3. Neoclassical Demand Theory

3.1 Preferences

- Not all preferences can be described by utility functions.
- This is inconvenient.
- We make the assumptions about preferences that utility functions require.
  - Not very realistic
  - No attention to psychology

- Preference relations $\succsim$ are defined on $X \subset \mathbb{R}^n_+$.
  - In order to use set notation, we describe $\succsim$ by the upper contour sets $G(x)$,
  $$y \in G(x) \Longleftrightarrow y \succsim x,$$
  - or by the lower contour sets $B(x)$,
  $$y \in B(x) \Longleftrightarrow x \in G(y) \text{ or } y \precsim x$$
  - We also define indifference sets $I(x) = G(x) \cap B(x)$.
    - If $y \in I(x)$ we write $y \sim x$.
  - Both set and preference notation can be used interchangeably.

NOTATION 3.1. ***Vector inequalities:***

$\gg$	*Strictly Greater*	*means* $>$ *in all components*
$>$	*Greater*	*means* $\geq$ *in all components but* $>$ *in some*
$\geq$	*Greater or Equal*	*means* $\geq$ *in all components*

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} \underset{\text{greater}}{\overset{\gg}{\text{strictly}}} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \underset{\text{greater}}{>} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \underset{\text{or equal}}{\overset{\geq}{\text{greater}}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

DEFINITION 3.1. $\succsim$ *is* **complete** *if for every* x *and* y, $y \in G(x)$ *or* $x \in G(y)$.

DEFINITION 3.2. $\succsim$ *is* **transitive** *if for every* x, y *and* z, *if* $z \succsim y$ *and* $y \succsim x$, *then* $z \succsim x$.

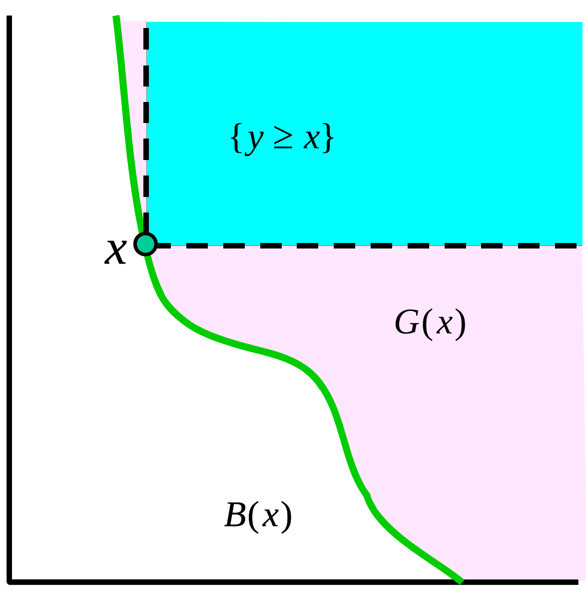
DEFINITION 3.3. $\succsim$ *is* **(strictly) convex** *if for every* x, $G(x)$ *is a (strictly) convex set.*

DEFINITION 3.4. $\succsim$ *is* **homothetic** *if* $y \succsim x \Longrightarrow$ *for all* $\alpha > 0$, $\alpha y \succsim \alpha x$ *(that is, if every indifference curve is a constant multiple of every other indifference curve.*

DEFINITION 3.5. **Monotonicity** *(more is better):*

- $\succsim$ *is* **strongly monotone** *if* $x > y \Longrightarrow x \succ y$
- $\succsim$ *is* **monotone** *if* $x \gg y \Longrightarrow x \succ y$
- $\succsim$ *is* **locally nonsatiated** *if every neighborhood of* x *contains a vector* y *such that* $y \succ x$*. [there is always a better point close by]*

- **Illustration of monotonicity:**



- Let $G^0(x) = \{z \mid z \in G(x), z \notin B(x)\}$ = the strictly-preferred set

- **Monotonicity**: More of all goods increases utility.

$$\left\{\begin{array}{c} \textit{the blue dark area} \\ \textit{not including } x \\ \textit{or the dotted lines} \end{array}\right\} \subset G^0(x).$$

- **Strong monotonicity**: More of any goods increases utility.

$$\left\{\begin{array}{c} \textit{the blue dark area} \\ \textit{including the dotted lines} \\ \textit{but not } x \end{array}\right\} \subset G^0(x).$$

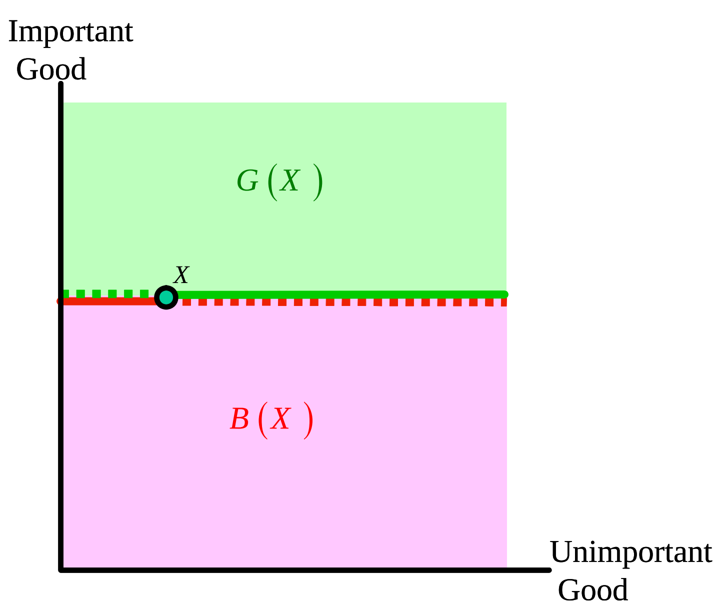
- **Local nonsatiation**: you can always increase utility by making a small change in your consumption bundle.
  - Examples?

