6. Production

DEFINITION 6.1. ***Production** is the **transformation** of a vector of commodities into a different vector of commodities.*

- **Production can be organized in either of two environments:**
  - **In hierarchical environments: within firms**
  - **In market environments: between firms**
  - **Important questions without clear answers:**
    - **Why do firms exist?**
    - **What determines their boundaries?**
- **Economists understand more about the market environment than about the hierarchical environment.**

- **We use a simple model of production within the firm in order to generate supply curves.**
- **Supply curves are analyzed within the market environment.**

DEFINITION 6.2. *A **production activity** or y is a vector of inputs (negative) and outputs (positive) that describes a transformation of some goods into others.*

DEFINITION 6.3. *A **production set** $Y \subset \mathbb{R}^n$ is the set of all activities that are physically possible to implement.*

- Suppose the commodities are given by:

$$\begin{bmatrix} \text{ice cream} \\ \text{cheese} \\ \text{milk} \\ \text{sugar} \end{bmatrix} \text{, and suppose } y = \begin{bmatrix} 2 \\ 1 \\ -6 \\ -8 \end{bmatrix} \in Y.$$

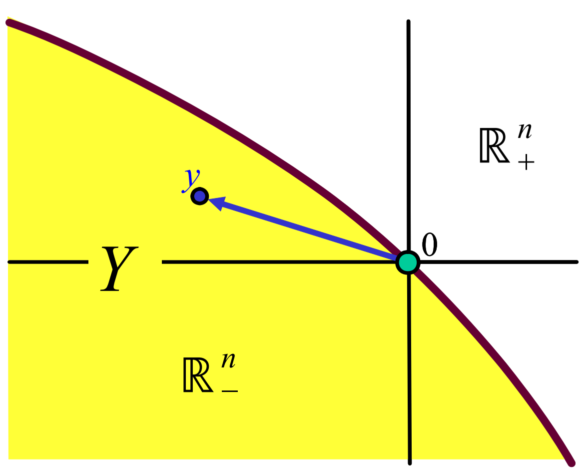
- Then it is physically possible to convert 6 units of milk and 8 units of sugar into 2 units of ice cream and 1 unit of cheese.
- Real production in modern economies is extraordinarily complicated.
- Vectors of inputs and outputs would be very large.

Think about the economics department at BU:

- Inputs:
  - Professors, each one with different skills, training.
  - Staff, again, each with different skills, training.
  - Structure (this building) is specialized.
  - Equipment, not too important.
- Outputs:
  - Undergraduate
  - Masters' training
  - Doctors' Training
  - Research of doctoral students
  - Research of professors

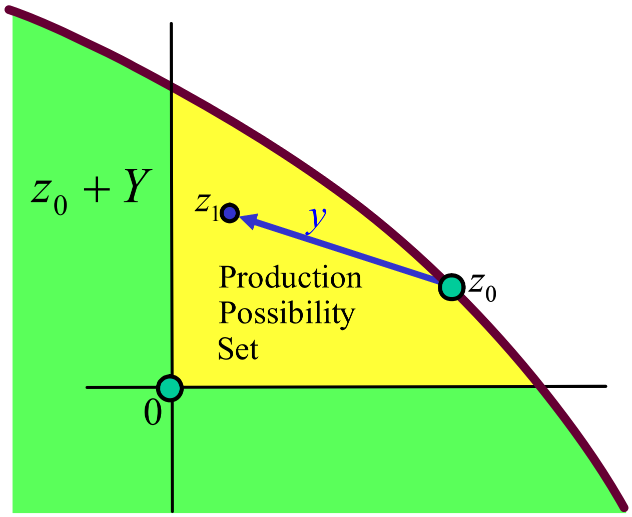
- **How could you represent in a vector what we do in the economics department?**
- **How could you represent all the *possible* production activities in the department, including those that have not been used here?**

- **The economists' model is extremely abstract!**
- **The set Y, below, is the production set.**
- **Y does not contain any points in $\mathbb{R}^n_+$ except 0. Why not?**
- **Think of the an activity vector $y \in Y$ as an arrow from the origin that indicates a decrease in the stocks of inputs and an increase in the stocks of outputs.**



- If a vector of initial resources, z_0, is added to each point in the set Y, a production possibility set (PPS) $Y + z_0 = \{y + z_0 \mid y \in Y\}$ is obtained.

- In the diagram below, the activity y transforms the initial resources z_0 to the the vector of resources z_1.



- The PPS lies in $\mathbb{R}^n_+$.

- Contains all the combinations of commodities that can be obtained in the economy.

- z_0 corresponds to 0 in the original production set.

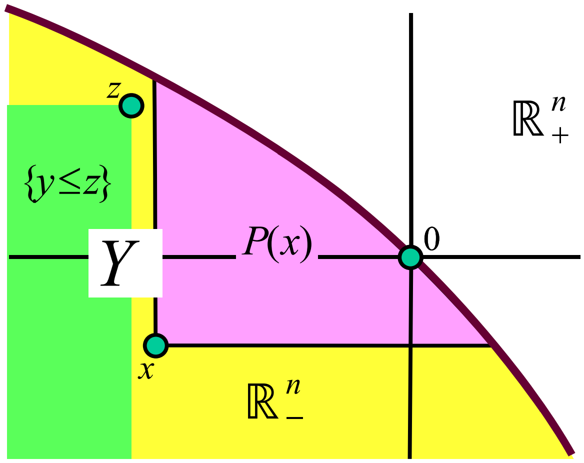
- z_0 is the vector obtained if no production occurs.

