4. Duality in Consumer Theory

DEFINITION 4.1. *For any utility function $U(x)$, the corresponding* indirect utility function *is given by:*

$$V(p,w) \equiv \max_x \{U(x) \mid x \geq 0,\ px \leq w\}$$
$$\equiv \max_x \{U(x) \mid x \in B_{p,w}\},$$

so that if x^* is the solution to the UMP, then $V(p,w) = U(x^*)$.

Note that

$$V(p,w) \equiv \max_x \{U(x) \mid x \geq 0,\ px \leq w\}$$

and

$$x(p,w) \equiv \operatorname*{argmax}_x \{U(x) \mid x \geq 0,\ px \leq w\},$$

so that

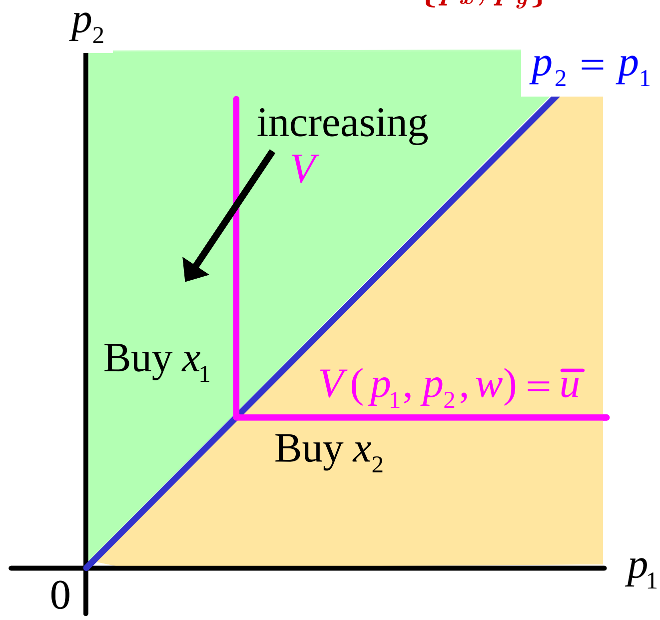
$$V(p,w) \equiv U(x(p,w)).$$

EXAMPLE 4.1. *Find the demand correspondence and the indirect utility function for the linear utility function $U = x + y$.*

- With the given utility function, x and y are perfect substitutes and the MU s are both *1* so the consumer will buy only the cheaper good.
- Let $p_m = \min\{p_x, p_y\}$. Demand for the cheaper good will be w/p_m and demand for the more expensive good will be *0*.
- If $p_x = p_y$ then demand for the goods can be any combination such that expenditures add up to w.

- The consumer will always buy w/p_m units of the cheaper good, so his utility must also be w/p_m. Therefore, the indirect utility function is

$$v(p_x, p_y, w) = \frac{w}{\min\{p_x, p_y\}}$$

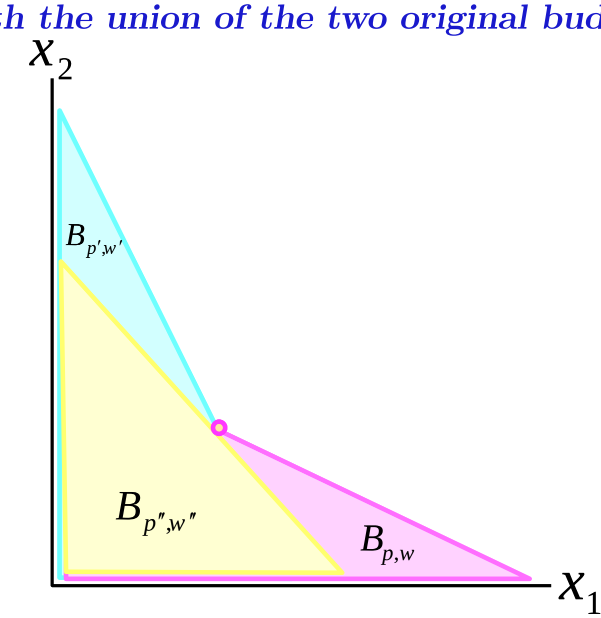


- ¿¿Quasiconcave??

Proposition 4.1. *Let $p'' = \alpha p + (1-\alpha)p'$ and $w'' = \alpha w + (1-\alpha)w'$ for $\alpha \in [0, 1]$. Then*

$$B_{p'',w''} \subset B_{p,w} \cup B_{p',w'}.$$

(If a new price and wealth vector is a convex combination of two price and wealth vectors, then the new budget set will be contained with the union of the two original budget sets.)



PROOF. We prove the contrapositive:

- If $x \notin B_{p,w}$ and $x \notin B_{p',w'}$, then $x \notin B_{p'',w''}$.
- But this must be true, because:
  - if $px > w$ and $p'x > w'$
  - then $[\alpha p + (1-\alpha)p']x > \alpha w + (1-\alpha)w'$.

■

PROPOSITION 4.2. *If U is continuous and locally nonsatiated (lns), then V is:*

i). Homogeneous of degree 0.

ii). Strictly increasing in w and monotonically decreasing in p.

iii). Quasiconvex (no-better-than sets, $B(p,w)$, are convex).

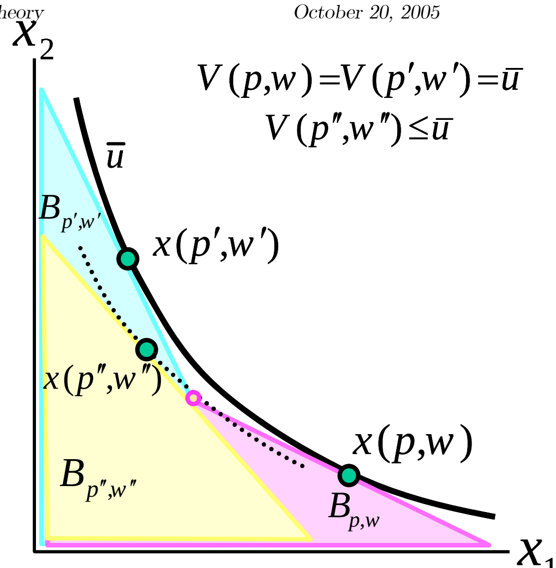
iv). Continuous in p and w.

