Tetrachordal Folding Operations

Jason Yust[1]

Boston University, Boston MA 02215, USA jyust@bu.edu

Abstract. Jonathan Bernard's trichordal folding operations relate trichords with a maximum of shared interval content. This paper generalizes this to any cardinality of chord, focusing on the case of tetrachordal folding. A tetrachordal folding holds one trichordal subset fixed and inverts another around a shared dyad, so that the two tetrachords share five interval classes and two trichordal subsets. These operations generalize naturally from pitch space to pitch-class space and to set classes. The last section of the paper demonstrates the analytical application of tetrachordal folding networks on Morton Feldman's "For Stephan Wolpe."

Keywords: Pitch-class set theory · folding · interval content · Morton Feldman.

1 Trichordal folding

In his work on Edgar Varése, Jonathan Bernard [2] defines "infolding" and "unfolding" operations that relate trichords of different types. He uses a successive-interval notation which I will adopt here, generalizing over transposition. The following definition is essentially Bernard's, with some new notation.

Definition 1 *Let a pitch-space trichord A be given by successive intervals (a_1, a_2), for $a_1, a_2 \in \mathbf{Z}$, then there are four unfolding operations.*

$$\begin{aligned} \mathrm{unf}_{1\mathrm{a}}(A) &= (a_1, a_1 + a_2) \\ \mathrm{unf}_{1\mathrm{b}}(A) &= (a_1 + a_2, a_1) \\ \mathrm{unf}_{2\mathrm{a}}(A) &= (a_1 + a_2, a_2) \\ \mathrm{unf}_{2\mathrm{b}}(A) &= (a_2, a_1 + a_2) \end{aligned} \tag{1}$$

There are also two infolding operations.

$$\begin{aligned} \mathrm{inf}_{\mathrm{a}}(A) &= \begin{cases} (a_1, a_2 - a_1), & \text{if } a_1 \leq a_2 \\ (a_2, a_1 - a_2), & \text{if } a_2 \leq a_1 \end{cases} \\ \mathrm{inf}_{\mathrm{b}}(A) &= \begin{cases} (a_2 - a_1, a_1), & \text{if } a_1 \leq a_2 \\ (a_1 - a_2, a_2), & \text{if } a_2 \leq a_1 \end{cases} \end{aligned} \tag{2}$$

Altogether these are the complete set of folding operations*.*

Figure 1 shows an example of these operations, which we can interpret as taking any one note from the trichord and inverting it around one of the other two notes (hence the six possibilities). The a and b versions of each operation are clearly always going to be related by inversion, so we can immediately simplify these operations by generalizing over inversions, as Bernard [2] does. Therefore if we consider unfolding and infolding as a relation, *folding*, we immediately have:

Proposition 1 *Folding is a symmetrical relation.*

Proposition 2 *If* $A = (a_1, a_2)$ *is in the folding relation with* B*, then so is the inversion of* A*,* (a_2, a_1)*.*

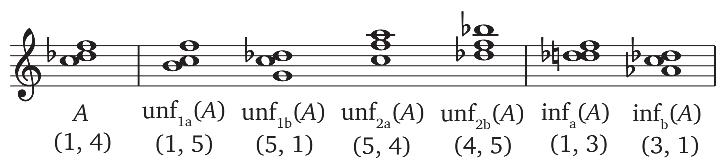

Fig. 1. Examples of unfoldings and infoldings of a chord.

This study pursues generalizations and applications of Bernard's idea taking the key feature to be that folding operations always preserve all the intervals of the set except at most one. These operations therefore express the minimum possible change in interval content (as represented for pitch-class sets, e.g., by Forte's interval vector [4]). While Definition 1 follows Bernard in defining foldings in pitch space (using intervals in $\mathbf{Z}$), I will be primarily interested in the generalization to pitch-class sets defined below. Most of the results in this section and the next nonetheless apply in both domains.

The next section will generalize folding operations to tetrachords and higher cardinalities with these priorities in mind, and the third section applies tetrachordal folding networks to an analysis of Morton Feldman's "For Stephan Wolpe."