The following are commonly assumed to be properties of Y:

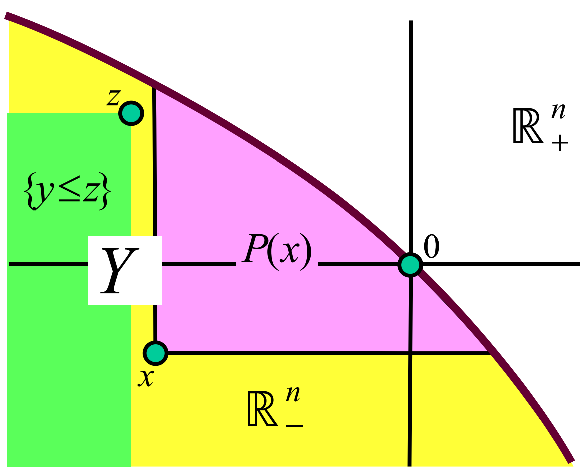
Assumption 6.1. ***Continuity in production: Y is closed.***

Assumption 6.2. ***Inactivity is possible, but there are no free products: $Y \cap \mathbb{R}_+ = \{0\}$.***

- ***You can't get outputs without using inputs.***
- ***But you can do nothing, so $0 \in Y$***



- **For any $x \in Y$, we define the higher-productivity set, $P(x)$, by $P(x) = \{y \mid y \in Y, \quad y \geq x\}$.**
  - **As compared with x, all the points in $P(x)$ (except for x itself) either**
    - **use smaller amounts of some inputs**
    - **and/or yield greater amounts of some outputs**



Assumption 6.3. ***Limited productivity:***

- ***Given any $x \in Y$, the higher-productivity set $P(x)$ is bounded. [Missing from M-C, but should be there.]***

Assumption 6.4. ***Free disposability:***

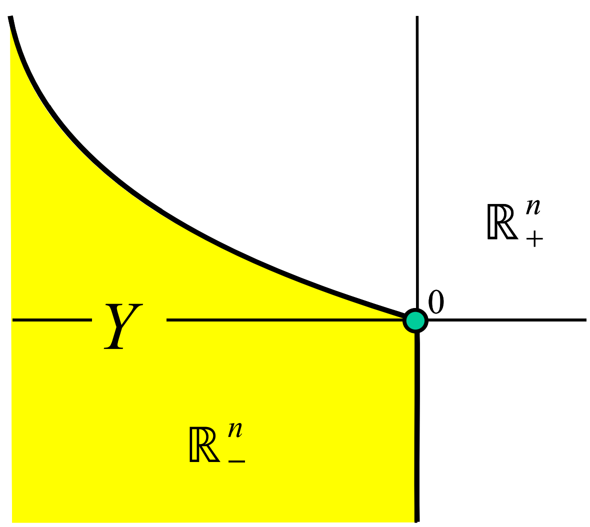
- ***If $z \in Y$, and $y \leq z$, then $y \in Y$.***
- ***It is always possible to waste inputs.***
- ***See graph.***

Assumption 6.5. ***Irreversibility:***

- ***if $y \in Y$, then $-y \notin Y$.***
  - ***Can't turn process around, produce inputs from outputs.***
  - ***Technical assumption: Labor, energy used in production, cannot be fully recovered.***

Assumption 6.6. ***Convexity: Y is a convex set.***

- ***Strong assumption.***
- ***Rules out economies of scale.***
- ***Graph below violates convexity, shows increasing returns.***

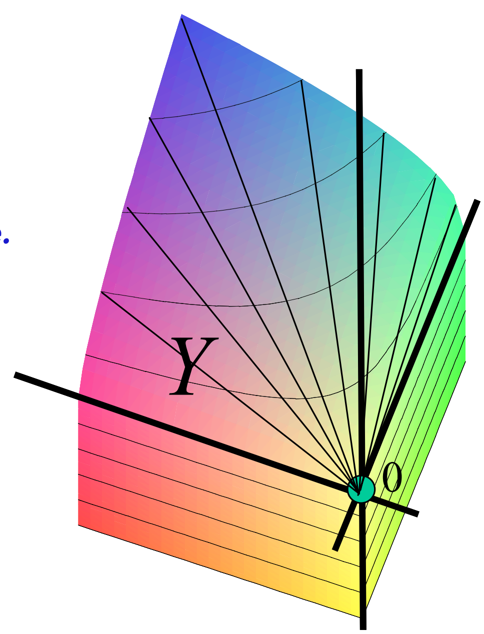


ASSUMPTION 6.7. ***Additivity:***

- ***If $x, y \in Y$, then $x + y \in Y$.***
  - ***Strong assumption.***
  - ***Rules out diseconomies of scale.***

ASSUMPTION 6.8. ***Y is a convex cone: If $x, y \in Y$, then for $\alpha, \beta > 0, \quad \alpha x + \beta y \in Y$.***

- ***Very strong assumption.***
- ***Implies additivity.***
- ***Mathematically, Y is closed under linear combinations.***
- ***Implies constant returns to scale.***



- **Replication argument:**
  - **Production sets have additivity property between two different production activities if they don't interfere with one another.**
  - **Production activities describe possible transformations, not resource constraints.**
  - **If a production activity exits, then it ought to be possible to replicate (duplicate) it exactly, as many time as desired.**
  - $\Longrightarrow$ **If** $y \in Y$**, then for any integer** n**,** $ny \in Y$.
  - $\Longrightarrow$ Y **has additivity.**

