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# NON-PRICE RATIONING OF INTERMEDIATE GOODS IN CENTRALLY PLANNED ECONOMIES

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How should rule-of-thumb priorities be assigned for the rationing of intermediate goods in an economy marked by shortages and non-scarcity prices? A method is developed for setting these priorities in an optimal way, and the method is applied to Soviet data as an illustration.

## 1. INTRODUCTION

THE DIFFERENT SECTORS of a modern industrial economy are highly dependent upon one another, directly or indirectly. The output of most sectors requires as inputs the products of many other branches of production. Some sectors, such as transportation and communications, trade, fuel, and power are vital to all industrial production. Should production in one of these sectors falter, the performance of the entire economy could be threatened. This is especially true in Soviet-type economies, which normally operate with little slack. Furthermore, production in these economies is frequently governed by relatively inflexible short term plans, which make it difficult to get around unforeseen shortages.

In any situation where demand for a good exceeds supply,[2] the good must be rationed. How the good is rationed will, to a large extent, determine the effect of the shortage of the good on the economy. Suppose, for example, there is a shortage of copper, and that copper is an essential input of two industries, toys and transportation. If copper is denied the toy industry, fewer toys will be produced, and it will end at that.[3] But if copper is denied the transportation sector, a resulting slowdown in freight delivery could have unfortunate repercussions throughout the economy. In the end, the indirect effect of denying copper to transportation could be more devastating to the toy industry than the effect of denying copper to the toy industry directly. Thus, it would seem that "basic" industries (in some sense of the word) should be given priority in rationing.

But there is another consideration which must be met. Suppose that certain toys each require 3 inches of copper wire, and that certain railroad signaling devices each require 1500 feet of the same wire. Even though the output of the

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[2] The terms "supply" and "demand" are used throughout this chapter in their operational sense. Given all the determinants of firm behavior (not specified in advance), the supply of a commodity (or quantity supplied) is the total amount of that commodity that firms offer to transfer to other firms; the demand for a commodity (or quantity demanded) is the total amount of that commodity that firms offer to receive from other firms.

[3] This is not to deny that a reduction in the output of toys may have some indirect effects on other productive activities via the consumer.

transportation industry is more basic than that of the toy industry, it may be unreasonable to sacrifice an extremely large quantity of toys for one signaling device. In our postulated case, the direct productivity of copper as an input into the toy industry is much higher than it is as an input into transportation. The productivity of an input in each of its possible uses turns out to be an important consideration in determining optimal priorities for its acquisition.

The scope of the theory developed below is limited to the development of a system for rationing intermediate goods in an economy with small shortages and absolutely no slack.[4] We shall devise a rationing system that is simple enough to be applied in a straight-forward manner by an administrative bureaucracy. The main task of the forthcoming analysis is the construction of a method for determining an optimal set of rationing priorities for use within this system. The analysis does not describe the actual procedures used in any existing planned economy; instead, it suggests for further consideration new procedures of a type which could be used in existing planned economies. Thus, the analysis is normative rather than descriptive.

The rationing system devised below is for use in the following situation. An annual plan of material supply (specifying output targets and input allotments for each producing unit) has been set and is in the process of being implemented. Each month (or other short period of time) commodities are distributed according to the annual plan on a prorated basis. Suddenly, a shortage appears. For some reason, there is not enough of a certain commodity available that month to satisfy final demand and producers' input requirements, as determined by prorating the annual plan. The effect that this shortage will have on current production is determined by how available quantities of the commodity are distributed. If the shortage is permitted to affect current production in the wrong way, new and more severe shortages will crop up in future months as a result. These recurring shortages could have a devastating effect on the fulfillment of the annual plan.

How, then, is the deficit commodity to be distributed? Since there is not enough of it to be distributed according to plan, some "second-best" method of distribution must be used. That is where a system of non-price rationing comes in.[5] It must provide relatively simple rules for distributing goods in short supply—rules that can be applied quickly by a decentralized bureaucracy, whenever a shortage occurs.

The system of rationing developed here, on the one hand, and construction of the annual plan, on the other, should not be confused. The plan is prepared and finalized before the production period under consideration begins. The rationing system has nothing to do with constructing the plan; rather, it is used during the

[4] The theory is easy to modify, however, so that economies with slack can be treated. See footnote 9.

[5] In theory, this second-best method of distribution could be a market-like method. For example, when shortages of an intermediate good arise, firms could bid for the available quantities of the scarce commodity. But there are two difficulties here. First, the bidding process may take more time than is desirable. Secondly, because output prices are not necessarily scarcity prices and because firms may not be profit maximizers, an allocation of the scarce commodity to firms which bid the most may be a very inefficient allocation.

implementation of the plan. The same rationing system cannot be directly applied to constructing the plane for two reasons. (i) The rationing system is based on the assumption that the supply is fixed, but in the planning process supply is the main variable. (However, to the extent that supplies are determined by limited productive capacities, it is conceivable that some elements of the rationing system could be used in the planning process.) (ii) The requisite simplicity of the rationing system makes it inappropriate for planning purposes.

Our discussion of non-price rationing of intermediate goods in the Soviet context is very closely tied in with the question of what happens when an economy tries to carry out an inconsistent or otherwise unfulfillable plan.

### *Glossary of Symbols*

Matrices and vectors are designated by upper-case letters; corresponding lower-case letters not otherwise defined represent elements of those vectors and matrices.

$A$: matrix of coefficients of input per unit output.
$D$: vector of final (net) outputs.
$\bar{D}$: vector of planned final outputs.
$\bar{d}_{ij}$: demand for commodity $i$ by sector $j$.
$e_i$: excess demand for commodity $i$.
$F$: final demand column of DMS matrix $H$.
$I$: identity matrix.
$H$: distribution-of-marginal-shortage matrix (DMS matrix).
$\bar{H}$: maximum-allotment-reduction matrix.
$\underline{H}$: minimum-allotment-reduction matrix.
$M$: interindustry section of DMS matrix $H$.
$n$: number of industries.
$P$: diagonal matrix of final-output prices.
$Q$: matrix of direct productivities of inputs.
$R$: priority ranking matrix.
$S$: vector of input shortages.
$T$: vector of inventory accumulations.
$V$: vector of eventual values per ruble.
$Y$: vector of actual gross outputs.
$\bar{Y}$: vector of planned gross outputs.
$y_{ij}$: quantity of commodity $i$ received by sector $j$.
$Z$: normal-distribution-of-product matrix.

## 2. A PRIORITY SYSTEM OF NON-PRICE RATIONING

### A. *Assumptions about the Economy*

Suppose we have an economy with a number of industries each producing a single product, and suppose that the production function of each product is of the fixed-proportions type so that there is no possibility of substitution between the different products when they are used as inputs. Under these circumstances, all of the production functions may be incorporated in a Leontief $A$ matrix, where $a_{ij}$ is the input of commodity $i$ required to produce a unit of commodity $j$.

An annual gross output target and a target for final demand is set for each commodity before the year under consideration begins. These targets may or may not be consistent, but they are final. Once the year starts, the targets cannot be changed. If the targets are not consistent, or, if there are production failures during the year, shortages will occur, rationing will be employed, and actual output will deviate from planned output.

Let $\bar{Y}$ and $\bar{D}$ be the vectors of planned annual gross output and planned final demand, respectively, and let $Y$ and $D$ be the vectors of *actual* annual gross output and final demand. In this economy, by definition, $Y$ and $D$ are related by

$$(1) \qquad Y = AY + D + T$$

where $T$ is the vector of annual unplanned changes in inventories. If $\bar{Y}$ and $\bar{D}$ are consistent, then $\bar{Y} = A\bar{Y} + \bar{D}$. But, given that shortages do occur (because of production breakdowns or other exogenous factors), there is no simple equation relating $Y$ and $D$ with $\bar{Y}$ and $\bar{D}$; there is no simple way of deriving the actual figures from the planned figures. One thing is clear, however: the vectors $Y$ and $D$ will have a crucial dependence on the system of rationing used to distribute goods in short supply.