EXAMPLE 3.1. *Lexicographical preferences:*

- **Commodities are ordered from most important to least important.**
- **At first, only the most important commodity is used to determine whether one bundle is more preferred than another**
- **If the amounts of the most important commodity in two bundles are exactly equal, then the next most important commodity is used to break the tie, etc.**
- **If x_2 is most important and x_4 is next then:**

$$\begin{bmatrix} 3.5 \\ 7.01 \\ 2 \\ 8 \end{bmatrix} \succ \begin{bmatrix} 350 \\ 7 \\ 200 \\ 18 \end{bmatrix} \succ \begin{bmatrix} 3500 \\ 7 \\ 200000 \\ 17 \end{bmatrix}$$

- **Illustration of Lexographical preferences.**

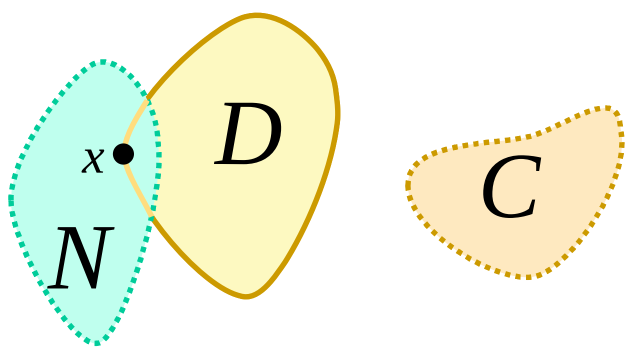


- **With lexicographical preferences, indifference curves have only one point. Why?**

3.2 Some informal topology.

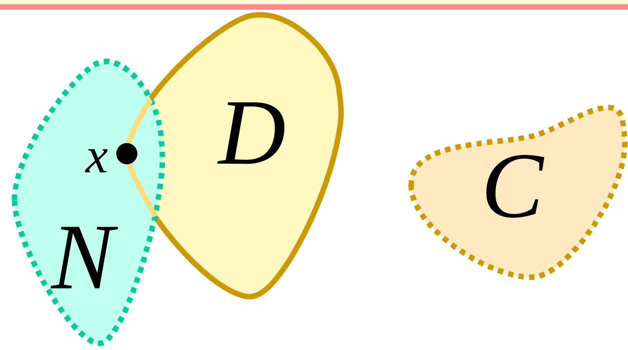
Definition 3.6. (Informal) ***The set N is a neighborhood of a point x if N contains a sphere (or circle or interval) around x.***

- **A neighborhood of x must include all points very close to x.**
- **A neighborhood can be as small as you like.**
- **If N is a neighborhood of x and $N \subset N'$, then N' is a neighborhood of x.**



- **N is a neighborhood of x**
- **C is not a neighborhood of x.**
- **D is not a neighborhood of x. Why not?**

DEFINITION 3.7. ***x* is a boundary point of a set *D* if every neighborhood of *x* (no matter how small) contains a point in *D* and a point outside of *D*.**



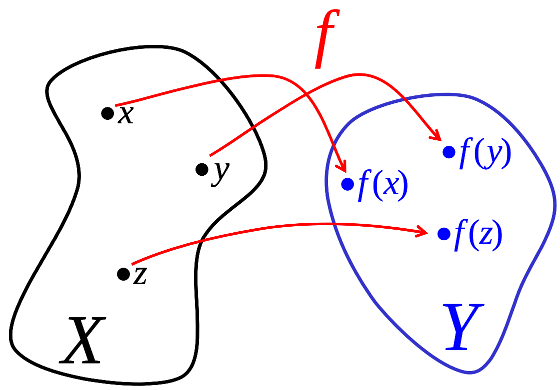
- **x is a boundary point of D, but not of N or C.**
- **Boundary points are the points at the edge of a set.**

DEFINITION 3.8. ***A set is open if it contains none of its boundary points.***

DEFINITION 3.9. ***A set is closed if it contains all of its boundary points.***

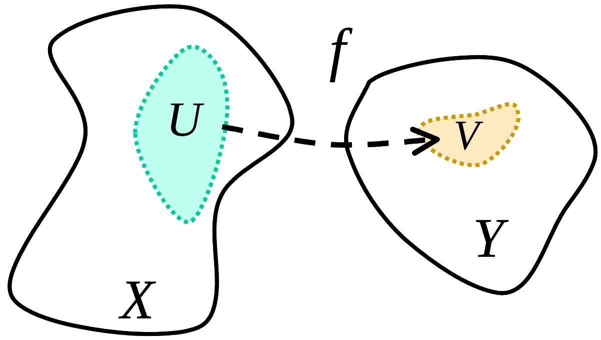
- **The complement of a closed set is open.**
- **The complement of an open set is closed.**
- **An open set is a neighborhood of each of its points.**
  - **Why?**

DEFINITION 3.10. *A* function f *maps a space* X *into a space* Y *(written* $f : X \to Y$*) if* f *assigns a point in* Y *to each point in* X*. The space* X *is called the* domain *of the* f*, and* Y *is called the* codomain*. The part of* Y *that is used is called the* range*.*



- **Functions can do things to spaces (transform them). Examples: shrinking, stretching, flattening, etc.**

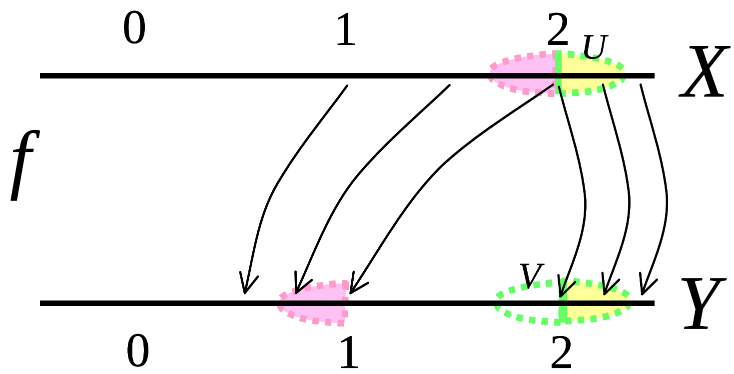
- **If $f : X \to Y$ and $S \subset X$, then the image of S (written $f(S)$) is the set of all points in Y that come from S.**
- **If $f : X \to Y$ and $R \subset Y$, then the inverse image of R (written $f^{-1}(R)$) is the set of all points in X that go into R.**



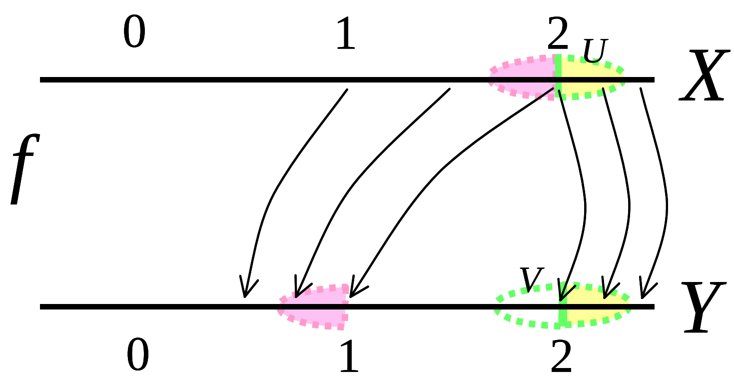
- **V is the image of U.**
- **$U \subseteq f^{-1}(V)$. Why not $U = f^{-1}(V)$?**

DEFINITION 3.11. ***(Informal) A function is continuous if all points that are near each other have images that are near each other.***

DEFINITION 3.12. (INFORMAL) ***A function is discontinuous if some points that are near each other have images that are far apart [see drawing below].***