PROPOSITION 6.1. ***If*** $0 \in Y$***, and*** Y ***is convex and has additivity, then*** Y ***is a convex cone.***

PROOF. **We have:**

- **By convexity, if** $y \in Y$**, then for** $\alpha \in [0, 1]$**,** $\alpha y = \alpha y + (1 - \alpha)0 \in Y$.
- **For** $\alpha > 1$**, find integer** $n \geq \alpha$.
  - **By additivity,** $ny \in Y$
  - **By convexity** $\frac{\alpha}{n} ny \in Y$.
  - $\Longrightarrow \alpha y \in Y$.
- **If** $x, y \in Y$**, then for** $\alpha, \beta > 0$**,** $\alpha x, \beta y \in Y$.
- **By additivity:** $\alpha x + \beta y \in Y$.

■

- Suppose $Y \subset \mathbb{R}^n$ is convex but not a convex cone.
- By assuming that there is a "hidden input," z, we can create Y' that is a convex cone.
- Assume that each $y \in Y$, uses also one unit of z.
- For each $y \in Y$, set $y' = \begin{bmatrix} y \\ -1 \end{bmatrix}$.
- Define $Y' = \left\{ \alpha \begin{bmatrix} y \\ -1 \end{bmatrix} \middle| \alpha \geq 0, \quad y \in Y \right\} \subset \mathbb{R}^{n+1}$.
- Y' is a convex cone.

DEFINITION 6.4. *If there is only one output, q, and a number of inputs, denoted by the vector z, then if Y is a closed set, has free disposability with $0 \in Y$, we define a* production function f *by*

$$f(z) = \max \left\{ q \;\middle|\; \begin{bmatrix} q \\ -z \end{bmatrix} \in Y \right\}$$

DEFINITION 6.5. *A production set Y exhibits* **increasing returns to scale** *if for any $y \in Y$ and $\alpha > 1$, there is a point $y' > ay$ with $y' \in Y$.*

- *If you increase the scale of production, then it is possible to increase productivity.*

DEFINITION 6.6. *A production set Y exhibits* **decreasing returns to scale** *if for any $y \in Y$ and $\alpha \in (0, 1)$, there is a point $y' > ay$ with $y' \in Y$.*

- *If you decrease the scale of production, then it is possible to increase productivity.*

DEFINITION 6.7. *A production set Y exhibits* **nonincreasing returns to scale** *if for any $y \in Y$ and $\alpha \in (0, 1)$, $ay \in Y$.*

- *It is always possible to decrease the scale of production.*

DEFINITION 6.8. *A production set Y exhibits* **nondecreasing returns to scale** *if for any $y \in Y$ and $\alpha > 1$, $ay \in Y$.*

- *It is always possible to increase the scale of production.*

DEFINITION 6.9. *A production set Y exhibits* **constant returns to scale** *if for any $y \in Y$ and $\alpha > 0$, $ay \in Y$.*

- *Nonincreasing and nondecreasing returns to scale.*

Proposition 6.2. *Let $f(x)$ with $x \in \mathbb{R}_+^n$ be a concave production function. Define a hidden input $z \in \mathbb{R}_+$ and a production function*

$$F(X, z) \equiv zf\left(\frac{1}{z}X\right).$$

Then F is concave, has constant returns to scale, and $F(x, 1) \equiv f(x)$.