For a trichord, (a_1, a_2), there are three possible ways to exchange one interval with a new one specified in Definition 1. Either we exchange a_2 for $2a_1 + a_2$ (unf$_1$), a_1 for $a_1 + 2a_2$ (unf$_2$), or $a_1 + a_2$ for $\pm(a_2 - a_1)$ (inf).

Notice that sets will be generalized over transposition throughout this study (hence the use of interval strings to define them) but not necessarily over inversion or octave equivalence. However, it is possible to transfer all of the definitions of foldings to pitch or pitch-class sets proper (not generalized over transposition) by fixing the transposition of the fixed dyad (or fixed trichord, in the case of tetrachordal foldings below), as the illustration in Figure 1 does.

In Bernard's applications of the folding operation the most important feature is that they are defined on interval strings, so he can interpret them as operating on trichords in pitch space. However they easily generalize to operations of set-classes. To do so, we reconceive the operations as acting on ordered sets as in [7], relaxing the assumption that order is conferred by registral position and allowing a_1 and a_2 to take negative values. Bernard's distinction between "unfolding" and "infolding" then becomes less meaningful, and we rename the operations simply as *foldings* indexed by the order position of the moving note followed by the order position of the note it is inverted around. Hence:

Definition 2 *There are six trichordal* folding *operations defined on interval strings,* (a_1, a_2)*, with* $a_1, a_2 \in \mathbf{Z_{12}}$*. These are defined as in Definition 1, but with sums taken modulo 12, as* $\mathrm{fold}_{21} = \mathrm{unf}_{1a}$*,* $\mathrm{fold}_{31} = \mathrm{unf}_{1b}$*,* $\mathrm{fold}_{23} = \mathrm{unf}_{2a}$*,* $\mathrm{fold}_{13} = \mathrm{unf}_{2b}$*,* $\mathrm{fold}_{12} = \mathrm{inf}_{a}$*,* $\mathrm{fold}_{32} = \mathrm{inf}_{b}$*,*

Definition 3 *Two trichordal set classes, A and B, relate by* folding*, or $A \asymp B$ if any pitch-space representatives of A and B relate by one of the folding operations from Definition 1.*

Proposition 3 *If two set classes, A and B, are related by folding, $A \asymp B$, then for any pitch-space representative of A, there is a pitch-space representative of B that relates to it by folding.*

Proof. It suffices to show that applying transposition, inversion, octave shift, or permutation to A results in the same operation applied to B (and possibly a change of the exact unfolding or infolding relation). The first two are straightforward. Octave shift refers to adding or subtracting multiples of 12 to/from individual pitches. For instance, replacing (a_1, a_2) with $(a_1, a_2 + 12)$ (last note moves up by octave). This clearly will simply induce some octave shift in B. Finally, given a permutation on A we can apply the same permutation to the indices of the folding operation and leave B unchanged. $\square$

Figure 2 displays the network of trichordal set classes relating by $\asymp$ using a 2-dimensional parameterization of the interval vector. I chose the parameterization arbitrarily with the goal of disambiguating all of the set classes and avoiding crossing edges. In addition this parameterization shows the symmetry of the network under the M5 automorphism of $\mathbb{Z}_{12}$ [8] by having the horizontal dimension dependent only on #ic1 and #ic5.

Note that we could extend the network in Figure 2 to include doubled ic1 and ic5 dyads, (001) and (005). However, other doubled dyads, whole tone chords, and diminished triads cannot exist in the same network. More generally, the sets in a given network have to have the same minimal embedding equal temperament. A whole-tone chord has 6-tET as a minimal embedding ET, and a doubled minor third, (003), has 4-tET as a minimal ET. This is because the interval of a semitone (or fourth or fifth, etc.) can never be produced by sums and differences of intervals in a smaller minimal embedding universe. This applies in pitch space as well as pitch-class space. For instance, the pitch-space chord (5,5) does not interact with the chords that have 12-tET as their minimal universe, even though its pitch-class equivalent (2,5) does.

