# Manove's Microeconomic Theory Slides

[Partly based on materials from the textbook for this course: Mas-Colell, Winston and Green, Microeconomic Theory, Oxford University Press, 1995]



# 1. Introduction

## 1.1 Administrative Matters

- Michael Manove / Jonathan Treussard
- My webpage

  http://www.bu.edu/econ/faculty/manove/
- Days and Times:
  - Classes: MW 5:00-6:30 room CAS 326
  - Problem Sections: F 9:00-10:30 CAS 326
  - My Office Hours: T 5:00-6:30 and F 3:30-5
  - Jonathan's Office Hours: M 3:30-5 and R 6-7:30

- 90 percent of what you learn in this course will be learned by doing problems.
  - Do problem sets;
  - Do the exams of previous years. On course webpage, see

Course Materials from Previous Years

- Textbook: Andreu Mas-Colell, Michael Winston, Jerry Green. *Microeconomics*.
- Please buy it. Excellent reference book.
- Please read the book and take your own class notes.
- But remember that problems are the most important part of the course.



## 1.2 How To Get an A in EC 701

- Do many problems.
- Attend class and discussion section.
- Go to bed by 11pm; get up by 7am.
- Do not memorize material.
- Do not copy the solutions to the problem sets.
- Do not solve problems in a mechanical way.
- Explain the material and problem solutions to your friends, your brothers, your dogs, and your aunts.
- Do economics, not mathematics.

## 1.3 Do Economics, not Mathematics

- **This is an example of an economics problem. We will review the material in detail during the next two weeks. If you cannot follow this example, do not worry!**

PROBLEM 1.1. ***Jordi wants to buy spoons $x_1$ and forks $x_2$. Each pair of one spoon and one fork gives Jordi $1$ unit of utility. But a spoon not matched with a fork gives him only $a$ units of utility, where $a < 1/2$. A fork not matched with a spoon also gives $a$. Let $p_1$ be the price of spoons, $p_2$ price of forks, and $w$, wealth. Find Jordi's demand function for spoons and forks.***

- **How should we think about this problem?**

- **Jordi wants to get the most utility for each dollar he spends.**



- **Which is the better purchase, singles (only spoons or only forks) or pairs?**
  - **The consumer gets $1$ unit of utility for each *pair* (one spoon and one fork) plus $a$ units of utility for each *single* (additional spoons or forks).**
  - **The price of pairs is $p_1 + p_2$, so the utility per dollar from buying pairs is**

$$\frac{1}{p_1 + p_2}.$$

- If Jordi buys singles, he will buy only the cheaper single. So assume that $p_1 \leq p_2$ so that Jordi will buy spoons (he will be indifferent if prices are the same).
- Then for singles, the utility per dollar is $a/p_1$.
- Therefore, Jordi will buy only singles if

$$\frac{a}{p_1} > \frac{1}{p_1 + p_2}$$

which is equivalent to

$$p_1 < \frac{a}{1-a} p_2.$$

Jordi will buy only pairs if the reverse is true.

- Does this make sense if $a = 0$? if $a = .49$?



- Demand:
  - If Jordi buys only singles, he will buy $w/p_1$ spoons,
  - but if he buys pairs, he will buy $w/(p_1 + p_2)$ pairs of spoons and forks.
- Therefore, assuming that $p_1 < p_2$, the demand function is given by

$$x_1 = \begin{cases} \frac{w}{p_1} & \text{for} \quad p_1 < \frac{a}{1-a} p_2 \\ \frac{w}{p_1+p_2} & \text{for} \quad p_2 > p_1 > \frac{a}{1-a} p_2 \end{cases}$$

and

$$x_2 = \begin{cases} 0 & \text{for} \quad p_1 < \frac{a}{1-a} p_2 \\ \frac{w}{p_1+p_2} & \text{for} \quad p_2 > p_1 > \frac{a}{1-a} p_2 \end{cases}$$

- **Challenge: solve this problem using formal mathematical tools!**
- **Can you define a utility function of spoons and forks?**



- **The utility function looks like this.**

$$U(x_1, x_2) = \min\{x_1, x_2\} + a\left(\max\{x_1, x_2\} - \min\{x_1, x_2\}\right)$$

**where**

$$0 < a < 1/2.$$

- **The budget constraint is**

$$p_1 x_1 + p_2 x_2 \leq w.$$

- **How would you solve the constrained maximization problem?**

$$\max_{x_1,x_2} U(x_1, x_2)$$

**subject to:**

$$p_1x_1 + p_2x_2 \leq w,$$

$$x_1, x_2 \geq 0.$$

  - **Would you take derivatives and set them to *0*?**



## 1.4 Why Study Micro Theory?

- **Basic tools for all of economics, including empirical economics.**
  - **Microeconomic theory itself doesn't say very much. You use theory as a tool so that you can say things about economy.**
  - **Tools are basic and general.**
  - **Tools are useful for building models**
  - **All of my own research uses the tools I will teach you.**

## 1.5 The principal topics in EC701

- Neoclassical Preference and Choice: how individuals make decisions.
  - Probably the weakest part of microeconomics. Complete rationality assumed.
  - Little psychology, little empirical evidence.
  - Source of preferences or reasons for choices are not explained.
  - Consistency in preference or choice is required.
  - Theory of demand.



- Modern fields, behavioral economics and behavioral finance, use psychological concepts, rather than a simple presumption of economic rationality.
  - Founders: Daniel Kahneman & Amos Tversky, among others.
  - Psychologists who changed economics.
- Modern decision theory also reflects these recent changes.
  - Psychological topics, e.g. temptation, are frequently modeled.

