with a payoff vector.



- **Every node x that is not a terminal node belongs to an information set K.**
- **If $x, y \in K$ then x and y must have the same possible actions: i.e.**

$$c(x) = c(y) \equiv c(K).$$



- **Each information set K is assigned to**
  - **either a player i who must choose an action $\alpha \in c(K)$**
  - **or nature, who chooses among the actions in $c(K)$ according to a probability distribution..**
- **Nodes in players' information sets are called decision nodes.**

- The intuition is this: the player knows the identity of his information set but not necessarily his node or the history.
  - If there is only one node in the information set, then the player knows the node and the history of the game up to that point.
  - But if there are several nodes, the player does not know at which node (or at which history) he is.
- A (pure) strategy for Player i is a function σ_i that for each of i's information sets K, selects an action $\sigma_i(K)$ with $\sigma_i(K) \in c(K)$.
- The strategy space for player i is the set of all such σ_i.

DEFINITION 8.4. *A* subgame *of an extensive-form game is formed by a part of the game tree with the following properties:*

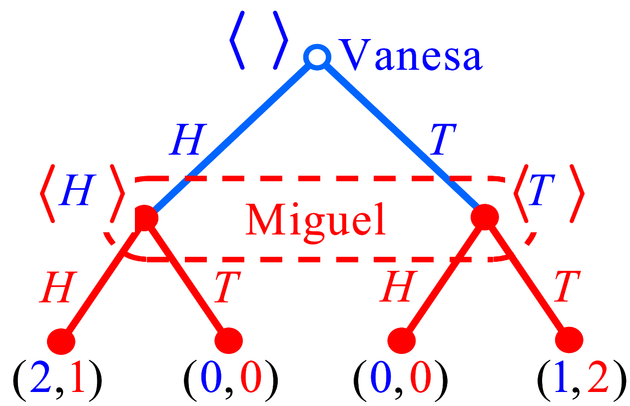
i). it begins with a decision node that is alone in its information set;

ii). it contains all the successor nodes, their information sets, and connecting branches, with the same permitted actions as in the original game;

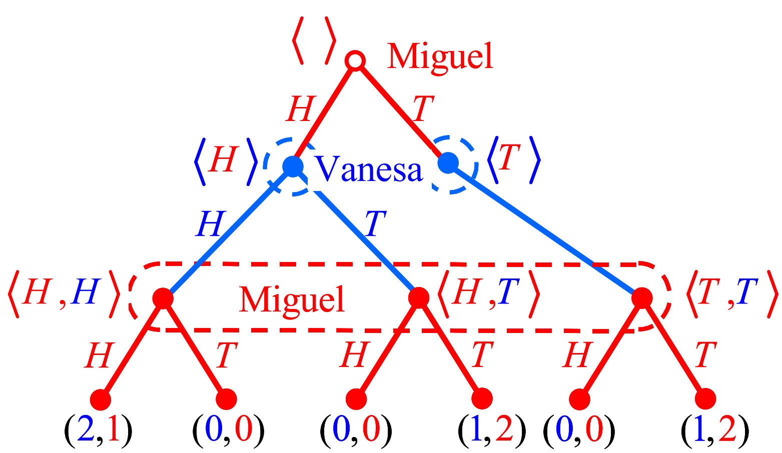
iii). if a subgame contains one node in an information set, it must contain all the nodes in that information set.

