Harmonic Qualities of Generic Sets

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Abstract. It is often useful to consider pitch-class sets at the *generic* level, i.e., as letter names or staff positions without accidentals. One way to represent this generic reduction mathematically is as a mapping to 7-tET sets, which can be understood as paths in voice-leading space. We can address generic reduction using *harmonic qualities*, derived from the Fourier transform on pitch-class sets, by direct comparison of Fourier coefficients of 12-tET and 7-tET sets. The first three coefficients correspond to the generic classification of sets as scalar, tertian, or quartal types. The diatonic harmonic quality is also strongly correlated with distance to the nearest generic set. I also propose a radial dimensional reduction of Fourier coefficients, the "star chart," to represent the harmonic-quality residuals of generic reduction. I conclude by using the various theoretical tools developed to analyze the introductory "Proverb" from Benjamin Britten's *Blake Songs*.

Keywords: Generic sets · Diatonic categories · Voice-leading spaces · Harmonic qualities · Fourier transform · Pitch-class sets.

The pitch-class content of music is often understood not only at the specific level of the designated 12-tET values of the notes, but also a generic level where pitch-class content is reduced to seven "step classes" per octave, represented by the letter names of notes or the lines and spaces of the staff. The topic has been approached from various angles. One is to develop methods and algorithms for performing these reductions, which can be understood as pitch-spelling algorithms [9, 8, 20, 21]. Another is to assume a spelling and to deal with the simplified generic pcsets as theoretical and analytical objects [11, 12, 15, 14, 17, 30] Recently Tymozcko [27] has proposed a classification system for chords based on generic reduction. The goal of this paper is to relate the generic reduction to recent work on Fourier transforms of pcsets and harmonic qualities.

Voice-leading geometry is one effective tool for understanding generic reduction. We may represent generic pcsets as 7-tET sets and the generic reduction of a pcset set as a voice leading from the 12-tET set to the 7-tET set. The voice-leading distance from one to the other reveals how good of a representative the generic set is. There are a few important advantages to this approach. The 7-tET sets can have doublings, where two notes in the specific set are understood as variants of a single step-class (e.g., chromatic semitones). We can recognize ambiguity, considering multiple possible generic representatives of a single specific pcset, distinguishing between them by voice-leading distances. For instance, a set like $(014)_{12}$[1] may be understood as ambiguous: The nearest generic set is $(002)_7$, which treats the semitone as a chromatic semitone and both other intervals as thirds (Ex.: the spelling CC♯E). But there are other reasonable possibilities, generic sets $(013)_7$ (Ex: CD♭E) and $(012)_7$ (Ex.: B♯C♯E). Section 1 will explain the voice-leading approach in more detail.

In adopting 7-tET sets as representatives of generic pcsets, we are not necessarily thinking of 7-tET as a tuning, or 7-tET sets as direct sonic objects in this sense. A better way to understand the status of 7-tET sets as theoretical objects is as a weighted average of all the real sets that reduce to them in voice-leading space. I will also use 7-tET sets as representatives of generic sets in this sense here, with the primary question being how the harmonic qualities change from the specific set to the generic set and what this tells us about the reduction. Section 2 will introduce Fourier transforms and the concept of harmonic qualities. Section 3 then compares the two approaches, showing that a heuristic using DFT magnitudes can largely reproduce the classification of 12-tET sets we get from the voice-leading approach, while also providing additional information about the sets. Section 4 builds on this to develop a visualization tool, the "star chart." I conclude with a brief analysis of the introductory passage from Benjamin Britten's *Blake Songs* as an illustrative application of the theory developed here.

1 Generic Reduction in Voice-Leading Space

Voice-leading geometries are cardinality-specific spaces in which chords correspond to points and voice leadings to paths. They begin by representing the chord as an ordered tuple of MIDI numbers and then reducing the space by applying any number of symmetries to the result, the most important being octave equivalence, permutational equivalence, and transpositional equivalence. For more thorough introductory treatments of voice-leading geometry see [7] and [18]. The following discussion assumes all three of these equivalences (set-type or OPT space), and sometimes generalizes over inversion also (i.e. operating at the set-class level). The two most common ways to calculate voice-leading distances in this space are the taxicab and Euclidean metrics. To calculate either of these the first step is to find transpositions of the two sets with the same pitch-class (pc) sum. The pc sum is the sum of the pc numbers mod 12, and sets matched by pc sum have purely balanced voice leadings (i.e. the sum of all the components of the voice leading is zero) [13, 26, 28]. Minimal voice leadings by the Euclidean and taxicab metrics are always perfectly balanced. Once the two sets are transposed to the same pc sum and ordered to match their nearest pcs, $A = a_1, a_2, \ldots, a_n$ and $B = b_1, b_2, \ldots, b_n$, the Euclidean distance between sets is $(\sum_{i=1}^{n}(a_i - b_i)^2)^{\frac{1}{2}}$ and the taxicab distance is $\sum_{i=1}^{n}|a_i - b_i|$.

Table 1 orders the twelve trichords by the Euclidean distance of the nearest 7-tET set and gives some alternative generic sets for sets that are potentially ambiguous. Although the sets are understood as chord types rather than set classes

[1] I will use subscripts like this to indicate the ET where this is not otherwise clear.

(generalized over transposition but not inversion), inversions are omitted from Table 1 because they will give exactly the same results. For instance, $(045)_{12}$ set (e.g. CEF) is also a Euclidean distance of 0.53 semitones from $(023)_7$. Sometimes the 12-tET set has symmetries not shared by the 7-tET set. For instance, $(012)_{12}$ is ambiguous because it is 0.73 semitones from $(001)_7$ and the inversionally related $(011)_7$ (e.g., CC♯D vs. CD♭D). The augmented triad is still more ambiguous, being 0.81 semitones from three different transpositions of $(024)_7$. Most of these inversionally related voice leadings are included in the table (with the exception of $(036)_{12}$ to $(023)_7$).

Table 1. The 12-tET trichords ordered by voice-leading distance to the nearest generic trichord (distances measured in semitones). Colors indicate the resulting classification of the trichords as scalar (green), quartal (blue), or tertian (violet).