- **Define a function that fits this drawing.**

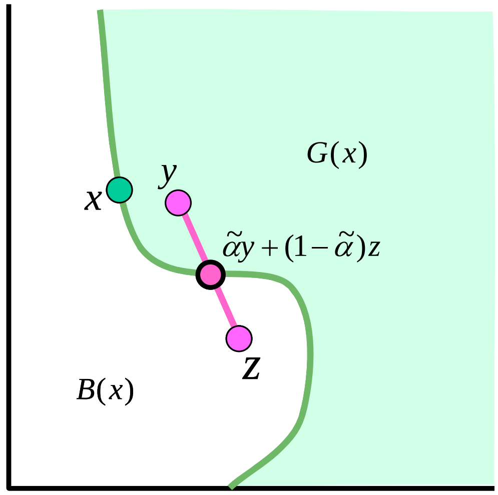


PROPOSITION 3.1. ***A function is continuous if and only if the inverse image of every open set is an open set.***

- **Above, the inverse image of V (set with green outline) is U, which is not open.**

DEFINITION 3.13. **A preference relation $\succsim$ is continuous** *if for all x, the sets $G(x)$ and $B(x)$ are closed.*

PROPOSITION 3.2. *If $\succsim$ is continuous and if $y \in G(x)$ and $z \in B(x)$ then for some $\tilde{\alpha} \in [0, 1]$ the point $\tilde{\alpha}y + (1 - \tilde{\alpha})z$ is indifferent to x.*

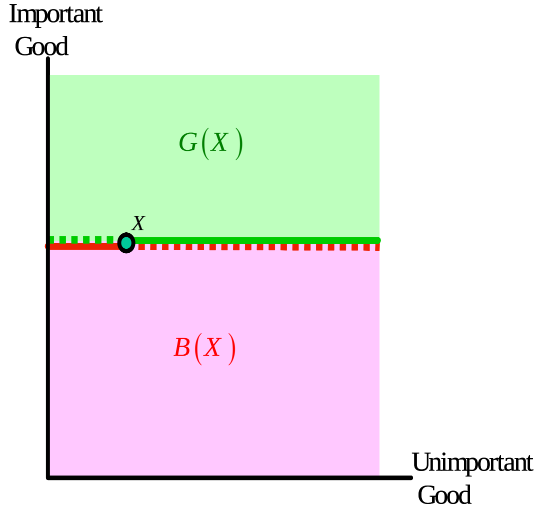


PROOF.

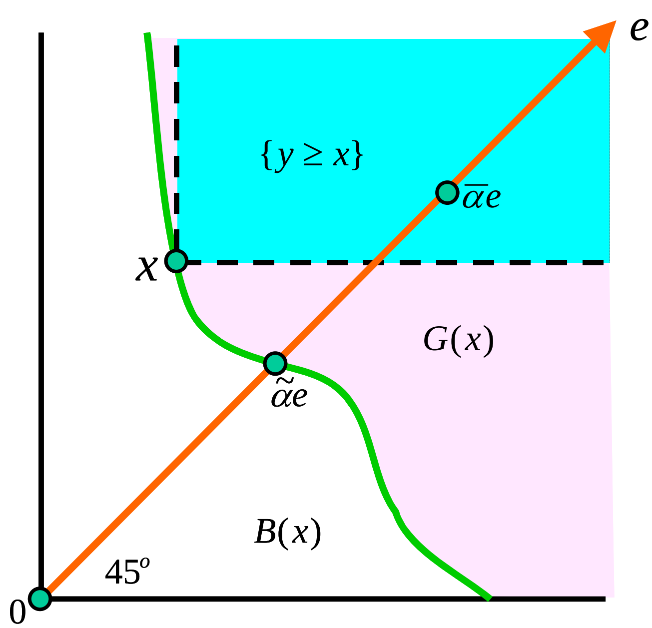
- By the continuity of $\succsim$, $B(x)$ and $G(x)$ are closed.
- Let $q(\alpha) = \alpha y + (1 - \alpha)z$.
- We have $q(0) = z \in B(x)$, $q(1) = y \in G(x)$.
- Let $\tilde{\alpha} = \sup\{\alpha \mid q(\alpha) \in B(x)\}$.
- $q(\tilde{\alpha})$ is a boundary point of $B(x)$ and of $B_c(x)$ [the complement of the set $B(x)$], because every neighborhood of $q(\tilde{\alpha})$ contains points in both sets [why?].
- But $B_c(x) \subseteq G(x)$ [why?]
- $q(\tilde{\alpha}) \in B(x)$ and $q(\tilde{\alpha}) \in G(x)$. [If $A \subseteq B$, B is closed and x is a boundary point of A, then $x \in B$.]
- Therefore $q(\tilde{\alpha}) \sim x$.

■

- **Lexicographical preferences are not continuous, so the proposition about indifference doesn't apply.**
- **Why not?**



PROPOSITION 3.3. ***If $\succsim$ is complete, transitive, monotonic and continuous, then it can be represented by a continuous utility function.***



PROOF (CONSTRUCTION).

1. Choose an $x \gg 0$. We will define $U(x)$.

2. Let e be the diagonal vector $\begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}$.

3. Choose $\bar{\alpha}$, such that $\bar{\alpha}e \gg x$.

4. By monotonicity, $\bar{\alpha}e \in G(x)$ and $0 \in B(x)$.

5. By continuity and the previous intermediate-value proposition, there is an $\tilde{\alpha}$ such that $\tilde{\alpha}e \sim x$.

6. Define the value of $U(x)$ to be $\tilde{\alpha}$.

7. We have: $x \sim U(x)e$.

■

PROOF OF REPRESENTATION. We show $U(x) \geq U(y) \iff x \succsim y$.

1. To show: $U(x) \geq U(y) \implies x \succsim y$:

(a) $U(x) = U(y) \implies x \sim U(x)e = U(y)e \sim y$ $\implies$[by transitivity] $x \sim y$.

(b) $U(x) > U(y) \implies U(x)e \gg U(y)e \implies$ [by monotonicity] $U(x)e \succ U(y)e \implies$ [by transitivity] $x \succ y$.