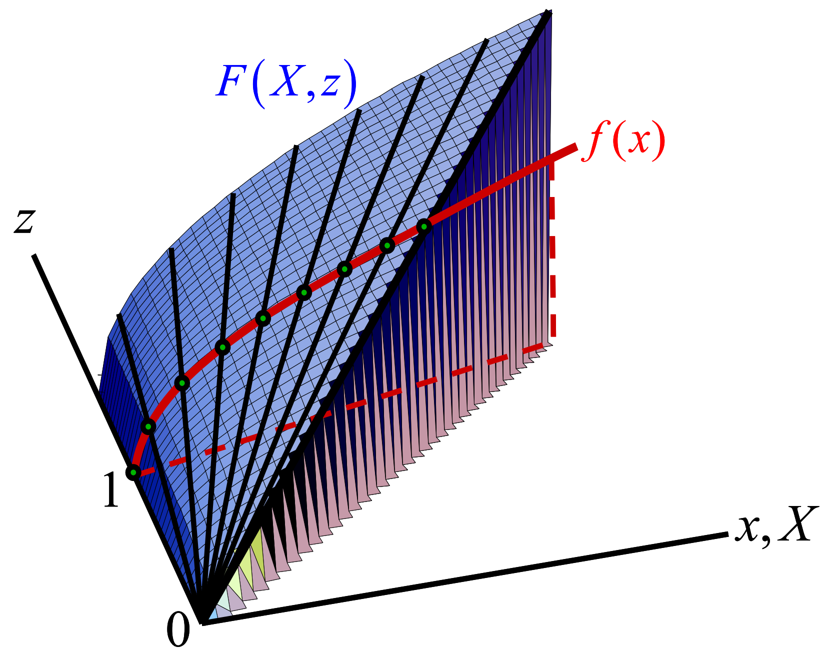
- The quantity

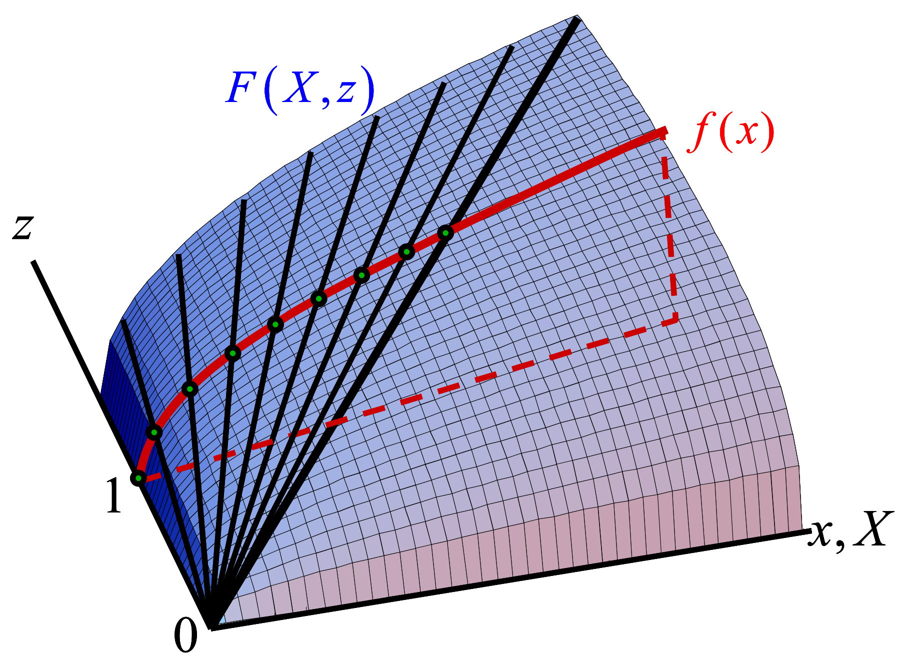
$$x \equiv \frac{1}{z}X$$

  should be interpreted as the quantity of X per unit of the hidden input.

- If you think of the hidden input z as management, this says that each separate unit of managment can operate the production process described by $f(x)$.
- Therefore, a total of z units of management and $X = z\,x$ units of the original input gives us $zf(x)$ units of output.
- This suggests that if a production function has decreasing returns to scale, it is because of a hidden input that is not increased as the scale of production increases.

- **Once the hidden input is introduced, the new production function is concave and has constant returns to scale.**
- **Equivalently, the set under the graph of the new production function is a convex cone.**





PROOF. We have

$$F(\alpha X, \alpha z) = \alpha z f\left(\frac{1}{\alpha z}\alpha X\right) = \alpha F(X, z),$$

which proves that F has constant returns to scale. In addition, letting $X'' \equiv \alpha X + (1-\alpha) X'$ and $z'' \equiv \alpha z + (1-\alpha) z'$, we have

$$\begin{aligned} F(X'', z'') &\equiv z'' f\left(\frac{1}{z''}X''\right) \equiv z'' f\left(\frac{\alpha}{z''}X + \frac{1-\alpha}{z''}X'\right) \\ &= z'' f\left(\frac{\alpha z}{z''}\frac{1}{z}X + \frac{(1-\alpha) z'}{z''}\frac{1}{z'}X'\right). \end{aligned}$$

And because

$$\frac{\alpha z}{z''} + \frac{(1-\alpha) z'}{z''} = 1,$$

the concavity of f now implies:

$$\begin{aligned} F(X'', z'') &\geq \frac{\alpha z}{z''} z'' f\left(\frac{1}{z}X\right) + \frac{(1-\alpha) z'}{z''} z'' f\left(\frac{1}{z'}X'\right) \\ &\geq \alpha z f\left(\frac{1}{z}X\right) + (1-\alpha) z' f\left(\frac{1}{z'}X'\right) \\ &= \alpha F(X, z) + (1-\alpha) F(X', z'), \end{aligned}$$

which proves that F is concave. ■

6.1 Profit Maximization and Cost Minimization

DEFINITION 6.10. *A firm is a* price-taker *if it assumes that all prices are fixed and that its own actions will have no effect on prices.*

- **This makes sense when a firm is very small compared to the size of the market.**
- **For now, assume that firms are price-takers.**
- **We also assume that there are n goods and that all production sets Y are closed, contain 0, and have free disposability.**
- **Let $\mathbb{R}^n_{++} \equiv \{p \gg 0\}$.**

DEFINITION 6.11. *A firm's* profit-maximization problem *(PMP) is:*

$$\max_y \{py \mid y \in Y\}$$

DEFINITION 6.12. *Suppose the PMP has a solution for every $p \gg 0$. Then, the* profit function *$\pi : \mathbb{R}^n_{++} \to \mathbb{R}$ is given by*

$$\pi(p) \equiv \max_y \{py \mid y \in Y\}$$

and the supply correspondence*, by*

$$y(p) \equiv \operatorname{argmax}_y \{py \mid y \in Y\}.$$

- **Note: y is both the supply of outputs and the derived demand for inputs.**

- Suppose there is one output q and a vector of inputs z, with scalar $p > 0$ the price of the output, and $w \gg 0$ the vector of input prices. Then, the profit function is given by:

$$\pi(p, w) = \max_z \{pf(z) - wz \mid z \geq 0\}$$

- and z, the derived demand for inputs, by

$$z(p, w) = \underset{z}{\operatorname{argmax}} \{pf(z) - wz \mid z \geq 0\}.$$

with the supply of output given by

$$q(p, w) = f(z(p, w)).$$

- From Kuhn-Tucker we know that

$$p\frac{\partial f}{\partial z_j} = w_j \text{ for } z_j(p, w) > 0$$

and

$$p\frac{\partial f}{\partial z_j} \leq w_j \text{ for } z_j(p, w) = 0$$

- This says that the value of the marginal product of an input must equal its price if the input is used and must be less or equal to its price if the input is not used.