### B. *How Not to Ration*

Some readers may feel that a sensible rationing scheme ought to fall right out of the $A$ matrix and, consequently, may be wondering what all the fuss is about. Suppose the planned outputs $\bar{Y}$ and final demands $\bar{D}$ have been set consistently, so that $\bar{Y} = A\bar{Y} + \bar{D}$, and suppose that after production begins an exogenously-induced shortage crops up. Could not the planners assume that the actual gross output $Y$ will equal $\bar{Y}$ minus the shortage, and then distribute inputs in accordance with the $A$ matrix as applied to $Y$? The answer is no. To see why, note that since $Y = AY + D + T$, we would have

$$(2) \qquad (\bar{D} - D) - T = (\bar{Y} - Y) - A(\bar{Y} - Y).$$

In other words, the loss in final product plus the unplanned decrease in inventory stocks equals the loss in gross output minus the amounts of the inputs not consumed as a result of the deviation from the plan. Unfortunately, the new final product may turn out to be negative in some sectors and therefore, assuming that the inventories and imports are insufficient to cover the deficit, the gross output $Y$ could not be produced. This system could not handle a shortage of abrasives, for example, since the planned final demand would be negligible and could not be reduced.

Even more importantly, however, setting actual gross output equal to the planned gross output minus the shortage could result in the final demand being reduced more than is desirable. An example might clarify this. Suppose that only two commodities are produced, coal and steel, and suppose that coal is produced using only labor, but each ton of steel requires a ton of coal as an input (this defines an $A$ matrix). Suppose planned final demand is set at 1000 tons of coal and 1000 tons of steel, i.e., $\bar{D} = (1{,}000, 1{,}000)$. Then, planned gross output, to be consistent, would have to be 2,000 tons of coal and 1000 tons of steel, or $\bar{Y} = (2{,}000, 1{,}000)$. Suppose, now, that a coal mine unexpectedly caved in during the year, so that only 1,000 tons of coal could be produced. The potential gross output would be reduced to $Y = (1{,}000, 1{,}000)$. Under the procedure described above, this new $Y$ would be produced, leaving a final product of $D = (I - A)Y$ (the potential gross output minus the required inputs) or $D = (0, 1{,}000)$. Thus under this system, the unexpectedly small output of 1,000 tons of coal would automatically be used entirely as an input for the production of the planned 1,000 tons of steel, leaving no coal as a final product. Since coal as a final product is used by the peasants to heat their homes in the bitter winter, the consequences would be dire indeed.

How else could a rationing system work? The authorities would simply give the peasants some of the coal they needed and leave the steel industry with less than all the inputs it requires to meet its target of 1,000 tons of steel. The existence of the $A$ matrix does not require the authorities to give the steel industry the inputs that the $A$ matrix and the planned output target indicate. Of course, if the steel industry does not get these inputs, it will not be able to meet its planned output target. But in our example, it would be better for the steel industry to fall short of its target, than for the peasants to freeze. (Please excuse the value judgement.)

### C. *Rationing and Unplanned Increases in Inventory Stocks*

Let $p$ designate a short time period, and let $Y_p$ be the vector which represents the quantities of the various commodities to be allotted during period $p$. Let $D_p$ represent the allotments to the final sector. Then, $Y_p$ and $D_p$ must be related by

$$Y_p = AY_p + D_p + T_p \tag{3}$$

where $T_p$ represents unplanned changes in inventories. The inventories subject to change are of two types: (*a*) inventories of unfinished commodities currently undergoing production, and (*b*) inventories of inputs which cannot be used because of the unavailability of complementary inputs.[6]

Equation (3) does not limit the specification of the rationing system in any way. It will be satisfied no matter how commodities are distributed (the equation is, in fact, true by definition). But equation (3) does point up the fact that some rationing procedures may cause, over the year, large unplanned build-ups of inventories, particularly those of type *b*. (If, as we shall assume, production processes are very

[6] In general, inventories also may be held as a hedge against uncertainty in demand, and they may accumulate ex post because of incorrect expectations about demand. In this model, however, the former category is ruled out and the latter category is narrowed to type *b* inventories by the no-slack assumption.

fast, so that at most a small quantity of goods is undergoing production at any given moment, the size of the possible change in the level of type *a* inventories over the year becomes insignificant.)

This leads to the following question: Would not any reasonable rationing system be such as to minimize or prevent an addition to these unusable inventories? I will argue that the answer is no. Consider the following example: half of the usual supply of iron is normally used to make sheet steel, and the other half of the iron is normally combined with nickel to make stainless steel. Because of an iron mine disaster, however, only half the usual supply of iron is available. If the available iron is all used to make sheet steel, the nickel that was to be used to make stainless will become a type *b* inventory; if all of the available iron is used to make stainless, there will be no immediate accumulation of inventories. But it would nevertheless be correct to use the iron for sheet steel if sheet steel is sufficiently more important than stainless to the functioning of the economy, and the nickel would be available for use at some later date.

So far, we have only discussed how not to ration. The remainder of this article is devoted to describing and analyzing the rationing system we propose.

## D. *The Constrained Priority Rationing Schedule*

The constrained priority rationing schedule (CPRS) provides relatively simple rules for distributing goods in short supply.

Let $\bar{d}_{ij}$ be the demand for commodity $i$ by sector $j$, let $y_i$ be the available supply of commodity $i$, and let $e_i = \Sigma_j \bar{d}_{ij} - y_i$ be the excess demand for commodity $i$. Let $y_{ij}$ be the amount of commodity $i$ that sector $j$ will actually receive. For each commodity $i$ to be distributed, a CPRS assigns in advance to each consuming sector $j$ (where $j$ indexes consumers of intermediate goods and the final demand sector) the following indices: (i) a ranking $r_{ij}$ which specifies sector $j$'s priority for receiving commodity $i$, (ii) an upper bound $\bar{h}_{ij}$ for the ratio $(\bar{d}_{ij} - y_{ij})/e_i$ (i.e., a maximum permissible allotment reduction per unit excess demand), and (iii) a lower bound $\underline{h}_{ij}$ for the ratio $(\bar{d}_{ij} - y_{ij})/e_i$ (i.e., a minimum permissible allotment reduction per unit excess demand).

The CPRS is applied in the following way. Suppose commodity $i$ is in short supply. The demanded allotment $\bar{d}_{ij}$ of each sector $j$ for commodity $i$ is first reduced to $\bar{d}_{ij} - \underline{h}_{ij}e_i$. Then all of the consuming sectors (including final demand) are ordered according to their priority rankings $r_{ij}$ for receiving commodity $i$. Beginning with the consuming sector with the lowest priority, the allotment of each succeeding sector $j$ is reduced in turn to $\bar{d}_{ij} - \bar{h}_{ij}e_i$, until the excess demand for commodity $i$ is eliminated.[7] This implies that all sectors, with possibly one exception, receive either the minimum or the maximum reduction in their allotments of commodity $i$. This sequence of adjustments is repeated whenever a shortage of commodity $i$ appears.

The CPRS is little more than a rule-of-thumb type system for determining to whom to give things when shortages exist. The CPRS priority ranking must be specified by the planners, but the minimum and maximum reductions per unit total excess demand may be, to a large extent, unspecified characteristics of the economy. In order to ensure that the size of an indicated reduction in a demanded allotment does not exceed the size of the allotment to begin with, the maximum reduction per unit total excess demand must not deviate by more than a reasonable amount from the ratio of the demanded allotment in question to the total demand for the input. It is not physically possible, for example, to allow ninety per cent of a ten per cent shortage of an input to fall on an industry which normally uses only one per cent of the total supply of the input.

Furthermore, in the real world, the freedom of planners to "distribute" shortages may be limited by non-homogeneity of the scarce input, pre-existing logistical arrangements, and political considerations, among other things. I am assuming that all of these constraints can and will be reflected by the minimum and maximum reduction specifications of the CPRS.

The CPRS has three major advantages: (i) it is practical—once a particular CPRS is specified, it can be applied by a decentralized bureaucracy to a wide variety of rationing situations within a given economy with little further effort; (ii) the CPRS class is quite general—different CPRS specifications approximate a wide variety of rationing systems; and (iii) the CPRS is easy to analyze.

On the other hand, the information required for the construction of the CPRS must be gathered, compiled, and forwarded to a central planning body by a large bureaucracy; and a large (though decentralized) bureaucracy must implement the CPRS. Consequently, many of the costs and biases associated with bureaucratic systems will be inherent in the CPRS.