2. To show: $U(x) \geq U(y) \impliedby x \succsim y$:

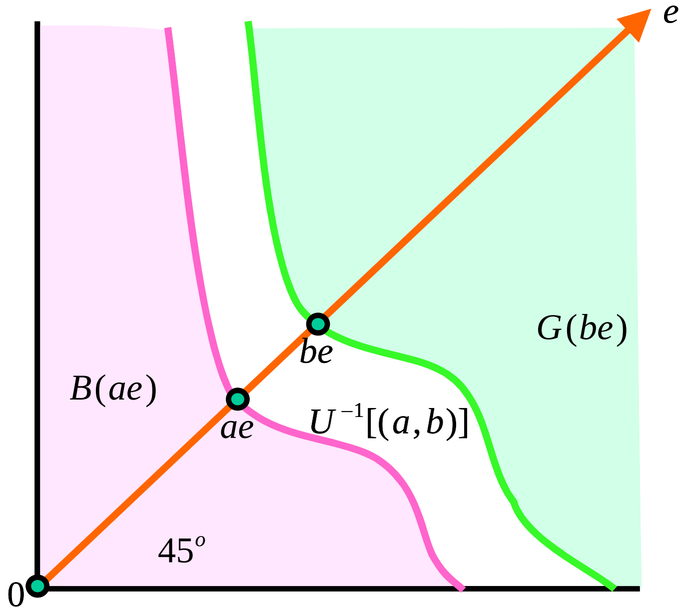
(a) $U(x) > U(y)$ and $y \succsim x$ contradict statement 1,

(b) so it follows that $y \succsim x \implies U(y) \geq U(x)$

(c) Switch x and y in the above statement.

■

PROOF OF THE CONTINUITY OF U. To show U is continuous, we show that the inverse image of any open interval (a, b) is an open set.



1. For this proof, we will need to use the following propositions. Let f be any function and let Y and Z be any sets in the codomain of f. Let S_C denote the complement of S.

(a) If $f^{-1}(Y)$ and $f^{-1}(Z)$ are closed sets, then $f^{-1}(Y \cup Z)$ is a closed set.

(b) $f^{-1}(Y_C) \equiv \left[f^{-1}(Y)\right]_C$

- *Prove these propositions as an exercise!*

2. $(a, b)_C = [0, a] \cup [b, \infty)$.

3. But $U^{-1}([0, a]) = B(ae)$, which, by the continuity of $\succsim$, must be closed,

4. and $U^{-1}([b, \infty)) = G(be)$, which must also be closed for the same reason.

5. Therefore, by 1a above, we know that $U^{-1}([0, a] \cup [b, \infty))$ is closed.

6. It follows that $U^{-1}([0,a] \cup [b,\infty))_C$ is open.

7. But by 1b, we have

$$U^{-1}((a,b)) \equiv U^{-1}(([0,a] \cup [b,\infty))_C) = U^{-1}([0,a] \cup [b,\infty))_C,$$

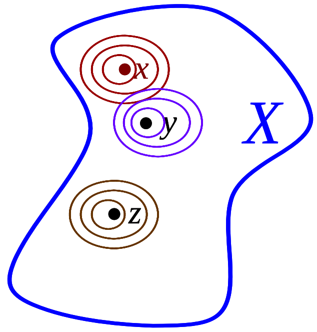
so that $U^{-1}((a,b))$ must be open.

■

EXAMPLE 3.2. *The utility function $U(x_1, x_2) = |x_2 - x_1|$ is continuous, but it represents preferences that lack monotonicity [why?]. However, those preferences can be represented by a utility function. Why is this possible? Can we use the above proof to demonstrate this fact?*

3.3 Sequences

- **The open sets of a space define the idea of "closeness" in a space.**



- **Think of it this way: open sets determine neighborhoods.**
- **And a point s is closer to x than other points are if it is in more (and smaller) neighborhoods of x.**

DEFINITION 3.14. ***A sequence is an ordered list of points, usually an infinite number of them. The list***

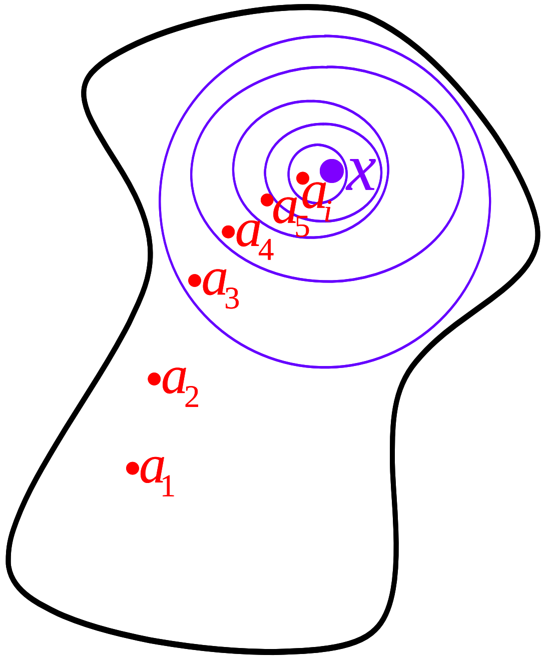
$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

is an example of an infinite sequence.

DEFINITION 3.15. ***A sequence is eventually in a set U if it starts either inside or outside of U, then goes inside and then doesn't come out again.***

DEFINITION 3.16. ***A sequence converges to a point x if the sequence is eventually in every neighborhood of x.***

- In the drawing below, the sequence $\{a_1, a_2, a_3, \ldots\}$ is converging to x, because it is eventually inside every neighborhood of x.



3.4 Compact Sets

Compact sets have a special kind of finiteness property.

"If you travel around a compact set long enough there will be a point in the set that you are frequently close to."

"If a set isn't compact, you can travel inside it forever, without being close to any point frequently."

DEFINITION 3.17. *A set C has the* convergent-subsequence property *if every sequence of points in C has a subsequence that converges to a point in C.*

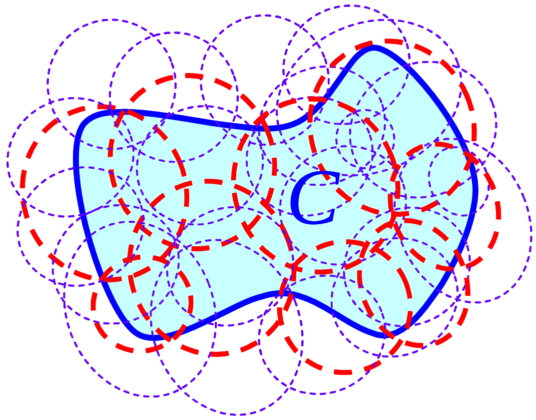
DEFINITION 3.18. *A set C is* compact *if has the convergent-subsequence property.*

EXAMPLE 3.3. *The set $\mathbb{R}_+ \equiv \{x \geq 0\}$ is not compact. Why not?*

- Consider the sequence $\{1, 2, 3, 4 \ldots\}$

Example 3.4. ***The interval $(0,\ 1)$ isn't compact. Why not?***

Definition 3.19. ***A set C has the* finite-subcover property *if the following is true:* `whenever` *an infinite number of open sets covers C, a finite collection of those open sets will be sufficient to cover C (the remaining sets are unnecessary).***



- **In the illustration, only the red sets (which are finite in number) are needed to cover C.**

Proposition 3.4. ***A set is compact if and only if it has the finite-subcover propery.***