Informal Proof.

i). Homogeneity: V **doesn't change if the budget set doesn't change.**

ii). Strictly increasing in w**; decreasing in** p**:**

- **nonsaturated preferences** $\Longrightarrow$ **strictly increasing in** w**.**
- **decreasing in** p**, because**
  - **increases in** p **make the budget set smaller**
  - **new budget set is inside the old one.**



iii). Quasiconvex: suppose $V(p, w) = V(p', w') = \bar{u}$**.**

- **Let** p'', w'' **be a convex combination of** p, w **and** p', w'**.**
- **From previous proposition, we know:**
  - **if** $x \in B_{p'',w''}$ **then it must be in either** $B_{p,w}$ **or** $B_{p',w'}$
  - **since** $\bar{u}$ **is the maximum utility available in those sets we have** $V(p'', w'') \leq V(p, w) = V(p', w')$**.**

iv). Continuity:

- $B_{p,w}$ is "continuous" in p and w
  - for small changes in p and w, additional and excluded commodity bundles are very close to the ones already there.
  - The continuity of U does the rest.
  - Yes, this is not really a proof, but the idea is the right one. ■

DEFINITION 4.2. *Given $U(x)$, the expenditure minimization problem (EMP) is*

$$\min_x px$$

$$s.t.\ U(x) \geq u$$

DEFINITION 4.3. *Given p, u, the expenditure function e is defined by*

$$e(p, u) = px^*,$$

where x^* solves EMP.

- The expenditure function yields the minimum expenditure required to reach utility u at prices p.
- More formally:

$$e(p, u) = \min_x \{px \mid U(x) \geq u\}$$

EXAMPLE 4.2. *Find the expenditure function for the linear utility function $U = x + y$. How much do we have to spend to get 100 units of utility if $p_x = 5$ and $p_y = 7$?*

- We already know that the indirect utility function is

$$v(p_x, p_y, w) = \frac{w}{\min\{p_x, p_y\}}.$$

- To find his expenditure function we set

$$u = \frac{w}{\min\{p_x, p_y\}}$$

and solve for w. We have

$$e(p_x, p_y, u) \equiv w = u \min\{p_x, p_y\}.$$

- Expenditure to get $u = 100$ when $p_x = 5$ and $p_y = 7$.

$$e(5, 7, 100) = 100 \min\{5, 7\} = 500.$$

PROPOSITION 4.3. (Duality) *Given $U(x)$, continuous and lns and a constant vector of prices $p \gg 0$, we have*

i). If x^* solves the UMP for $w > 0$,

- *then x^* solves the EMP when u is set to $U(x^*)$*
- *and $e(p, v(p, w)) = w$.*

ii). If y^* solves the EMP for $u > U(0)$,

- *then y^* solves the UMP when w is set to py^**
- *and $v(p, e(p, u)) = u$.*

Informal Proof.

i). Given that x^* solves UMP for prices p and income w,

- suppose that x^* does not solve the EMP for prices p and utility $U(x^*)$.
- Then there is an x' that gives as much utility as x^* but costs strictly less ($px' < px^*$),
- Thus, in the UMP, we can spend a little more than px' while spending less than w.
- Therefore by nonsatiation, we can find x'' with $px'' < w$ and $U(x'') > U(x') \geq U(x^*)$, a contradiction.
- Therefore $e(p, v(p,w)) = px^*$,
- and nonsatiation of U implies that $px^* = w$.

ii). Given that y^* solves EMP,

- we know that $U(y^*) \geq u$.
- Suppose that y^* does not solve the UMP.
- Then there is a y' with $py' \leq w\ (\equiv py^*)$ such that $U(y') > U(y^*) \geq u$
- Therefore because U is continuous, we can choose $y'' < y'$ with $U(y'') > U(y^*)$ but $py'' < py' \leq py^*$, a contradiction (y^* didn't minimize expenditure as assumed).
- The continuity of U implies that $U(y^*) = u$, for otherwise money could be saved by allowing $U(y^*)$ to fall without violating the utility constraint of the EMP.

■

Proposition 4.4. ***For U continuous and nonsatiated, $e(p,u)$ is***

i). Homogeneous of degree 1 in p.

ii). Strictly increasing in u; increasing in p.

iii). Concave in p.

iv). Continuous in p, u.

Informal Proof.

i). Homogeneous in p:

$$\begin{aligned} e(\alpha p, u) &= \min_x \{\alpha p x \mid U(x) \geq u\} \\ &= \alpha \min_x \{p x \mid U(x) \geq u\} \\ &= \alpha e(p, u). \end{aligned}$$

ii). Strictly increasing in u and increasing in p:

- For utility:
  - By definition $e(p,u)$ is the required expenditure to obtain u.
  - Suppose $u' > u$ could be obtained by consuming x' without increasing expenditures.
  - By continuity of u we could obtain $u'' > u$ even if we consume a little less than x', that is at a lower expenditure than $e(p,u)$, a contradiction.
- For prices:
  - Suppose that $p' > p$.
  - Then if x' solves the EMP at prices p' with $U(x') \geq u$, we have
    $$e(p', u) = p'x' \geq px' \geq e(p, u)$$
  - [Why not $p'x' > px'$?]

iii). Concave in p. Think of a consumer who normally consumes x'' at prices p''

- Suppose she spends a day at prices p and another day at prices p', where
$$p'' = \frac{1}{2}(p + p').$$
- Could reach same utility at same average expense by consuming x'' both days.
- But can save money by adapting her choice of goods to the current prices.
- By substituting cheap for expensive goods, you can get same utility for less money at more extreme prices than at average prices.