PROPOSITION 6.3. *If Y has nondecreasing returns to scale (which includes the case of constant returns to scale), then $\pi(p)$ is not defined for some $p \gg 0$.*

PROOF. Suppose $y \in Y$ with $y \nleq 0$.

- Choose $p \gg 0$ such that $py > 0$.
  - To do this choose p_i small for $y_i < 0$, large for $y_i > 0$.
- For any $\alpha > 1$, $\alpha y \in Y$.
- $p(\alpha y) = \alpha(py) \to \infty$ as $\alpha \to \infty$.
- By definition

$$\pi(p) \equiv \max_y \{py \,|\, y \in Y\},$$

  but py is unbounded for $y \in Y$, so the maximum doesn't exist. ■

- In what follows, we assume that $\pi(p)$ is defined for all $p \gg 0$, and that Y is closed, contains 0, and satisfies free disposal.

PROPOSITION 6.4. *π is homogeneous of degree 1 and convex in prices.*

PROOF. The proof of homogeneity is the same as for the expenditure function. For convexity:

- Suppose $p, p' \gg 0, \alpha \in [0, 1]$ and $p'' = \alpha p + (1 - \alpha)p'$.
- For $y'' \in y(p'')$ we have

$$\begin{aligned} \pi(p'') &= p''y'' \\ &= \alpha p y'' + (1-\alpha)p'y'' \\ &\le \alpha\pi(p) + (1-\alpha)\pi(p'). \text{ [Why?]} \end{aligned}$$

■

- **What is the economic interpretation of the convexity of the profit function?**
  - **The average of profits at two different prices is at least as much as profits at the average prices.**
  - **Profits are higher at extreme prices than at average prices.**

6.2 The separating hyperplane theorem and duality in production.

- **Suppose you have a production set and a point outside the production set.**
- **How can you find a price vector that makes the outside point more profitable than any point in the production set?**
- **The** ***separating hyperplane theorem*** **answers that question.**

Definition 6.13. *The set*

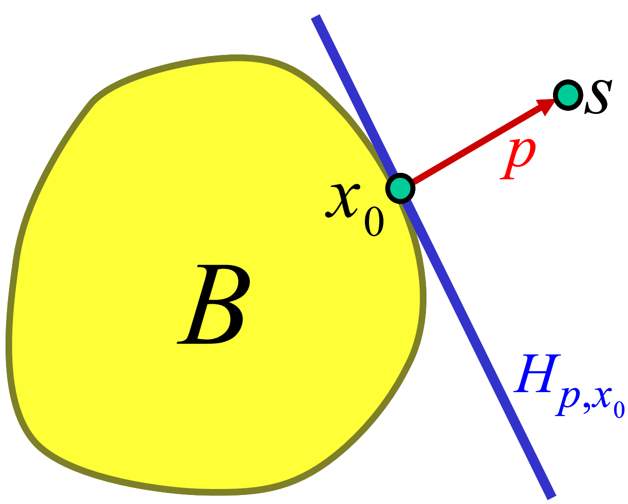
$$H_{p,x_0} \equiv \{x \mid x \text{ and } x_0 \in \mathbb{R}^n,\ p(x - x_0) = 0\}$$

is a hyperplane of dimension $n-1$ orthogonal to p through the point x_0.

- A hyperplane is the union of all lines orthogonal to a fixed vector and passing through a given point.
- The budget frontier is a hyperplane orthogonal to the price vector and passing through a any point x_0, such that $px_0 = w$.

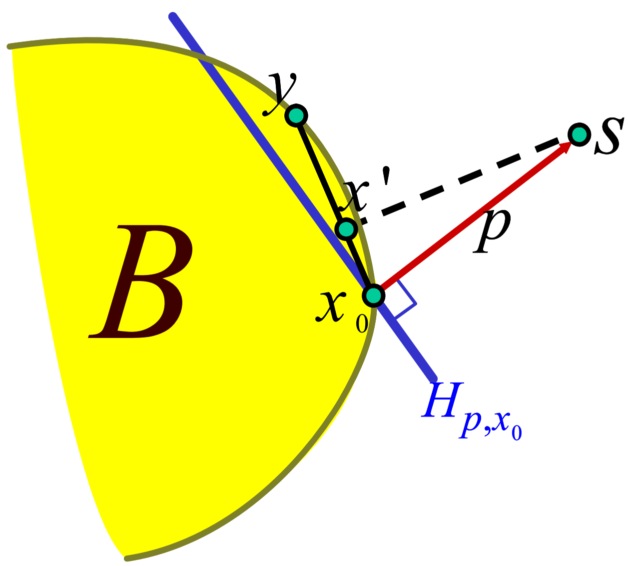
Proposition 6.5. *(Separating Hyperplane Theorem) If $B \subset \mathbb{R}^n$ is convex and closed, and if $s \notin B$, then*

- *there is a hyperplane H of dimension $n-1$ such that B lies on one side of H and s lies on the other;*
- *more precisely, there is some x_0 and $p \neq 0$, such that for all $b \in B$, $pb \leq px_0 < ps$.*



PROOF.