Proposition 3.5. ***If C is compact and f is a continuous function, then $f(C)$ is compact.***

Proposition 3.6. ***Suppose C is compact and suppose F is closed and $F \subset C$. Then F is compact.***

DEFINITION 3.20. *A set X is* bounded *if it is contained in a sphere (of finite radius).*

PROPOSITION 3.7. (BOLZANO-WEIERSTRASS THEOREM) *In Euclidean vector spaces (including the real line), a set is compact if and only if it is closed and bounded.*

PROPOSITION 3.8. *If the image of a set S under a continuous function is unbounded, then S is NOT compact.*

PROPOSITION 3.9. *Any compact set $C \subset \mathbb{R}$ contains a maximum point (and a minimum point).*

PROPOSITION 3.10. (MAXIMUM VALUE THEOREM) *Suppose $C \subset \mathbb{R}^n$ is compact and suppose $f : C \to \mathbb{R}$ is continuous. Then f takes a maximum value on C.*

PROOF. Because C is compact and f is continuous, we know that $f(C)$ is a compact subset of $\mathbb{R}$, so that $f(C)$ has a maximum point.■

3.5 Utility Maximization

- **Finding the most preferred point in budget set:**

DEFINITION 3.21. *The* constrained utility-maximization problem *(UMP) is given by:*

$$\max u(x)$$
$$s.t.$$
$$x \geq 0$$
$$px \leq w.$$

PROPOSITION 3.11. *If u is continuous, then for any p, w, UMP has a solution. We write the set of solutions for each p, w as $x(p, w)$ [the demand correspondence].*

PROOF.

- The budget set $B_{p,w} \equiv \{x \mid px \leq w\}$ is compact.
- Because u is continuous, $u(B_{p,w})$ is also compact.
- $u(B_{p,w}) \subset \mathbb{R}$.
- Therefore $u(B_{p,w})$ contains its maximum value, $\bar{u}$.
- The set $u^{-1}(\bar{u}) \equiv \text{argmax}_x\{U(x) \mid px \leq w\} \equiv x(p, w)$.

■

PROPOSITION 3.12. *The demand correspondence $x(p, w)$ is homogeneous of degree 0*

PROOF. Budget set is unchanged when p, w is multiplied by a positive constant. ■

PROPOSITION 3.13. *If utility u is continuous, locally nonsatiated (lns), then $x(p, w)$ exists and satisfies Walras Law.*

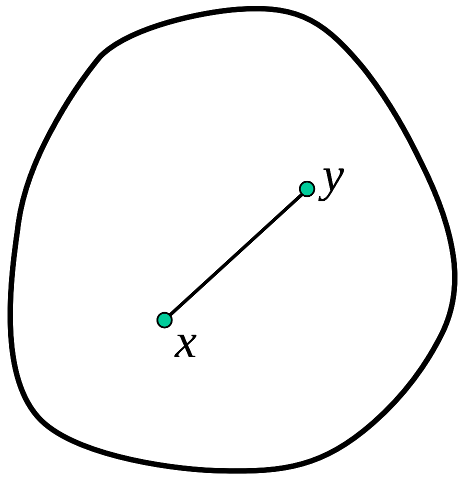
PROOF.

- By the continuity of u, we know that $x(p, w)$ exists.
- Suppose $x^* \in x(p, w)$ and $px^* < w$.
- Then there is a neighborhood N of x^* such that for any $x \in N$, $px < w$ (why?).
- By local non-satiation there is $\hat{x} \in N$, with $u(\hat{x}) > u(x^*)$, a contradiction to utility maximization.

■

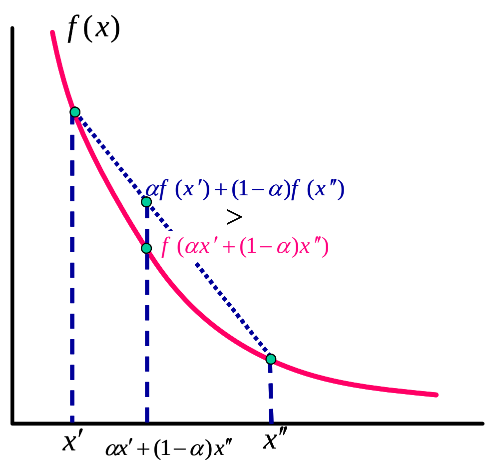
3.6 Convexity, Concavity and Quasiconcavity

DEFINITION 3.22. ***A set X in a vector space is convex if for any $x, y \in X$, the line $\overline{xy} \subset X$, that is if $\alpha x + (1-\alpha)y \in X$ whenever $0 \leq \alpha \leq 1$.***



DEFINITION 3.23. ***A function $f : X \to \mathbb{R}$ is convex if***

- ***for all $x, y \in X$, and $\alpha \in [0, 1]$,*** $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-a)f(y)$
- ***that is, the line between any two points on the graph of the function, lies on or above the graph.***



PROPOSITION 3.14. ***A function is convex if and only if the set above its graph, $F = \{(x, z) \mid x \in X, z \geq f(x)\}$, is convex.***

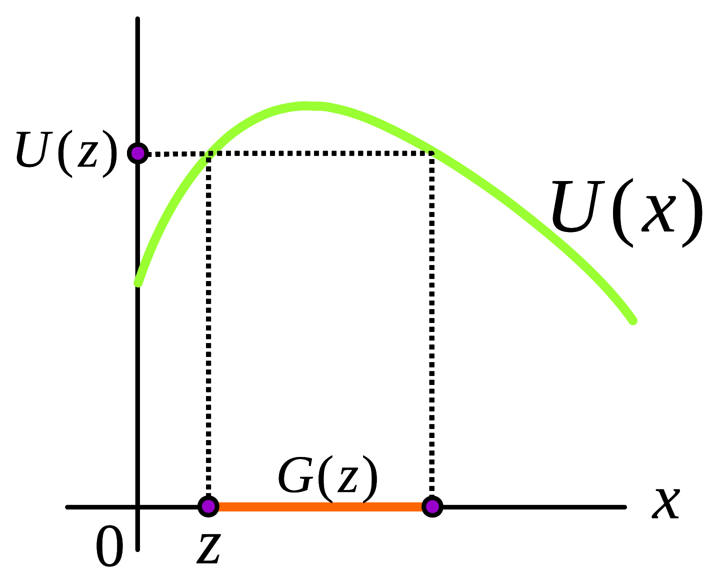
Definition 3.24. *A function $g : X \to \mathbb{R}$ is concave if and only if $-g$ is a convex function.*

Definition 3.25. *A function U is quasiconcave if for all x the sets $G(x) = \{y \mid U(y) \geq U(x)\}$ are convex. A function V is quasiconvex if for all p the sets $B(p) = \{q \mid V(q) \leq V(p)\}$ are convex.*