- **Production: decision-making by owners of firms.**
  - **Strong rationality more reasonable, especially for large firms: built into firm structure, entrepreneurs, managers, accountants, outside directors, etc.**
  - **Revenues, costs, profit maximization**
  - **Theory of supply**



- **Equilibrium: a stable economic state.**
- **Theory of Strategic Interaction (Game Theory)**
  - **Tool for defining and finding various types of equilibria.**
  - **When all agents are small (competitive markets) or when only one agent is large, price mediates all interactions between agents.**
    - **Little interesting strategic interaction.**
    - **If you know the price, you don't care what the other agents are doing.**
  - **More than one large agent $\Longrightarrow$ strategic interaction is usually important**
    - **You want to respond to the actions of other agents.**
  - **Game theory is important tool for models.**

**Types of equilibria:**

- **Competitive equilibria**
  - **Prices summarize strategies of all agents.**
  - **Equilibrium exists when prices and strategies are consistent.**
- **Many other types of equilibria are defined by game-theoretical solution concepts.**



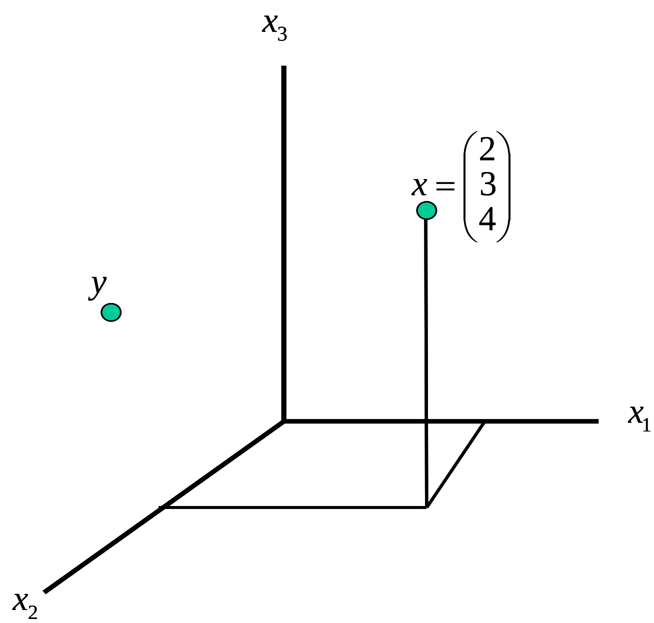
# 2. Preference and Choice

## 2.1 Preferences

- **Preferences are psychological entities.**
- **Most aspects of preferences are usually ignored by economists.**
  - **Origin ignored**
  - **Causes ignored**
  - **Intensity ignored**
  - **Dynamics ignored**
    - **What causes preferences to change?**
    - **What are the effects of changing preferences?**
    - **Equilibria of preferences.**

## General representation of preferences in economics:

- **Commodity space: each point is a bundle of goods.**






- **Binary relation $\succsim$ defined on commodity space $X$**
- **$x \in X$ is a vector of quantities of each commodity in existence.**
- **For $x, y \in X$, $y \succsim x$ means that $y$ is at least as preferred (desired) as $x$.**

- Rationality
  - Completeness: Either $y \succsim x$ or $x \succsim y$ or both.
    - Obviously false for real people:
    - People don't know characteristics of most goods.
    - People don't know how characteristics will affect them.
    - Worse: people make decisions without knowing their preferences.
  - Transitivity (consistency): If $y \succsim x$ and $x \succsim v$ then $y \succsim v$.



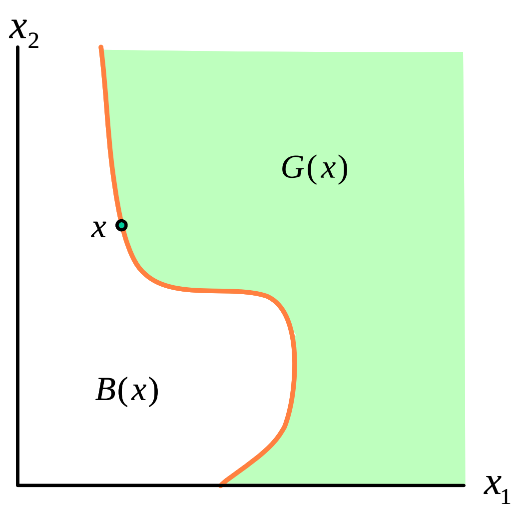
- Failure of rationality:
  - (read pp 7-8 Mas-Colell small print); Matthew Rabin, JEL
  - Framing, and other heuristic biases
  - Taste a function of state (dynamics)
- Unrealistic models may be useful. Why?

## Contour Sets

- **Upper contour sets** (as-good-as sets): for each $x \in X$, define $G(x) \equiv \{y \mid y \succsim x\}$
  - $G(x)$ is the set of all bundles that are as good (preferred or desired) as $x$.
  - $G(x) \subset X$
  - $G$ is a function that maps points in $X$ into subsets of $X$.
    - $\mathcal{P}(X)$ denotes the power set of $X$, which is the set of all subsets of $X$
    - $G : X \to \mathcal{P}(X)$
  - The function $G$ is a general, precise and complete mathematical description of $\succsim$.



- For well behaved $G$, boundaries of the sets $G(x)$ are the indifference curves. Why?




- We also define **lower contour sets** (no-better-than sets), $B(x) \equiv \{y \mid x \in G(y)\}$.
- Indifference curves are given by $I(x) \equiv B(x) \cap G(x)$.

## Utility Functions

- If $U : X \to \mathbb{R}$ has the property:

$$U(x) \geq U(y) \iff x \succsim y.$$

then $U$ is a utility function that describes the preferences $\succsim$.

  - For example, if $U(x) = -3.7$ and $U(y) = -4$, then $x \succsim y$.
  - The utility values provide an order for the commodity bundles but have no other meaning.
  - In classical theory, utility was intended to be a measure of consumer satisfaction (a psychological construct) ...
  - ...in the same way that IQ is intended to be a measure of intelligence.



