Flexibly Defined Tuning Systems Using Continuous Fourier Transforms

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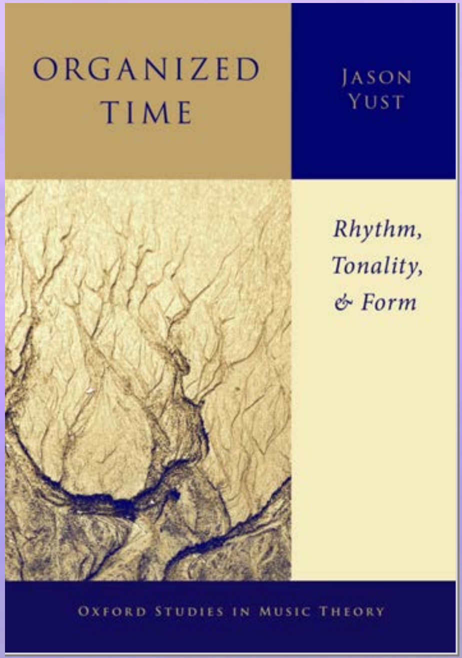


Mathematical and Computational Models in Music Workshop

University of Pavia, June 2021

Remembering Jack Douthett (1942–2021)

Two important ideas:

- Evenness
- Continuous functions underlying discrete phenomena

Outline

(1) Flexible tuning systems

(2) Periodic functions:
heptatonicity, triadicity, etc.

(3) Fourier coefficients, spectra, phases

(4) Application: Balinese Pelog

(5) Application: Persian Dastgah tuning

Flexibly Defined Tuning Systems

A critique of defining tunings as idealized pitch sets

Tuning Theory

The usual framework:

- A scale or tuning can be represented by a set of points in frequency space.
- Pitches occurring in practice are approximations of these — i.e., these are the *intended* frequencies, realized to within some tolerance (degree of precision).

What's wrong here?

Tuning Theory

The usual framework:

- A scale or tuning can be represented by a set of points in frequency space.
- Pitches occurring in practice are approximations of these — i.e., these are the *intended* frequencies, realized to within some tolerance (degree of precision).

Tuning variability is viewed as *error*. This is a distinctly classical-European attitude. Other traditions have a *positive* attitude towards tuning flexibility.

Tuning Theory

Example, Violinist vs. fiddler:

In both cases the tuning system is considered to be 12-tET.

The violinist prizes accuracy of intonation and recognizes theoretical point in frequency space as ideal representations of notes.

However, the violinist also recognizes the possibility of expressive intonation (frequency vibrato, sharpened leading tones), within narrow constraints.

The fiddler also recognizes note identities within a 12-t system, tied to regions of frequency space.

The fiddler requires a wider range of tuning flexibility for expressive intonation, such as portamento.

Tuning Theory

An alternative framework:

- A scale or tuning is defined by flexible interval categories.
- Within some scale-identifying constraints, tuning flexibility is a *resource* of the scale system.

Example, violinist and fiddler:

Both require scale-defining tuning constraints.

Both also prize flexibility of intonation within that system (to differing degrees).

Pythagorean and just intonation fall within that range of flexibility. Rather than distinct *tuning systems,* these are intonational variations of a system that can be applied *ad hoc* as the musical situation demands (tuning up a chord in a string quartet, making a melody stand out, etc.).

Tuning Theory

Interval categories are transposable.

A *closed* system is therefore *periodic* (in pitch / log-freq.)

An *open* system can be understood as a subset of closed systems.

Multiple "grains" of interval categorization can exist simultaneously. (Ex.: Generic and specific intervals.)