### E. *An Example of a Constrained Priority Rationing Schedule*

As an example of a CPRS, consider the following economy which produces coal and textiles. Suppose the following CPRS were specified:

Minimum Allotment Reduction Matrix: $[\underline{h}_{ij}]$

| *Inputs* | Consumers: Coal Industry | Consumers: Textile Industry | Consumers: Final Demand |
|---|---|---|---|
| Coal | .1 | .1 | .2 |
| Textiles | 0.0 | .1 | .1 |

Let us consider how a *unit shortage* of coal would be "distributed" to the coal-using sectors. The consuming sectors, coal, textiles, and final demand, would first be assigned the specified minimum reduction of .1, .1, and .2 units of coal respectively. This would account for a total of .4 of the unit shortage with .6 left to be distributed. Since final demand has the lowest priority rank as a consumer of coal,

[7] Both $\bar{d}_{ij} - \underline{h}_{ij}e_i$ and $\bar{d}_{ij} - \bar{h}_{ij}e_i$ must be non-negative. See explanation in next paragraph.

Maximum Allotment Reduction Matrix: $[\bar{h}_{ij}]$

| *Inputs* | | *Consumers* | |
|---|---|---|---|
| | Coal Industry | Textile Industry | Final Demand |
| Coal | .5 | .5 | .5 |
| Textiles | .5 | .8 | 1.0 |

Priority Ranking Matrix: $[r_{ij}]$

| *Inputs* | | *Consumers* | |
|---|---|---|---|
| | Coal Industry | Textile Industry | Final Demand |
| Coal | 3 | 2 | 1 |
| Textiles | 2 | 3 | 1 |

the part of the shortage assigned to final demand would be increased by .3 to the specified maximum reduction .5. The part of the shortage assigned to textiles is next in line for an increase, and the textile industry can absorb the entire remaining shortage by having its assignment set to .4. Thus, the coal industry, textile industry, and final demand would have their demanded allotments of coal reduced by .1, .4, and .5 units respectively. A shortage of textiles would be analogously handled. The total effect of the CPRS is the same for each unit of a shortage and can be summed up by the following matrix:

DMS Matrix—Distribution of Marginal Shortage $[h_{ij}]$

| *Inputs* | | *Consumers* | |
|---|---|---|---|
| | Coal Industry | Textile Industry | Final Demand |
| Coal | .1 | .4 | .5 |
| Textiles | 0.0 | .1 | .9 |

The distribution of marginal shortage matrix (DMS matrix) simply describes how the associated constrained priority rationing schedule "distributes" a one-unit shortage of each product. It completely determines the outcome of any rationing by the CPRS. Suppose that, during some short period of time (say, a month) the demanded allotments of inputs, based on prorated annual output targets, were as follows:

Demanded Allotments of Inputs

| *Inputs* | *Consumers* | | | |
|---|---|---|---|---|
| | Coal Industry | Textile Industry | Final Demand | Total Demand |
| Coal | 8 | 6 | 6 | 20 |
| Textiles | 2 | 1 | 7 | 10 |

Then if the available supply of coal were, say, 19 and that of textiles 8, there would be shortages of 1 and 2, respectively. Distributing the shortages according to the DMS matrix and subtracting from the demanded allotments yields:

Rationed Allotments

| *Inputs* | *Consumers* | | | |
|---|---|---|---|---|
| | Coal Industry | Textile Industry | Final Sector | Total Allotted |
| Coal | 7.9 | 5.6 | 5.5 | 19 |
| Textiles | 2.0 | .8 | 5.2 | 8 |

### F. *A Formal Representation of the CPRS*

Suppose an economy contains $n$ single-product industries and a final-demand sector. A constrained priority rationing schedule may be formally represented by a triplet of $n \times (n + 1)$-dimensional matrices $(\underline{H}, \overline{H}, R)$, the minimum allotment-reduction matrix, the maximum allotment-reduction matrix, and the priority ranking matrix, respectively. (The columns of the matrices represent the $n + 1$ consuming sectors—the $n$ industries and the final demand sector, respectively.) The matrix elements $\underline{h}_{ij}$ and $\overline{h}_{ij}$ represent the minimum and maximum permissible reduction of the allotment of input $i$ to sector $j$ per unit excess demand of product $i$, so that we have

$$
(4) \qquad \begin{aligned} &0 \leqslant \underline{h}_{ij} \leqslant 1, \\ &0 \leqslant \overline{h}_{ij} \leqslant 1, \\ &\sum_{j=1}^{n+1} \underline{h}_{ij} \leqslant 1, \\ &\sum_{j=1}^{n+1} \overline{h}_{ij} \geqslant 1. \end{aligned}
$$

If $r_{is} > r_{it}$, then sector $s$ has higher priority for the acquisition of input $i$ than sector $t$ has.

The matrix $H$ is said to be a DMS matrix associated with $(\underline{H}, \bar{H}, R)$ if and only if (i) $\underline{H} \leqslant H \leqslant \bar{H}$, (ii) for each $i$, $\Sigma_j h_{ij} = 1$, and (iii) for each $i$, there exists some value $k$ of $j$, such that $h_{ij} = \bar{h}_{ij}$ whenever $r_{ij} < r_{ik}$, and $h_{ij} = \underline{h}_{ij}$ whenever $r_{ij} > r_{ik}$.

The existence of an associated DMS matrix is guaranteed by (4).[8]

## 3. OPTIMIZING PRIORITY RANKINGS

### A. *The Effect of Shortages on Production: Some Assumptions*

Let us assume that the priority rankings are the policy variables available to the planners, and that the minimum and maximum allotment reduction matrices are predetermined by constraints inherent in the organization of the economy or by political considerations. How should the priority rankings be set? An optimization problem presents itself: Determine a priority ranking which, under the given conditions, will minimize the loss in GNP resulting from any shortage which might arise.

In order for this problem to be well defined, it is necessary to define the effect of shortages of necessary inputs on production of a given product. For this purpose we define the "bottleneck-productivity" $q_{ij}$ of input $i$ in industry $j$ to be the productivity of input $i$ when it alone is scarce. Note that $q_{ij} = 1/a_{ij}$ for $a_{ij} > 0$ and $q_{ij} = 0$ when $a_{ij} = 0$, where $a_{ij}$ is the required input of product $i$ per unit output of product $j$.

The following principles will be adopted:

(i) Suppose industry $j$ receives as an input a smaller quantity of product $i$ than is required to produce its output target $y_j$. If all other inputs are present in the full amount, the loss in the output of product $j$ as a result of the input deficiency is given by $s_{ij}q_{ij}$, where $s_{ij}$ is the size of the deficiency of product $i$. More simply, if four nails are needed to make a chair, and if the chair industry receives eight nails less than its output target requires, the output of chairs will be diminished by two as a result.

(ii) If industry $j$ receives smaller quantities of several inputs than are required to produce its output target $y_j$, then the loss in output of product $j$ is given by $\Sigma_i s_{ij}q_{ij}$. In other words, if four nails and two boards are required to make a chair, and if the chair industry receives both four nails and two boards less than is indicated by its output plan, then the output of chairs will be diminished by two (not one) chairs, as a consequence.

This assumption implies that when a sector of the economy is short of many different inputs, the loss of output will be the sum of the losses that each shortage would cause separately, and not the maximum of the losses that each would cause separately. In order for the total loss of output to equal the maximum loss that each input shortage would cause separately, the "distribution" of the shortages

[8] If two elements in a row of $R$ are equal, the first of the two elements is considered to be greater. This guarantees the uniqueness of $H$.

would have to be coordinated. In general, the same *producing units* whose allotments of one scarce input are reduced must also receive smaller quantities of any complementary inputs in short supply. But there are several justifications for assuming that such coordination does not occur or is negated:

(a) There may be a lack of information during the period of plan execution. Let us assume that centrally processed information concerning current economic activity is available only in time for the construction of future annual economic plans, but not in time for guidance in the execution of the current plan. Consider the case of chair production. If one enterprise is short of boards and another is short of nails, the effects of the shortages would be additive. Given that shortages are marginal, this situation may well arise, because the distributors of nails may not know which of the chair producers have had their allotments of boards reduced. Recall that under traditional Soviet-type conditions, the enterprise has little incentive to reveal any information that would result in reduced allotments.

