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MATHEMATICAL NOTES

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NORMABILITY OF WEAKNORMED LINEAR SPACES

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1. Introduction. The concept of a weaknormed real linear space is the generalization of the concept of a normed linear space which results when the homogeneity property of the norm, $\|\alpha x\| = |\alpha| \cdot \|x\|$ for any scalar α, is weakened by restricting α to the nonnegative real numbers. Some of the general properties of such spaces have been discussed by Leichtweiss [2]. Quasimetric spaces, which are related to weaknormed linear spaces in the same manner as metric spaces are related to normed linear spaces, are discussed by Ribeiro [3] and by Balanzat [1].

This note contains several theorems about topologies on weaknormed real linear spaces. We show that a weaknorm, in much the same manner as a norm, induces a topology on a real linear space, though the space with this topology is not necessarily a topological linear space, nor even a topological group. A real linear space with a topology is said to be weaknormable if a weaknorm can be assigned to the space such that the topology induced by the weaknorm is identical with the given topology. We shall state necessary and sufficient conditions for weaknormability of a real linear space with a topology, and the conditions necessary and sufficient for normability of a space with a topology induced by a weaknorm. Two examples are given of a weaknormed linear space with a topology that is not normable.

Throughout this paper, the function s denotes vector addition; that is, $s(x, y) = x + y$, where x and y are members of a vector space. The function m denotes multiplication of a vector by a nonnegative scalar; i.e., $m(\alpha, x) = \alpha x$, where α is a nonnegative scalar and x is a vector. R' denotes the set of nonnegative reals.

2. Quasi-topological linear spaces and weaknormability. A *quasi-topological linear space* (X, τ) is a linear space X with a topology τ such that the function $s: X \times X \rightarrow X$ is continuous in the usual product topology, and the function $m: R' \times X \rightarrow X$ is also continuous. A set S is said to be *semi-balanced* if $E \cdot S = S$, where E is the real interval $[0, 1]$. The standard theorem characterizing all topological linear spaces in terms of local properties ([4], Theorem 3.3-F, for example) holds for quasi-topological linear spaces as well, with "balanced" replaced by "semi-balanced." If $x_0 \in X$, $\alpha > 0$, then the functions $x \rightarrow x_0 + x$, $x \rightarrow \alpha x$ are homeomorphisms.

A *weaknorm* is a real valued functional $\|\cdot\|$ defined on a linear space, where $\|\cdot\|$ has the following properties:

(1) $\|x\| \geqq 0$, (2) $\|x\| + \|y\| \geqq \|x+y\|$,
(3) $\alpha\|x\| = \|\alpha x\|$ for $\alpha \geqq 0$, (4) $\|x\| = 0$ if and only if $x = 0$.

A linear space X with its weaknorm $\|\cdot\|$ is called a *weaknormed linear space* and denoted by $(X, \|\cdot\|)$. By the remarks in the preceding paragraph, a weaknormed linear space is a quasi-topological linear space with the sets of the form $\{x: \|x\| < r\}$, $r > 0$, as an open neighborhood base at 0. A linear space with a topology is said to be *weaknormable* if a weaknorm can be assigned to the space such that the topology induced by the weaknorm is identical with the given topology.

If X is a linear space and $0 \in K \subset X$ and K is absorbing, the well-known *Minkowski functional* of K is given by $p(x) = \inf \{\alpha: \alpha > 0, x \in \alpha K\}$. For properties of p, see e.g. [4], pp. 134–35. A set S in a quasi-topological linear space is *bounded* if for every neighborhood U of 0, there exists $\gamma > 0$ such that $S \subset \gamma U$.

THEOREM 1. *A quasi-topological linear space X is weaknormable if and only if X is a T_1-space and there exists in X a convex and bounded open neighborhood K of* 0.