- Let x_0 be the closest point in B to s.
  - [How do we know that a "closest point" exists? See the problem set.]
- set $p \equiv s - x_0$
- so that $0 < p \cdot p = p(s - x_0)$.
- Therefore $px_0 < ps$.
- Suppose for some $y \in B$, $py > px_0$.
- Then $p(y - x_0) > 0$.
- Set $x' = \alpha y + (1 - \alpha)x_0$ where $\alpha \in (0, 1)$.
- By convexity, $x' \in B$.



- We show that for small enough α, x' is closer to s than x_0 is, a contradiction:

$$\begin{aligned} s - x' &= s - \alpha y - (1 - \alpha)x_0 \\ &= (s - x_0) - \alpha(y - x_0) \\ &= p - \alpha(y - x_0) \end{aligned}$$

so that for very small α we have,

$$\begin{aligned}\|s-x'\|^2 &= (s-x')(s-x') \\ &= (s-x_0)(s-x_0) \\ &\quad +\alpha^2(y-x_0)(y-x_0) \\ &\quad -2\alpha p(y-x_0) \\ &< \|s-x_0\|^2 \text{ [Why?]}\end{aligned}$$

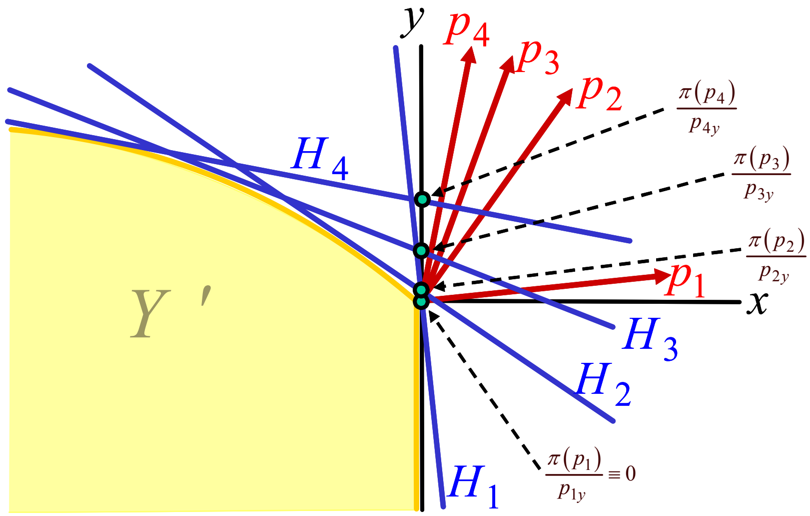
■

DEFINITION 6.14. *The dual production set Y' of a production set Y is given by*

$$Y' = \{y \mid py \le \pi(p) \text{ for all } p \gg 0\},$$

where $\pi(p)$ is the profit function derived from Y.

- The dual production set is bounded by the lower envelope of the hyperplanes defined by the various price vectors and points on the output axis as shown in the drawing below.



- **For many commonly used production functions, the profit function is undefined at price vectors p that contain input prices equal to 0.**

EXAMPLE 6.1. *Consider the production function $q(\ell) = 2\sqrt{\ell}$, where ℓ is the input of labor.*

- *Let the price of output be 1 and let w denote the wage rate, so that $\pi = 2\sqrt{\ell} - w\ell$.*
- *The demand for labor $\ell(w)$ is unbounded at $w = 0$, and $\pi(0)$ is undefined.*
- *But for all $w > 0$, we have $\ell(w) = 1/w^2$, and $\pi(w) = 2/w - 1/w = 1/w$.*

PROBLEM 6.1. *Prove more generally that for well-behaved homogeneous production functions with decreasing returns to scale, $\pi(p)$ is defined for all $p \gg 0$ but is generally undefined at price vectors p that contain zero prices for one or more inputs.*

- **For this reason, it makes sense to define the dual production set in terms of $p \gg 0$.**

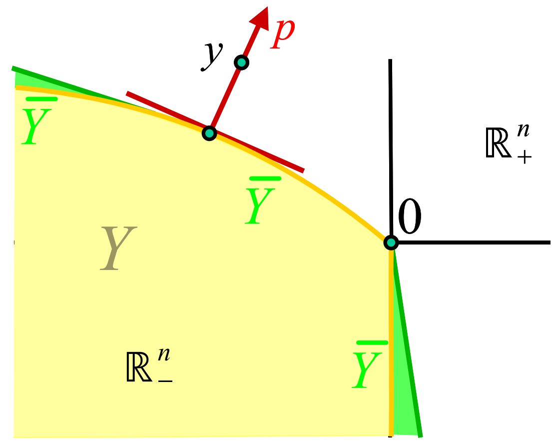
Proposition 6.6. ***(Duality Theorem for Production Sets). Let Y' be the dual dual production set of Y. If Y closed, contains 0, has free disposability and is convex, then $Y' = Y$.***

- **The theorem says that just as you can derive the profit function from the technology, so you can find the technology from knowledge of the profit function.**

Proof.