- Not all preferences can be represented by a utility function, but:

PROPOSITION 2.1. *If the preference relation $\succsim$ can be described by a utility function $U$, then $\succsim$ is rational.*

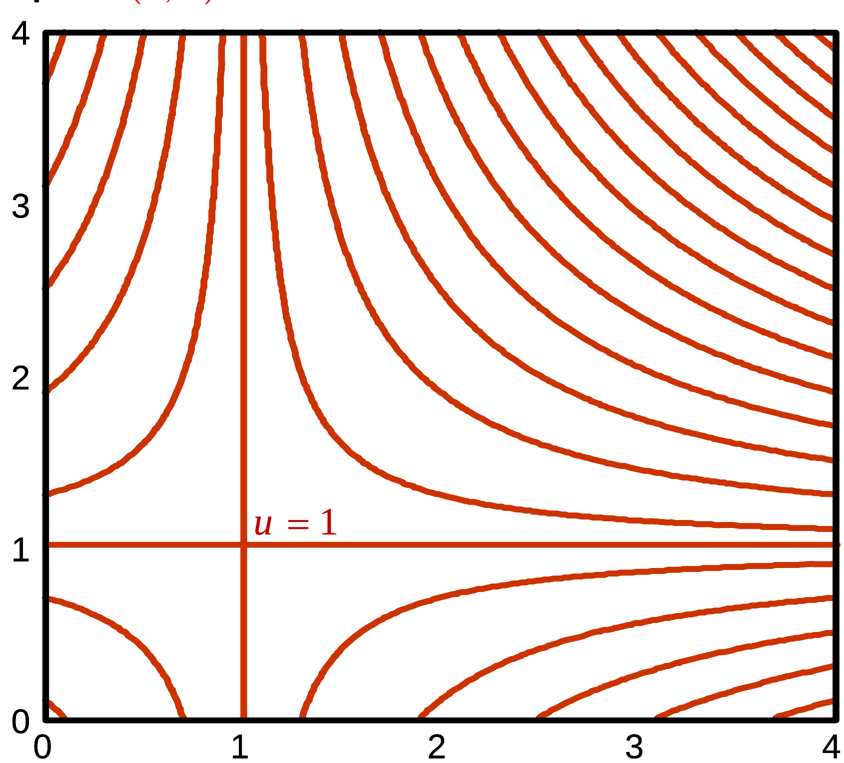
- Prove this as an exercise.

**Example 2.1.** ***Inma likes wine, and Inma likes beer. The more the better. But Inma doesn't like them together. Construct a continuous utility function and upper contour sets for this example.***

- Beer $(c)$ and wine $(v)$ are both good, but when both are present, there is a negative effect.
- How can we represent this?
  - One possibility is a negative product term, $-cv$, in the utility function.
    - If $c, v > 0$, some utility is lost.
    - But if $c = 0$ and $v > 0$ (or $c > 0$ and $v = 0$), then no utility is lost.

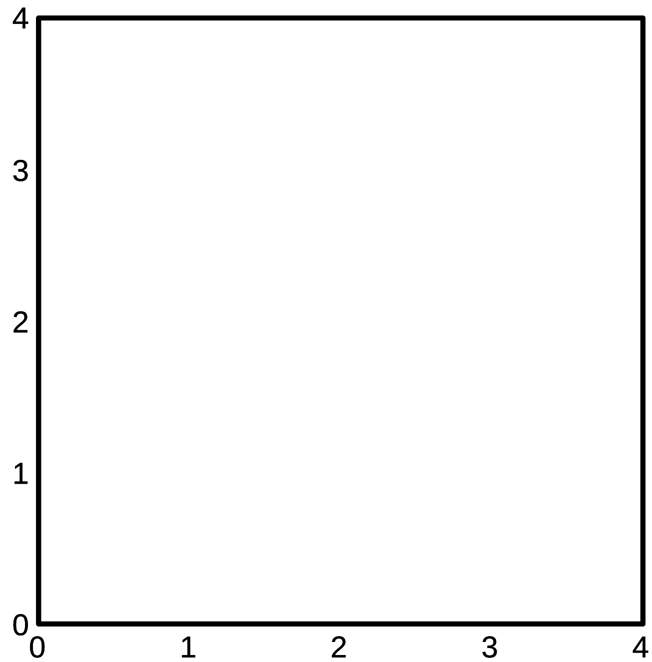


- For example $U(c, v) = c + v - cv$




  - What are the utility values of the various indifference curves?

- **Another example:** $U(c, v) = |c - v|$.






## 2.2 Consumer Choice (Neoclassical Model)

- **Notation**

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_L \end{bmatrix}$$, **commodity vector (consumption bundle)**

- **Commodity space:** $X = \{x \mid x \in \mathbb{R}^L_+\}$, **positive orthant of** $L$**-dimensional real Euclidean space.**

- **We assume linear prices: price per unit not a function of how much you buy.**
- **Nonlinear prices?**
  - **Price of leisure?**
  - **Mobile phone calls? (not truly competitive)**



- **Price space, $p \in \mathbb{R}^L_+$.**

$$p = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix} \equiv \textbf{price vector}$$

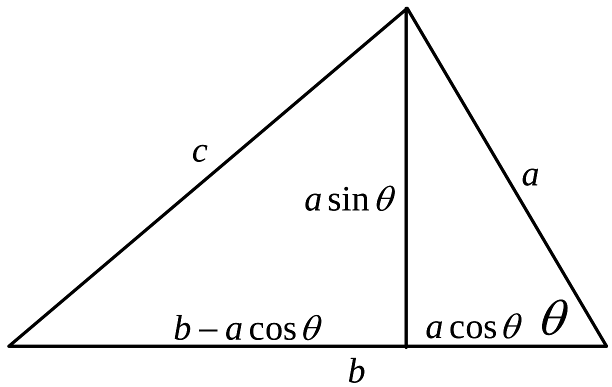
- **$px$ = expenditure required to buy $x$ [$px \equiv p \cdot x \equiv$ dot product].**
  - **Example:**

$$p = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \ x = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 8 \end{bmatrix}$$

**then**

$$px = (2 \times 3) + (1 \times 5) + (3 \times 2) + (4 \times 8) = 49.$$

- **An important fact about dot products.**
- **In the triange below** $c^2 = a^2 + b^2 - 2ab\cos\theta$. **Why?**




PROBLEM 2.1. ***Show that if*** $x$ ***and*** $y$ ***are vectors, then*** $xy = |x|\,|y|\cos\theta$.



PROBLEM 2.2. ***Consider three points in space,*** $v$, $x$ ***and*** $y$***,and consider the lines*** $\overline{vx}$ ***and*** $\overline{vy}$***. Show that*** $\overline{vx}$ ***and*** $\overline{vy}$ ***are orthogonal (perpendicular) if and only if the inner product*** $(x - v)(y - v) = 0$.