(b) Shortages involving a non-storable commodity may occur at different times. Suppose transportation services are also required for the production of chairs. The chair producer may be short of boards one month, and short of transportation services during the next month. The effect of the shortages would be additive.

(c) Shortages may be confined to differing localities. Suppose the production of glass requires both electric power and transportation services. If there were an electric power shortage in the locality of one glass factory, and a shortage of transportation services in the locality of another glass factory, the effect of the shortages on output would be additive.

Conditions (a), (b), and (c) imply that the maximum of the losses that would be caused by each of a series of shortages would understate the total loss caused by all of the shortages. In addition, they provide some justification for the mathematically tractable assumption that the total loss of output will be the sum of the losses that would be caused by each of the shortages separately.

(iii) Inventories of inputs accumulating at a producing unit as a result of the failure of that unit to meet production plans will not be available for use by another unit during the current planning period.

These principles are designed to deny to the planning organization the benefit of any doubt, in order that we may analyze the workings of the CPRS under extremely adverse circumstances. They should be considered as supplementary to the previously stated assumptions of fixed-proportion production functions and no input substitutability.

## B. *Prices and Eventual Values*

Up to now, we have been discussing the problem in physical rather than value terms. Indeed, in any situation where rationing is necessary, prices, by definition, do not reflect true exchange values; and for most purposes, it would not make sense to use them. In order to have the concept of GNP needed for the optimization problem, however, it is necessary to use some sort of prices to compare the values of final goods. It should be noted that the prices to be introduced are used only to determine a preference ordering for different assortments of final goods, and never to compare or ration intermediate goods. In fact, it turns out that the solution to the optimization problem is quite insensitive to the prices used.

A second concept that we need will be referred to as the *eventual value* of a product. The eventual value of a product is simply the loss of GNP which would result from the loss of a marginal unit of the product. The eventual value of a product can also be thought of as the value of a certain collection of final goods which incorporates exactly one unit of the product as an input. Clearly, the eventual value of a product is heavily dependent on the rationing system employed to ration scarce goods, since the rationing system determines the assortment of final goods which are dependent on the existence of a marginal unit of the product in question.

### C. *Deriving the Eventual Values*

If we ration with a CPRS, there is a fairly simple relationship between eventual values and the DMS matrix. Suppose $H$ is the DMS matrix associated with the CPRS $(\underline{H}, \bar{H}, R)$. Let $M$ be the interindustry part of $H$, with $m_{ij}$ equal to the fraction of the shortage of product $i$ "distributed" to industry $j$; and let $F$ be the final demand column of $H$, with $f_i$ the fraction of a shortage of product $i$ falling directly on final demand. (These identities shall be expressed as $H \equiv (M, F)$.)

Suppose that $S$ is a vector of shortages, $s_i$ being the magnitude of the shortage of product $i$. Then, if $P$ is a diagonal matrix, with $p_{ii}$ the final-sector price of product $i$, the short-run *direct* loss in GNP caused by the shortage $S$ is given by $\Sigma_{i=1}^{n} p_{ii} f_i s_i$, or by $S'PF$ in matrix notation. But $S$ will also result in reduced outputs in various industries consuming the scarce commodities as inputs, and this will cause further losses in GNP. The quantity of input lost to industry $j$ as a result of the shortage of product $i$ is given by $s_i m_{ij}$. The resulting loss in output of product $j$ is $s_i m_{ij} q_{ij}$, where $q_{ij}$ is the productivity of the input $i$ in industry $j$. Since losses caused by shortages are additive, the total loss of output of product $j$ caused by the shortages $S$, is given by $\Sigma_{i=1}^{n} s_i q_{ij} m_{ij}$ or by the product $S'(Q \times M)$ where $(Q \times M)$ is the matrix obtained from element-by-element multiplication of $Q$ by $M$. Because we have assumed that there are no inventories, $S'(Q \times M)$ is actually a vector of secondary shortages caused by the original shortages $S$.[9] The secondary shortages $S'(Q \times M)$ will cause both a direct loss to GNP $S'(Q \times M)PF$ and tertiary shortages $S'(Q \times M)^2$, and so forth. Thus the total loss to GNP caused by the original shortages $S$ can be expressed by

$$S'PF + S'(Q \times M)PF + S'(Q \times M)^2PF + \dots$$

provided that the series converges and that $S$ is small enough so that the loss of output of each sector is less than the original target output for that sector.

Let $V$ be the vector in which $v_i$ denotes eventual value of product $i$, i.e., $v_i$ is the loss of GNP per unit shortage of product $i$. From the preceding paragraph it

[9] We may modify these calculations to treat the contingency that stocks of commodity $i$ exist by setting all the elements of the $i$th row of $(Q \times M)$ equal to zero.

follows that if $V$ is finite, $V$ is given by

$$(5) \qquad V = \sum_{r=0}^{\infty} (Q \times M)^r PF.$$

Note that $v_i$ represents the *total* loss in final products of *all* sectors that would result per unit shortage of product $i$; consequently, it does not make any sense to add up the components of $V$.

### D. *The Eventual-Value Ranking Algorithm*

The purpose of this analysis is to determine a priority ranking which, given minimum and maximum allotment reductions per unit shortage, will minimize the loss of GNP which would result from a shortage. We can now state the problem more precisely as follows.

Find the priority ranking matrix $\tilde{R}$ with an associated DMS matrix $\tilde{H} \equiv (\tilde{M}, \tilde{F})$ that simultaneously minimizes each component of

$$\sum_{r=0}^{\infty} (Q \times M)^r PF$$

subject to (i) $\underline{H} \leqslant H \leqslant \overline{H}$ and (ii) for each $i$, $\Sigma_{j=1}^{n+1} h_{ij} = 1$. This is a non-linear programming problem, and it can be attacked from a strictly mathematical point of view. There is, however, an iterative procedure for solving this problem that has an economic rationale. The procedure, to be called the *eventual-value ranking algorithm*, is explained below.

Equation (5) gives us the eventual values $V$ of the products of the economy as a function of the DMS matrix $H \equiv (M, F)$ which is associated with a priority ranking matrix $R$. The eventual-value ranking algorithm uses $V$ in a natural way to derive a new priority ranking matrix $R'$ from $R$.

In the discussion opening this chapter, two factors were mentioned in connection with determining a basis for setting priorities in rationing: (i) the direct productivity of the scarce input in the consuming industry, and (ii) the importance of the output of the consuming industry to the general functioning of the economy. But direct productivity of an input is the reciprocal of the appropriate entry in the $A$ matrix, and the importance of an output of an industry is given in the relevant sense by its eventual value. In order to set priorities for acquisition of scarce inputs, it is logical to compare the total effects of withholding a unit of the input from each of the industries in question. Suppose a unit of input $i$ is withheld from industry $j$. Then the output of product $j$ will be lowered by $q_{ij}$. Let $l_{ij}$ be the direct and indirect loss to GNP resulting from this loss of output of product $j$, given a priority ranking schedule $R$ and its associated eventual value vector $V$. Since the eventual value $v_j$ of product $j$ gives the loss in GNP per unit shortage of product $j$, it follows that $l_{ij} = v_i q_{ij}$, where $j \leqslant n$. For the sake of consistency, we may define $l_{i,n+1}$ to be loss of GNP resulting from withholding a unit of product $i$ from final demand. Clearly, $l_{i,n+1} = p_{ii}$.

We define a new priority ranking schedule $R'$ in such a way as to minimize the estimated losses to GNP that would result from a shortage, where estimated losses are imputed by using the eventual values $V$ associated with the previous priority ranking $R$. Thus for any input $i$, the sector $j$ for which $l_{ij}$ is the largest is the sector which will be given the highest priority for acquiring input $i$, because, given $V$, withholding a unit of product $i$ from this sector would cause the greatest estimated loss of GNP. Likewise, the sector with the second largest value of $l_{ij}$ will have second priority, and so forth. More formally, a new priority ranking matrix $R'$ is defined by setting $r'_{ij} = l_{ij}$, or by

$$(6) \qquad r'_{ij} = \begin{cases} v_i q_{ij} & \text{for} \quad j \leqslant n, \\ p_{ii} & \text{for} \quad j = n + 1. \end{cases}$$

Of course, only the ordinal relationships between the elements on each row of $R'$, and not the cardinal values of the elements, are significant.