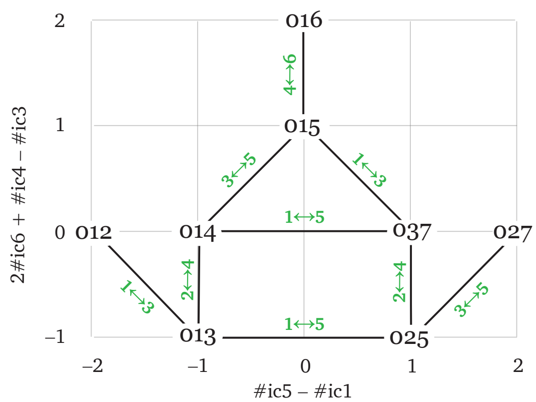

Fig. 2. Trichordal folding network.

2 Tetrachordal folding

Bernard [1, 2] explores the possibility of extending the folding operations to tetrachords, but his definition is far too loose, resulting in an unwieldy number of relations. If our goal is to preserve the property of maximizing similarity of interval content, however, an effective generalization is ready at hand. In the trichordal foldings, we hold one dyad constant while moving the third note so as to preserve one of the intervals it makes with the other two notes, by inverting it. This process guarantees that two of the three intervals will remain the same. This immediately generalizes to tetrachords: hold three of the notes constant, and choose another trichordal subset containing the fourth note. Invert this trichordal subset around the dyad it shares with the first trichord. Then exactly one note moves, preserving two of the trichordal subsets (up to inversion), and by extension, five of the six intervals.

This definition of tetrachordal folding makes a useful and manageable relation. The possibilities are listed in Table 1 and Figure 3 provides an example on a set in pitch space. The choice of two out of four trichordal subsets leads to twelve possible folding relations, which can be reduced to six by inversional equivalence. We can immediately see that the properties of trichordal folding described in the previous section generalize to tetrachordal folding, including Propositions 1, 2, and 3.

A similar generalization to larger sets is immediately evident, but there is an important caveat. For a set of size n, when we choose two $(n-1)$ subsets, their intersection (of size $n-2$) must be inversionally symmetrical for the operation to be well defined. For $n-2=1,2$ this is guaranteed, but for $n \geq 5$ it becomes a significant restriction, more so for larger n. Interestingly for $n=5$ in 12-tET all set classes have at least one inversionally symmetrical trichordal subset, so pentachordal folding operations are worth exploring, but I will pursue this no further at present.

Table 1. Tetrachordal folding operations on an intervallically defined set (a_1, a_2, a_3), permuted to keep all the intervals positive for $a_3 > a_1$ and $a_1 > a_3$ respectively.

Operation	Definition	Operation	Definition	Interval change
$\text{fold}_{2(13)}$	(a_2, a_1, a_3)	$\text{fold}_{4(13)}$	(a_3, a_1, a_2)	$(a_2 + a_3) \to (a_1 + a_3)$
$\text{fold}_{3(24)}$	(a_1, a_3, a_2)	$\text{fold}_{1(24)}$	(a_2, a_3, a_1)	$(a_1 + a_2) \to (a_1 + a_3)$
$\text{fold}_{1(23)}$	$(a_2, a_1, a_3 - a_1)$	$\text{fold}_{4(23)}$	$(a_3 - a_1, a_1, a_2)$	$(a_1 + a_2 + a_3) \to \lvert a_3 - a_1 \rvert$
	or $(a_2, a_3, a_1 - a_3)$		or $(a_1 - a_3, a_3, a_2)$	
$\text{fold}_{1(34)}$	$(a_2, a_3, a_1 + a_2)$	$\text{fold}_{2(34)}$	$(a_1 + a_2, a_3, a_2)$	$(a_1) \to (a_1 + 2a_2 + a_3)$
$\text{fold}_{3(12)}$	$(a_2, a_1, a_2 + a_3)$	$\text{fold}_{4(12)}$	$(a_2 + a_3, a_1, a_2)$	$(a_3) \to (a_1 + 2a_2 + a_3)$
$\text{fold}_{3(14)}$	$(a_1, a_3 - a_1, a_1 + a_2)$	$\text{fold}_{2(14)}$	$(a_1 + a_2, a_3 - a_1, a_2)$	$(a_2) \to \lvert a_3 - a_1 \rvert$
	or $(a_3, a_1 - a_3, a_2 + a_3)$		or $(a_2 + a_3, a_1 - a_3, a_3)$	