- **Budget set:** $B_{p,w} = \{x \mid px \le w\}$, **where** $w =$ **income or wealth.**
- **Budget frontier (line or hyperplane)** $\bar{B} = \{x \mid px = w\}$

PROPOSITION 2.2. ***$p$ is orthogonal (perpendicular) to the budget frontier $\bar{B}$ (that is, to any line in $\bar{B}$ )***

PROOF. **If $x, y \in \bar{B}$, then the inner product is $p(y-x)$.**

$$p(y-x) = py - px = w - w = 0.$$

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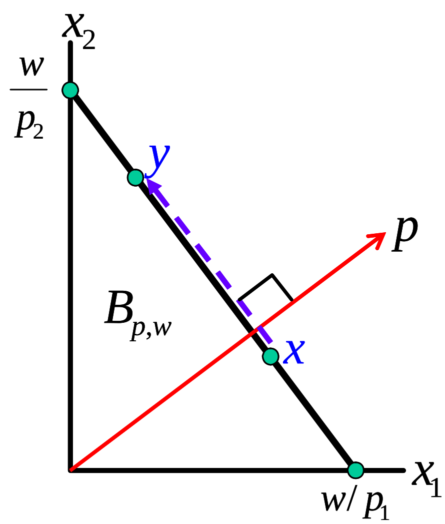
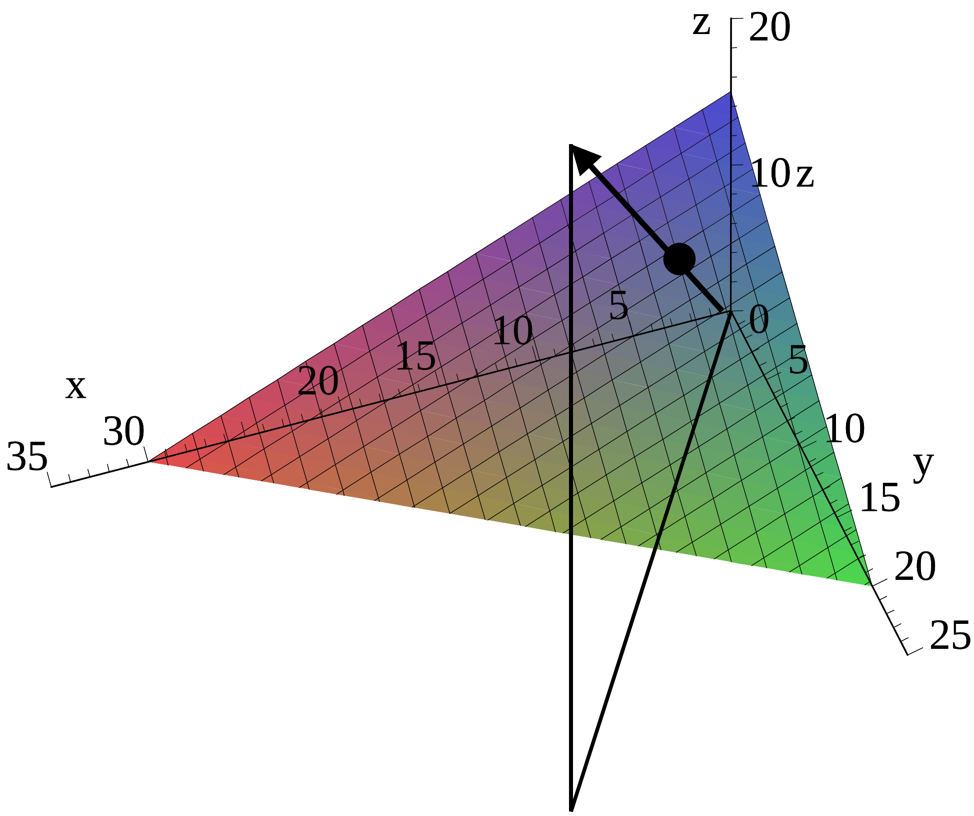





EXAMPLE 2.2. ***Suppose we have commodities in amounts $x, y$ and $z$, with $p = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, and $w = 60$. Draw a graph of the price vector and the budget set.***

- **The intercepts of the budget set are they amounts of each good the consumer could buy if he bought nothing else: $60/2$, $60/3$ and $60/4$.**
- **The budget set is bounded by the two-dimensional plane with those intercepts.**
- **The price vector must be orthogonal to the buget set.**






## 2.3 Choice and Revealed Preference

- **Can preferences or utility be oberved or measured directly?**
  - **Could a psychologist discover the nature of a person's utility function by asking her good questions?**
  - **Does your aunt get more utility from green tea or from black tea?**

- Economists like to think that "you cannot get inside a person's head" so that utility is fundamentally unobservable.
- Therefore, we have developed a system for inferring utility from a person's behavior in the market place.
- Econometricians have tried to construct utility functions by analyzing market data.
  - see Richard W. Blundell, Martin Browning, Ian A. Crawford, 2003, "Nonparametric Engel Curves and Revealed Preference," *Econometrica*, vol. 71(1), pages 205-240.
  - They use revealed-preference theory (explained below) to do that.



- Let $P \equiv \mathbb{R}^L_+$ denote the space of prices, $W \equiv \mathbb{R}_+$, the space of income values (or wealth) and $X \equiv \mathbb{R}^L_+$, commodity space.
- Let $\succsim$ define preferences on $X$, and for each $x \in X$, and let $G(x)$ and $B(x)$ represent the corresponding upper and lower contour sets of $x$.
- Let $\mathcal{S}(X)$ denote a set of subsets of commodity space $X$.
  - If $S \in \mathcal{S}(X)$, then $S \subseteq X$.
  - [Hint: Think of $S$ as a budget set .]