Proof. We prove sufficiency; the proof of necessity is standard. Let p be the Minkowski functional of K. It is easy to see that, as in the standard situation, p is a weaknorm for X. We prove that p induces the given topology as follows. Suppose $x \in K$. Then the pair $(1, x) \in m^{-1}(K)$, which is open since K is. Choose neighborhoods A of 1 and U of x with $A \cdot U \subset K$. A contains some $\alpha > 1$ for which $\alpha x \in K$; thus $p(x) \leqq \alpha^{-1} < 1$. This shows that $K = \{x: p(x) < 1\}$, so that $\{\gamma K: \gamma > 0\}$ is a base for the topology induced by p, which, by the boundedness of K, is thus identical with the given topology.

3. Normability of weaknormed linear spaces. A weaknormed linear space is *normable* if there exists a norm for the space such that the topology induced by the norm is identical with the topology induced by the weaknorm. The major theorem of this section states necessary and sufficient conditions for normability of a weaknormed linear space.

LEMMA. *Let $(X, \|\cdot\|)$ be a weaknormed linear space with the following property: for any sequence $\{x_n\}$ such that $\|x_n\| \to 0$, we also have $\|-x_n\| \to 0$. Then there exists $M > 1$ such that for all $x \in X$, we have $\|x\| \leqq M\|-x\|$.*

Proof. Suppose that no such M exists. Then for each $n > 0$ there exists $x_n \in X$ with $\|x_n\| > n\|-x_n\|$. Let $y_n = (n\|-x_n\|)^{-1} \cdot x_n$. Then $\|-y_n\| = 1/n$ and $\|y_n\| > 1$, contradicting the hypothesis.

THEOREM 2. *A weaknormed linear space $(X, \|\cdot\|)$ is normable if and only if, for any sequence $\{x_n\}$ in X, $\|x_n\| \to 0$ implies $\|-x_n\| \to 0$.*

Proof. Since multiplication by -1 is a homeomorphism in a normed linear space, the condition is necessary. To prove sufficiency we first note that the function $p(x)=\max\{\|x\|, \|-x\|\}$ is a norm on X. Next, given $r>0$, we see that $\|x\|<r/M$ implies $p(x)<r$, where M is given by the lemma; so that the topology induced by $\|\cdot\|$ is the same as the topology induced by p.

Note that for any weaknormed linear space $(X, \|\cdot\|)$, the functional p, as defined above, is a norm on X, and the p-topology is stronger than the $\|\cdot\|$-topology. Later, we shall give some examples in which it is properly stronger.

THEOREM 3. *Let $(X, \|\cdot\|)$ be a weaknormed linear space which is not normable. If S is any sphere in X, the set $-S=\{x: -x\in S\}$ is unbounded.*

Proof. It suffices to prove that the negative of any sphere $U=\{x: \|x\|<r\}$ about 0 is unbounded. Choose a sequence $\{x_n\}$ in U such that $\|x_n\|\to 0$ while $\|-x_n\|\not\to 0$. Let $y_n=r(2\|x_n\|)^{-1}\cdot x_n\in U$. The sequence $\{-y_n\}$ is unbounded, and the theorem follows.

THEOREM 4. *Every finite-dimensional weaknormed linear space is normable.*

Proof. Let $(X, \|\cdot\|)$ be a finite-dimensional weaknormed linear space, and let $\{v_i\}$ $(i=1, \cdots, n)$ be a basis for X. Consider a function f on the real space $l^1(n)$ into the non-negative reals, given by $f(\alpha)=\|\sum\alpha_i v_i\|$, where α denotes the n-tuple $(\alpha_1, \cdots, \alpha_n)$. The function f is continuous, for

$$\begin{aligned} |f(\alpha)-f(\beta)| &= \left|\,\|\textstyle\sum\alpha_i v_i\| - \|\sum\beta_i v_i\|\,\right| \\ &\leqq \max\{\|\textstyle\sum(\alpha_i-\beta_i)v_i\|, \|\sum(\beta_i-\alpha_i)v_i\|\} \\ &\leqq \textstyle\sum|\alpha_i-\beta_i|\cdot\max\{\|v_i\|, \|-v_i\|\} \leqq N\cdot(\sum|\alpha_i-\beta_i|), \end{aligned}$$