Every subgame that is not the original game is called a proper *subgame.*

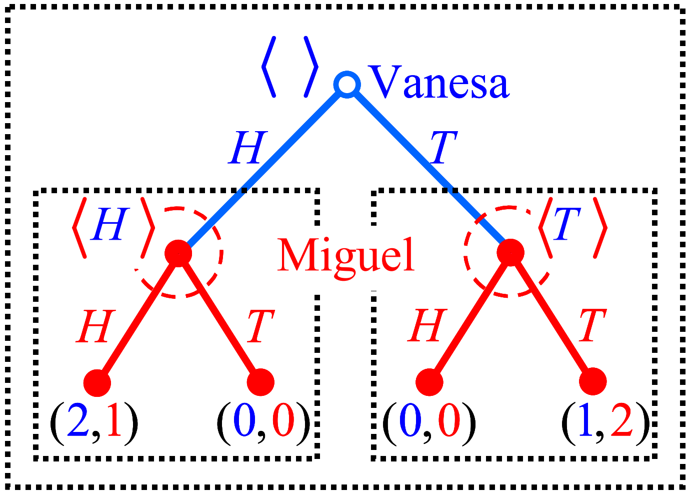
- **Every subgame begins with full information about the history of the entire game up to that time.**
- **In the simultaneous-move coordination game, the entire game is the only subgame, because only the initial node is contained by itself in an information set.**
- **Miguel's nodes do not form subgames, because Miguel does not know the history of the game at either of his nodes.**



- **In the game below, neither of Vanesa's nodes form subgames, because each potential subgame would contain only part of an information set**

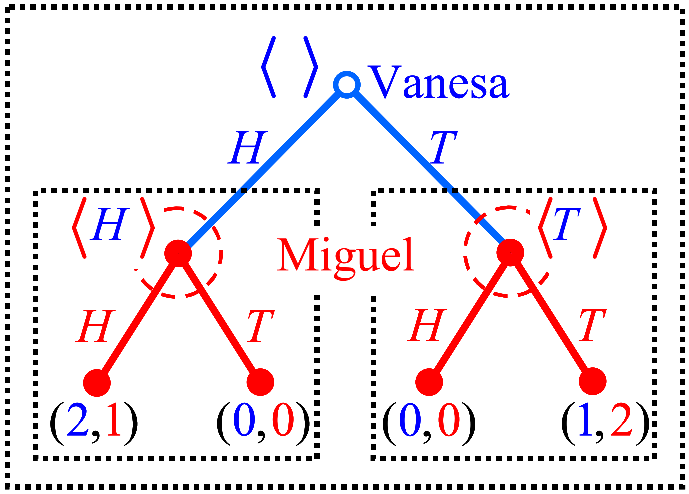


- **In the sequential-move coordination game has 3 subgames: Vanesa's initial node and each of Miguel's nodes begin subgames.**
- **Each node can be labeled by its history.**



- **Each strategy for the entire game induces a strategy on every subgame.**

Definition 8.5. ***Suppose n is the number of players. A strategy profile $\langle \sigma_1, \ldots, \sigma_n \rangle$ is a Nash equilibrium of the extensive-form game if each player's strategy σ_i in $\langle \sigma_1, \ldots, \sigma_n \rangle$ is a best response to the other players' strategies $\sigma_{\neq i}$, in $\langle \sigma_1, \ldots, \sigma_n \rangle$.***



- **In the coordination game, Vanesa has two strategies:**

i). H: $\langle\rangle \rightarrow \boldsymbol{H}$

ii). T: $\langle\rangle \rightarrow \boldsymbol{T}$

- **Miguel has four:**

i). Always H: $\{\langle H\rangle \rightarrow H,\ \langle T\rangle \rightarrow H\}$

ii). Copy: $\{\langle H\rangle \rightarrow H,\ \langle T\rangle \rightarrow T\}$

iii). Opposite: $\{\langle H\rangle \rightarrow T,\ \langle T\rangle \rightarrow H\}$

iv). Always T: $\{\langle H\rangle \rightarrow T,\ \langle T\rangle \rightarrow T\}$

- **As in the normal-form game, there are three Nash equilibria:**

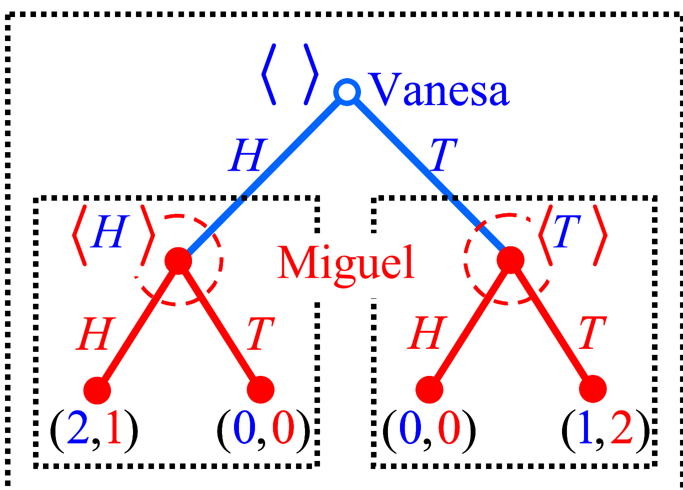
i). ⟨H, Always H⟩.