Definition 2.1. ***The function $C$ is a choice function on the sets $\mathcal{S}(X)$ if for any set $S \in \mathcal{S}(X)$, $C(S)$ is a nonempty subset of $S$. Think of $C$ as a function that identifies the commodity bundles a person would choose within a set of commodity bundles.***

Definition 2.2. ***The choice function $C$ is said to correspond to preferences $\succsim$ if for any $S \in \mathcal{S}(X)$, $C(S)$ is given by***

$$C(S) \equiv \{x \in S \mid S \subset B(x)\},$$

***where $B(x)$ is the lower contour set of $x$.***

- **This definition says that no points in $S$ are better than the points in $C(S)$.**
- **When $C$ corresponds to preferences, you can think of $C$ as a function that chooses the best commodity bundles in $S$.**



- **Note that $C(S) \subset S$.**
- **If $\bar{x} \in C(S)$ and $x \in S$, then $\bar{x} \succsim x$.**
- **How would you illustrate $C(S)$ by using indifference curves?**

DEFINITION 2.3. *If for some $S \subset X$, we have $x, y \in S$ and $x \in C(S)$, we say "$x$ is revealed as good as $y$."*

DEFINITION 2.4. *If for some $S \subset X$, we have $x, y \in S$ and $x \in C(S)$ and $y \notin C(S)$, we say "$x$ is revealed preferred to $y$."*

DEFINITION 2.5. *The choice function $C$ satisfies the weak axiom of revealed preference if whenever $x$ is revealed as good as $y$, then $y$ cannot be revealed preferred to $x$.*

- The weak axiom of revealed preference assures consistency in choice.



PROPOSITION 2.3. *If $\succsim$ is rational, then $C$ satisfies the weak axiom of revealed preference.*

PROOF. Suppose $x$ is revealed as good as $y$.
- Then for some $S$, we have $x, y \in S$ and $x \in C(S)$.
- Therefore $x \succsim y$.
- Let $\hat{S}$ be a different set, and suppose $x, y \in \hat{S}$ and $y \in C(\hat{S})$.
- We show that $x \in C(\hat{S})$ [otherwise $y$ would be preferred to $x$]
  - Let $z \in \hat{S}$.
  - We know that $y \succsim z$.
  - By transitivity $x \succsim z$.
  - But this is true for all $z \in \hat{S}$, which implies that $x \in C(\hat{S})$.

Therefore $y$ is not revealed preferred to $x$, and the weak axiom is satisfied. ■

# 2.4 Demand Functions and Comparative Statics

## The Definition of Demand

DEFINITION 2.6. *In the neoclassical model of consumer behavior, a demand function $x : P \times W \to X$, maps prices and income (or wealth) into commodity-choice vectors as follows: $x(p, w) = C(B_{p,w})$.*

- **In general, demand is a correspondence; $x(p, w) \subset X$, but we usually assume that $C(B_{p,w})$ contains only one point so that demand is a function.**
- **Definition of demand implies that its value is completely determined by the budget set $B_{p,w}$ and the consumer-choice function $C$.**
  - **No money in this model.**
  - **Price and wealth determine budget set, nothing more.**



- **What kind of information about the consumer does the demand function contain?**
  - **How much a consumer will buy?**
  - **How much a consumer will buy when prices are at the market equilibrium?**
  - **How much a consumer would want to buy at every reasonable combination of income and prices?**

## Homogeneity of Demand

DEFINITION 2.7. *Let $X$ and $Y$ be vector spaces. A function $f : X \to Y$ is* homogeneous of degree $n$ *if for all $x \in X$ and $\alpha > 0$, $f(\alpha x) = \alpha^n f(x)$.*

- The function behaves like polynomial of the form $as^n$ along any ray coming out from the origin. (If $n = 0$, it is constant along any ray.)
- Example: Is $f(x, y) \equiv xy$ a homogeneous function? Of what degree?



PROPOSITION 2.4. *If $f$ homogeneous of degree $n$, then for $n > 0$*

$$f(x) \equiv \frac{1}{n}\frac{\partial f(x)}{\partial x}x,$$

*and for $n = 0$,*

$$\frac{\partial f(x)}{\partial x}x \equiv 0.$$

EXAMPLE 2.3. *$f(x) = x^n$ is a scalar function homogeneous of degree $n$, because $f(ax) \equiv a^n f(x)$. Proposition implies:*

$$f(x) = \frac{1}{n} f'(x) x$$

*or*

$$x^n = \frac{1}{n} n x^{n-1} x,$$

*which is clearly correct.*



PROOF. **Start with $y = f(x)$. Let $x = \alpha z$. For $n > 0$,**

$$\frac{\partial y}{\partial \alpha} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} = \frac{\partial f}{\partial x} z \text{ [chain rule]}$$

**By homogeneity, we also have $y = f(\alpha z) = \alpha^n f(z)$, so**

$$\frac{\partial y}{\partial a} = n\alpha^{n-1} f(z),$$

**and we can write:**

$$n\alpha^{n-1} f(z) \equiv \frac{\partial f(\alpha z)}{\partial x} z$$

**True for all $\alpha$, so set $\alpha = 1$. Then $x = z$, and we have,**

$$f(x) \equiv \frac{1}{n}\frac{\partial f}{\partial x} x.$$

For $n = 0$, $f(\alpha z) \equiv f(z)$, so differentiating both sides with respect to $\alpha$ gives

$$\frac{\partial f(\alpha z)}{\partial x} z = 0 \text{ [chain rule]}$$

and setting $\alpha = 1$

$$\frac{\partial f(x)}{\partial x} x \equiv 0.$$

■



- Results for homogeneous functions expressed in scalar notation:
  - For $n > 0$,

$$f_i(x_1, ..., x_n) = \frac{1}{n} \sum_j \frac{\partial f_i(x_1, ..., x_n)}{\partial x_j} x_j.$$

  - For $n = 0$,

$$\sum_j \frac{\partial f_i(x_1, ..., x_n)}{\partial x_j} x_j \equiv 0.$$

    - If $f$ is constant on $\alpha x$, then

$$\sum_j \frac{\partial f_i(x_1, ..., x_n)}{\partial x_j} \Delta x_j \equiv 0.$$

    - But moving along the ray $\alpha x$, $\Delta x_j$ is proportional to $x_j$, so we can substitute $x_j$ for $\Delta x_j$.
    - Weighted average of derivatives along path must be $0$ for function constant along the path.