	Closest generic set			Alternative 1			Alternative 2		
12t-ET set	7-tET set	Ex.	VL dist.	7-tET set	Ex.	VL dist.	7-tET set	Ex.	VL dist.
(027)	(014)	CDG	0.2						
(025)	(013)	CDF	0.31						
(024)	(012)	CDE	0.4						
(037)	(024)	CE♭G	0.42						
(013)	(012)	CD♭E♭	0.51						
(015)	(013)	CD♭F	0.53						
(036)	(024)	CE♭G♭	0.61	(013)	CD♯F♯	0.93			
(026)	(013)	CDF♯	0.62	(014)	CDG♭	0.84			
(016)	(014)	CD♭G♭	0.65	(003)	CC♯F♯	0.76			
(014)	(002)	CC♯E	0.71	(013)	B♯C♯E	0.82	(012)	CD♭E	0.91
(012)	(001)	CC♯D	0.73	(011)	CD♭D	0.73	(012)	B♯C♯D♭	1.01
(048)	(024)	CEG♯	0.81	(025)	CEA♭	0.81	(035)	CF♭A♭	0.81

A nice feature of generic sets is their simplicity: with the assumption of permutational, transpositional, and inversional equivalence (i.e. the set class level) there are three generic intervals (step, third, fourth) and only four generic trichords: (012), (013), (014), and (024). Three of the four generic trichords are *generated* by one of the three intervals (i.e. the trichord results from two iterations of the interval): (012) by (01), (024) by (02), and (014) by (03). The remaining trichord (013) is a special “FLID” (flat interval distribution) set, having exactly one instance of each interval (see [3–5, 30]). Tymoczko [27] promotes a classification system for pc sets based on this fact, as scalar (step-based), tertian (third-based), quartal (fourth-based), or mixed.[2] The colors (green, violet,

[2] An important difference in Tymoczko’s [27] approach and the present one is that he only considers generic sets without doublings. For trichords, as evident in Table 1, this is not too problematic because there is always at least a first or second alternative generic set without doublings, but for larger sets this is not true, and we would have to consider much larger voice leadings to avoid generic sets with doublings. Many classifications, especially for larger sets, change if doublings are not allowed.

blue) in Table 1 show this classification. Mixed chords (black) can be mixed in two different senses: their closest generic set can be the mixed set, $(013)_7$ (e.g., $(025)_{12}$), or they can be ambiguous and close to multiple types. For example, $(014)_{12}$ could be considered a third with doubling (tertian), a mixed trichord, or a scalar trichord. If a set is ambiguous, but all its neighbors are of the same type, as with $(012)_{12}$, then the classification is unambiguous.

Generic tetrachords are similarly simple, since they are the complements of the trichords: (0123) = scalar, (0134) = quartal, (0135) = tertian, and (0124) = mixed. A caveat, however, is there are many more possibilities of trichords with doublings included with the cardinality-4 chords. Nonetheless, the trichords with doublings can be readily classified by removing the doubling. Table 2 classifies the 12-tET tetrachords using the same system, again ordered by voice-leading distance to the nearest generic tetrachord. Note that the tertian chords include all of the traditionally recognized seventh-chord types (in order of appearance: minor, major, dominant/half-diminished, augmented/minor-major, diminished) and the scalar types include all possible arrangements of whole and half-steps (2-1-2, 1-2-2 / 2-2-1, 2-2-2, 1-1-2 / 2-1-1, 1-1-1, 1-2-1).

2 Harmonic Qualities of Generic and Specific Sets

Harmonic qualities were originally defined by Quinn [22] as a way of describing and classifying pcsets in 12-tET. A pcset is represented by a 12-place *indicator vector*, $A = (a_0, a_1, \ldots, a_{11})$, with one place per pc, starting from C, with a 1 indicating if the pc is present and a 0 if it is absent. The DFT of the pcset is a vector of complex numbers $\hat{A} = (\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{11})$ defined by:

$$\hat{a}_k = \sum_{j=0}^{11} a_j e^{-i2\pi kj/12} = \sum_{j=0}^{11} a_j(\cos(2\pi kj/12) + i\sin(2\pi kj/12)) \qquad (1)$$

These twelve complex numbers reduce to six harmonic qualities on the grounds that $\hat{a}_0$ is equal simply to the cardinality of the pcset and so is ignored, and $\hat{a}_{12-k}$ is the conjugate of $\hat{a}_k$ (this is known as *aliasing*), so $\hat{a}_7$ through $\hat{a}_{11}$ can also be ignored.[3] This leaves six qualities represented by $\hat{a}_1$ through $\hat{a}_6$. Typically it is the *magnitudes* of these qualities, which are invariant with respect to transposition, inversion, homometry (Z-relation), and complementation, which are of interest. Theorists have generally agreed on the naming of qualities 4–6 as octatonicity, diatonicity, and whole-tone quality respectively, based on the scale types that maximize the given quality (what Quinn [22] calls a *prototype*). There is less agreement about the first three, with the first variously referred to as "chromaticity" or "clustered" quality, the second as "dyadicity," "tritone" or "quartal" quality, and the third as "triadicity," "hexatonicity," or "augmentedness" [2, 3, 6, 10, 23, 31, 34, 37] Comparing these 12-tET qualities to generic 7-tET qualities

[3] This and other mathematical properties of the DFT are explained in more detail in [2, 6, 31, 33].

Table 2. The 12-tET trichords ordered by voice-leading distance to the nearest generic trichord (distances measured in semitones). Colors indicate the resulting classification of the trichords as scalar (green), quartal (blue), or tertian (violet).