ii). ⟨H, Copy⟩.

iii). ⟨T, Always T⟩.

Definition 8.6. *A strategy profile $\langle\sigma_1, \ldots, \sigma_n\rangle$ is a subgame-perfect equilibrium (SPE) of the extensive-form game if the strategies induced by each σ_i are a best response to the other induced strategies $(\sigma_{\neq i})$ in every subgame.*

- **A Nash equilibrium requires each strategy to be a best response in every subgame played in the equilibrium.**
- **A subgame-perfect equilibrium requires each strategy to be a best response in every subgame, whether or not it is played in equilibrium.**



- **Miguel has two subgames: $\langle H\rangle$ and $\langle T\rangle$.**
- **In $\langle H\rangle$, Miguel's best response is either Always H or Copy.**
  - **Both of these strategies induce the strategy H in the subgame.**
- **In $\langle T\rangle$, Miguel's best response is either Copy or Always T.**
  - **Both of these strategies induce the strategy T in the subgame.**

- **Only Copy is a best response in both of Miguel's subgames.**
  - **Only Copy can be part of a subgame-perfect equilibrium.**
- **Vanesa's best response to Copy is H.**
- **$\langle$H, Copy$\rangle$ is a unique subgame-perfect equilibrium.**

- **The method of solution used above is called backward induction.**
- **The principle of backward induction is this:**
  - **First find the SPE of every proper subgame.**
  - **Then find the SPE of the entire game.**
  - **Only strategies consistent with an SPE of every proper subgame will be consistent with an SPE of the entire game.**
  - **Note: to find the SPE's of a proper subgame, you will first have to find the SPE's of its proper subgames.**

8.6 Bargaining in 2-Player Games

- Bargaining is voluntary: each party is free to walk away with what she has.
- The total value obtainable with cooperation will be at least as much as the sum of what the parties each have without cooperation. Why?
- The difference between the total values with and without cooperation is called the **surplus.**
- The parties may bargain to divide the surplus between them.
- If they fail, they will have only their initial values (sometimes called "threat values")

The Ultimatum Game

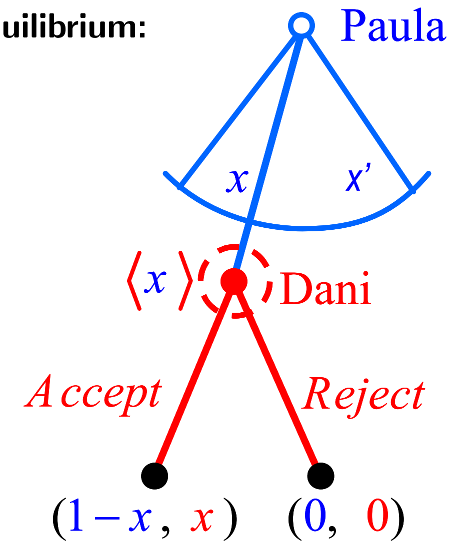
- Suppose your mother gave $1000 to your older sister as a present for both of you.
- And suppose your sister offered you the proportion x.
- You can accept your sister's offer, or you can reject it.
- But if you reject the offer, you and your sister both receive 0, because Mom takes the money and burns it!
- How much is the surplus in this game?