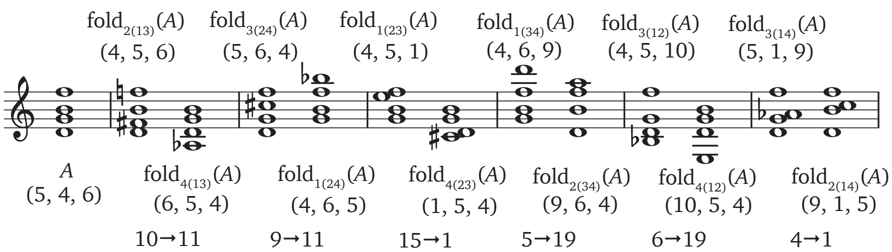

Fig. 3. An example of tetrachordal folding operations on an open-position dominant seventh chord in pitch space.

Figure 4 shows the network of tetrachordal folding operations on non-degenerate tetrachords without doublings. Again, I choose an arbitrary parameterization that disambiguates all of the set classes, avoids crossing edges, and shows the M5 automorphism as a mirror symmetry around a vertical axis. Note that the network is not planar, so it is impossible to eliminate all crossing edges. It is also impossible to disambiguate all of the set classes based on the interval vector alone, because of the all-interval tetrachords (0146) and (0137), which have the same interval vector. Therefore, I also include (in the horizontal dimension) a count of two trichord types, the major/minor triad (037) and its M5 partner (014). This means that edges representing the same change of interval classes are not always exactly the same distance in the horizontal dimension.

The tetrachordal folding operations always preserve two out of four trichordal subsets and five out of six intervals by design. A natural question is whether they are the only such operations. In pitch space, this is in fact the case, but not in pitch-class space.

Proposition 4 *Two tetrachordal pitch sets A and B, not related by transposition or inversion, are related by a folding operation if and only if they share two trichord types (generalized over inversion) as subsets and five interval types.*

Proof. The forward implication (only if) is true by construction. It is only necessary to prove the converse.

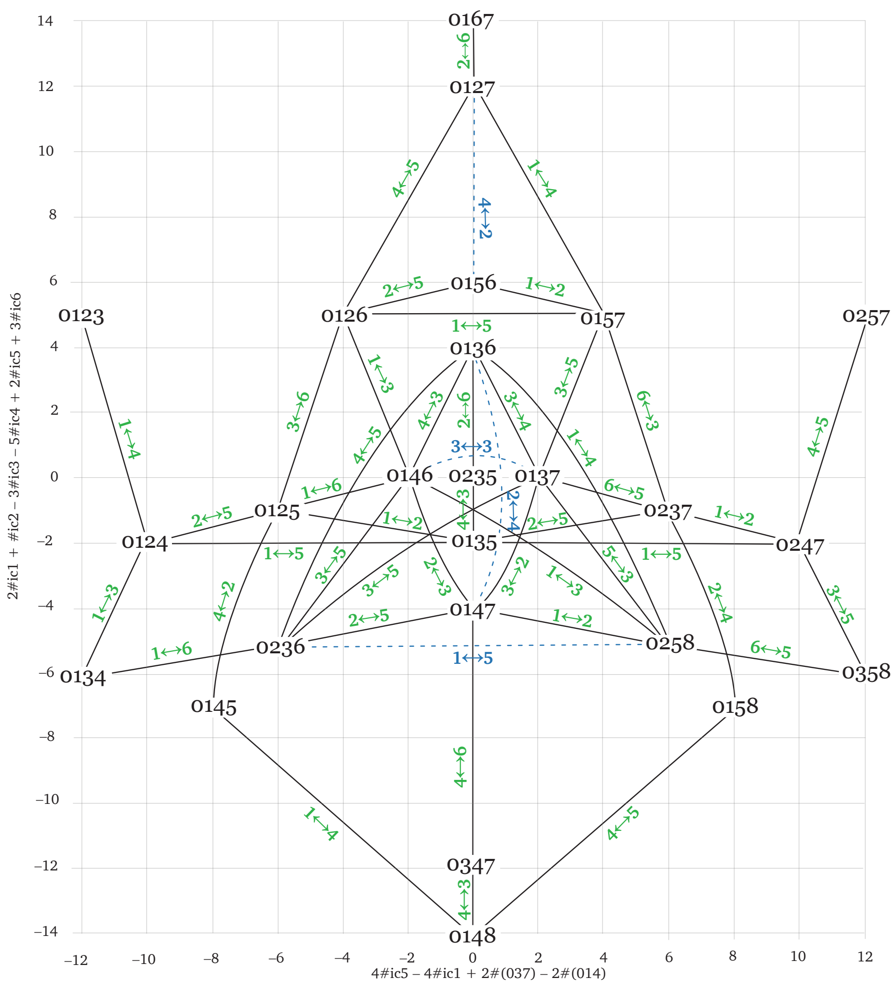

Fig. 4. Tetrachordal folding network.