	Closest generic set			Alternative 1			Alternative 2		
12t-ET set	7-tET set	Ex.	VL dist.	7-tET set	Ex.	VL dist.	7-tET set	Ex.	VL dist.
(0257)	(0134)	CDFG	0.32						
(0247)	(0124)	CDEG	0.42						
(0358)	(0135)	CE♭FA♭	0.45						
(0235)	(0123)	CDE♭F	0.52						
(0237)	(0124)	CDE♭G	0.53						
(0135)	(0123)	CD♭E♭F	0.55						
(0158)	(0135)	CD♭FA♭	0.59						
(0258)	(0135)	CDFA♭	0.62						
(0246)	(0123)	CDEF♯	0.64						
(0157)	(0134)	CD♭FG	0.65						
(0136)	(0124)	CD♭E♭G♭	0.65						
(0137)	(0124)	CD♭E♭G	0.68						
(0347)	(0224)	CE♭EG	0.71	(0124)	CD♯EG	1.00	(0234)	CE♭F♭G	1.00
(0156)	(0134)	CD♭FG♭	0.72						
(0125)	(0113)	CD♭DF	0.73	(0013)	CC♯DF	0.88	(0123)	B♯C♯DE♯	1.12
(0124)	(0012)	CC♯DE	0.74	(0112)	CD♭DE	0.96	(0123)	B♯C♯DE	1.08
(0146)	(0023)	CC♯EF♯	0.77	(0134)	B♯C♯EF♯	0.84	(0123)	CD♭EF♯	1.20
(0127)	(0014)	CC♯DG	0.77	(0114)	CD♭DG	0.77	(0124)	CD♭E♭♭G	1.25
(0123)	(0112)	CD♭DE♭	0.77	(0012)	CC♯DE♭	1.04	(0011)	CC♯DD♯	1.04
(0147)	(0024)	CC♯EG	0.78	(0124)	CD♭EG	0.92	(0134)	B♯C♯EG	1.05
(0126)	(0013)	CC♯DF♯	0.82	(0114)	CD♭DG♭	0.96	(0113)	CD♭DF♯	1.13
(0148)	(0135)	B♯C♯EG♯	0.82	(0024)	CC♯EG♯	0.89			
(0134)	(0123)	B♯C♯D♯E	0.83	(0012)	CC♯D♯E	0.97	(0022)	CC♯E♭E	1.09
(0248)	(0124)	CDEG♯	0.85	(0125)	CDEA♭	0.85	(0135)	CDF♭A♭	1.10
(0236)	(0124)	CDE♭G♭	0.87	(0123)	CDE♭F♯	0.93			
(0167)	(0034)	CC♯F♯G	0.87	(0144)	CD♭G♭G	0.87	(0134)	CD♭F♯G	1.12
(0268)	(0134)	CDF♯G♯	0.90	(0135)	CDF♯A♭	1.03	(0245)	B♯DF♯G♯	1.03
(0145)	(0123)	CD♭EF	0.91	(0133)	CD♭F♭F	0.91	(0022)	CC♯EE♯	1.15
(0369)	(0135)	CD♯F♯A	0.95	...					

will throw light onto the conceptual distinctions implicit in these terminological differences.

We can ascribe harmonic qualities to generic sets using exactly the same procedure, but with indicator vectors of length 7 instead of 12. The qualities $\hat{a}_0 \ldots \hat{a}_6$ are defined by

$$\hat{a}_k = \sum_{j=0}^{6} a_j e^{-i2\pi kj/7} \tag{2}$$

And the same mathematical properties hold that enable us to reduce these seven qualities to three, $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$. For present purposes, the important question is how these three qualities relate to the six 12-tET qualities. The answer is simple: we can directly compare same-numbered qualities. This is best appreciated by noting that these discrete Fourier transforms are instances of a more general continuous FT, in which the number of notes (and the octave) must be fixed but the notes can be arbitrarily tuned. To perform a continuous FT we represent a chord A as a vector of length n where each element of the vector is a pitch belonging to the chord (and n is the cardinality). Then for each k, $0 \leq k < \infty$, supposing the pitches to be measured on a scale of cents, $8^{\text{ve}} = 1200$,

$$\hat{a}_k = \sum_{j=0}^{n} e^{-i2\pi k a_j/1200} = \sum_{j=0}^{n} (\cos(2\pi k a_j/1200) + i\sin(2\pi k a_j/1200)) \tag{3}$$

Some of the terms that have been used for $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ make sense in this more general setting: clustered quality, dyadicity or quartal quality, and triadicity. Others, chromaticity, tritone quality, augmentedness, and hexatonicity, apply only in the 12-tET context. Terms like “dyadicity” and “triadicity” have therefore been preferred by theorists working in the context of music in which diatonic or diatonic-like scales serve as intermediaries between chords and the full chromatic. This is similar to associating chords with a generic type.

3 Fourier Transform and Voice Leading

We can make some broad observations about how voice leading affects harmonic qualities, following [25, 28]. Suppose we have a voice leading from A to B in which only one voice, u, moves. Then

$$\hat{b}_k - \hat{a}_k = e^{-i2\pi k b_u/1200} - e^{-i2\pi k a_u/1200} \tag{4}$$

In other words, the vector on the complex plane from $\hat{a}_k$ to $\hat{b}_k$ is the same as that between two unit vectors on the complex plane, whose distance is determined by the size of the voice leading times k, ranging from 0 to 2. Depending on the direction of the voice leading, this change could affect the size of the coefficient ($|\hat{b}_k| - |\hat{a}_k|$), the phase of the coefficient ($\arg(\hat{b}_k) - \arg(\hat{a}_k)$) or some mixture of both. If multiple voices move, these vectors simply add together, and they can to varying extents reinforce or cancel one another out, depending on whether they they are similar or opposing in direction. Putting these complications aside, we can say at least that if the voice leading times k, $k(b_u - a_u)$, is relatively small, then the difference between $\hat{a}_k$ and $\hat{b}_k$ will necessarily be small, both in magnitude and phase.

Both of these conditions are met for $1 \leq k \leq 3$ for the specific-to-generic voice leadings towards the top of Tables 1 and 2. To illustrate, Figure 1 plots the distances on the complex plane against taxicab voice-leading distance between the generic and specific sets. (Taxicab distance makes it simpler to calculate a theoretical maximum distance for the given voice-leading distance.) Some distances do get close to the theoretical maximum, especially for the smaller voice leadings. However, these displacements may affect the phase more than the magnitude of the coefficient. Figure 2 is a similar plot giving the change of magnitude from the specific to the generic set with the same theoretical maximum lines (which can be positive or negative changes). A few of the displacements are still close to the theoretical maxima, but they can also often be much smaller.