Towards Fourier Theory, Some Concepts

Heptatonicity
Chromaticity

Triadicity

Dyadicity

Diatonicity

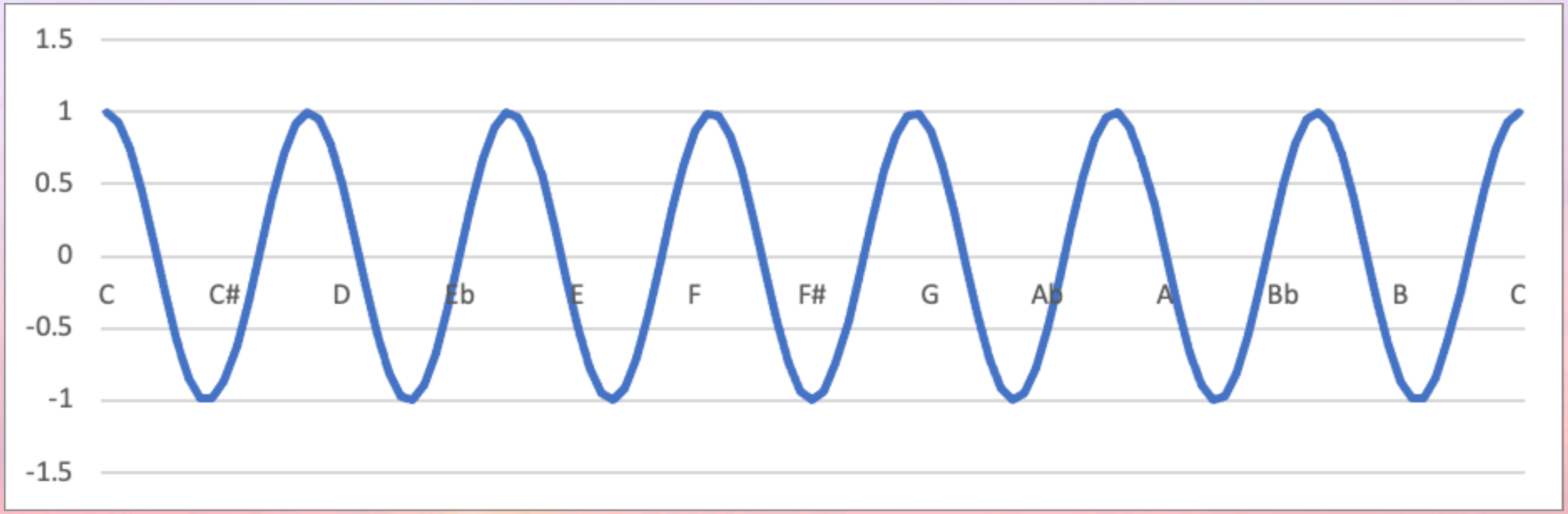
Hexatonicity

Octatonicity

Heptatonicity

Heptatonic: Division of the 8ve into 7 equally spaced bins.

Heptatonicity: Pitch collection viewed through such a division.

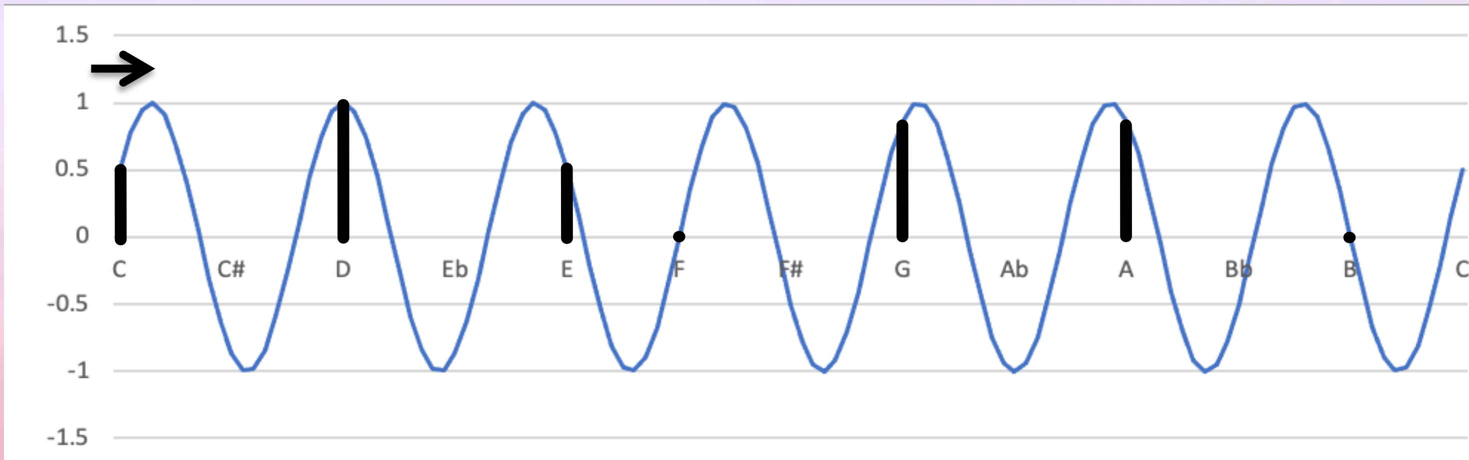


Heptatonicity

Example: C Diatonic scale

Phase:

$2\pi/6$



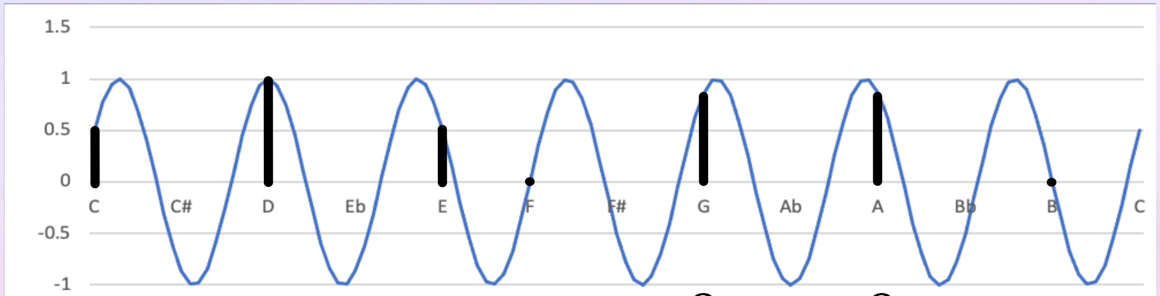
Magnitude: 0.5 + 1.0 + 0.5 + 0 + 0.87 + 0.87 + 0 = 2.28

Formula: $f_7 = \sum \cos(2\pi x_i \cdot 7/8\text{ve}) + j \sum \sin(2\pi x_i \cdot 7/8\text{ve})$