Assume then that A has trichordal subsets, α and β, and B has subsets of the same types, and the two sets share five interval types. Let B' be either B itself or an inversion of B that contains α, and let β' be the subset of B' related to β by transposition and/or inversion. Either $\beta' = \mathrm{T}_\tau(\beta)$ for some interval τ or $\beta' = \mathrm{I}_\sigma(\beta)$ for some inversional index σ. For both cases, let α have the interval series (a_1, a_2) and A be (a_1, a_2, a_3) such that β is (a_2, a_3). Note that there is no loss of generality through free choice of permutation (allowing, e.g., that a_1, a_2, a_3 can take negative values).

By construction, β' must share a dyad with α; let this be an interval b. The first possibility is that β' shares the same dyad with α as β, and $b = a_2$, which means that the five intervals shared by A and B, $a_1, a_2, a_3, a_1 + a_2$, and

a_2+a_3, and the one belonging only to A, $a_1+a_2+a_3$, are all potentially unique. Otherwise, there is a redundancy in the interval β' shares with α. Either $b = (a_1$ or $a_1+a_2)$ (as a subset of α) $= (a_3$ or $a_2+a_3)$ (as a subset of β'), or $a_1 = a_2$ or $a_1+a_2 = a_2$.

First consider the last two possibilities. If $a_1 = a_2$, then α is symmetrical. Transpose or invert B' to map β' onto β. α and its transposition or inversion will still have a common dyad, so the sets will relate by a flip of α. If $b = a_1+a_2 = a_2$, then $a_1 = 0$ and α has a doubled note. Freely choosing between the doubled notes, this situation then coincides with the the regular $b = a_2$ scenario.

For the remaining possibilities first assume that $\beta' = \mathrm{T}_\tau(\beta)$. If $b = a_2 \neq a_1$ then $B' = A$, contradicting the premise. For all of the other possibilities, we can find an additional inversion of β that shares a dyad with β', or an inversion of α sharing a dyad with α. (I leave it to the reader to work out the details.) Therefore $A \asymp B$.

Finally assume that $B' = \mathrm{I}_\sigma(\beta)$. For $b = a_2 \neq a_1$ we have $A \asymp B$. For $a_1 = a_3$ or $a_1+a_2 = a_2+a_3$, we have that α is an inversion of β, and hence also $A \asymp B$. In the remaining two cases we violate the interval-sharing premise. Consider $b = a_1+a_2 = a_3$; B then has two copies of a_2 whereas A has two copies of a_1+a_2, in addition to the other distinct intervals $2a_2 \in B$ and $2(a_1+a_2) \in A$. The remaining case $b = a_1 = a_2+a_3$ is essentially the same swapping the roles of a_3 and a_1. $\square$

An interesting consequence of this proof is in the last condition: it is possible for two tetrachords to share two trichordal subsets but not five intervals, in which case they are not directly related by folding, specifically when there is some duplicated interval in the one set and a different duplicated interval in the other. An example would be (0135) and (0136), which each have (013) and (025) subsets, but the first has an (024) subset with two copies of ic2 and an ic4, and the other an (036) subset with two copies of ic3 and an ic6.

This proof only holds in pitch space. Most of it transfers to pitch-class space, except one conclusion: it is possible for β' to overlap α in the interval that makes a_2 and be a non-trivial transposition of β if a_2 is a tritone. Therefore, when dealing with set classes, there is one operation that is not a folding but has the same properties of preserving five intervals and two trichord types. Specifically, for a tetrachord containing a tritone, transpose one of the notes not belonging to it by tritone. This preserves both intervals with the tritone, and just changes one, the interval between the two non-tritone notes.