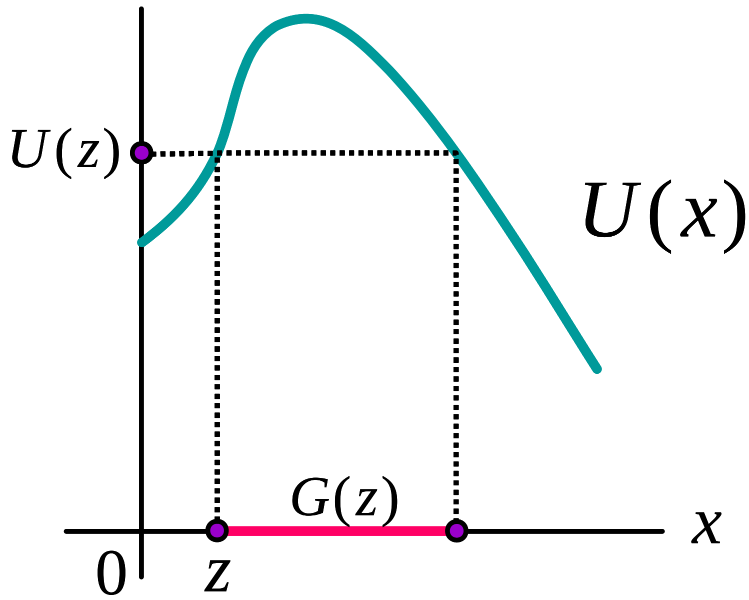
Proposition 3.15. *All convex functions are quasiconvex, and all concave functions are quasiconcave (but not the other way round).*

Proof. Try it as an exercise. ■

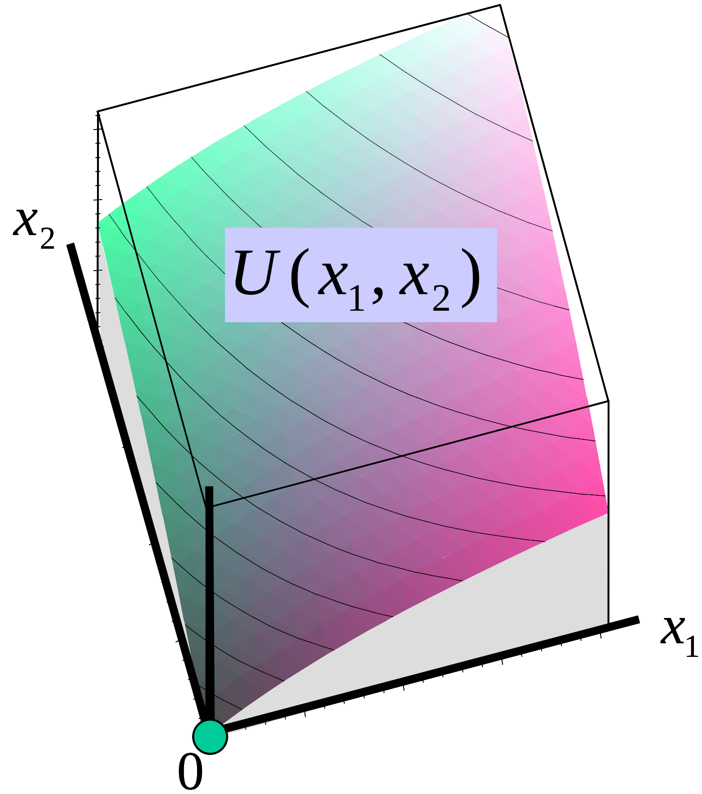
- Concavity is observed in the graph of a function.
- Quasiconcavity is observed in the domain of the function.
- Concave and quasiconcave.



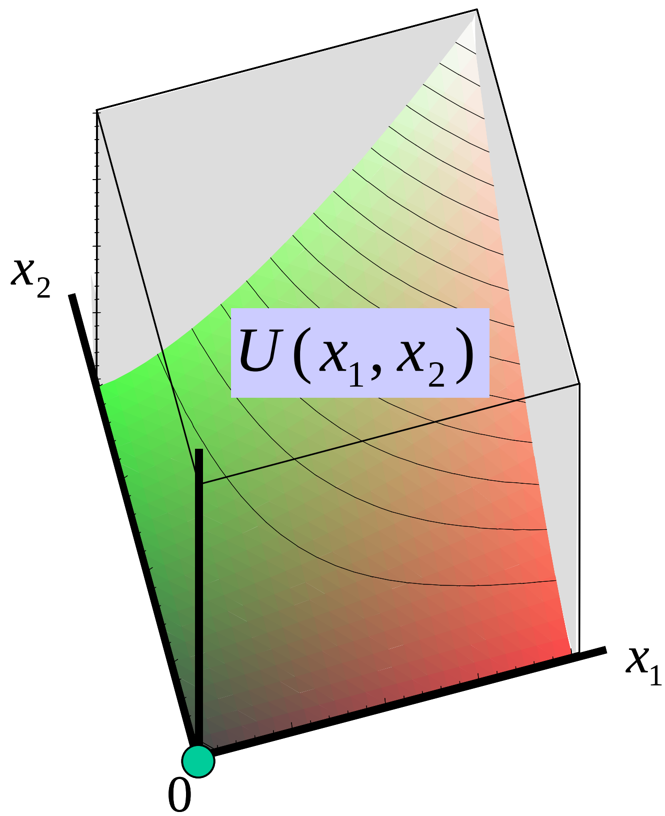
- **Quasiconcave but not concave.**



- **The following are 3-dimensional graphs of two utility functions**
- **The graph below represents $U(x_1, x_2) \equiv 3 \ln(x_1 + 1)(x_2 + 1)$.**
  - **This utility function is concave, therefore, quasiconcave.**
  - **The surface curves downwards not only from side-to-side as above, but also along a ray from the origin.**