Magnitude = $|f_7|$, Phase = $\arg(f_7)$

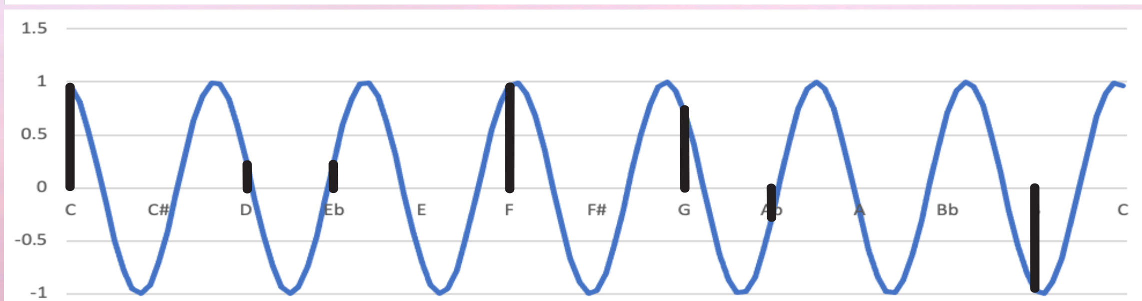
Heptatonicity

Heptatonicity is an evenness measure for scales



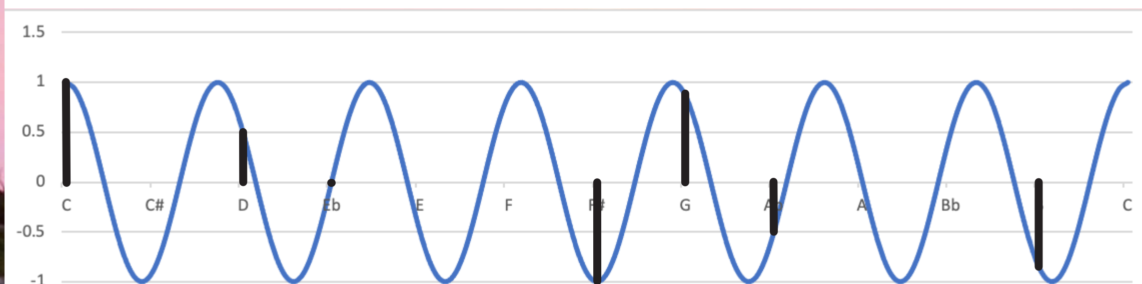
C diatonic scale

0.5 + 1.0 + 0.5 +0 + 0.87 + 0.87 + 0 = 3.73



C harmonic minor

0.97+0.26+0.26 +0.97 +0.71−0.26 − 0.97 = 1.93



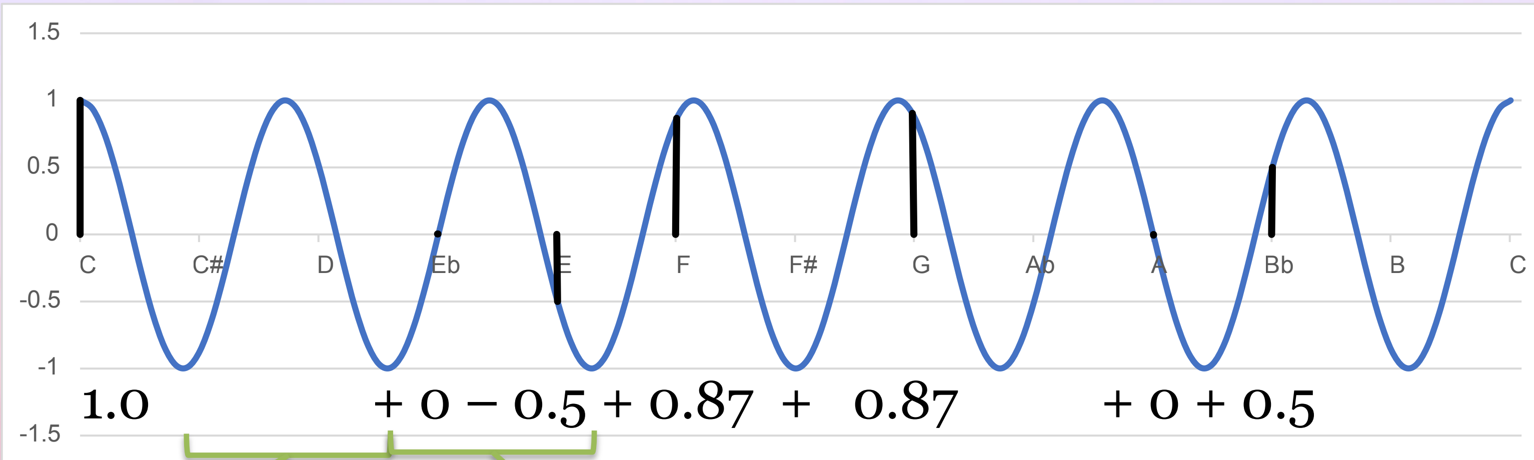
C Hungarian minor (CDE♭F♯GA♭B)

1.0 + 0.5 + 0 − 1.0+0.87−0.5 − 0.87 = 0

Heptatonicity

But: Notes are not forced to cover scale degree "bins"