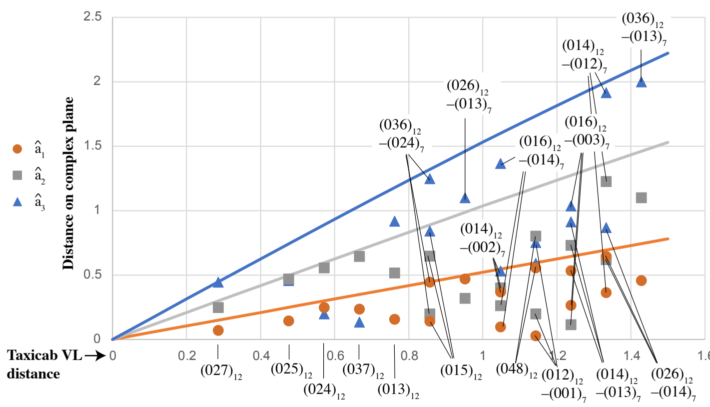

Fig. 1. Distance between points on the complex plane for $\hat{a}_1$–$\hat{a}_3$ by taxicab voice-leading distance for the specific-to-generic mappings in Table 1. The lines show theoretical maximum values, assuming that the voice leading is evenly split between two voices, if there is no cancellation between vectors resulting from each displacement (theoretically the distances could be slightly larger if the voice leading is divided between three voices, but because they are constrained to be balanced voice leadings, assuming that only two voices move is a sufficient approximation.) Data labels for 12-tET sets that map to multiple generic sets give both the specific and generic sets, using subscripts 12 and 7 to distinguish them.

Coefficient $\hat{a}_7$ presents a special case: for the generic set, the vectors for every note of the chord are at phase zero. This means all components of the voice leading tend to point in similar directions (towards the positive real axis) and mostly combine constructively. As a result $\hat{a}_7$ distance is highly correlated

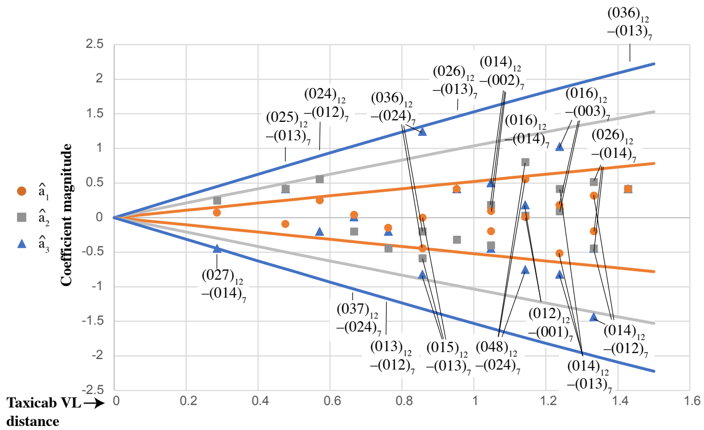

Fig. 2. Coefficient magnitudes, $|\hat{a}_1|$–$|\hat{a}_3|$, by taxicab voice-leading distance for the specific-to-generic mappings in Table 1. The lines show the theoretical maxima as in Figure 1.

with the voice-leading distance. In addition, $|\hat{a}_7|$ is a maximum possible value (equal to the cardinality) for 7-tET sets, so the $\hat{a}_7$ distance between the specific and generic sets is primarily determined by the size of $|\hat{a}_7|$ for the specific set. (Recall that the two sets are matched in voice-leading balance, so the phases of $\hat{a}_7$ cannot differ by much unless $|\hat{a}_7|$ for the specific set is small.) Therefore $|\hat{a}_7|$ (which is equal to $|\hat{a}_5|$ for a 12-tET set) is strongly correlated with the voice-leading distance to the nearest generic set, as illustrated for the 12-tET tetrachords in Figure 3. The logic explaining this correlation is strong when $|\hat{a}_7|$ is large, and gets weaker as $|\hat{a}_7|$ gets smaller. For example, if a set contains a tritone, the two notes making the tritone will cancel out in $\hat{a}_7$, making the coefficient necessarily relatively small. We can transpose the tritone to get a different set, and this can affect the voice-leading distances to the generic set, but will have no effect on the $\hat{a}_7$ distance. The same can be said, to a lesser extent, about two notes that make a semitone (which nearly cancel one another out in $\hat{a}_7$). This is clearly visible in Figure 3. The correlation is nearly perfect as $|\hat{a}_7|$ approaches the maximum value of 3 or 4 (the cardinality), and gradually breaks down as it approaches zero.

Tymoczko [25] makes a more general version of this same observation, namely that the magnitude of Fourier coefficient k is closely correlated with voice-leading distances from the nearest k-tET set.[4] His conclusion is that a DFT-based theory of harmonic qualities can be replaced by a theory of voice-leading distances, and are therefore superfluous. Of course, one can draw precisely the opposite conclusion by the same logic, that the voice-leading theory of Section 1 is effectively subsumed by the Fourier approach. The choice between one conclusion

[4] See also Amiot [3], who shows how to prove this mathematically.