- More formally:
  - Suppose that for $\alpha \in (0, 1)$, $p'' = \alpha p + (1 - \alpha)p'$
  - and suppose that x'' solves the EMP with utility u, so that $U(x'') \geq u$.
  - Then $px'' \geq e(p, u)$ and $p'x'' \geq e(p', u)$ [why?]
  - Therefore

$$\begin{aligned} e(p'', u) &= p''x'' = (\alpha p + (1 - \alpha)p')x'' \\ &= \alpha p x'' + (1 - \alpha)p'x'' \\ &\geq \alpha e(p, u) + (1 - \alpha)e(p', u). \end{aligned}$$

iv). Continuous in p, u. Follows from:

- continuity of the constraint set $\{x \mid U(x) \geq u\}$ as a function of u
- and continuity of the objective function px in x and p.

■

DEFINITION 4.4. ***Hicksian demand $h(p,u)$ is a consumption vector x^* that solves the EMP.***

- **We have**

$$e\,(p,u) = \min_x \{px \mid U(x) \geq u\}$$

and

$$h(p,u) = \operatorname*{argmin}_x \{px \mid U(x) \geq u\}$$

- **Also,** $e(p,u) = ph(p,u)$.

Because $x(p,w)$ solves the UMP and $h(p,u)$ solves the EMP, the proposition on utility duality tells us:

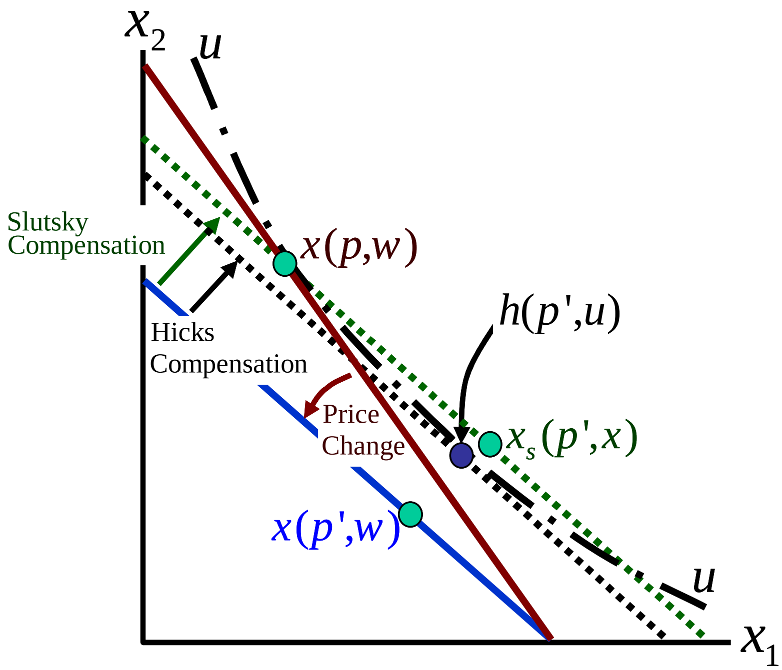
PROPOSITION 4.5. ***Given that U is continuous, $u > U(0)$ and $w > 0$, we have***

i). $x(p,w) = h(p, v(p,w))$,

- ***Walrasian demand at wealth w = Hicksian demand at utility level produced by w.***

ii). $h(p,u) = x(p, e(p,u))$,

- ***Hicksian demand at utility u = Walrasian demand with wealth required to reach u.***



- **Graph above shows the difference between**
  - **Slutsky compensated demand** $x_s(p', x)$
  - **and Hicksian demand** $h(p', u)$**.**

Suppose a consumer has consumption vector $x(p, w)$ **and utility** $u = U(x(p, w))$**,**

- **and then prices change from** p **to** p'**.**
- **We have**

$$\begin{aligned} x_s(p', x) &= x(p',\ p'x(p, w)) \\ h(p', u) &= x(p',\ e(p', u)) \end{aligned}$$

- **Slutsky compensated demand = Walrasian demand when the consumer is given sufficient wealth to buy his original consumption vector** $x(p, w)$**.**
- **Hicksian demand = Walrasian demand when the consumer is given sufficient wealth to reach his original utility level,** $u = U(x(p, w))$**.**

- We know $U(x_s(p', x)) \geq U(h(p', u))$. [Why?]
- As the price change $p' - p$ gets small, difference between Hicksian demand and Slutsky demand becomes second-order small.
- We will show that

$$S(p, w) \equiv \left.\frac{\partial x_s(p', x)}{\partial p'}\right]_{p'=p} \equiv \left.\frac{\partial h(p', u)}{\partial p'}\right]_{p'=p}$$

- Both have the same derivatives at $p' = p$.
- Therefore, the Slutsky Equation is true for Hicksian compensated demand.
- "Compensated demand" usually refers to Hicksian demand
- Slutsky demand is rarely used.