where

$$N = \max_{1\leqq i\leqq n}\max\{\|v_i\|, \|-v_i\|\}.$$

Consider the set $S=\{\alpha: \sum|\alpha_i|=1\}\subset l^1(n)$. S is compact, so that the non-negative set $f(S)$ has a least member, say r. Should $r=0$, then for some $\alpha\in S$ we would have $f(\alpha)=\|\sum\alpha_i v_i\|=0$, implying $\sum\alpha_i v_i=0$, a contradiction since $\{v_i\}$ are linearly independent. Therefore, $r>0$. But for any $\sum\beta_i v_i$ in X, we have

$$\|\textstyle\sum\beta_i v_i\| \geqq r\cdot(\sum|\beta_i|)$$

since, if some $\beta_i\neq 0$, $\|\sum\beta_i v_i\|=(\sum|\beta_i|)\|\sum(\sum|\beta_j|)^{-1}\beta_i v_i\|$, where

$$\|\textstyle\sum(\sum|\beta_j|)^{-1}\beta_i v_i\|\in f(S).$$

Furthermore, $N\cdot(\sum|\beta_i|)\geqq\|\sum-\beta_i v_i\|$. The conclusion follows by Theorem 2.

That any finite-dimensional weaknormed linear space is topologically isomorphic to the Euclidean space of the same dimension, is now an immediate consequence of the theorem that all finite-dimensional normed linear spaces of a given dimension are topologically isomorphic (see [4], p. 95).

4. Examples of nonnormable weaknormed linear spaces.

Example 1. Let X be the set of all functions in $L^1[1, \infty)$ (identifying, in the usual way, functions equal almost everywhere). For any function $f\in X$, we define $\|f\| = \int_1^\infty f^*(t)dt$, where $f^*(t) = f(t)$ when $f(t) \geqq 0$, and $f^*(t) = |f(t)|/t$ when $f(t) < 0$. The space $(X, \|\cdot\|)$ is a weaknormed linear space. To show that X is not normable, consider the sequence $\{f_n\}$, where $f_n(t) = 0$ for $t < n+1$ and

$$f_n(t) = -(t-n)^{-2} \quad \text{for } t \geqq n+1.$$

It is clear that $\|f_n\| \to 0$, while $\|-f_n\| = 1$ for all n, and the conclusion follows by Theorem 2.

Example 2. Let (Y, μ) be a nonatomic measure space with $\mu(Y) < \infty$. With each real-valued measurable function f on Y (identifying, in the usual way, functions equal almost everywhere) we associate two functions f_+ and f_- defined as follows: $f_+(x) = \max\{f(x), 0\}$, and $f_-(x) = \max\{-f(x), 0\}$. Let

$$p(f) = \int f_+^2 d\mu + \int f_- d\mu.$$

Then the set $K = \{f : p(f) < \infty, p(-f) < \infty\}$ is a vector space; indeed, K contains the same functions as $L^2(Y)$.

We will consider the Minkowski functional of the set $S = \{f \in K : p(f) \leqq 1\}$. S contains 0, for $p(0) = 0 < 1$. We prove that S is absorbing. Take any function $f \in K$ where $f \neq 0$. Pick α such that $|\alpha| < \min\{1/p(f), 1/p(-f), 1\}$. Then, if $\alpha \geqq 0$, we have

$$p(\alpha f) = \alpha^2 \int f_+^2 d\mu + \alpha \int f_- d\mu \leqq \alpha p(f). \tag{1}$$

But $\alpha < 1/p(f)$, and thus, by (1), $p(\alpha f) \leqq \alpha p(f) < 1$. If $\alpha < 0$, we have by (1), $p(\alpha f) \leqq p(|\alpha|(-f)) = |\alpha| p(-f) < 1$. Therefore, $\alpha f \in S$; S is absorbing.