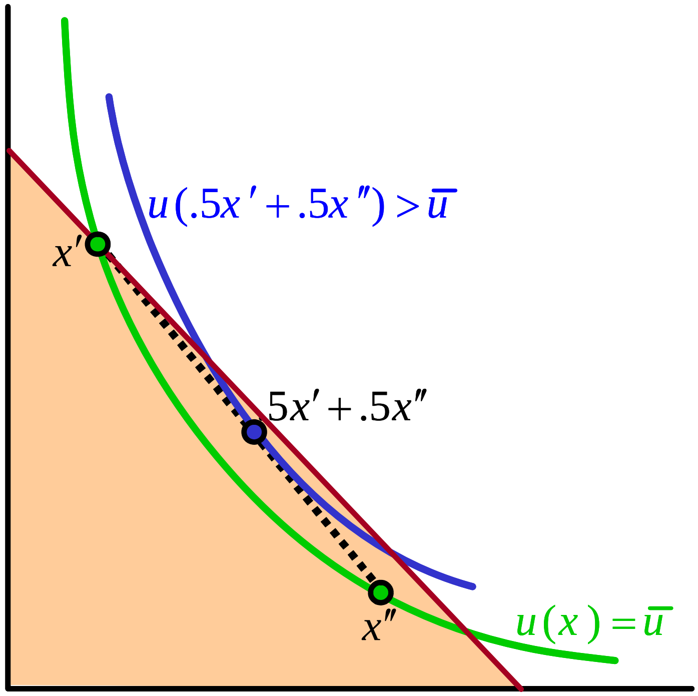


- **The graph below represents $U(x_1, x_2) \equiv (x_1 x_2)^{1.5}$.**
  - **The function is quasiconcave, because the G sets are convex sets.**
  - **However, the function is not concave, because the set under the surface is not convex.**
  - **The reason is that the surface curves up as you move away from the origin along any ray.**



- **Notice that both utility functions represent very similar preferences.**
- **Quasiconcave functions can be made concave by monotonic transformations, so the preferences that they represent need not change.**

Proposition 3.16. ***If u is strictly quasiconcave, $x(p,w)$ is a function.***



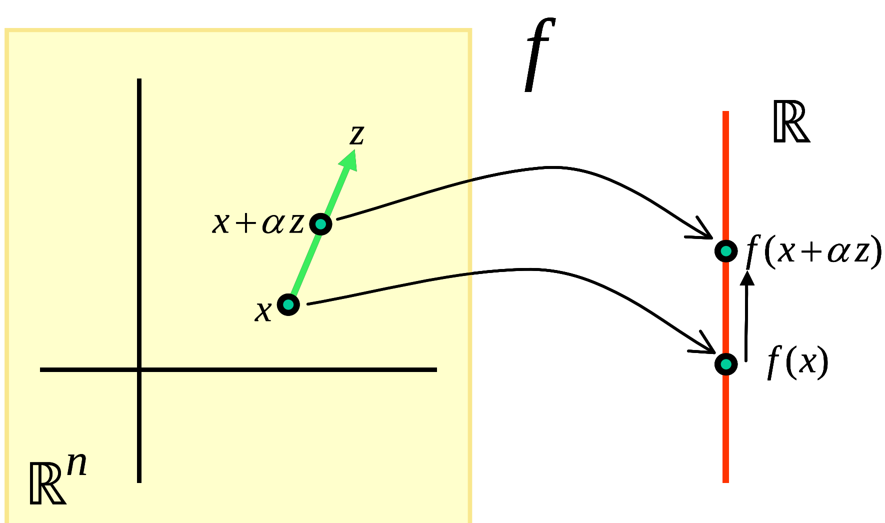
Proof.

- **Suppose $x', x'' \in x(p,w)$.**
- **Then $.5x' + .5x''$ is in $B_{p,w}$ (budget sets are convex).**
- **But by strict q.c. $u(.5x' + .5x'') > u(x') = u(x'')$**
  - **remember: with strict q.c., lines lie above indifference curves**
- **This implies x' and x'' are not utility-maximizing, a contradiction.** ∎

3.7 Multivariate Optimization and Utility Maximization

DEFINITION 3.26. ***Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then $\frac{\partial f(x)}{\partial x} \equiv \nabla f(x) \equiv [\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n}]$ is called the gradient of f at x.***

DEFINITION 3.27. ***Let z be a vector of unit length ($\|z\| = 1$) and let x be a point in $\mathbb{R}^n$. Define $h(\alpha) = f(x + \alpha z)$. The directional derivative of f at x in the direction z, is given by $h'(0)$.***



PROPOSITION 3.17. ***If z is any vector of unit length, then $\nabla f(x) \cdot z$ is the directional derivative of f in the direction z, so that***

$$f(x + \alpha z) = f(x) + (\nabla f(x) \cdot z)\alpha + \text{(second-order-small error)}$$

The derivative of f in the direction $\nabla f(x)$ is the maximum directional derivative for all directions, which means that $\nabla f(x)$ is the direction of the maximum rate of change of f at x.

PROOF. **First, we show that $\nabla f(x) \cdot z$ is the directional derivative:**

- **By the chain rule,**

$$h'(\alpha) = \sum_i \frac{\partial f(x + \alpha z)}{\partial x_i} \frac{\partial (x_i + \alpha z_i)}{\partial \alpha}$$

$$\equiv \sum_i \frac{\partial f(x + \alpha z)}{\partial x_i} z_i \equiv \nabla f(x + \alpha z) \cdot z,$$

- **so that $h'(0) \equiv \nabla f(x) \cdot z$.**
- $h(\alpha) =$**(approx)** $h(0) + h'(0)\alpha$,
- **Therefore,**

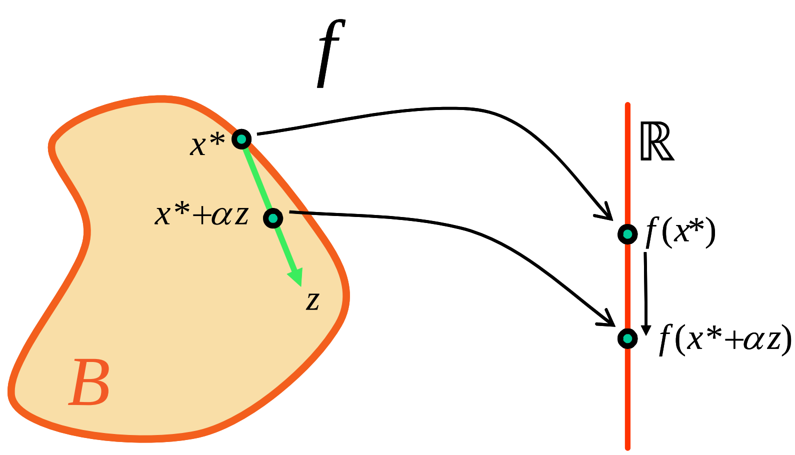
$$f(x + \alpha z) = \textbf{(approx)}\ f(x) + (\nabla f(x) \cdot z)\alpha.$$

Now we show that the directional derivative is greatest in direction ∇f

- $u \cdot v = \|u\| \, \|v\| \cos\theta$, where θ is the angle between u and v.
- Let $\bar{z} = \frac{1}{\|\nabla f\|}\nabla f$,
  - direction of $\bar{z}$ = direction of ∇f
  - $\|\bar{z}\| = 1$
  - so that $\nabla f \cdot \bar{z} = \|\nabla f\| \, \|\bar{z}\| \cos 0 = \|\nabla f\|$
- If $\|z\| = 1$, $\nabla f \cdot z = \|\nabla f\| \, \|z\| \cos\theta \leq \|\nabla f\|$
- which means that the derivative in the direction $\bar{z}$ (= direction ∇f) gives the maximum directional derivative.