PROPOSITION 4.6. *(M-C 3.E.4; Law of Demand) On average, when prices rise, the substitution effect is negative. More formally:*

- *If $U(x)$ is continuous and lns, and*
- *$h(p, u)$ is a function,*
- *then for all p'' and p'*

$$(p'' - p')[h(p'', u) - h(p', u)] \leq 0.$$

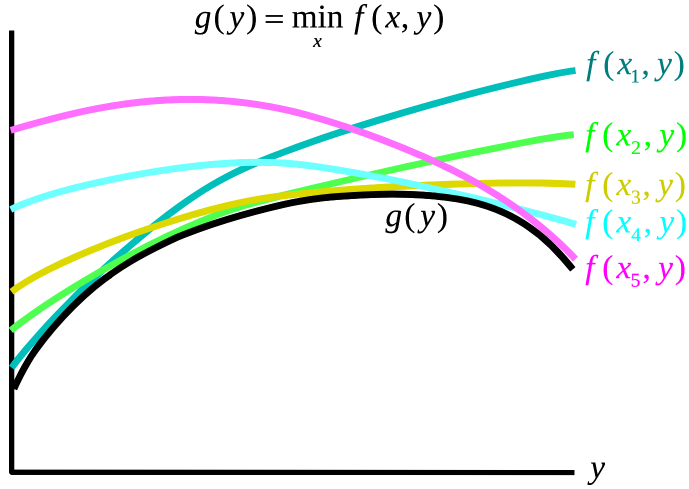
PROOF. We have

- (1) $p''h(p'', u) \leq p''h(p', u)$ [why?]
  - $\Longrightarrow$(2) $p''h(p'', u) - p''h(p', u) \leq 0$
- (3) $p'h(p', u) \leq p'h(p'', u)$. [why?]
  - $\Longrightarrow$(4) $p'h(p', u) - p'h(p'', u) \leq 0$
- Add (2) and (4) and factor the results.

■

4.1 The Envelope Theorem

- Suppose that a family of functions is described by $f(x, y)$ for different fixed parameters x and a variable y.
- At each point, we compare the values of all functions in the family, and choose the minimum value.
- This creates a new function $g(y) \equiv \min_x f(x, y)$. The function $g(y)$ is called the lower envelope of $f(x, y)$.
- In the figure, the family members and the lower envelope are plotted as functions of y.



- The theorem says that the slope of the envelope at any point is the same as the slope of the member of the family that it touches.
- M-C has a more general version of the theorem: don't worry about it, because it is quite messy.

Proposition 4.7. (Envelope Theorem) *Let $g(y) = \min_x f(x,y)$, where $f(x,y)$ is differentiable. Then*

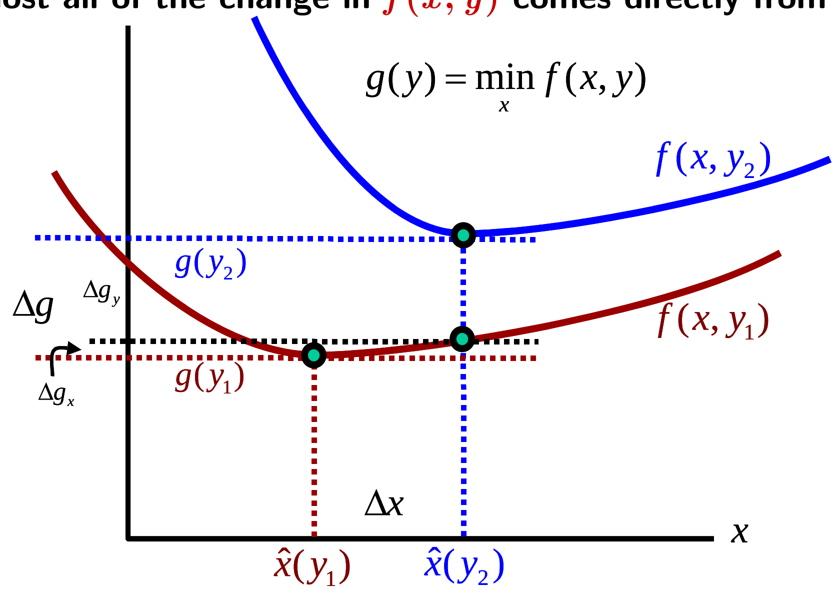
$$g'(y) = \frac{\partial f(x,y)}{\partial y}\bigg]_{x=\hat{x}(y)}$$

where $\hat{x}(y)$ is the value of x that minimizes $f(x,y)$.

The intuition

- As y changes x also must change because x must always minimizes $f(x,y)$.
- If y changes by Δy, the change $\Delta g(y)$ comes from two sources
  - directly from Δy
  - and from Δx (which is caused by Δy).

- The envelope theorem says that if Δy is small, the part of Δg that comes from Δx (labeled Δg_x on the graph) is near *0*, because...
  - The curves are flat at $\hat{x}(y)$, because $\hat{x}(y)$ minimizes $f(x,y)$.
  - So at $x = \hat{x}(y)$, if y is held constant, Δx produces a small change in $f(x,y)$.
  - Almost all of the change in $f(x,y)$ comes directly from Δy.



PROOF. Let $\hat{x}(y)$ be the solution of $\min_x f(x, y)$. The f.o.c for $\hat{x}(y)$ is

$$\left.\frac{\partial f}{dx}\right]_{x=\hat{x}(y)} = 0.$$

We can now write:

$$g(y) = f(\hat{x}(y), y)),$$

so, by the chain rule,

$$g'(y) = \left.\frac{\partial f}{dx}\right]_{x=\hat{x}(y)} \hat{x}'(y) + \frac{\partial f}{\partial y}.$$

The first term is *0*. ■

PROPOSITION 4.8. *If $U(x)$ is continuous and lns, and $h(p,u)$ is a function, then*

$$h(p,u) = \nabla_p e(p,u).$$

PROOF.

- We know that

$$e(p,u) = \min_x \{px \mid U(x) = u\}$$

- Notice the equality constraint [why equality?]
- We can write this as a saddle-point problem:

$$e(p,u) = \max_\lambda \min_x \{px - \lambda[u - U(x)]\}$$

- Envelope theorem says: in calculating $\partial e/\partial p$, λ and x can be treated as constants at their optimal values.
- The only term that contains p explicitly is px.
- Thus $\nabla_p e(p,u) \equiv \partial e/\partial p = x^* \equiv h(p,u)$.