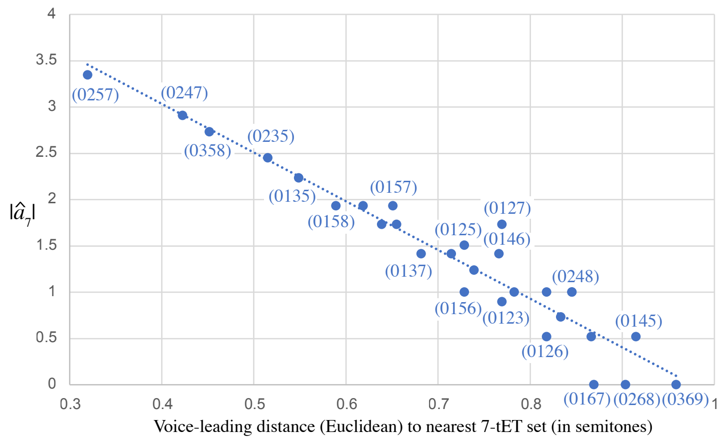

Fig. 3. Distance to the nearest 7-tET set by $|\hat{a}_7|$ for all 12-tET tetrachords

and the other rests upon the virtues of the two theories, and which better match the goals at hand. Tymoczko's main argument in favor of voice-leading theory is that the DFT is, as he describes it, a "black box." This characterization can no longer be seriously maintained in light of the many subsequent publications that have explained the theoretical basis for the procedure in detail. Also in favor of voice-leading distances is that they may tie into other aspects of voice-leading theory (for instance much of the theory presented in [26]). It is interesting to note, however, that evenness (i.e. distance from perfectly even collections) remains central to many of the theoretical uses of voice-leading geometry. On the side of Fourier theory, we may note the fact that it applies across cardinalities (the value of which is extensively explored in [32]) and ease of calculation. For example, it may be applied to scores in an automated fashion making questions amenable to corpus analysis as in [1, 6, 19, 29, 35–37]. There are also many theorems in Fourier theory that can lead directly to useful observations. For example, it is not obvious from the theory of voice-leading geometry that the proximity of a set to some k-tET set implies anything about its proximity to some $(12 - k)$-tET set, or the proximity of its complement to some k-tET set, which immediately follow from the aliasing property and invariance of DFT magnitudes under complementation.

Returning to the classification system described in Section 1, we may now observe that it can also be deduced from DFTs without necessarily finding all the nearest generic sets. Since $|\hat{a}_1|$, $|\hat{a}_2|$, and $|\hat{a}_3|$ are as a rule minimally affected by the displacement to nearby generic sets, we can classify the sets based on whether any of these three qualities predominate. I will propose a heuristic that takes advantage of another useful property of DFTs, the Parseval-Plancherel (conservation of power) theorem (for n-tET pc vector $A = (a_0, \ldots, a_n)$):

$$\sum_{k=0}^{n-1} a_k^2 = \frac{1}{n}\sum_{k=0}^{n-1} \hat{a}_k^2 \tag{5}$$

Note that if there are no doublings or weightings the quantity on the left is simply equal to the cardinality $(= 1^2 + \cdots + 1^2)$. Therefore a good way to normalize coefficient magnitudes is to consider their *percent share of power*, $p_k = 100\hat{a}_k^2/np$, with $p = \sum_{k=0}^{n-1} a_k^2$. Table 3 gives the percent share of power for all of the 12-tET trichords and highlights in boldface those that are relatively large. If exactly one of p_1, p_2, p_3 takes a substantially larger share of power than the others, we classify the set based on that, as indicated by the color coding. Two cases are worthy of mention: for $(036)_{12}$, p_1, p_2, and p_3 are all small because $(036)_{12}$ is actually a 4-tET set and therefore has a maximum possible value for p_4. I classify this as tertian on the grounds that $\hat{a}_4$ is the 7-tET complement of $\hat{a}_3$. In other words, $(036)_{12}$, not especially triad-like in the sense that it is not especially close to an even threefold division of the octave, obtains its status as a triad through the aliasing property of 7-tET, as a subset of a typical 7th chord. The $(026)_{12}$ trichord is the only one classified differently in Table 3 than in Table 1, on the grounds that p_2, though not especially large in absolute terms, is somewhat large in the context of a whole-tone chord, where $\hat{a}_6$ takes up most of the power. Table 1 identifies $(026)_{12}$ as mixed on the grounds that the mixed generic trichord, $(013)_{12}$, is the closest, although the quartal trichord, $(014)_{12}$, is also relatively close. Regardless, the classification of $(026)_{12}$ is weakly determined by both methods.

Besides essentially reproducing the generic classifications we obtained by the voice-leading method, the Fourier coefficients provide additional information. The mixed trichords, for instance, are all mixed for somewhat different reasons: for $(025)_{12}$ the primary quality is $\hat{a}_5$, making it a good representative of the mixed trichord. For $(015)_{12}$ the competing qualities are quartal and tertian, whereas for $(014)_{12}$ the competing qualities are scalar and tertian.

Table 3. The 12-tET trichords and the percent share of power of coefficients $\hat{a}_1$–$\hat{a}_6$. Colors indicate the resulting classification of the trichords as scalar (green), quartal (blue), or tertian (violet) based on harmonic qualities. Relatively large power values are given in boldface.

Set	p_1	p_2	p_3	p_4	p_5	p_6	Set	p_1	p_2	p_3	p_4	p_5	p_6
(027)	1.5	**11.1**	2.8	0.0	**20.7**	2.8	**(036)**	2.8	2.8	2.8	**25.0**	2.8	2.8
(025)	6.3	2.8	2.8	8.3	**15.9**	2.8	**(026)**	2.8	**8.3**	2.8	8.3	2.8	**25.0**
(024)	**11.1**	0	2.8	0.0	**11.1**	**25.0**	**(016)**	2.8	**19.4**	2.8	8.3	2.8	2.8
(037)	0.7	2.8	**13.9**	8.3	**10.4**	2.8	**(014)**	**10.4**	2.8	**13.9**	8.3	0.7	2.8
(013)	**15.9**	2.8	2.8	8.3	6.3	2.8	**(012)**	**20.7**	**11.1**	2.8	0.0	1.5	2.8
(015)	5.6	**11.1**	**13.9**	0.0	5.6	2.8	**(048)**	0.0	0.0	**25.0**	0.0	0.0	**25.0**