We shall show that S is convex. Suppose $f, g \in S$ and $\alpha + \beta = 1$, where $\alpha, \beta \geqq 0$. Then,

$$\begin{aligned} p(\alpha f + \beta g) &= \int (\alpha f + \beta g)_+^2 d\mu + \int (\alpha f + \beta g)_- d\mu \\ &\leqq \int (\alpha f_+ + \beta g_+)^2 d\mu + \int (\alpha f_- + \beta g_-) d\mu. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} (\alpha f_+ + \beta g_+)^2 &= \alpha^2 f_+^2 + \beta^2 g_+^2 + 2\alpha\beta f_+ g_+ \\ &\leqq \alpha^2 f_+^2 + \beta^2 g_+^2 + \alpha\beta f_+^2 + \alpha\beta g_+^2 = \alpha f_+^2 + \beta g_+^2. \end{aligned}$$

Hence,

$$\begin{aligned} p(\alpha f + \beta g) &\leqq \alpha \int f_+^2 d\mu + \beta \int g_+^2 d\mu + \alpha \int f_- d\mu + \beta \int g_- d\mu \\ &= \alpha p(f) + \beta p(g) \leqq 1, \end{aligned}$$

since $p(f)$, $p(g) \leqq 1$. It follows that S is convex. By the properties of Minkowski functionals, we see that $(K, \|\cdot\|)$ is a weaknormed linear space, where $\|\cdot\|$ is the Minkowski functional of S.

To show that K is not normable, it is sufficient, by Theorem 2, to show that for any $n>0$, there exists an $f \in K$ such that $\|f\| \geqq n\|-f\|$. Given n, define f as follows: $f(x)=n$ on some set of measure $1/n^2$, and $f(x)=0$ everywhere else. We have $p(f)=1$ and thus $\|f\|=1$. However, $p(-nf)=1$, so that $\|-nf\|=n\|-f\|=1=\|f\|$. Thus f is as required, and the conclusion follows.

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References

1. M. Balanzat, On the metrization of quasimetric spaces, Gaz. Mat. Lisboa 12, no. 50 (1951) 91–94.

2. K. Leichtweiss, Sur les espaces de Banach autoadjoints. Séminaire C. Ehresmann, 1957/58, exp. no. 4, Fac. des Sci. de Paris, 1958.

3. H. Ribeiro, Sur les espaces à métrique faible, Portugal. Math., 4 (1943) 21–40, 65–68.

4. A. E. Taylor, Introduction to Functional Analysis. Wiley, New York, 1959.

HADAMARD MATRICES AND SOME GENERALISATIONS

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For any given integer, $n \geqq 2$, the matrix H_n is called a *Hadamard Matrix* if H_n is an $n \times n$ matrix with elements ± 1 which satisfies $H_n H_n' = nI_n$. It is known that such a matrix can exist only if either $n=2$ or n is a multiple of four. It has been conjectured that these necessary conditions are also sufficient, but this has never been proved. A list of the numerous partial results available is given in [**1**] and [**4**], and in [**3**] and [**5**] it has also been proved true for the cases $n=92$ and 156.

Generalisations are possible in several directions. Following [**1**], we say that $H(p; n)$ is a generalised Hadamard Matrix of order n, if $H(p; n)$ is an $n \times n$ matrix all of whose elements are pth roots of unity and such that $H\overline{H}' = nI_n$. In this notation the ordinary Hadamard Matrix is an $H(2; n)$.

Another generalisation, without departing from the elements ± 1 arises as follows. Let $g(n)$ equal, for each n, the value of the greatest $n \times n$ determinant with elements ± 1. Then it follows by Hadamard's Theorem that $g(n) \leqq n^{n/2}$, where equality is possible if and only if the matrix G_n associated with $g(n)$ is an $H(2; n)$. However, we may ask for all values of n, and not only for multiples of four, what the value of $g(n)$ is, and also ask for the structure of G_n. It is possible to show, in addition to the fact that $g(n) = n^{n/2}$ whenever an $H(2; n)$ exists, that $g(3)=4$, $g(5)=48$ and $g(6)=160$, and also it has been proved in [**2**] that given any positive ϵ, $g(n) > n^{(1/2-\epsilon)n}$ for all sufficiently large n. If the conjecture