■

PROPOSITION 4.9. *(M-C 3.G.2) For the Jacobian matrix $\partial h(p,u)/\partial p$ we have:*

i). $\partial h(p,u)/\partial p = \partial^2 e(p,u)/\partial p^2$

ii). $\partial h(p,u)/\partial p$ *is negative semidefinite,*

iii). $\partial h(p,u)/\partial p$ *is symmetric, and*

iv). $[\partial h(p,u)/\partial p] \cdot p = 0$

PROOF. We have:

i). 2nd derivative of $e(p,u)$: Immediate from $h(p,u) = \partial e(p,u)/\partial p$

ii). Negative semidefinite: From concavity of expenditure function.

iii). Symmetric:

- The off-diagonal elements of $\partial h(p,u)/\partial p$ are the cross-partial derivatives of $e(p,u)$.
- But well-behaved functions have symmetric cross-partial derivatives (i.e. $\partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x$).

iv). $[\partial h(p,u)/\partial p] \cdot p = 0$

- $h(p,u)$ is homogeneous of degree 0 in p.
- Result follows from Euler's formula.

■

PROPOSITION 4.10. *(Slutsky equation for Hicksian demand.) Given $U(x)$ strictly quasiconcave and well-behaved and the corresponding indirect utility function $V(p,w)$, we have*

$$\frac{\partial x_i(p,w)}{\partial p_j} = \frac{\partial h_i(p,u)}{\partial p_j} - \frac{\partial x_i(p,w)}{\partial w} x_j(p,w)$$

where $u = V(p,w)$.

PROOF. The proof depends on the previously-established identity $h(p,u) \equiv x(p, e(p,u))$.

- By chain rule:

$$\frac{\partial h_i(p,u)}{\partial p_j} \equiv \frac{\partial x_i(p,w)}{\partial p_j} + \frac{\partial x_i(p,w)}{\partial w}\frac{\partial e(p,u)}{\partial p_j}$$

- But

$$\frac{\partial e(p,u)}{\partial p_j} \equiv h_j(p,u) \equiv h_j(p, V(p,w)) \equiv x_j(p,w)$$

- Substitution completes the proof.

■

PROPOSITION 4.11. ***(Roy's Identity) Given $U(x)$ strictly quasiconcave and well-behaved and the corresponding indirect utility function $V(p, w)$, we have***

$$x_j(p, w) = -\frac{\partial V(p, w)}{\partial p_j} \Big/ \frac{\partial V(p, w)}{\partial w}.$$

PROOF.

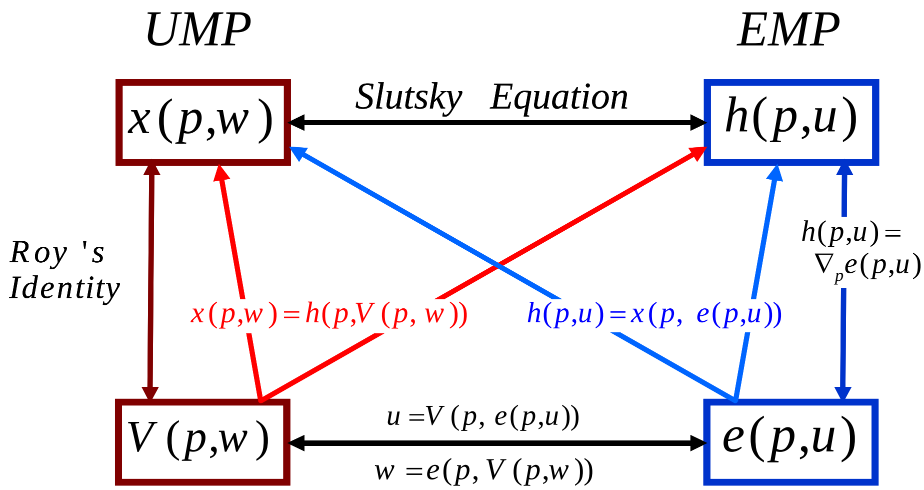
- **First, the intuition:**
  -
$$\frac{\partial V(p, w)}{\partial p_j} \Big/ \frac{\partial V(p, w)}{\partial w} \mathrel{\hat{=}} \frac{\Delta u}{\Delta p_j} \Big/ \frac{\Delta u}{\Delta w} = \frac{\Delta w}{\Delta p_j} = -x_j(p, w)$$
  - **We overlooked some little details:**
  - **for example, $x_j(p, w)$ changes when p changes,**
  - **but $x_j(p, w)$ is a utility maximizer, so the envelope theorem tells us that we can ignore this change.**

- **Formal proof:**
  - **Let $u = V(p, w)$, so that $w = e(p, u)$**
  - **We have $V(p, e(p, u)) \equiv u$**
  - **Hold u constant. By the implicit-function theorem, we have:**
$$\frac{\partial e(p, u)}{\partial p_j} \equiv -\frac{\partial V(p, w)}{\partial p_j} \Big/ \frac{\partial V(p, w)}{\partial w}$$
  - **but**
$$\frac{\partial e(p, u)}{\partial p_j} \equiv h_j(p, u) = h_j(p, V(p, w)) \equiv x_j(p, w).$$

■

- **The chart below summarizes the duality between the UMP and the EMP.**

- **It is taken (with editorial errors corrected) from M-C, p. 75.**



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5. Welfare Analysis

- Changes in price (and incomes) lead to changes in level of utility that consumers can obtain.
- Many economic policies affect prices:
  - competition policy
  - foreign trade policy
  - tax law
  - business regulations
- It would be useful to be able to measure the effect of price changes on utility.