Table 4 gives the same information for all of the 12-tET tetrachords. Again the classification, on the grounds of the relative values of p_1, p_2, and p_3, largely reproduces the voice-leading-based classification of Table 2, with, again, differences appearing in the marginal cases. One of these is $(0126)_{12}$, which Table 4 classifies as quartal. The status of this chord as mixed by voice-leading standards is marginal: it is almost as close to a doubled quartal trichord, $(0114)_7$, as it is to a doubled mixed trichord, $(0013)_7$. Another difference is the whole-tone set, $(0248)_{12}$, which Table 4 classifies as tertian on the grounds that it is the whole-tone complement of a major third, even though it is substantially closer in voice leading to the mixed tetrachord. The $(0248)_{12}$ as mixed tetrachord has no special status in tonal theory, whereas it does appear as a kind of seventh chord, an augmented dominant (dominant seventh with augmented fifth). Another interesting case is the French augmented sixth, $(0268)_{12}$, whose quartal classification by the DFT method matches its closest voice-leading neighbor, but the generic seventh chord is a close second. The DFT shows that, after taking into account that it is a whole-tone chord and therefore most of the power is in $\hat{a}_6$, it has a high $\hat{a}_4$ in addition to $\hat{a}_2$, and therefore can be interpreted as a seventh chord, which is precisely its status in tonal theory as a kind of augmented sixth chord. The last difference is $(0145)_{12}$, which is classified as tertian by the DFT method, even though the doubled generic third is a fairly distant second alternative voice-leading neighbor.

Table 4. The 12-tET tetrachords and the percent share of power of coefficients $\hat{a}_1$–$\hat{a}_6$. Colors indicate the resulting classification of the tetrachords as scalar (green), quartal (blue), or tertian (violet) based on harmonic qualities. Relatively large power values are given in boldface.

Set	p_1	p_2	p_3	p_4	p_5	p_6	Set	p_1	p_2	p_3	p_4	p_5	p_6
(0257)	1.7	**6.3**	0.0	2.1	**23.3**	0.0	**(0124)**	**17.6**	2.8	4.2	2.8	3.2	8.3
(0247)	3.2	2.1	4.2	2.1	**17.6**	8.3	**(0146)**	4.2	8.3	4.2	8.3	4.2	8.3
(0358)	1.1	0.0	**8.3**	8.3	**15.6**	0.0	**(0127)**	6.3	**18.8**	0.0	2.1	6.3	0.0
(0235)	**12.5**	0.0	0.0	8.3	**12.5**	0.0	**(0123)**	**23.3**	6.3	0.0	2.1	1.7	0.0
(0237)	4.7	6.3	8.3	2.1	**11.9**	0.0	**(0147)**	2.1	6.3	8.3	**14.6**	2.1	0.0
(0135)	**10.4**	2.1	4.1	2.1	**10.4**	8.3	**(0126)**	7.8	**14.6**	4.2	2.1	0.6	8.3
(0158)	0.6	6.3	**16.7**	2.1	7.8	0.0	**(0148)**	2.1	2.1	**20.8**	2.1	2.3	8.3
(0258)	0.6	2.1	**4.2**	**14.6**	7.8	8.3	**(0134)**	**15.6**	0.0	8.3	8.3	1.1	0.0
(0246)	**6.3**	2.1	0.0	2.1	6.3	**33.3**	**(0248)**	2.1	2.1	**8.3**	2.1	2.1	**33.3**
(0157)	0.6	**14.6**	4.2	2.1	7.8	8.3	**(0236)**	7.8	2.1	4.2	**14.6**	0.6	8.3
(0136)	6.3	6.3	0.0	**14.6**	6.3	0.0	**(0167)**	0.0	**25.0**	0.0	8.3	0.0	0.0
(0137)	4.2	8.3	4.2	8.3	4.2	8.3	**(0268)**	0.0	**8.3**	0.0	8.3	0.0	**33.3**
(0347)	4.2	0.0	**16.7**	8.3	4.2	0.0	**(0145)**	7.8	6.3	**16.7**	2.1	0.6	0.0
(0156)	2.1	**18.8**	8.3	2.1	2.1	0.0	**(0369)**	0.0	0.0	0.0	**20.8**	0.0	0.0
(0125)	**11.9**	6.3	8.3	2.1	4.7	0.0							

4 The Star Chart

The association of 12-tET harmonic qualities with those of nearby generic sets outlined in the previous section can serve as the basis for a visualization of the harmonic qualities using the *RadViz* dimensionality reduction method of [16]. Pereira, Bernardes, and Martins, [23], recently proposed using this method for visualizing Fourier coefficient magnitudes. I will define a visualization using the normalized *star coordinate* method, which is mathematically equivalent to *RadViz* [24].

We begin by arranging the six coefficients at equally spaced points around the unit circle by selecting a permutation of the six share-of-power values defined in the previous section. I will use $P = (p_5, p_6, p_3, p_2, p_1, p_4)/\sum_{k=1}^{6} p_k$. The denominator normalizes the values so that they sum to 1.[5] The star coordinate for a given set A, $S(A)$, is then given by multiplying each normalized p_k by its corresponding unit vector:

$$\begin{gathered} S(A) = (P(A) \cdot X, P(A) \cdot Y) \\ \text{with } x_k = \cos(2\pi k/6) \text{ and } y_k = \sin(2\pi k/6) \end{gathered} \tag{6}$$

Figures 4 and 5 plot $S(A)$, the "star chart," for all the trichords and tetrachords. The choice of permutation is crucial for this visualization, and the results must be interpreted in light of the theoretical rationale for the permutation.[6] For instance, if a point is on the left towards the $|\hat{a}_5|$ pole, this does not necessarily mean that it has a high $|\hat{a}_5|$; it means that $|\hat{a}_5|$ *or* the combination of $|\hat{a}_4|$ and $|\hat{a}_6|$ is large relative to $|\hat{a}_2|$ and the combination of $|\hat{a}_1|$ and $|\hat{a}_3|$. For the star chart, the arrangement of Fourier coefficients is selected according to two criteria: (1) mod-7 complements are always opposite (as indicated by the dashed lines) meaning that all 7-tET sets are at the center of the space. Therefore deviations from the origin show the differences between the set and its possible generic identities. (2) The odd and even coefficients are evenly spaced, as shown by the triangles. Even coefficients are maximally reinforced by tritones, while tritones cancel out for odd coefficients. By balancing these, differences that have to do with the presence of tritones, the interval with purely ambiguous generic status, are eliminated from the star coordinates. Figures 4 and 5 reintroduce this factor by taking the difference $((p_1 + p_3 + p_5) - (p_2 + p_4 + p_6))/\sum_{k=1}^{6} p_k$ and representing this with color. The green sets owe their position primarily to nearby coefficient axes on the green triangle, which the purple sets owe their position

[5] For a 7-tET set this normalization converts the values to *non-zero share of power*. For a 12-tET set normalizing to non-zero share of power (as in [4]) would require also including p_7 through p_{11} in the sum, which is equivalent to doubling the values of the first five coefficients. Therefore, the sixth coefficient is, in a sense, over-represented from the 12-tET perspective.