We can also generalize this to other cardinalities, as *T-shift.* Specifically:

Definition 4 *Let pitch-class set A of size n have a T_x-symmetrical subset of size $n-2$. A* T-shift *of A fixes a size $n-1$ subset that includes the T_x-symmetrical subset, and moves the remaining note by some multiple of x.*

Figure 4 shows the T-shift operations on tetrachords not equivalent to foldings with dashed lines.

This motivates the following, which I leave as conjectures.

Conjecture 1. Let A and B be pitch sets of cardinality n. Then A and B share two subsets of cardinality $n-1$ and $\binom{n}{2}-1$ intervals if and only if they relate by a folding operation.

Conjecture 2. Let A and B be pitch-class sets of cardinality n. Then A and B share two subsets of cardinality $n-1$ and $\binom{n}{2}-1$ intervals if and only if they relate by a folding operation or a T-shift operation.

3 Application to Morton Feldman's "For Stephan Wolpe"

The first 12 minutes of Morton Feldman's work for chorus and vibraphones, "For Stephan Wolpe," repeat a single progression of ten chords in four-part chorus with small variations, primarily in rhythm, transposition, and voicing (changing the registral ordering and octaves of the notes without changing the set type), reflecting his concept of "crippled symmetry" [3, 5]. The set classes that Feldman uses exist in a relatively compact region of the folding network, as shown in Figure 5. Successive chords are usually two or three steps apart, the only exception being both progressions involving the exceptional (0057) chord. If we skip over this chord then the entire progression is in steps of 2 or 3.

The special property of these progressions is that exactly one trichord is shared between successive chords in all cases except the last progression, (0136)-(0135), which share two. In almost all cases the shared trichord is (015) – here again the exception is the penultimate (0136) chord. This shares an (016) with the preceding (0156) and (013) and (025) with the following (0135). Figure 5 shows regions defined by shared trichords between the chord types that Feldman uses.

The first version of the progression, given in Figure 6 is representative, containing the voicings used most frequently in all subsequent versions of the progression. Although the trichordal subsets are never voiced in exactly the same way from one chord to the next, they usually involve a single octave adjustment so that individual intervals are preserved in their exact pitch distance. In particular, the characteristic 4-semitone interval of (015) is almost always present, usually as E-G♯ or B-D♯. The 7-semitone interval ties together instances of (027), (016), and (015), and the 14-semitone interval ties together (027) and (025). The final chord, which changes the voicing and transposition of the initial (0135) but preserves its bass note, serves as a kind of summary of the whole progression, including all of these intervals.

The second half of the piece regularizes the rhythm and leaves behind the ten-chord progression in favor of a series of repeated three-chord progressions. These are organized into eight phrases by means of the punctuating vibraphone passages. The first six phrases explore small regions of the tetrachord network, as shown in Figure 7. Adjacent chords in this section are often related directly by folding and never by more than two links in the network (with the exception of the last chord of phrase 6, an isolated whole-tone chord which does not occur in the network). The first four phrases rely exclusively on chords with (014) subsets.