■

Proposition 3.18. **(Kuhn-Tucker Necessary Conditions)** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable and let $B \subset \mathbb{R}^n$ with $x^* \in B$. Then $f(x^*) = \max\{f(x) \mid x \in B\}$ only if the directional derivative of $\nabla f(x^*) \cdot z \leq 0$ for every direction z within B (that is, for every direction z such that for all positive and sufficiently small α, we have $x^* + \alpha z \in B$).*



PROOF. **We prove the equivalent contrapositive proposition.**

- **Suppose that for some z we have $x^* + \alpha z \in B$ for all α sufficiently small.**
- **Then if $\nabla f(x^*) \cdot z > 0$, we have $f(x^* + \alpha z) > f(x^*)$ for small α.**
- **Therefore $f(x^*)$ is not a maximum for $x \in B$.**

■

- **Now we can apply this to the UMP.**
- **Assume U is well-behaved (continuous and twice differentiable).**
- **Then we have...**

PROPOSITION 3.19. ***If $x^* \gg 0$ is the solution to***

$$\max_x U(x)$$

$$\text{such that}$$

$$x \geq 0$$

$$px = w$$

then for some $\lambda > 0$, $\nabla U(x^*) = \lambda p$ or, equivalently

$$\frac{\partial U}{\partial x_1} = \lambda p_1$$

$$\ddots$$

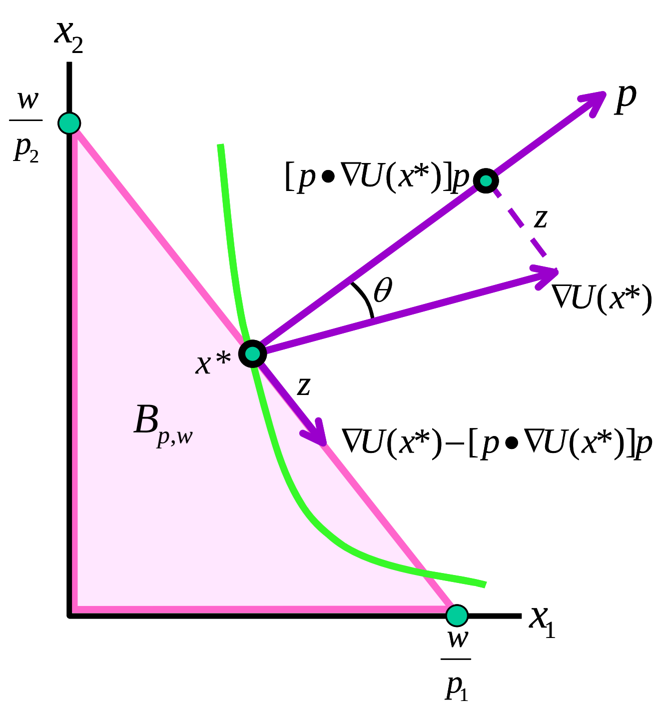
$$\frac{\partial U}{\partial x_L} = \lambda p_L$$

[Economic interpretation: If a positive quantity of every good is demanded, then marginal utility per dollar must be the same for every good.]

Informal economic proof:

- Suppose $\mathrm{MU}_i / p_i > \mathrm{MU}_j / p_j$.
- Decrease demand for j by Δx_j.
  - Utility loss = $\mathrm{MU}_j \, \Delta x_j$
  - Money saved = $p_j \Delta x_j$
- Use money save to by x_i
  - can buy $\Delta x_i = p_j \Delta x_j / p_i$
  - Utility gain = $\mathrm{MU}_i \, p_j \Delta x_j / p_i >$ utility loss.
- Utility was not maximized.

PROOF.



- Suppose $\nabla U(x^*)$ is not a multiple of p.
- Assume $\|p\| = \sqrt{p \cdot p} = 1$ (also $p \cdot p = 1$).
- The projection of $\nabla U(x^*)$ onto p is $[p \cdot \nabla U(x^*)]p$
- So $z = \nabla U(x^*) - [p \cdot \nabla U(x^*)]p$ is orthogonal to p. [Why?] [See illustration]

- For α small enough, $x^* + \alpha z \gg 0$
- $p(x^* + \alpha z) = px^* = w$
- Therefore $x^* + \alpha z$ is in the budget set.
- But the derivative of U in the direction z is

$$\nabla U(x^*)z = \nabla U(x^*) \cdot \nabla U(x^*) - (p \cdot \nabla U(x^*))^2 > 0$$

[why?]

- Conclusion: x^* does not maximize U.

■

EXAMPLE 3.5. ***Construct a utility function that corresponds to the following demand schedule:***

$$x_1 = \frac{2w}{3p_1}$$

$$x_2 = \frac{w}{3p_2}.$$

- Solution strategy for example above:
  - The proposition implies that any indifference curve must be orthoginal to p at the quantities demanded x. Why?
  - We will try to find a function $x_2 = x_2(x_1)$ that describes a curve with that property. This will be our candidate indifference curve.
  - We then find a utility function $U(x_1, x_2)$ that is constant everywhere on the candidate indifference curve.
  - Finally, we check to see if that utility function produces the desired demand curve.

- Note that $x_1, x_2 \gg 0$ at any positive prices, so that the UMP must have only interior solutions.
- Solving for prices we find the demand-price function:

$$p_1 = \frac{2w}{3x_1}$$

$$p_2 = \frac{w}{3x_2}.$$

- The slope of the price vector is p_2/p_1, so if the curve $x_2(x_1)$ is orthoginal to p, then the slope of $x_2(x_1)$ must be $-p_1/p_2$.

- **If we apply the demand-price function, we have the equation**

$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2} = -\frac{2x_2}{x_1}.$$

This can be represented as the differential equation

$$\frac{dx_2}{x_2} = -2\frac{dx_1}{x_1},$$

which must be true for $x_2(x_1)$, our candidate indifference curve.

- **Integrating both sides gives us the following implicit formula for $x_2(x_1)$:**

$$\log x_2(x_1) = -2\log x_1 + C,$$

or

$$C = 2\log x_1 + \log x_2(x_1),$$

where C is the constant of integration.

- **If we define a utilty function by**

$$u(x_1, x_2) \equiv 2\log x_1 + \log x_2,$$

then $u(x_1, x_2(x_1)) = C$ for all combinations $(x_1, x_2(x_1))$, so that this definition of u satisfies the requirement that it be constant on the candidate indifference curve.

- **Using a monotonic transformation, we can transform $u(x_1, x_2)$ to**

$$U(x_1, x_2) = x_1^{2/3} x_2^{1/3},$$

which is a Cobb-Douglas function.

- **Finally, it is clear that this utility function yields the desired demand function.**