[6] Thus the star chart is significantly different from the FQS space of [23] and so I refer to it by a different name. Another, less consequential, difference is my use of power values rather than raw magnitudes.

primarily to nearby axes on the purple triangle. A biproduct of property (2) is that every coefficient axis is adjacent to its neighbors ($k \pm 1$), with the exception of the mod-7 complements 3 and 4.

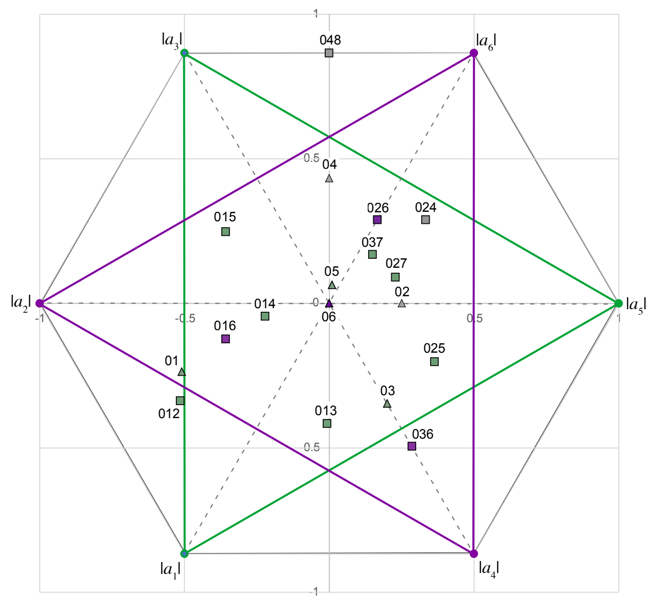

Fig. 4. Star chart with dyads and trichords.

By property (1), we can interpret positions in the star chart as reflecting the way in which a particular set deviates from nearby generic sets, as residuals of generic reduction. (This is also true of the color: 7-tET sets always balance odd and even because the complement of an odd coefficient is always even and vice versa.) Sets furthest from the nearest 7-tET sets are among those farthest from the center: (0369), (0145), (0268), (0167), etc. One feature of interest is that the smaller values of k (1, 2, 3) are all on the left and the larger (4, 5, 6) on the right. Therefore sets with semitones tend to be farther to the left, and more even sets further to the right. The design of the space is such that sets of each generic type are spread out in regions, and one good use of the space is to compare different voice leadings from the same generic set.

Sets close to generic sets with doublings are concentrated in the lower left. This may be understood through a principle of *effective cardinality*, equal to $\hat{a}_0^2/p$. Effective cardinality disambiguates the cardinality ($\hat{a}_0$) from the $\hat{a}_0$ share of power. A doubling increases p without changing $\hat{a}_0$, which decreases the effective cardinality and increases the share of power for all other coefficients. Therefore a voice leading from a generic 7-tET set with a doubling to a 12–tET set without

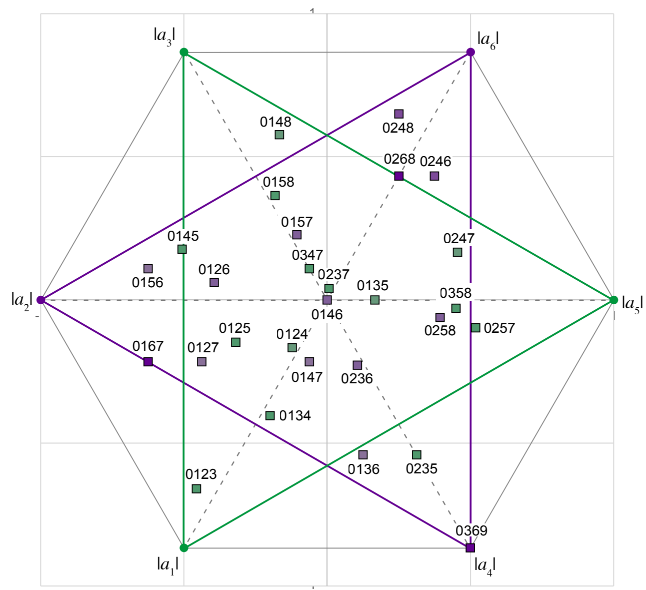

Fig. 5. Star chart with tetrachords.

one will increase the share of power for all coefficients, with the increase having a larger effect for lower k.

5 Analysis of Britten's *Blake Songs*, "Proverb I"

Figure 6 presents the first Proverb from Benjamin Britten's *Blake Songs*. The piano states a tetrachordal partition of the aggregate, returning to the initial tetrachord in m. 4. The first and third tetrachords are inversionally related (0125)s and the second is (0127), symmetrical around the axis that relates the (0125)s. Tables 2 and 4 both classify (0125) as mixed, leaning towards scalar, and (0127) as quartal. The reciting tones of the voice can be understood as an (0248) tetrachord, and this is classified as tertian according to Table 4. Taken together, then, the passage balances the three types. Aspects of balance and completion (aggregate partition, representation of all generic types) reflect the subject of the text, which advances the simple idea that the divine is everything, including things that we are disinclined to associate with it: vanity, violence, sexual desire. The piano's tetrachords, (0125) and (0127), have similar star coordinates in Figure 5, in the region of the space containing the sets whose nearest 7-tET neighbors have doublings. They differ, however, in the odd–even dimension, with (0127) on the even side and (0125) on the odd side. The voice tetrachord is opposite both of them along the $\hat{a}_1$-$\hat{a}_6$ axis, illustrating another form of balance-through-opposition in the passage. One way to understand this balance as a metaphor for the balance of good and evil in the universe is to consider the status of major seconds: in the piano they are always uncanny augmented sixths, pulled away from their diatonic nature, where the voice presents them as major seconds. Another form of balance is represented by the C-F♯ tritone, which occurs in the central (0127) tetrachord, is the inversional axis of the tetrachordal partition, and is also the final interval in the voice.