Given a priority ranking matrix $R$ and the associated vector of eventual values $V$, the cost in potential GNP of an exogenously-induced shortage of one unit of input $i$ is given by $\Sigma_j m_{ij} q_{ij} v_j + p_{ii} f_i$, or by $(Q \times M)V + PF$, in matrix notation. Since $R'$ is constructed so as to minimize this cost, it follows that

$$(7) \qquad (Q \times M')V + PF' = \min_{M,F} \{(Q \times M)V + PF\}$$

in every component simultaneously, where $H' \equiv (M', F')$ is the DMS matrix associated with $R'$.

The eventual-value ranking algorithm is formally defined as the process of generating the sequence of triplets, $(R_0, H_0, V_0)$, $(R_1, H_1, V_1)$, $(R_2, H_2, V_2), \ldots,$ where $H_i \equiv (M_i, F_i)$ and $V_i$ are the DMS matrix and eventual value vector associated with $R_i$ and where $R_{i+1} = R'_i$. A triplet $(\tilde{R}, \tilde{H}, \tilde{V})$ is said to be optimal if $\tilde{V} = \Sigma_{r=0}^{\infty} (Q \times \tilde{M})^r P\tilde{F}$ has the property that $\tilde{V} = \min_{M,F} \{\Sigma_{r=0}^{\infty} (Q \times M)^r PF\}$ in every component simultaneously. The following theorem asserts that such an optimal triplet $(\tilde{R}, \tilde{H}, \tilde{V})$ exists and that the sequence generated by the eventual-value ranking algorithm converges to that optimal triplet in a finite number of steps. In addition, it is shown that the sequence $\{V_i\}$ converges to $\tilde{V}$ monotonically.[10]

The proof of this theorem requires the assumption that any vector of eventual values is greater than zero in every component. This assumption is an explicit restatement of our previous no-slack assumption; it means that no matter what priority ranking is used, a shortage of any product will cause some loss in GNP.

THEOREM: *Given that $R_0$ is chosen so that $V_0 = \Sigma_{r=0}^{\infty} (Q \times M_0)^r PF_0$ exists, and given that any eventual value vector $V$ is strictly positive, it follows that (i) each successive eventual-value ranking iteration yields $R_i$ and $H_i \equiv (M_i, F_i)$ for which $V_i = \Sigma_{r=0}^{\infty} (Q \times M_i)^r PF_i$ exists, and $V_i \leqslant V_{i-1}$; (ii) if $V_{i-1} \neq \tilde{V}$, then $V_i < V_{i-1}$;*[11] *(iii) $\tilde{V}$ exists, and for some $m > 0$, $V_m = \tilde{V}$.*

[10] Martin Weitzman suggested to me that such a strong theorem could be proved. A theorem with a similar mathematical structure appears in his paper [**3**]. See also Bellman and Dreyfus [**1**, p. 78].

[11] The inequality "$<$" as applied to vectors indicates "$\leqslant$" in each component and "$<$" in at least one component.

PROOF: (i) Assume $V_{i-1} = \Sigma_{r=0}^{\infty} (Q \times M_{i-1})^r PF_{i-1}$ exists. Then $(Q \times M_{i-1})V_{i-1} = \Sigma_{r=1}^{\infty} (Q \times M_{i-1})^r PF_{i-1}$, so that $V_{i-1} - (Q \times M_{i-1})V_{i-1} = PF_{i-1}$ or $V_{i-1} = (Q \times M_{i-1})V_{i-1} + PF_{i-1}$. By equation (7), it follows that $V_{i-1} \geqslant (Q \times M_i)V_{i-1} + PF_i$. By repeatedly substituting $(Q \times M_i)V_{i-1} + PF_i$ for $V_{i-1}$, we have

$$V_{i-1} \geqslant (Q \times M_i)V_{i-1} + PF_i \geqslant (Q \times M_i)[(Q \times M_i)V_{i-1} + PF_i] + PF_i$$
$$\geqslant, \ldots, \geqslant (Q \times M_i)^k V_{i-1} + \left[\sum_{r=0}^{k-1} (Q \times M_i)^r\right] PF_i.$$

Since $(Q \times M_i)^k V_{i-1} \geqslant 0$, it follows that $[\Sigma_{r=0}^{k-1} (Q \times M_i)^r] PF_i \leqslant V_{i-1}$ for any $k$. It follows that $V_i = \Sigma_{r=0}^{\infty} (Q \times M_i)^r PF_i$ exists and that $V_i \leqslant V_{i-1}$.

(ii) Suppose $V_i = V_{i-1}$. Then, by (7), $V_{i-1} = \min_{M,F} \{(Q \times M)V_{i-1} + PF\}$ so that $\min_{M,F} \{(Q \times M)V_{i-1} + PF - V_{i-1}\} = 0$. Likewise

$$\min_{M,F} \{(Q \times M)^{r+1} V_{i-1} + (Q \times M)^r PF - (Q \times M)^r V_{i-1}\} = 0.$$

Thus, for any $k > 0$,

$$\min_{M,F} \left\{ \sum_{r=0}^{k} \left[ (Q \times M)^{r+1} V_{i-1} + (Q \times M)^r PF - (Q \times M)^r V_{i-1} \right] \right\} = 0.$$

Simplification yields

$$\min_{M,F} \left\{ (Q \times M)^{k+1} V_{i-1} + \sum_{r=0}^{k} (Q \times M)^r PF - V_{i-1} \right\} = 0$$

so that

$$(8) \qquad V_{i-1} = \min_{M,F} \left\{ (Q \times M)^{k+1} V_{i-1} + \sum_{r=0}^{k} (Q \times M)^r PF \right\}.$$

Suppose that for some $\hat{V} = \Sigma_{r=0}^{\infty} (Q \times \hat{M})^r P\hat{F}$, $V_{i-1} \nleqslant \hat{V}$. By assumption $\hat{V}$ must be strictly positive, so that $\lim_{r\to\infty} (Q \times \hat{M})^r = 0$.[12] Thus, for a sufficiently large $k$, $V_{i-1} \nleqslant (Q \times \hat{M})^{k+1} V_{i-1} + \hat{V}$, so that $V_{i-1} \nleqslant (Q \times \hat{M})^{k+1} V_{i-1} + \Sigma_{r=0}^{k} (Q \times \hat{M})^r P\hat{F}$. But this contradicts (8), and thus $V_{i-1} = \min_{M,F} \{\Sigma_{r=0}^{\infty} (Q \times M)^r PF\} = \tilde{V}$.

We have shown that if $V_i = V_{i-1}$, then $V_{i-1} = \tilde{V}$. Therefore, if $V_{i-1} \neq \tilde{V}$, then $V_{i-1} \neq V_i$. Since (i) requires that $V_i \leqslant V_{i-1}$, we have that if $V_{i-1} \neq \tilde{V}$, then $V_i < V_{i-1}$.

(iii) Since there are only a finite number of distinct priority ranking matrices $R$ (of any given dimension), there can be only a finite number of distinct eventual-value vectors in the sequence $\{V_i\}$. Thus, for some $m$, we must have $V_m = V_i$, where $i > m$. But by (ii) this is impossible unless $V_m = \tilde{V}$.

[12] In general, if $A$ is any non-negative square matrix and $X$ is a non-negative vector of the same dimension, and if $Y = \Sigma_{r=0}^{\infty} A^r X$ is strictly positive (in every component), then $\lim_{r\to 0} A^r = 0$.

The proof is as follows: For sufficiently large $k$, $\Sigma_{i=0}^{k} A^i X$ also must be strictly positive. Suppose $\lim_{r\to\infty} A^r \neq 0$. Then $\lim_{r\to\infty} A^r \Sigma_{i=0}^{k} A^i X \neq 0$, so that $\lim_{r\to\infty} \Sigma_{i=r}^{r+k} A^i X \neq 0$. But this contradicts the assumption that $\Sigma_{r=0}^{\infty} A^r X$ converges.