- **First, suppose $y \in Y$.**
  - **Because $\pi(p) = \max\{py \mid y \in Y\}$, we have $py \leq \pi(p)$ for all $p \gg 0$.**
  - **Therefore $y \in Y'$, so that $Y \subset Y'$.**

- **Now suppose $y \notin Y$.**
  - **Define vectors of the form $\delta_j \equiv \begin{bmatrix} \delta \\ \delta \\ -1 \\ \delta \end{bmatrix}$, where $\delta > 0$ and -1 is the jth component.**
  - **Set $\bar{Y} = Y + \sum \alpha_j \delta j$ for all $\alpha_j \geq 0$.**
  - **Note that as δ goes to 0, $\bar{Y}$ looks more and more like Y.**
  - **Set δ small enough so that $y \notin \bar{Y}$.**



- Apply separating-hyperplane theorem to $\bar{Y}$ and y.
- Find p and x_0 such that $px \leq px_0 < py$ for all $x \in \bar{Y}$.
- Note $p \gg 0$. Why?
- $\pi(p) \leq px_0 < py$
- $y \notin Y'$.

■

PROPOSITION 6.7. *(From M-C 5.C.1. pp. 138.) Suppose Y is closed, contains 0, and has the free-disposal property, and suppose π is defined for all $p \gg 0$.*

i). the supply (and derived demand) relation $y(\cdot)$ is homogeneous of degree 0.

ii). If Y is convex, then for each p, the set $y(p)$ is convex.

iii). If Y is strictly convex, then $y(\cdot)$ is a function.

iv). (Hotelling's Lemma) If $y(p)$ is a single point, then the profit function π is differentiable at p and $y(p) = \nabla\pi(p)$.

v). If $y(\cdot)$ is a function and is differentiable at p, then $\partial y(p)/\partial p$ is a symmetric positive-semidefinite matrix with

$$\frac{\partial y(p)}{\partial p} p = 0.$$

- The proofs are similar to the corresponding proofs for the expenditure function.
- There are no income or wealth effects with supply and derived demand.
- Inputs are purchased, converted to outputs and then sold, all simultaneously.
- No wealth is required in these models.

PROPOSITION 6.8. *(Law of Supply) Suppose $y(\cdot)$ is a supply (derived demand) function. Then for any two price vectors p' and $p \gg 0$, we have*

$$(p' - p)(y(p') - y(p)) \geq 0.$$

PROOF. Because $y(p)$ maximizes profits at prices p, we know that

$$p'y(p') \geq p'y(p)$$

and

$$py(p) \geq py(p')$$

∎

- If only one price changes, then the corresponding quantity changes in the same direction
- Because $\partial y(p)/\partial p$ is a symmetric positive-semidefinite matrix, own-price elasticity of supply is positive, and that of derived demand is negative (why?).

- Suppose we are given a production set Y and a price vector p.
- Sequential maximization is possible.
  - We can find a short-run profit function holding capital fixed, ...
  - and from that we can find a long-run profit function allowing capital to vary.

Example 6.2. *Suppose the technology is defined by*

$$\mathbf{Y} = \left\{ \begin{bmatrix} q \\ -k \\ -\ell \end{bmatrix} \middle| \; k, \ell \geq 0 \text{ and } q \leq \sqrt{k\ell} \right\},$$

and suppose p, r and w, are the prices of q, k and ℓ. Then find the short- and long-run profit functions if k is fixed in the short run.

- The short-run passive profit function is:

$$\pi\left(p, r, w \mid k\right) = \max_{\ell} \left\{ p\sqrt{k\ell} - rk - w\ell \right\}$$

[Why don't we have to maximize over q?]

- The foc is

$$0 = \frac{\partial \pi}{\partial \ell} = \frac{p}{2}\sqrt{\frac{k}{\ell}} - w$$

so that

$$\ell = \frac{k}{4} \Big/ \left(\frac{w}{p}\right)^2.$$

- Substituting into the passive profit function gives

$$\begin{aligned} \pi(p, r, w \mid k) &= \left[p\sqrt{k\ell} - rk - w\ell\right]_{\ell = \frac{k}{4}/\left(\frac{w}{p}\right)^2} \\ &= \left(\frac{p^2}{4w} - r\right)k. \end{aligned}$$

- What happens if

$$\frac{p^2}{4w} < r \text{ ?}$$

- The long-run profit function is

$$\pi(p, r, w) = \max_k \left(\frac{p^2}{4w} - r\right)k = ???$$

6.3 Cost Minimization

- If some commodities are always inputs and others are always outputs, then we can break the profit maximization problem into two parts:
  - The minimization of costs over input vectors z conditional on the output vector q.
  - The maximization of profits over all output vectors q.
- Similar to short-run and long-run profits, above.
- For simplicity we assume that there is only one output q.
- Let $q = f(z)$ be the production function, p, the price of the output, and w, the vector of input prices.

Definition 6.15. *The* passive cost function *is defined by* $C_P(z \mid w) = wz$.

- The passive cost function adds up the cost of the shopping list.
- It is of no use unless you know z.
- We would like to know how much it cost to produce a given quantity of output.