Example: C-D♯-E-F-G-A-B♭



= 2.73

This bin is empty

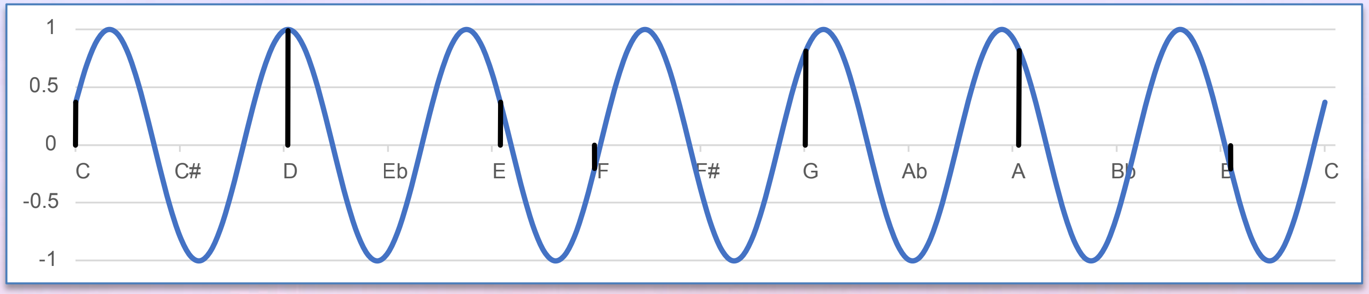
This bin contains two notes

The heptatonic *spans* of this scale are 2-0-1-1-1-1-1

Heptatonicity only measures evenness when interval spans are 1-1-1-1-1-1-1

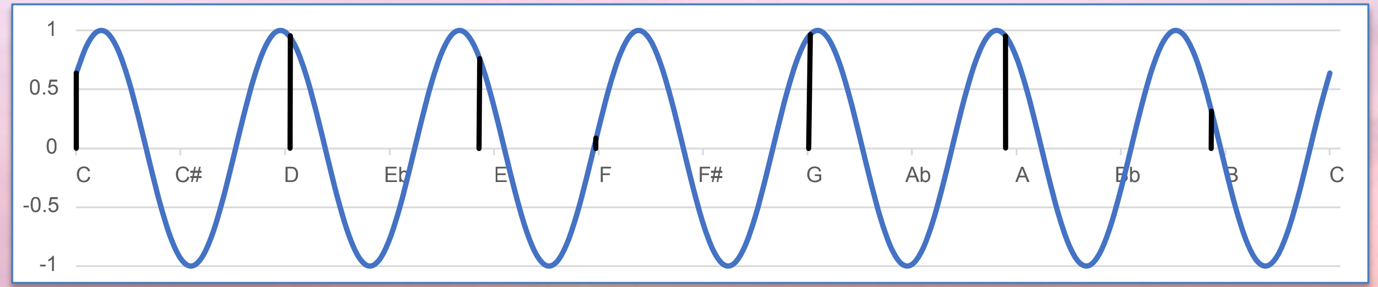
Heptatonicity

Heptatonicity does not assume any temperament



Pythagorean diatonic

0.37 + 1.0 + 0.37 − 0.21 + 0.82 + 0.82 − 0.21 = 2.97

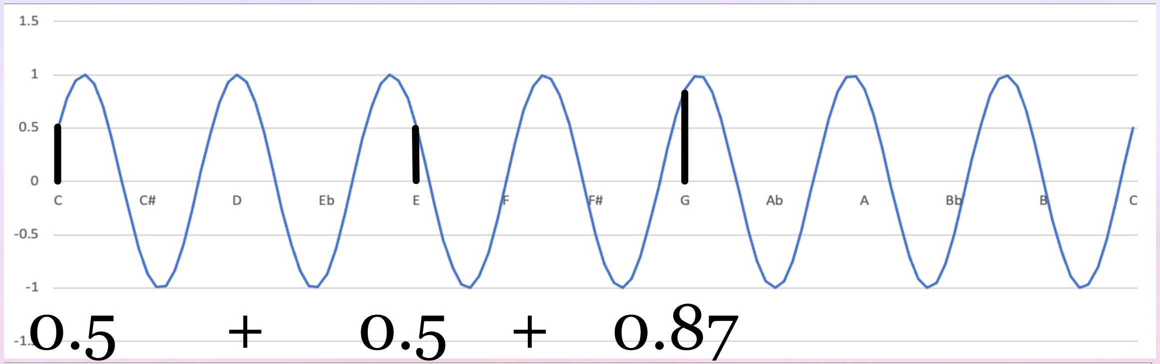