## 4. AN ILLUSTRATIVE APPLICATION

Several constrained priority rationing schedules were determined for a 17-sector breakdown of the Soviet economy. The data used were aggregated from the 1959 Soviet ruble input-output table as adjusted by V. Treml [2]. In each case, the rationing constraints were specified by a simple rule, and an optimal priority ranking matrix was determined by the eventual-value ranking algorithm defined in Section 3.

In deriving the optimal CPRS presented in Tables III to V, the parameters were specified as follows:

The rationing constraints were given by $\underline{H} = 0$ and $\bar{H} = [\min\{2z_{ij}, \sqrt{z_{ij}}\}]$ (see Table II), where $\underline{H}$ and $\bar{H}$ are the minimum and maximum allotment reduction matrices respectively (see Section 2, subsections D–F), and where $z_{ij}$ is the fraction of the output of sector $i$ consumed by sector $j$ (see Table I) as derived from the input-output table.

Prices for final outputs were specified to be the standard prices that had been used in constructing the input-output table ($p_{ii} = 1$) in the case of consumption goods, and three times the standard prices ($p_{ii} = 3$) in the case of capital goods (machinery and construction). The additional value inputed to capital goods is meant to reflect a common planners' preference in Soviet-type economies.

The optimal priority-ranking matrix corresponding to these parameters is given in Table III. Each row in this matrix specifies a priority ordering for the acquisition of a given product. The sectors with the smallest numbers along the row have the lowest priorities. If, for example, there were a shortage of fuel, then, according to the priority ranking matrix, final users, the construction industry, agriculture, the construction-materials industry, etc., would have their allotments of fuel reduced in turn by the maximum allowable amount, until the shortage of fuel was eliminated.

It is noteworthy that the structures of the priority ranking matrices turned out to be remarkably insensitive to the prices used to compute GNP. As an experiment, an optimal priority ranking matrix was computed with 8 of the 17 final-product prices set equal to 5 times their standard prices and with the remaining prices left unchanged (i.e., $p_{ii} = 5$ for $i$ even, and $p_{ii} = 1$ for $i$ odd). This matrix differed only slightly from the matrix computed using the standard final-product prices for every commodity ($p_{ii} = 1$ for all $i$). This insensitivity to final-product prices is a result of the fact that the ultimate effects on GNP of cutting the allotment of an input to each of two different sectors frequently differs by a large order of magnitude, rendering differences in relative prices of the components of final product unimportant.

The distribution-of-marginal-shortage (DMS) matrix appears in Table IV. Each row in this matrix describes the "distribution" of a shortage of a given product. The entry, say, in the "fuel" row and in the "food-processing" column indicates the fraction of a shortage of fuel that would have to be absorbed by the food-processing industry.

The eventual value per ruble and the bottleneck-loss elasticity of each sector, as well as the final-product price, are given in Table V. The eventual value of a good is

TABLE I

NORMAL DISTRIBUTION OF INPUTS ($Z$) (IN PERCENTAGE)

| | Metallurgy | Fuel | Electric Power | Machinery | Abrasives | Chemicals | Wood Products | Construction Materials | Glass | Light Industry | Food Processing | Construction | Agriculture | Forestry | Transportation and Communication | Trade and Procurement | Other | Final Products |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Metallurgy | 34 | 1 | 0 | 32 | 0 | 2 | 1 | 2 | 0 | 0 | 1 | 12 | 0 | 0 | 1 | 1 | 1 | 13 |
| Fuel | 13 | 14 | 10 | 3 | 0 | 2 | 3 | 3 | 0 | 1 | 3 | 3 | 8 | 0 | 13 | 1 | 0 | 23 |
| Electric Power | 11 | 9 | 0 | 10 | 0 | 5 | 3 | 6 | 0 | 5 | 3 | 6 | 2 | 0 | 4 | 1 | 2 | 32 |
| Machinery | 1 | 1 | 0 | 14 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 8 | 6 | 0 | 2 | 1 | 1 | 61 |
| Abrasives | 7 | 0 | 0 | 62 | 7 | 1 | 4 | 1 | 1 | 1 | 0 | 8 | 4 | 0 | 0 | 1 | 0 | 2 |
| Chemicals | 2 | 1 | 0 | 15 | 0 | 27 | 2 | 0 | 0 | 8 | 1 | 4 | 5 | 0 | 5 | 1 | 6 | 23 |
| Food Products | 1 | 3 | 0 | 4 | 0 | 1 | 25 | 1 | 0 | 1 | 3 | 25 | 1 | 0 | 1 | 6 | 3 | 24 |
| Construction Materials | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 17 | 0 | 0 | 0 | 73 | 0 | 0 | 1 | 0 | 0 | 6 |
| Glass | 0 | 0 | 0 | 6 | 0 | 3 | 5 | 0 | 1 | 0 | 6 | 36 | 1 | 0 | 1 | 0 | 0 | 40 |
| Light Industry | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 36 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 57 |
| Food Processing | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 23 | 0 | 3 | 0 | 0 | 0 | 1 | 71 |
| Construction | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 100 |
| Agriculture | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 31 | 0 | 23 | 0 | 0 | 0 | 0 | 37 |
| Forestry | 0 | 0 | 0 | 0 | 0 | 0 | 46 | 0 | 0 | 0 | 0 | 22 | 0 | 2 | 0 | 0 | 5 | 25 |
| Transportation and Communication | 7 | 25 | 0 | 7 | 0 | 5 | 15 | 15 | 1 | 4 | 11 | 0 | 7 | 0 | 0 | 1 | 1 | 0 |
| Trade and Procurement | 4 | 7 | 0 | 4 | 0 | 1 | 4 | 3 | 1 | 12 | 32 | 0 | 25 | 0 | 0 | 0 | 6 | 0 |
| Other | 2 | 0 | 0 | 3 | 0 | 1 | 1 | 2 | 0 | 2 | 1 | 3 | 2 | 0 | 0 | 0 | 3 | 80 |

TABLE II

MAXIMUM ALLOTMENT REDUCTIONS ($\bar{H}$) (IN PERCENTAGE)

| | Metallurgy | Fuel | Electric Power | Machinery | Abrasives | Chemicals | Wood Products | Construction Materials | Glass | Light Industry | Food Processing | Construction | Agriculture | Forestry | Transportation and Communication | Trade and Procurement | Other | Final Products |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Metallurgy | 58 | 1 | 0 | 57 | 0 | 4 | 2 | 5 | 0 | 1 | 1 | 23 | 0 | 0 | 2 | 1 | 1 | 26 |
| Fuel | 26 | 29 | 19 | 6 | 0 | 4 | 6 | 6 | 1 | 2 | 6 | 6 | 15 | 0 | 25 | 1 | 1 | 46 |
| Electric Power | 22 | 19 | 0 | 20 | 0 | 11 | 5 | 13 | 1 | 9 | 6 | 11 | 5 | 0 | 8 | 2 | 4 | 56 |
| Machinery | 2 | 2 | 1 | 29 | 0 | 1 | 2 | 2 | 0 | 1 | 3 | 15 | 12 | 0 | 4 | 1 | 2 | 78 |
| Abrasives | 14 | 1 | 0 | 79 | 14 | 2 | 8 | 2 | 2 | 3 | 0 | 15 | 8 | 0 | 0 | 2 | 0 | 5 |
| Chemicals | 4 | 2 | 0 | 29 | 0 | 52 | 4 | 1 | 0 | 15 | 2 | 7 | 10 | 0 | 10 | 2 | 11 | 45 |
| Wood Products | 2 | 7 | 0 | 7 | 0 | 3 | 50 | 2 | 1 | 2 | 6 | 49 | 2 | 0 | 3 | 13 | 5 | 48 |
| Construction Materials | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 35 | 0 | 0 | 1 | 86 | 0 | 0 | 2 | 0 | 0 | 11 |
| Glass | 1 | 0 | 0 | 13 | 0 | 7 | 9 | 0 | 3 | 0 | 11 | 60 | 1 | 0 | 1 | 0 | 0 | 63 |
| Light Industry | 1 | 1 | 0 | 2 | 0 | 2 | 2 | 0 | 0 | 60 | 1 | 3 | 1 | 0 | 1 | 2 | 0 | 75 |
| Food Processing | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 46 | 0 | 5 | 0 | 0 | 0 | 3 | 84 |
| Construction | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 100 |
| Agriculture | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 17 | 55 | 0 | 46 | 0 | 0 | 0 | 1 | 61 |
| Forestry | 0 | 0 | 0 | 0 | 0 | 0 | 68 | 0 | 0 | 0 | 0 | 43 | 0 | 4 | 0 | 0 | 10 | 50 |
| Transportation and Communication | 13 | 49 | 0 | 15 | 0 | 11 | 30 | 31 | 2 | 8 | 22 | 1 | 14 | 0 | 1 | 2 | 2 | 0 |
| Trade and Procurement | 7 | 15 | 0 | 7 | 0 | 2 | 9 | 6 | 2 | 25 | 57 | 0 | 50 | 0 | 0 | 0 | 13 | 0 |
| Other | 4 | 0 | 0 | 6 | 0 | 2 | 2 | 3 | 0 | 3 | 2 | 6 | 3 | 0 | 0 | 0 | 7 | 90 |