- We apply this to demand functions:

PROPOSITION 2.5. *A demand function is homogeneous of degree 0 (in $p$ and $w$).*

PROOF. For any $p, w$ and $\alpha > 0$,

$$B_{p,w} = \{x \mid px \leq w\} = \{x \mid \alpha px \leq \alpha w\} = B_{\alpha p, \alpha w}.$$

Therefore,

$$x(p, w) \equiv C(B_{p,w}) = C(B_{\alpha p, \alpha w}) \equiv x(\alpha p, \alpha w).$$

∎

- Homogeneity of degree zero means that the absolute level of prices and wealth doesn't matter. No money illusion.

- Only the relative values have an effect.

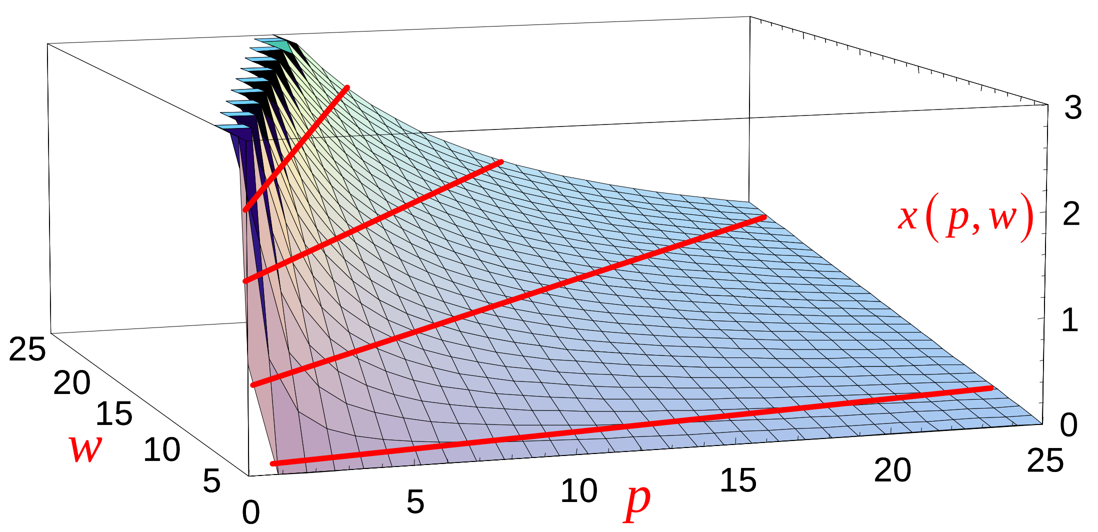


EXAMPLE 2.4. *3 dimensional plot of $x(p, w) = w/p$*

- Discontinuous at origin

- Not graphed near origin.

- Constant along every ray.



# Elasticity of Demand

DEFINITION 2.8. *The* price-elasticity *of demand $\varepsilon_{ij}$ of commodity $i$ to the price of commodity $j$ is given by*

$$\varepsilon_{ij} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i}.$$

- Therefore,

$$\varepsilon_{ij} = \text{approximately } \frac{\Delta x_i}{x_i} \Big/ \frac{\Delta p_j}{p_j} = \frac{\%\Delta x_i}{\%\Delta p_j},$$

the ratio of the percentage change in quantity to the percentage change in price.



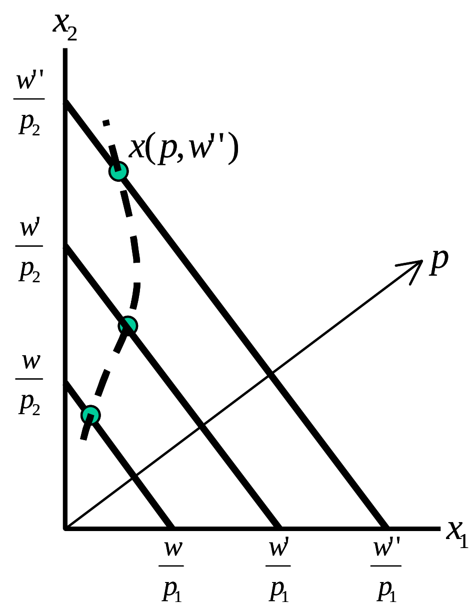
DEFINITION 2.9. *The* income-elasticity *of demand $\varepsilon_{iw}$ is given by*

$$\varepsilon_{iw} = \frac{\partial x_i}{\partial w}\frac{w}{x_i}$$

- Why useful?
- Homogeneity $\Longrightarrow$ for each commodity $i$,
  $\varepsilon_{i1} + \varepsilon_{i2} + \ldots + \varepsilon_{iL} + \varepsilon_{iw} = 0.$
- Why?

- **Income (Wealth) -Consumption Curve**

- **Expansion path.**






## Walras Law

Assumption 2.1. ***Walras Law:*** $px = w$ ***(all wealth is spent).***

Proposition 2.6. ***Walras Law implies***

$$-\sum_i p_i \frac{\partial x_i}{\partial p_j} = x_j.$$

- **Intuition: if the price of good $j$ increases by \$1, you would need $x_j$ more dollars to buy the original bundle.**

- **...Since you do not have $x_j$ more dollars, your demand must adjust so as to allow you save the $x_j$ dollars.**

INFORMAL PROOF. **Suppose price of $j$ increases by $\Delta p_j$.**

- **Then your demand will change by** $\Delta x_i = \dfrac{\partial x_i}{\partial p_j}\Delta p_j$
- **This creates an expenditure reduction on commodity $i$ of** $-p_i\Delta x_i = -p_i\dfrac{\partial x_i}{\partial p_j}\Delta p_j$.
- **Total expenditure reduction for all commodities is** $-\sum_i p_i\dfrac{\partial x_i}{\partial p_j}\Delta p_j$
- **But the price increase created an added expense for $j$ equal to** $x_j\Delta p_j$.
- **By Walras Law the expenditure reduction must equal the added expense:**

$$-\sum_i p_i\frac{\partial x_i}{\partial p_j}\Delta p_j = x_j\Delta p_j.$$

- **Cancel the $\Delta p_j$.**

■



PROPOSITION 2.7. ***Walras law implies***

$$\sum_i p_i\frac{\partial x_i}{\partial w} = 1.$$

- **Intuition: If wealth increases by \$1, your expenditures must increase to absorb that dollar.**