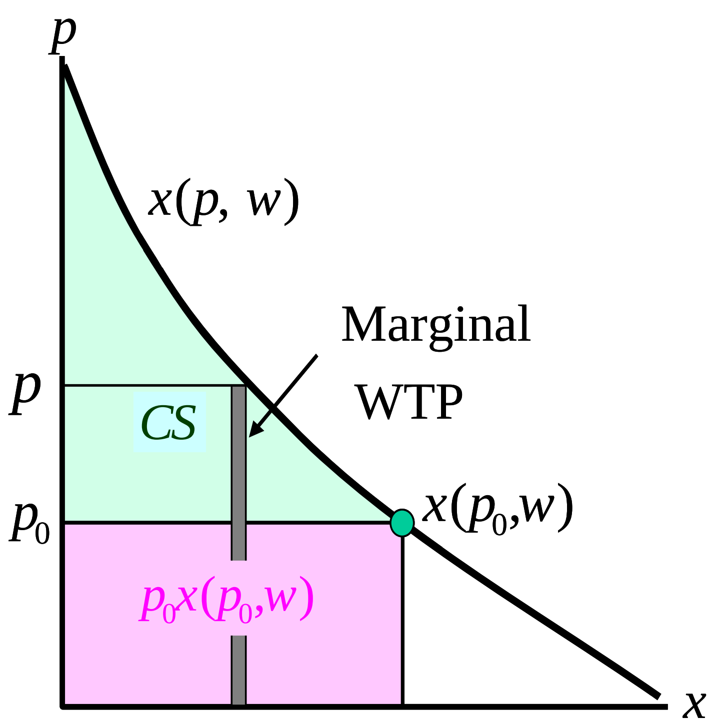
- Can we use the indirect utility function?
  - If prices and incomes change from p_0, w_0 to p_1, w_1,
  - then utility increases by $\Delta U = v(p_1, w_1) - v(p_0, w_0)$.
  - But not very useful for evaluating policy:
    - Utility is not an observable economic variable.
    - Psychologists have done little to create tools for measuring utility.
    - Most economists don't trust the measurements of psychologists.
    - Most economists consider utility to be only an ordinal ranking.

- All is not lost.
- Economists can count money.
- Traditional (Marshall's) monetary measure of utility change from additional goods: willingness-to-pay (WPT).
- Suppose the consumer is willing to pay (at most) \$100 for 40 kilograms of rice. Then he is indifferent between:
  - (40 kilograms of rice and \$100 less)
  - and no change
- Conclusion: WTP is a measure of utility gain from goods.

DEFINITION 5.1. *The* `willingness to pay` $WTP(x)$ *for a commodity vector* x *is the maximum amount the consumer would voluntarily pay for* x.

DEFINITION 5.2. `Consumer surplus (CS)` *gained from* x *is given by* $WTP(x) - px$.

- WTP can be measured in the market.
- $WTP \mathrel{\mathring{=}}$ area under demand curve



- **More precisely:**
  - **Let $p(x, w)$ be the demand-price function**
    - **inverse of demand function**
    - **prices p at which the consumer would demand x**
  - **We have**

$$\mathrm{WTP}(x) \circeq \int_0^x p(x, w)\,dx$$

  - **or, equivalently**
  - **the willingness to pay for the marginal unit of a good at demand $x(p, w)$ is given by**

$$\mathrm{MWTP}(x) = p(x, w).$$

  - **¿¿¿Why???**

- Consumer does not want to buy unit x when $p > p(x, w)$,
- but DOES buy it when $p = p(x, w)$.

- But the measurement is not exact.
  - Suppose x is selling at prices p_0
  - We would like to know the MWTP(x) at prices p_0,
  - but $p(x, w)$ gives us the MWTP(x) at prices $p = p(x, w)$.
  - Why are they different?
  - Because of the income effect.

- We can find other monetary equivalents of the utility changes caused by price changes.
- Theoretically more sound, [Robert Willig argued otherwise]
- but more difficult to measure and use.
- Suppose a consumer faces prices p_0 and has wealth w.
- Think of a possible price change from the price level p_0 to the level p_1.
- The price change would change utility from $u_0 = v(p_0, w)$ to $u_1 = v(p_1, w)$.

Definition 5.3. *The* compensating variation *[in wealth], $CV(p_0, p_1, w)$ is the amount of money that a consumer would have to pay after a price change from p_0 to p_1 in order to revert to her original level of utility.*

- *Mathematically:*

$$v(p_1, w - CV(p_0, p_1, w)) = v(p_0, w) = u_0$$

Definition 5.4. *The* equivalent variation *[in wealth], $EV(p_0, p_1, w)$ is the amount of money a consumer would have to receive in place of a price change from p_0 to p_1 in order to reach the level of utility that the price change would have created.*

- *Mathematically:*

$$v(p_0, w + EV(p_0, p_1, w)) = v(p_1, w) = u_1$$

- **Both compensating and equivalent variation are positive for a price change that increases utility.**

- With CV we are increasing wealth as we increase prices in order to keep the consumer at her original level of utility, u_0.
- With EV we are decreasing wealth instead of increasing prices in order to keep the consumer at the new lower level of utility, u_1, that would have been created by the price increase.
- Both measures determine the change in wealth that is precisely equivalent to a change in prices,
- but at different levels of utility.

PROPOSITION 5.1. *Both CV and EV can be expressed by use of the expenditure function as follows:*

i). $CV(p_0, p_1, w) = e(p_1, u_1) - e(p_1, u_0)$, *and*

ii). $EV(p_0, p_1, w) = e(p_0, u_1) - e(p_0, u_0)$,

where $u_0 = v(p_0, w)$ *and* $u_1 = v(p_1, w)$.