Definition 6.16. *The* cost minimization problem *(CMP) is*

$$\min_z C_P(z \mid w)$$
$$s.t.\ f(z) \geq q$$
$$z \geq 0$$

DEFINITION 6.17. *The* **cost function** $c(w \mid q)$ *is the solution of CMP, that is*

$$c(w \mid q) = \min_{z \geq 0}\{wz \mid f(z) \geq q\}.$$

DEFINITION 6.18. *The* **conditional derived-demand function** $z(w \mid q)$ *solves the CMP, that is*

$$z(w \mid q) = \underset{z \geq 0}{\operatorname{argmin}}\{wz \mid f(z) \geq q\}.$$

PROPOSITION 6.9. *The profit function is given by*

$$\pi(p, w) = \max_{q}\{pq - wz(w \mid q)\}$$

PROOF. **Apply sequential maximization.** ■

- **If w is assumed fixed and q is assumed variable, then we use:**

DEFINITION 6.19. *The* **total cost function** *is defined by* $C(q) = c(w \mid q)$.

DEFINITION 6.20. *The* **marginal cost function** *is defined by* $MC(q) \equiv C'(q)$.

DEFINITION 6.21. *The* **average cost function** *is defined by* $AC(q) \equiv C(q)/q$.

PROPOSITION 6.10. *Suppose* $q^* > 0$ *minimizes* $AC(q)$*. Then* $MC(q^*) = AC(q^*)$.

- **Why?**

PROPOSITION 6.11. ***(M-C.5.C.2) Suppose that $c(w, q)$ is the cost function of a single-output technology Y, closed with free disposal and with production function $f(z)$, and suppose that $z(w \mid q)$ is the associated derived-demand correspondence. Then***

i). $c(w \mid q)$ is homogeneous of degree 1 in w and nondecreasing in q.

ii). $c(w \mid q)$ is a concave function of w.

iii). (Duality theorem for cost functions) If $\{z \mid z \geq 0, f(z) \geq q\}$ is convex for every q, then

$$Y = \{(-z, q) \mid wz \geq c(w \mid q) \text{ for all } w \gg 0\}.$$

iv). $z(w \mid q)$ is homogeneous of degree 0 in w.

v). If $\{z \mid z \geq 0, f(z) \geq q\}$ is convex, then $z(w \mid q)$ is convex. If $\{z \mid z \geq 0, f(z) \geq q\}$ is strictly convex then $z(w \mid q)$ is single valued. If $\{z \mid z \geq 0, f(z) \geq q\}$ is strictly convex for all q, then z is a function.

vi). (Shepard's Lemma) If $z(w \mid q)$ is a function, then $c(w \mid q)$ is differentiable, and

$$\nabla_w c(w \mid q) \equiv z(w \mid q).$$

vii). If $z(w \mid q)$ is differentiable, then

$$\frac{\partial z(w \mid q)}{\partial w} \equiv \frac{\partial^2 c(w \mid q)}{\partial w^2}$$

so that $\partial z(w \mid q)/\partial w$ is a symmetric negative-semidefinite matrix with

$$\frac{\partial z(w \mid q)}{\partial w} w = 0.$$

viii). If $f(z)$ is homogeneous of degree 1 (has constant returns to scale), then so are $c(w \mid q)$ and $z(w \mid q)$ in q.

ix). If $f(z)$ is concave, then $C(q)$ is a convex function of q (MC is increasing).

Example 6.3. Find conditional derived demand, the cost function, supply and the profit function when $Q \equiv F(K, L) \equiv AK^{\alpha}L^{1-\alpha}$, where $0 < \alpha < 1$, with prices p, r, w.

- Both capital and labor must be used to obtain positive output.
- Therefore

$$\begin{aligned}
\frac{\frac{\partial F}{\partial K}}{r} &= \frac{\frac{\partial F}{\partial L}}{w} \\
\frac{A\alpha L^{1-\alpha}}{rK^{1-\alpha}} &= \frac{A(1-\alpha)K^{\alpha}}{wL^{\alpha}} \\
\frac{wL}{1-\alpha} &= \frac{rK}{\alpha} \\
K &= \frac{\alpha}{1-\alpha}\frac{w}{r}L.
\end{aligned}$$

- This shouldn't be surprising, because with a Cobb-Douglas production function, the share of expenses for labor and capital must be proportional to the exponents.

- Substituting into the production function yields the conditional derived demands:

$$Q = A\left(\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^{\alpha} L$$
$$L^{*} = \frac{1}{A}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\left(\frac{r}{w}\right)^{\alpha} Q$$
$$K^{*} = \frac{1}{A}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\left(\frac{w}{r}\right)^{1-\alpha} Q.$$

- Therefore the cost function is

$$C(r, w, Q) = rK^{*} + wL^{*}$$
$$C(r, w, Q) = \bar{A} r^{\alpha} w^{1-\alpha} Q$$

where

$$\bar{A} = \frac{1}{A}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha} + \frac{1}{A}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}.$$

- We have

$$AC = MC = \bar{A} r^{\alpha} w^{1-\alpha}$$

- If $p < AC$, nothing will be produced and profits are zero.
- If $p > AC$, production will be unbounded and so will profits.

6.4 Do firms behave according to the neoclassical model?

- Profit maximization may share some of the same problems as utility maximization.
  - The maximization problem is very difficult
    - Firms don't know their costs.
    - Firms don't know which production activities are possible.
  - Questions of psychology
    - Entrepreneurs are biased by optimism and hubris.
    - Time inconsistencies

- Firms have problems that individuals do not.
  - Principal/agent questions.
    - How to get workers to work hard?
    - How to get managers to pursue the interests of the firm's owners?
      - Problems of managerial control of firms.
      - Private benefits.
  - Owners of firms may maximize their utility rather than profits.
    - Risk aversion.