DEFINITION 2.10. ***A price change $\Delta p$ (a vector) is "Slutsky compensated" if income (or wealth) is adjusted by the amount $\Delta w_s = x(p,w)\Delta p$ (so that the consumer can still afford to buy the quantity demanded before the price change).***

PROPOSITION 2.8. ***Suppose***

- ***a demand*** `function`, ***$x(p, w)$, satisfies Walras Law,***
- ***price changes from $p$ to $p'$***
- ***and Slutsky compensation changes $w$ to $w'$***
- ***Let $x \equiv x(p, w)$ and $x' \equiv x(p', w')$.***
- ***Assume $x \neq x'$.***
- ***Then***

$$(p' - p)(x' - x) < 0$$
$$\Updownarrow$$
$$\left\{\begin{array}{l}x \text{ satisfies the weak axiom} \\ \text{of revealed preference}\end{array}\right.$$



PROOF.

**Compensation + Demand Function + Walras Law**

$$\Updownarrow$$

$$p'x = p'x'$$

**which means that $x'$ is revealed preferred to $x$**

**Revealed preference axiom**

$$\Updownarrow$$

$$px' > px$$

**(otherwise $x$ would be revealed preferred to $x'$)**

**Therefore:**

**Compensated price change + Walras Law + Revealed Preference**

$$\Updownarrow$$

$$p'x = p'x'$$

and $px' > px$

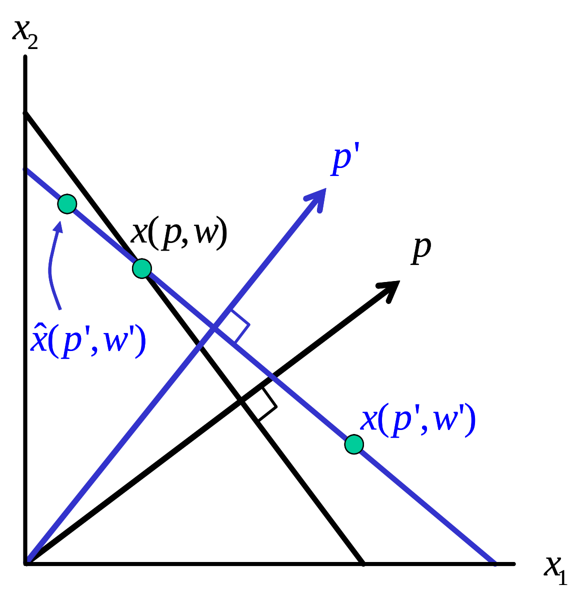
$$\Updownarrow$$

$$(p' - p)(x' - x) < 0$$

■

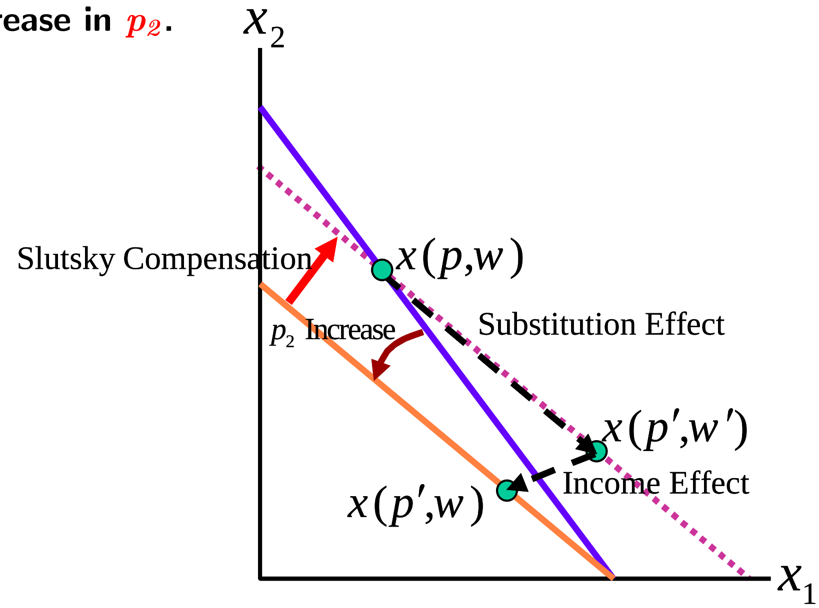


- $x\,(p', w')$ **satisfies the week axiom of revealed preference, because new demand was not in budget set at the old prices.**
- **But** $\hat{x}\,(p', w')$ **violates the week axiom.**



## 2.5 Income and Substitution Effect of Uncompensated Price Change.

- For simplicity assume $x(p, w) \neq x(p', w')$.
- The graph below illustrates the income and substitution effects of an increase in $p_2$.






Definition 2.11. *Suppose price changes from $p$ to $p'$. The substitution effect is*

$$\Delta x_s = x(p', w') - x(p, w),$$

*where $w'$ reflects Slutsky compensation. The substitution effect allows prices to change while holding buying power constant.*

- From previous proposition:

Weak axiom of revealed preference $\Longleftrightarrow$ $\Delta x_s \Delta p < 0$ for all $\Delta p$.

DEFINITION 2.12. *The* income effect *is given by*

$$\Delta x_w = x(p', w) - x(p', w'),$$

*the difference between demand at the new prices without and with Slutsky compensation.*

- Net (uncompensated) change in demand is

$$\Delta x = \Delta x_s + \Delta x_w = x(p', w) - x(p, w)$$



EXAMPLE 2.5. *Suppose that Pierre's utility is given by*

$$U(x, y) = x + y,$$

*and suppose prices and wealth are $p_x = 4$, $p_y = 5$ and $w = 60$. If $p_x$ changes to $p'_x = 6$, find the change in the quantities demanded and decompose them into the substitution and income effects.*

## Solution:

- Before the price change, Pierre buys only good $x$. Why?
- Therefore $x = 15$ and $y = 0$.
- After the price change, Pierre buys only good $y$.
- Therefore $x = 0$ and $y = 12$.
- Slutsky compensation for the price change would enable Pierre to buy $x = 15$ at the price $p'_x = 6$, so the compensation must be $\Delta w = 30$.
- Pierre would then purchase $y = 18$ and $x = 0$. Why?
- This means that the substitution effect is $\Delta x_s = -15$ and $\Delta y_s = 18$.
- Therefore the income effect is $\Delta x_w = 0$ and $\Delta y_w = -6$.