Just diatonic

0.64 + 0.95 + 0.76 + 0.10 + 0.96 + 0.99 + 0.26 = 4.65

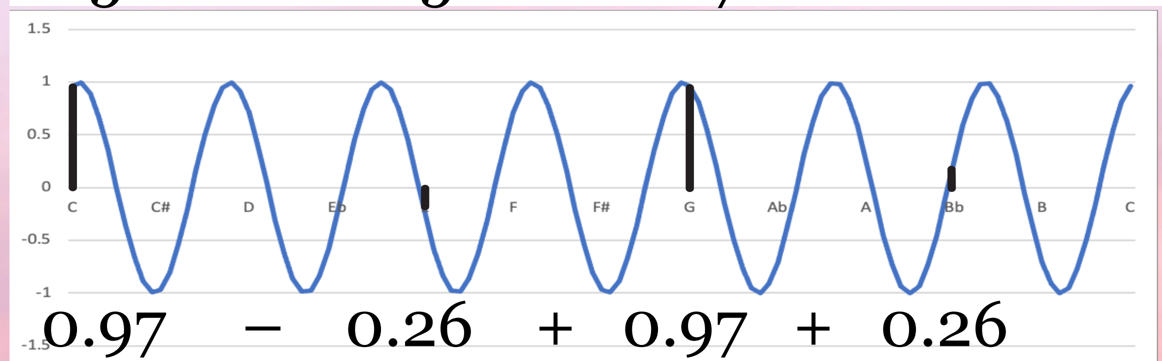
Heptatonicity

Heptatonicity can also be measured for subsets of scales



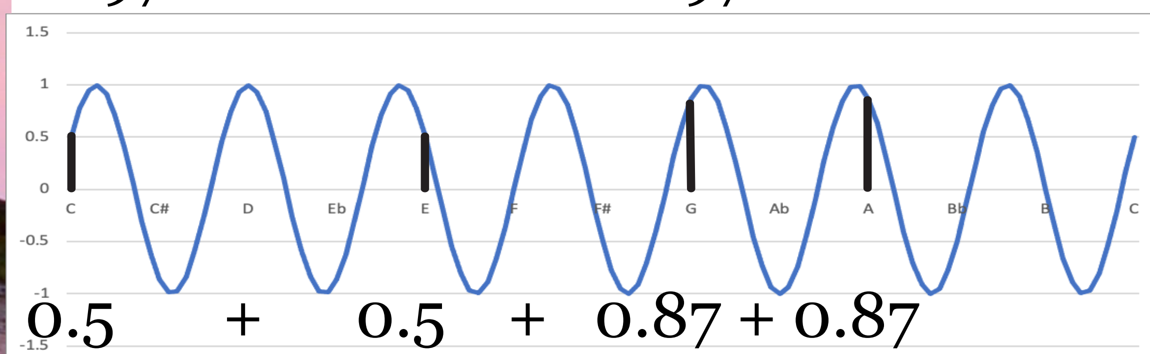
0.5 + 0.5 + 0.87 = 1.87

C major triad
2-2-3



0.97 − 0.26 + 0.97 + 0.26 = 2.46

C dominant seventh
2-2-2-1

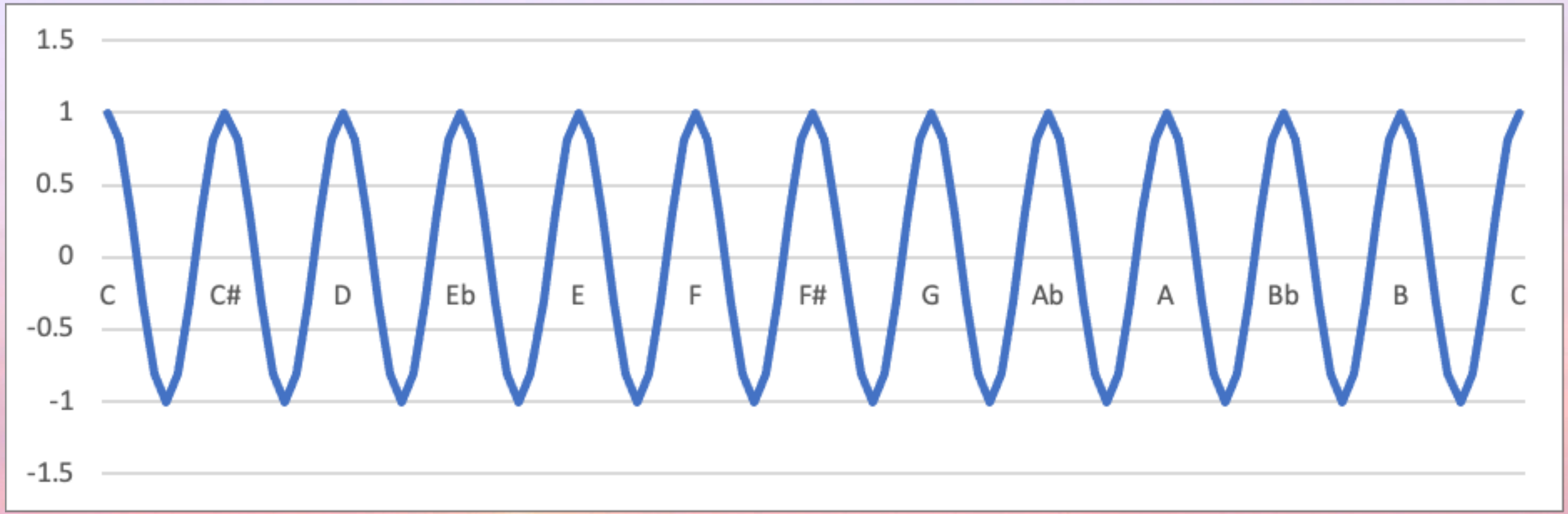


0.5 + 0.5 + 0.87 + 0.87 = 2.73

C add 6
2-2-1-2

Chromaticity

We can similarly define ***chromaticity*** as **approximation to a subset of a 12-tone equal tuning**