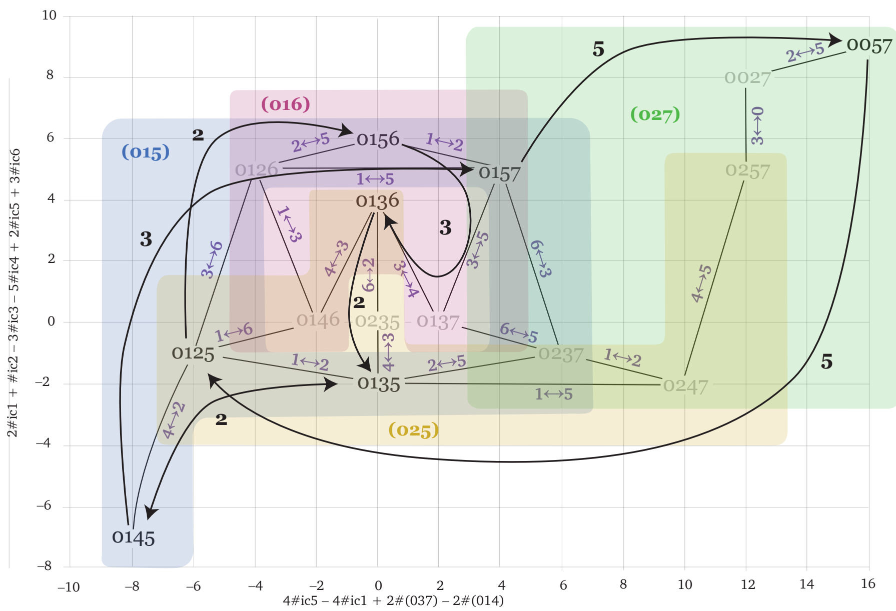

Fig. 5. Tetrachordal folding network for the first part of Feldman's "For Stephan Wolpe." Shaded regions show significant shared trichords..

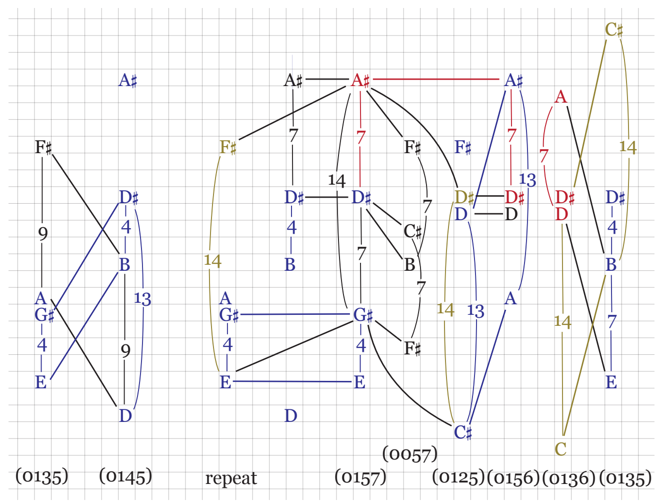

Fig. 6. The first (and representative) version of the chord progression that defines the first half of Feldman's "For Stephan Wolpe" in graph notation. Pitch-space intervals shared between successive chords are highlighted. Colors selectively indicate membership in different trichordal subsets, with blue for (015), red for (016), gold for (025).

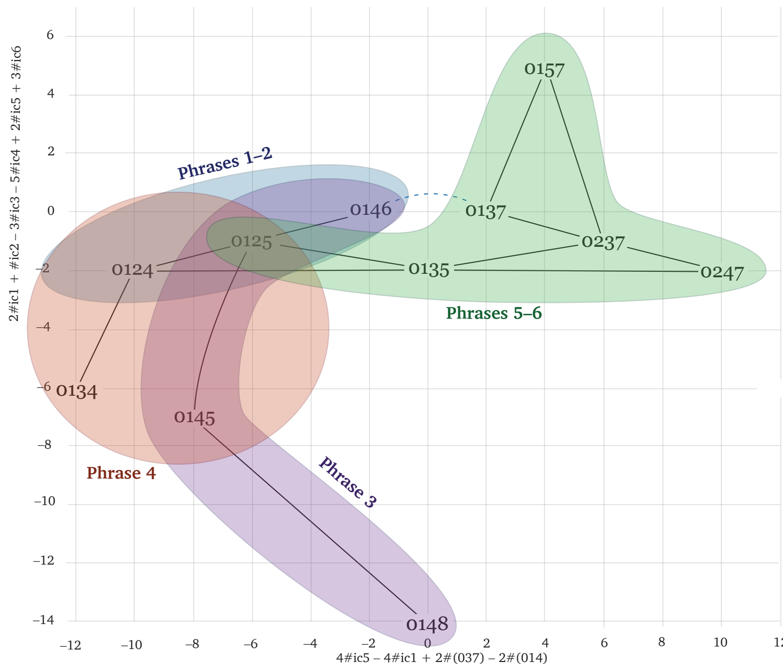

Fig. 7. Tetrachordal folding network for the first six phrases of the second part of Feldman's "For Stephan Wolpe."