We define substitution and income effects in terms of derivatives.

DEFINITION 2.13. *The* Slutsky compensated demand function $x_s(p', z)$ *is the demand at prices* $p'$ *when when the consumer is always provided with the wealth needed to buy the consumption vector* $z$*; that is:*

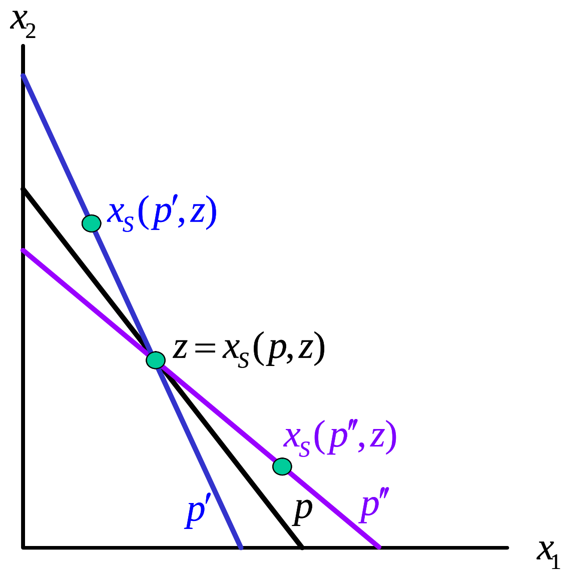
$$x_s(p', z) \equiv x(p', p'z)$$

PROPOSITION 2.9. *Given* $p$ *and* $w$*, we have*

$$x_s(p, x(p, w)) = x(p, w).$$

- Why?

- Illustration of a Slutsky demand function.
- The consumer can always buy $z$, if she wants to.
- In the graph below we illustrate the Slutsky compensated budget sets that correspond to different prices, and we illustrate Slutsky-compensated demand at each of the prices.






- Let $\bar{w}$ denote the level of buying power (real income).

**Definition 2.14.** *The Slutsky derivative*

$$\left.\frac{\partial x(p, w)}{\partial p}\right]_{\bar{w}}$$

*is the derivative of compensated demand $x_s(p', z)$ with respect to prices $p'$ evaluated at $p' = p$ while holding buying power constant at $z = x(p, w)$; that is:*

$$\left.\frac{\partial x(p, w)}{\partial p}\right]_{\bar{w}} = \left.\frac{\partial x_s(p', z)}{\partial p'}\right]_{p'=p,\ z=x(p,w)}.$$

PROPOSITION 2.10. *The* **Slutsky Equation** *(in scalar notation):*

$$\frac{\partial x_i(p,w)}{\partial p_j} \equiv \left.\frac{\partial x_i(p,w)}{\partial p_j}\right]_{\bar{w}} - \frac{\partial x_i(p,w)}{\partial w} x_j(p,w)$$

- **The first right-hand term describes the substitution effect, and the second, the income effect (in the limit as price changes become small).**



PROOF. **Let $z = x(p,w)$ and $w' = p'z$. Then,**

$$\frac{\partial x_{si}(p',z)}{\partial p'_j} = \left.\frac{\partial x_i(p',w')}{\partial p'_j}\right]_{p'_j,w' \text{ varying}}$$

$$= \frac{\partial x_i(p',w')}{\partial p'_j} + \frac{\partial x_i(p',w')}{\partial w'}\frac{\partial w'}{\partial p'_j} \quad \text{[chain rule]}$$

$$= \frac{\partial x_i(p',w')}{\partial p'_j} + \frac{\partial x_i(p',w')}{\partial w'} z_j$$

**Replacing $z_j$ by $x_j(p,w)$, setting $p' = p$ (and $w' = w$) and denoting $\left.\frac{\partial x_{si}(p',z)}{\partial p'_j}\right]_{p'=p}$ by $\left.\frac{\partial x_i(p,w)}{\partial p_j}\right]_{\bar{w}}$ yields**

$$\left.\frac{\partial x_i(p,w)}{\partial p_j}\right]_{\bar{w}} = \frac{\partial x_i(p,w)}{\partial p_j} + \frac{\partial x_i(p,w)}{\partial w}\frac{\partial w}{\partial p_j}.$$

**We obtain the Slutsky equation by rearranging terms.** ■

DEFINITION 2.15. *A quadratic expression, in which every term is quadratic, is positive definite if its value is positive for any nonzero values of the variables. The expression is negative definite if its value is negative for any nonzero values of the variables.*

EXAMPLE 2.6. $-x^2 + xy - y^2$ *is negative definite. Why?*
*Hint: Find the maximum value the expression can take for a given $x$.*

- Note: a quadratic expression can be written by use of a symmetric matrix.

$$\begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -x^2 + xy - y^2$$

- The matrix is defined to be positive (negative) definite when the expression is.



PROPOSITION 2.11. *The Slutsky matrix* $\left.\frac{\partial x(p,w)}{\partial p}\right]_{\bar{w}}$ *is negative definite.*

INFORMAL PROOF.

- $\Delta x_s \mathrel{\mathring{=}} \left.\frac{\partial x(p,w)}{\partial p}\right]_{\bar{w}} \Delta p$
- The weak axiom of revealed preference implies $\Delta p \Delta x_s < 0$.
- Therefore, for all sufficiently small $\Delta p$,

$$\Delta p \left.\frac{\partial x(p,w)}{\partial p}\right]_{\bar{w}} \Delta p < 0.$$

- But the sign of the left-hand side stays the same if $\Delta p$ is multiplied by any scalar. Why?
- Therefore, the inequality is true for any $\Delta p$, and the matrix must be negative definite.

■

- Note that

$$\left.\frac{\partial x(p,w)}{\partial p}\right]_{\bar{w}}$$ is negative definite

$\Longrightarrow$ all diagonal terms are negative (among other things)

$\Longrightarrow$ "own" substitution effect is always negative.