- **Would you reject an offer of**
  - **40 percent of the money?**
  - **30 percent of the money?**
  - **10 percent of the money?**
  - **5 percent of the money?**
  - **1 percent of the money?**
- **Obviously, there is no correct answer. So please tell the truth.**

A Formal Model of the Ultimatum Game:

- **Mamá gives Paula $1000 to divide between her younger brother, Dani, and herself.**
- **Paula must offer Dani a fraction x of the money, where $0 \leq x \leq 1$.**
- **Dani can accept or reject Paula's offer.**
- **If Dani accepts the offer,**
  - **Dani receives x**
  - **Paula receives $1 - x$**
- **If Dani rejects Paula's offer, both receive 0, and Mamá burns the money.**

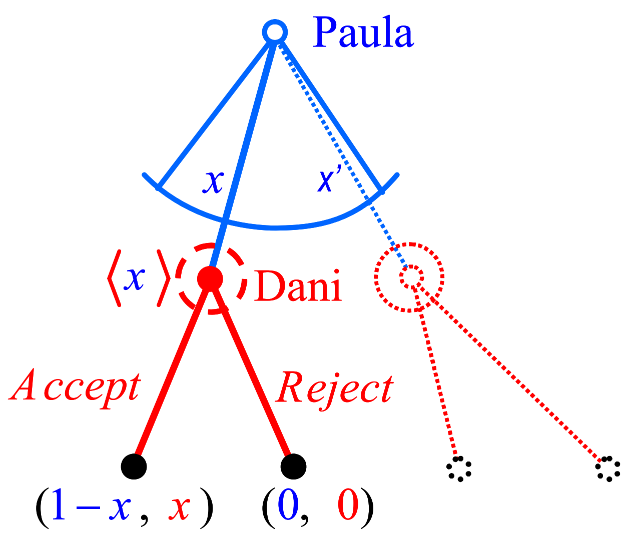
Subgame perfect equilibrium:



- Dani's strategy space is the set of functions $\{f : [0, 1] \to \{A, R\}\}$.
  - Example of one of Dani's possible strategies:

$$f(x) = \begin{cases} A \text{ for } x \text{ rational} \\ R \text{ for } x \text{ irrational} \end{cases}.$$

- What are Dani's subgames?
- Dani has a subgame $\langle x \rangle$ for each $x \in [0, 1]$.



- What strategies in the entire game are Dani's best responses in all his subgames of the form $\langle x \rangle$?

- either

$$f_a(x) = \begin{cases} A \text{ for } x > 0 \\ R \text{ for } x = 0 \end{cases}$$

- or

$$f_b(x) = A$$

- Paula's strategy space: $\{x \mid 0 \leq x \leq 1\}$
- What is Paula's best response to $f_a(\cdot)$?
  - There is none! Why not?
- What is Paula's best response to $f_b(\cdot)$?
  - $x = 0$.
- Unique SPE: $\langle x = 0, \quad f_b(\cdot)\rangle$
- Does this agree with what people do?

Directed Ultimatum Game:

- Suppose your mother gives $1000 to your older sister.
- Mom tells your sister that she must offer you between 0 and 5 percent of the money, but not more than 5 percent.
- And suppose your sister offers you exactly 5 percent.
- Would you reject that offer?

- **Why does behavior in the directed ultimatum game differ from behavior in the ordinary ultimate game?**
  - **How does the structure of the game differ from that of ordinary ultimatum game?**
  - **What are the effects of that difference?**
  - **Is the difference in behavior a question of motives?**

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8.7 Two-thirds-of-the-mean Game

(from Jörgen Weibull's minicourse)

- There are n players.
- Each player i simultaneously chooses a number $x_i \in [0, 100]$
- The players whose numbers are closest to $2/3$ the mean of all selected numbers are the winners.
  - i.e, the winners are

$$i^* \in \underset{i}{\operatorname{argmin}} \left\{ \left| x_i - \frac{2}{3}\bar{x} \right| \right\}$$

  where

$$\bar{x} \equiv \sum_{j=1}^{n} x_j$$

- The winners receive a total of \$1000 (if there is more than one winner, the prize is divided equally between them.)

- What is the strategy space in this game?
  - $[0, 100]$
- Find a Nash equilibrium. Is it unique?
  - Do this formally as an exercise.
- Can we predict that members of the species *homo economicus* will play an equilibrium strategy?
  - Yes. Do this formally as an exercise.