A distinct shift happens in the second-to-last phrase, which is considerably longer than any other, at 27 measures with repeats. Now adjacent chords are 2–4 steps apart except for two instances where (0135) and (0235) are adjacent. The progression includes isolated extreme chords, (0123) and (0127), along with (0257), which is more connected with other chords in the passage though it is never directly adjacent to (0247) in the progression. Figure 8 shows these chords in the tetrachordal folding network along with those of the last phrase. The last phrase continues to focus on progressions between chords that are 2–4 steps apart, but overall relies upon a more connected set of chords, excluding (0123) and (0257). The central harmony of phrases 1–6, (0125), returns in this last phrase after being absent for all of the penultimate phrase.

The tetrachordal folding networks thus help us circumnavigate some of the usual problems of analysis and form in Feldman's late music. Hanninen [6], for example, points out that pervasive repetition and lack of textural changes in these pieces inhibits segmentation. While this description appears to characterize the second half of "For Stephan Wolpe" well, a closer look at the use of chord types and their arrangement in the folding network reveals a more definite plan in distinct stages. A relatively limited subnetwork first expands, and then moves back towards a region familiar from the first part of the piece. At the last stage, Feldman also returns to progression types familiar from the first part, charac-

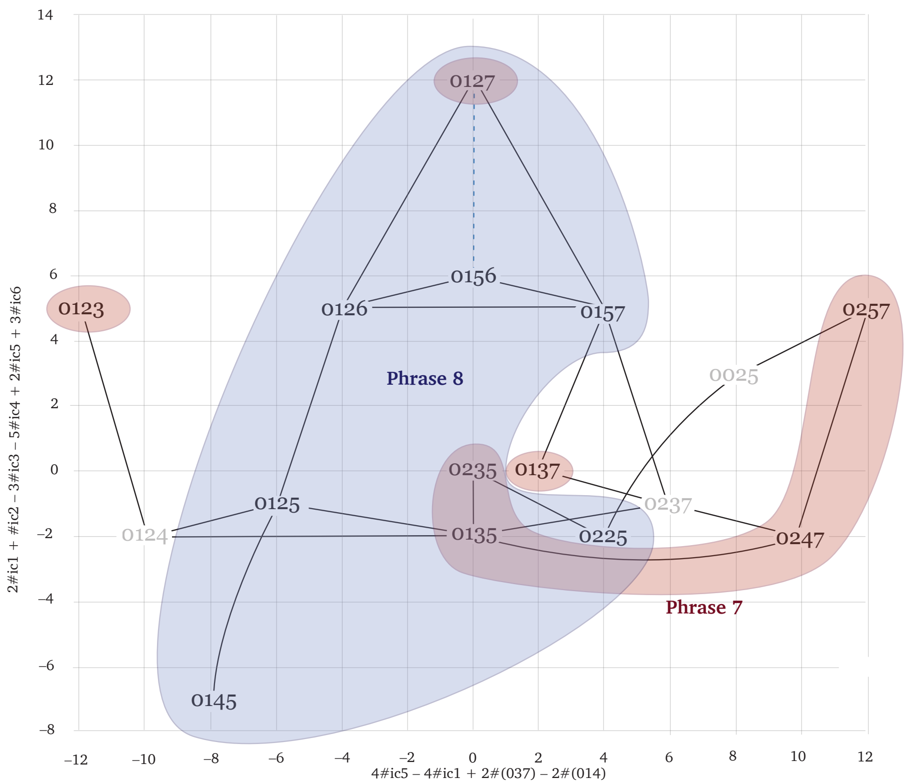

Fig. 8. Tetrachordal folding network for the last two phrases of the second part of Feldman's "For Stephan Wolpe."

terized by larger distances in the network. At the same time, the chord palette expands to the far reaches of the network. The piece thus exhibits a cogent form that combines principles of return and terminal expansion.

4 Conclusion

Folding operations, a generalization of Bernard's [2] unfolding and infolding operations, can be defined on sets of any size, and can operate in pitch-space or pitch-class space, with or without generalizations over inversion. Networks of these operations are useful for mapping out distances between chord-types based on interval and subset content. The application of these networks on tetrachords to Feldman's "For Stephan Wolpe" shows that distances in these networks are musically meaningful and that chord types limited to connected subnetworks can be compositionally useful. Feldman's manipulation of distances between adjacent chords and connectedness and location of his subnetworks illustrate his sensitivity to these properties, and reveals a shape to this piece that is not immediately apparent on its rather simple and static surface.

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