TABLE III

Priority Ranking for Rationing of Inputs ($R$)

| | Metallurgy | Fuel | Electric Power | Machinery | Abrasives | Chemicals | Wood Products | Construction Materials | Glass | Light Industry | Food Processing | Construction | Agriculture | Forestry | Transportation and Communication | Trade and Procurement | Other | Final Products |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Metallurgy | 4 | 15 | 17 | 3 | 16 | 11 | 7 | 5 | 9 | 8 | 6 | 2 | 14 | 18 | 13 | 12 | 10 | 1 |
| Fuel | 6 | 12 | 9 | 11 | 18 | 14 | 7 | 4 | 13 | 8 | 5 | 2 | 3 | 17 | 10 | 15 | 16 | 1 |
| Electric Power | 7 | 13 | 17 | 5 | 16 | 12 | 8 | 3 | 10 | 4 | 6 | 2 | 9 | 0 | 14 | 15 | 11 | 1 |
| Machinery | 11 | 14 | 16 | 3 | 18 | 15 | 7 | 6 | 10 | 8 | 5 | 2 | 4 | 17 | 12 | 13 | 9 | 1 |
| Abrasives | 7 | 0 | 0 | 2 | 6 | 10 | 4 | 8 | 5 | 0 | 0 | 3 | 0 | 0 | 0 | 9 | 0 | 1 |
| Chemicals | 13 | 15 | 18 | 4 | 17 | 6 | 8 | 10 | 12 | 2 | 9 | 3 | 5 | 16 | 11 | 14 | 7 | 1 |
| Wood Products | 13 | 12 | 17 | 8 | 18 | 14 | 3 | 6 | 11 | 5 | 4 | 2 | 10 | 16 | 15 | 7 | 9 | 1 |
| Construction Materials | 11 | 12 | 15 | 5 | 16 | 13 | 7 | 3 | 8 | 6 | 4 | 2 | 9 | 14 | 10 | 0 | 0 | 1 |
| Glass | 10 | 0 | 11 | 6 | 12 | 7 | 4 | 8 | 5 | 0 | 3 | 2 | 0 | 0 | 9 | 0 | 0 | 1 |
| Light Industry | 12 | 15 | 17 | 6 | 16 | 10 | 4 | 9 | 11 | 2 | 5 | 3 | 8 | 18 | 13 | 7 | 14 | 1 |
| Food Processing | 12 | 15 | 16 | 10 | 14 | 7 | 8 | 9 | 13 | 4 | 2 | 6 | 3 | 0 | 0 | 11 | 5 | 1 |
| Construction | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Agriculture | 12 | 13 | 0 | 0 | 0 | 10 | 6 | 0 | 11 | 4 | 2 | 7 | 3 | 8 | 9 | 0 | 5 | 1 |
| Forestry | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 2 | 0 | 5 | 0 | 0 | 4 | 1 |
| Transportation and Communication | 10 | 8 | 18 | 7 | 17 | 12 | 3 | 2 | 9 | 5 | 4 | 11 | 6 | 16 | 15 | 14 | 13 | 1 |
| Trade and Procurement | 10 | 11 | 13 | 9 | 0 | 12 | 6 | 5 | 8 | 3 | 2 | 0 | 4 | 0 | 0 | 0 | 7 | 1 |
| Other | 10 | 0 | 0 | 7 | 0 | 12 | 9 | 4 | 11 | 3 | 8 | 2 | 6 | 0 | 0 | 0 | 5 | 1 |

TABLE IV

DISTRIBUTION OF MARGINAL SHORTAGES ($H$) (IN PERCENTAGE)

| | Metallurgy | Fuel | Electric Power | Machinery | Abrasives | Chemicals | Wood Products | Construction Materials | Glass | Light Industry | Food Processing | Construction | Agriculture | Forestry | Transportation and Communication | Trade and Procurement | Other | Final Products |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Metallurgy | 0 | 0 | 0 | 51 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 23 | 0 | 0 | 0 | 0 | 0 | 26 |
| Fuel | 20 | 0 | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 0 | 6 | 6 | 15 | 0 | 0 | 0 | 0 | 46 |
| Electric Power | 0 | 0 | 0 | 11 | 0 | 0 | 0 | 13 | 0 | 9 | 0 | 11 | 0 | 0 | 0 | 0 | 0 | 56 |
| Machinery | 0 | 0 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 15 | 0 | 0 | 0 | 0 | 0 | 78 |
| Abrasives | 0 | 1 | 0 | 79 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 4 | 8 | 0 | 0 | 0 | 0 | 5 |
| Chemicals | 0 | 0 | 0 | 29 | 0 | 0 | 0 | 0 | 0 | 15 | 0 | 7 | 3 | 0 | 0 | 0 | 0 | 45 |
| Wood Products | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 49 | 0 | 0 | 0 | 0 | 0 | 48 |
| Construction Materials | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 86 | 0 | 0 | 0 | 0 | 0 | 11 |
| Glass | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 35 | 1 | 0 | 0 | 0 | 0 | 63 |
| Light Industry | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 75 |
| Food Processing | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | 0 | 0 | 0 | 0 | 84 |
| Construction | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 100 |
| Agriculture | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 39 | 0 | 0 | 0 | 0 | 0 | 0 | 61 |
| Forestry | 0 | 0 | 0 | 0 | 0 | 0 | 7 | 0 | 0 | 0 | 0 | 43 | 0 | 0 | 0 | 0 | 0 | 50 |
| Transportation and Communication | 0 | 0 | 0 | 0 | 0 | 0 | 30 | 31 | 0 | 8 | 22 | 0 | 10 | 0 | 0 | 0 | 0 | 0 |
| Trade and Procurement | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 | 57 | 0 | 18 | 0 | 0 | 0 | 0 | 0 |
| Other | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 3 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 90 |

TABLE V

| | Eventual Value per Ruble | Bottleneck-Loss Elasticity | Final Output Price |
|---|---|---|---|
| Metallurgy | 63.3 | 3.6 | 1 |
| Fuel | 184.9 | 10.4 | 1 |
| Electric Power | 315.2 | 4.6 | 1 |
| Machinery | 14.8 | 1.8 | 3 |
| Abrasives | 4978.0 | 2.5 | 1 |
| Chemicals | 175.6 | 5.7 | 1 |
| Wood Products | 16.9 | 0.8 | 1 |
| Construction Materials | 16.9 | 0.5 | 1 |
| Glass | 96.4 | 0.3 | 1 |
| Light Industry | 2.4 | 0.4 | 1 |
| Food Processing | 2.7 | 0.6 | 1 |
| Construction | 3.0 | 0.4 | 3 |
| Agriculture | 4.3 | 0.9 | 1 |
| Forestry | 671.6 | 0.8 | 1 |
| Transportation and Communication | 135.2 | 6.2 | 1 |
| Trade and Procurement | 57.7 | 2.7 | 1 |
| Other | 50.6 | 1.7 | 1 |

the shadow price that would obtain if the good were in short supply, no slack existed anywhere in the economy, the demand-supply relationships for all other goods were perfectly balanced, and rationing were subject to the stated CPRS constraints. It should be noted that eventual value per ruble is a measurement which depends strongly on the assumed prices of intermediate goods, some of which cannot be scarcity prices since shortages exist. Therefore, it is of doubtful validity to compare eventual values per ruble of the outputs of the different sectors.