PROOF. **Remember that v and e are inverses. We have:**

i).

$$\begin{aligned} & e(p_1, u_1) - e(p_1, u_0) \\ = \; & e(p_1, v(p_1, w)) \\ & - e(p_1, v(p_1, w - CV(p_0, p_1, w))) \\ = \; & w - w + CV(p_0, p_1, w) \\ = \; & CV(p_0, p_1, w) \end{aligned}$$

ii).

$$\begin{aligned} & e(p_0, u_1) - e(p_0, u_0) \\ = \; & e(p_0, v(p_0, w + EV(p_0, p_1, w))) \\ & - e(p_0, v(p_0, w)) \\ = \; & (w + EV(p_0, p_1, w)) - w \\ = \; & EV(p_0, p_1, w) \end{aligned}$$

■

- **Keep in mind that**

$$e(p_1, u_1) \equiv e(p_0, u_0) \equiv w \text{ [why?]}$$

so that we can write

i). $CV(p_0, p_1, w) = e(p_0, u_0) - e(p_1, u_0)$, **and**

ii). $EV(p_0, p_1, w) = e(p_0, u_1) - e(p_1, u_1)$.

- **Because** $h(p, u) \equiv \nabla_p e(p, u)$, **we know:**

PROPOSITION 5.2. ***CV and EV are given by the path-independent line integrals***

$$CV(p_0, p_1, w) = \int_{p_1}^{p_0} h(p, u_0) \cdot dp,$$

and

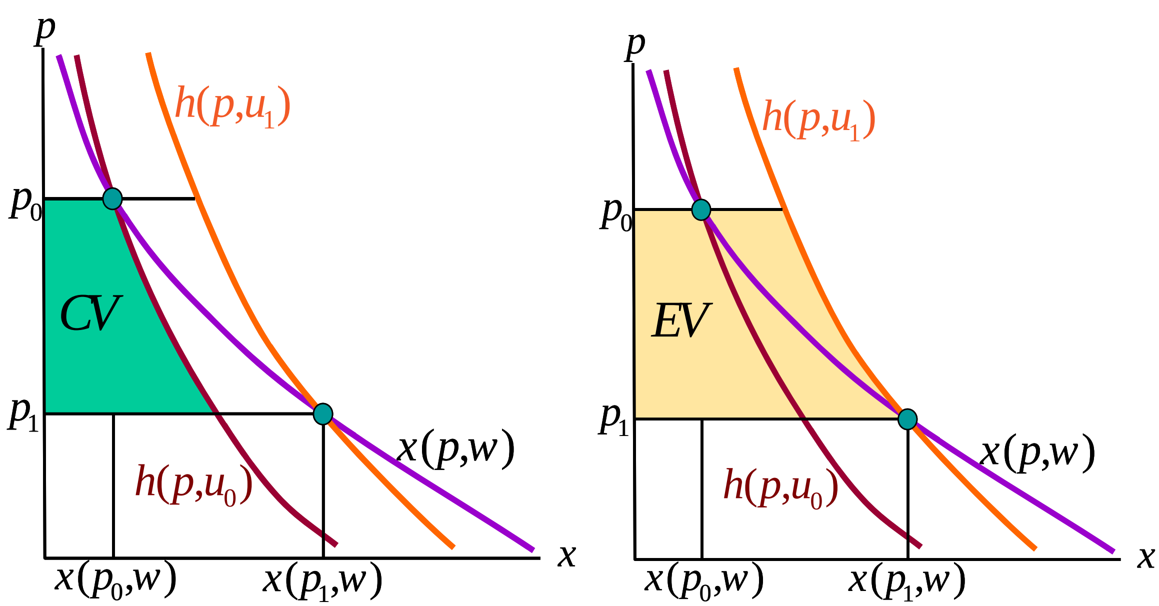
$$EV(p_0, p_1, w) = \int_{p_1}^{p_0} h(p, u_1) \cdot dp.$$

where p_1 ***and*** p_2 ***are price vectors (not scalers) and*** $h(p, u) \cdot dp$ ***(a scalar) is the inner product of the vectors*** $h(p, u)$ ***and*** dp ***as*** p ***moves along a path from*** p_1 ***to*** p_0 ***in price space.***

- The expression $h(p, u) \cdot dp \equiv de(p, u)$ is a scalar quantity that represents the change in expenditure in all markets as the price vector is continuously changed.

- We have assumed that e is such that the integral is not path-dependent, which means it doesn't matter how we get from p_1 to p_0. So if $p_1 = (2, 4)$ and $p_0 = (6, 7)$ we could integrate as we go from $(2, 4)$ to $(6, 4)$ to $(6, 7)$, or we could integrate as we go from $(2, 4)$ to $(2, 7)$ to $(6, 7)$, and the value of the integral would be thes same.

- The following graph illustrates the idea when all but one price is kept constant.



- Suppose we cut price $1 at a time.
- The wealth released by each price cut is equal to the `current` amount we are buying.
- But we are holding utility constant as we cut price,
- by taking this money away from the consumer as it is released.
- So, as the price changes, the amount purchased remains on the compensated demand curve $h(p, u)$.
- The total amount of money taken away is the compensating variation.
- Compensating variation accumulates only in those markets in which price is changed. For a small price change, it is proportional to the amount of the purchase in the corresponding market at the current price.

- We can also use changes in consumer surplus as a monetary measure of a welfare change caused by a price change.
- More specifically, think of ΔCS the amount of additional money made available to the consumer as the price is gradually lowered.
- As with CV, ΔCS accumulates only in those markets in which price is changed. For a small price change, it is proportional to the amount of the purchase in the corresponding market at the current price..
- But the difference between CV and ΔCS, is that with CV we take away the funds released as the price falls, and with ΔCS we let the consumer keep them.