Fig. 6. Benjamin Britten, "Proverb I" from *Blake Songs*

The three tetrachords also share an exceptional property in Figure 3: All three are among those furthest above the trend line, meaning their $|\hat{a}_5|$ value is higher than what would be expected from their voice-leading distance from the nearest generic set. This indicates an uncanny diatonicity that reflects the prominent theme of the uncanny presented in the text. This is further emphasized by Britten's spellings in the piano part, which do not match the nearest neighbors in Table 2 and imply an even higher voice-leading distance from the generic sets than the minimum values represented in Figure 3. Instead his spellings match the considerably more distant second alternatives of both types, which sacrifice proximity to avoid doublings – in other words, Britten avoids chromatic seconds in his spellings. The high-diatonicity subsets, (025) and (027), are represented with diminished thirds rather than major seconds, presenting the (025) trichords as false triads.

A closer look at the diatonicity of the passage reveals that there is a more specific logic centering on the bass notes of each chord. Here are the notes of each measure located on a circle of fifths oriented by Britten's spellings:

F♭ C♭ G♭ D♭ A♭ **E♭** B♭ F C **G** **D** A

D♭ A♭ E♭ B♭ F **C** G D A E **B** **F♯**

B♭ **F** C G D **A** E B F♯ C♯ **G♯** D♯

F♭ C♭ F♯ D♭ A♭ **E♭** B♭ F C **G** **D** A

The bass notes, indicated in red, are always at the center of the segment of the circle of fifths. The other notes of the tetrachord (boldface) are on the periphery, while the reciting tone in the voice (blue) is closer to the bass note, first on the sharp side, then on the flat side. The last reciting tone is spelled as F♯ instead of G♭, putting it outside the segment of the line of fifths implied by the spellings in the piano.

The $\hat{a}_5$ orientation of the piano tetrachords is therefore consistently opposed to that of the bass notes, as illustrated by the solid arrows crossing the center of the space in Figure 7. The voice reciting tones instead tend to reinforce the diatonicity of the bass, especially in mm. 1–2, as shown by the dotted arrows. In mm. 1–2 the voice is close to the bass on the sharp side (counterclockwise) while in mm. 3–4 it is slightly farther away on the flat side. Fig. 7 makes the symmetry of the tetrachordal partition around C, the bass note of the symmetrical (0127) trichord, especially evident. This is especially interesting in that the song that follows, "London," the first song of the set, has a 2-flat key signature. C is the inversional axis of the 2-flat diatonic scale and has an $\hat{a}_5$ value with the same angle from the origin (phase).

6 Conclusion

This paper began by using voice-leading geometry to model the generic representation of 12-tET pitch-class sets and the classification of sets based on generic set-class space into scalar, tertian, and quartal, a method that reflects prior approaches. We subsequently found that many of the conclusions of the voice-leading approach are reflected in Fourier coefficients of 12-tET sets. Specifically, the size of the fifth (seventh) coefficient is closely related to the voice-leading distance from the nearest 7-tET set, and the sizes of the first three coefficients indicate the classification of a set as scalar, tertian, or quartal. Therefore an approach based on harmonic qualities can reproduce many of the useful features of the voice-leading approach, while also introducing significant advantages such as the ability to compare sets across cardinalities, and ease of calculation and

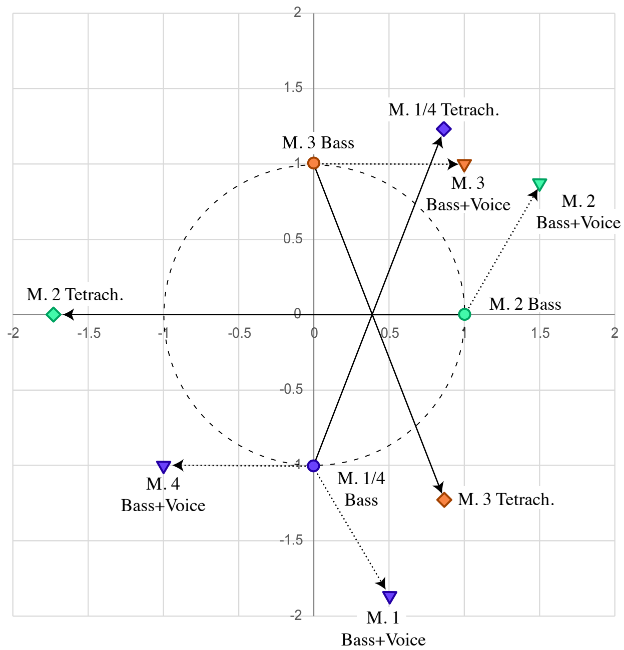

Fig. 7. Sets from Britten's "Proverb I" in $\hat{a}_5$ space.

the ability to consider weighted sets (making corpus-analysis applications feasible). Section 4 proposes an additional tool, the star chart, that complements the generic classification system by distinguishing different ways that the harmonic qualities can diverge from the generic images of a 12-tET set. The brief analysis of Britten's "Proverb I" in the last section illustrates some uses of analytical tools derived from this theory and the ways in which they can be coupled with other aspects of Fourier analysis and music analysis to lead to a holistic interpretation of music at the margins of tonal and atonal practices.

Acknowledgments. Thanks to the editors and anonymous reviewers for their suggestions and corrections.

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