The bottleneck-loss elasticity of a sector is given by the product of the eventual value per ruble and the gross output of the sector, divided by the total value of final output (GNP). It measures the fractional fall in final output that would result from a marginal fractional fall in the output of the given sector under the above stated conditions. The fact that the bottleneck-loss elasticity of fuel is 10.4, for example, implies that in a perfectly balanced but completely taut economy in which the specified rationing constraints prevailed, a 1 per cent decrease in the output of fuel would necessarily cause GNP to fall by roughly 10.4 per cent, in the short run. Thus the bottleneck-loss elasticity could be used as an index of the direct and indirect *short-run* contribution of a sector to the final product. Unlike the eventual value per ruble, the bottleneck-loss elasticity does not depend on assumed prices for intermediate goods, so that comparison of bottleneck-loss elasticities of different sectors is meaningful.[13]

The theory developed in Sections 2 and 3 and applied here contains several simplifying assumptions which do not apply in this example, so that the practical

[13] The bottleneck-loss elasticity of a commodity depends on final output prices only as weights in comparing different final product assortments with one another. But the eventual value per ruble of a commodity also depends on the price of the commodity that was used to define a unit (one ruble's worth) of the commodity.

TABLE VI

TECHNOLOGICAL COEFFICIENTS ($A$)

| | Metallurgy | Fuel | Electric Power | Machinery | Abrasives | Chemicals | Wood Products | Construction Materials | Glass | Light Industry | Food Processing | Construction | Agriculture | Forestry | Transportation and Communication | Trade and Procurement | Other |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Metallurgy | .337 | .005 | .002 | .150 | .083 | .035 | .011 | .045 | .030 | .001 | .002 | .055 | .000 | .001 | .011 | .008 | .012 |
| Fuel | .129 | .143 | .375 | .014 | .061 | .031 | .033 | .060 | .068 | .003 | .007 | .015 | .020 | .017 | .157 | .009 | .006 |
| Electric Power | .028 | .024 | .001 | .012 | .071 | .024 | .007 | .031 | .015 | .004 | .002 | .007 | .002 | .000 | .013 | .004 | .007 |
| Machinery | .026 | .024 | .026 | .143 | .034 | .016 | .027 | .040 | .045 | .003 | .007 | .080 | .034 | .021 | .049 | .019 | .044 |
| Abrasives | .001 | .000 | .000 | .003 | .069 | .000 | .000 | .000 | .002 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 |
| Chemicals | .010 | .007 | .002 | .039 | .034 | .275 | .015 | .005 | .016 | .014 | .001 | .010 | .008 | .009 | .037 | .007 | .053 |
| Wood Products | .008 | .030 | .001 | .014 | .007 | .021 | .254 | .020 | .041 | .003 | .006 | .103 | .002 | .005 | .015 | .069 | .038 |
| Construction Materials | .001 | .001 | .001 | .002 | .004 | .001 | .001 | .174 | .008 | .000 | .000 | .184 | .000 | .003 | .005 | .000 | .000 |
| Glass | .000 | .000 | .000 | .002 | .001 | .004 | .003 | .000 | .015 | .000 | .001 | .011 | .000 | .000 | .001 | .000 | .000 |
| Light Industry | .010 | .012 | .003 | .013 | .104 | .063 | .030 | .007 | .017 | .356 | .003 | .019 | .003 | .001 | .013 | .043 | .004 |
| Food Processing | .001 | .000 | .000 | .001 | .022 | .054 | .003 | .001 | .001 | .014 | .231 | .003 | .030 | .000 | .000 | .001 | .096 |
| Construction | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 |
| Agriculture | .000 | .000 | .000 | .000 | .000 | .001 | .003 | .000 | .000 | .103 | .285 | .000 | .232 | .015 | .002 | .000 | .024 |
| Forestry | .000 | .000 | .000 | .000 | .000 | .000 | .012 | .000 | .000 | .000 | .000 | .002 | .000 | .020 | .000 | .000 | .002 |
| Transportation and Communication | .054 | .200 | .001 | .028 | .036 | .075 | .138 | .236 | .098 | .010 | .021 | .001 | .014 | .016 | .004 | .010 | .015 |
| Trade and Procurement | .030 | .061 | .002 | .013 | .000 | .011 | .040 | .046 | .101 | .033 | .064 | .000 | .053 | .000 | .000 | .000 | .086 |
| Other | .013 | .000 | .000 | .008 | .000 | .011 | .006 | .020 | .012 | .003 | .001 | .009 | .003 | .000 | .000 | .000 | .034 |

usefulness of these results is limited. The two most important of the simplifying assumptions are (i) the homogeneity of the output of each sector, and (ii) the impossibility of input substitution.

The marginal product of an input into a sector was derived from an aggregated input-output table by dividing the total value of the input into the sector by total value of the output of the sector. However, because the output of a sector is composed of many products, it is perfectly possible that the input is actually used in the production of only part of the output of the sector. Therefore, the apparent marginal product of an input into a sector may be much larger than the actual marginal product of the input. On the other hand, differences within each of the input categories could have exactly the opposite effect. The former of these effects is probably pronounced in the calculation of the marginal product of the output of a small sector as an input into a large sector, so that the bottleneck-loss elasticities of small sectors are probably exaggerated.

The assumption that an available input can never be substituted for one in short supply also exaggerates the importance of each sector to the production process. It also tends to increase the apparent size of the bottleneck-loss elasticities, and to distort the results of the algorithm for finding optimal priority rationing schedules. The amount of substitution actually possible depends on how the sectors of the model are defined and the time period under consideration. Obviously, the more aggregated (and, thus different from one another) the sectors are, and the shorter the time period covered, the smaller the amount of possible input substitution between sectors. For this reason, it seems unlikely that the distortion caused by input substitutability would be a major factor in this example. However, if this theory were applied to more detailed data (with potentially more useful results) distortion caused by input substitutability could be of major proportions. Therefore, it is necessary to generalize the theory to take account of input substitutability before it can have a practical use in rationing.

Another problem in applying this method is the specification of the constraints. For purposes of simplicity, the maximum allotment reduction constraints $\overline{H}$ were specified as the minimum of a linear and power function of $Z$, the share of total output of one sector normally consumed by another. In particular, the maximum allotment reduction as a percentage of the size of the shortage of an input was set at twice the percentage of the input normally consumed by the sector when that percentage is small, but this ratio is smaller when the percentage of the input normally consumed is larger.

Despite the above reservations, the results obtained from this illustrative example have some interesting implications.

First, the results indicate that sectors using a small fraction of total production of an input product normally ought to be given a high priority for its acquisition, and that large users of an input should usually bear the brunt of CPRS rationing. This is a result of the fact that given a fixed-proportions production function, inputs generally have a high productivity in sectors which use small amounts of them.

A second conspicuous result of this example is that the final-product sector (consumption and investment) is invariably assigned the lowest priorities for the acquisition of goods. This occurs because the loss to GNP as a result of withdrawing a scarce good from final use is merely the value of the good, while the loss to GNP as a result of withdrawing a scarce good from intermediate uses is magnified by the tendency of a shortage of an input into a production process to cause a shortage of the output of that process, and so on. This fact may partly explain the tendency of planners in Soviet-type economies to use the level of consumption as a cushion for unexpected shortages.

A third result, the computed bottleneck-loss elasticities, was more reassuring than informative. The sectors with the highest elasticities were fuel, transportation and communications, chemicals, and electric power, in that order. The reader may wish to ask himself if these sectors correspond to those he would judge most vital to the maintenance of GNP *in the short run.* An analogous question may be asked about glass, construction, light industry, construction materials, and food[14] which have the lowest bottleneck-loss elasticities.

All of this short-run information was obtained by applying the theory developed in Sections 2 and 3 to static input-output data. Results pertaining to the longer run, could be obtained by applying the theory to dynamic input-output data.

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[3] WEITZMAN, MARTIN L.: "On Choosing an Optimal Technology," *Management Science*, 13 (1967), 413–428.

[14] Note that a large percentage of food produced has been treated in this paper as a final product, with the implication that a reduction in food supply would not have repercussions in the productive sphere. This treatment of food (and of other consumer goods) is justifiable only if it is also assumed, as it is here, that the economy is far above the subsistence level and that the reductions in supply will be relatively small.