- **We have:**

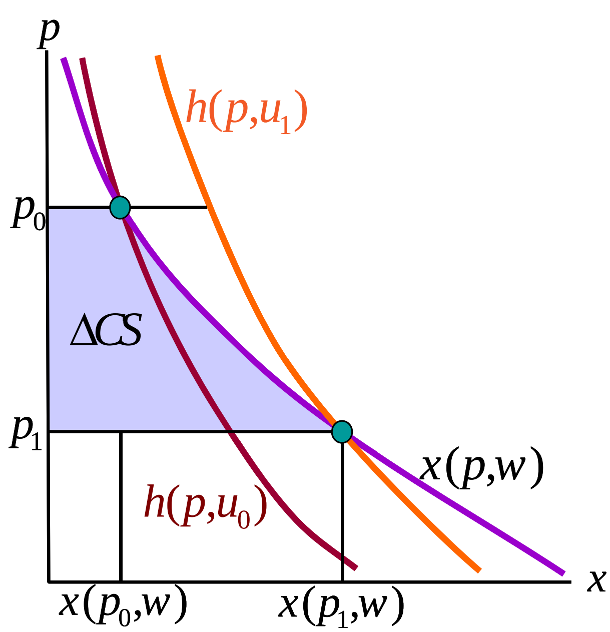
$$CS(p_0) = \int_0^{x(p_0,w)} (p(x,w) - p_0)\,dx$$
$$CS(p_1) = \int_0^{x(p_1,w)} (p(x,w) - p_1)\,dx$$

so that

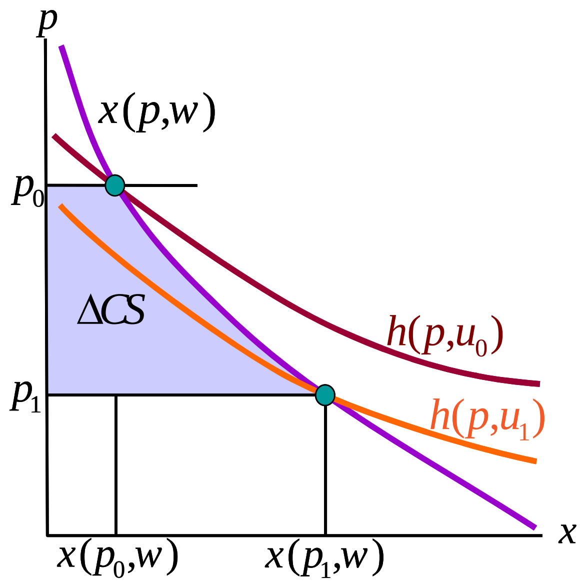
$$\Delta CS = CS(p_1) - CS(p_0) = \int_{p_1}^{p_0} x(p,w)\,dp$$

- **For normal goods, the various measures of welfare change are related as follows:**

$$CV(p_0, p_1, w) < \Delta CS < EV(p_0, p_1, w)$$



- **For inferior goods the inequalities are reversed. See below.**



- **Are the differences large?**

EXAMPLE 5.1. ***For the linear utility function $U = x + y$ find the demand correspondence, the indirect utility function, the expenditure function, and the Hicksian compensated demand. Then, for $p_y = 2$ and $w = 60$ find the compensating variation, the equivalent variation and the change in consumer surplus if p_x changes from 3 to 1.***

- **With the given utility function, x and y are perfect substitutes and the MU s are both 1 so the consumer will buy only the cheaper good.**

- **Let $p_m = \min\{p_x, p_y\}$. Demand for the cheaper good will be w/p_m and demand for the more expensive good will be 0.**

- **If $p_x = p_y$ then demand for the goods can be any combination such that expenditures add up to w.**

- The consumer will always buy w/p_m units of the goods, so his utility must also be w/p_m. Therefore, the indirect utility function is

$$v(p_x, p_y, w) = \frac{w}{\min\{p_x, p_y\}}$$

- To find his expenditure function we set

$$u = \frac{w}{\min\{p_x, p_y\}}$$

and solve for w. We have

$$e(p_x, p_y, u) \equiv w = u \min\{p_x, p_y\}.$$

- We have

$$h_x(p_x, p_y, u) = \frac{\partial e}{\partial p_x} = \begin{cases} u & \text{for } p_x < p_y \\ 0 & \text{for } p_x > p_y \end{cases};$$

likewise for h_y.

- Note that $\partial e/\partial p_x$ is undefined at $p_x = p_y$, but in that case it is clear that h_x and $h_y \in [0, u]$ and $h_x + h_y = u$.

- Before the price change, when $p_x = 3$, $p_y = 2$ and $w = 60$, we have

$$u_0 = \frac{w}{\min\{p_x, p_y\}} = 30,$$

and after p_x changes to 1, we have

$$u_1 = \frac{w}{\min\{p_x, p_y\}} = 60.$$

-

$$\begin{aligned} e(p_0, u_0) &\equiv e(3, 2, 30) = 30 \cdot 2 = 60 \\ e(p_1, u_0) &\equiv e(1, 2, 30) = 30 \cdot 1 = 30 \\ CV &\equiv e(p_0, u_0) - e(p_1, u_0) = 30 \end{aligned}$$

-

$$\begin{aligned} e(p_0, u_1) &\equiv e(3, 2, 60) = 60 \cdot 2 = 120 \\ e(p_1, u_1) &\equiv e(1, 2, 60) = 60 \cdot 1 = 60 \\ EV &\equiv e(p_0, u_1) - e(p_1, u_1) = 60 \end{aligned}$$

- **Consumer surplus changes as p_x changes but only while $p_x < p_y$, that is as p_x goes from 2 to 1. In that range we have $x(p_x, p_y, w) = w/p_x$ so**

$$\Delta \mathrm{CS} = \int_1^2 \frac{60}{p_x} dp_x = 60 [\log 2 - \log 1]$$
$$= 60 \log 2 = 41.6.$$

Note that 41.6 is between 30 and 60.