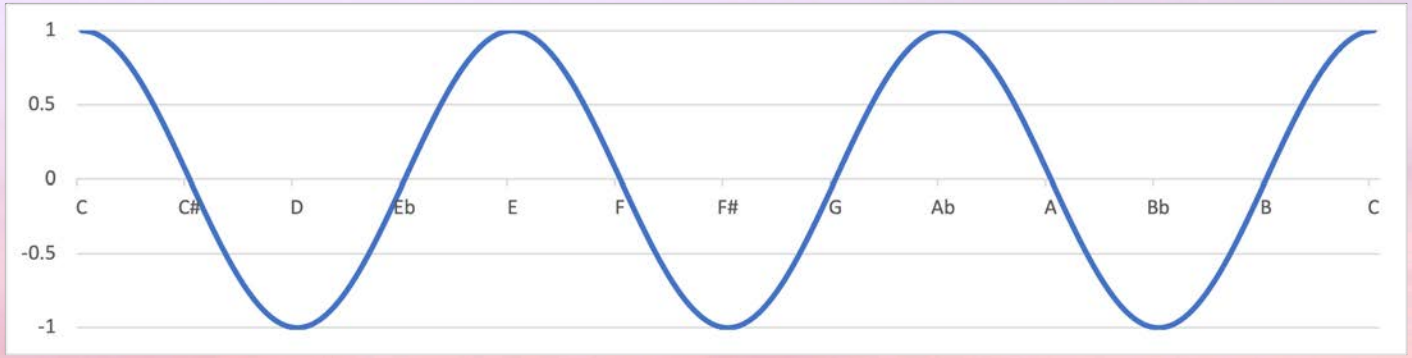


All ordinary pcsets have perfect chromaticity

Triadicity

Triad: Relatively even spacing of three pitch-classes

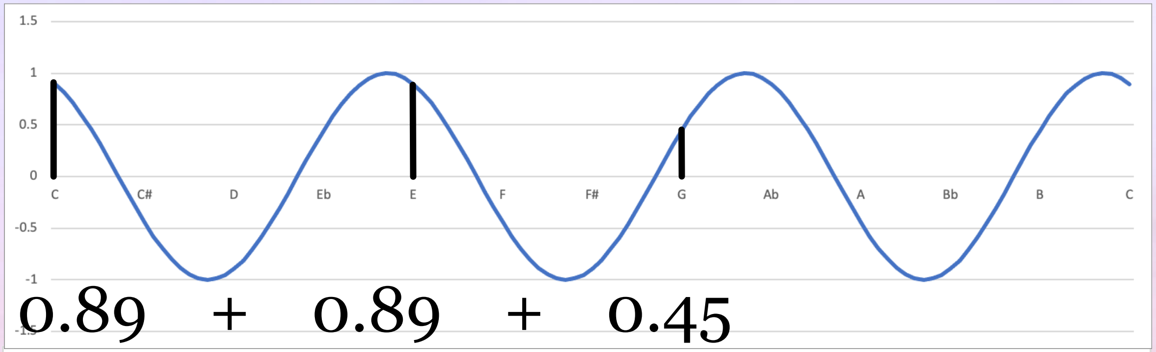
Triadicity: A cosine function over the pitch-classes with frequency 8ve/3:



- Positive values are good representatives of the triadicity; negative values are poor representatives.
- The curve can vary in phase (different triadicities).

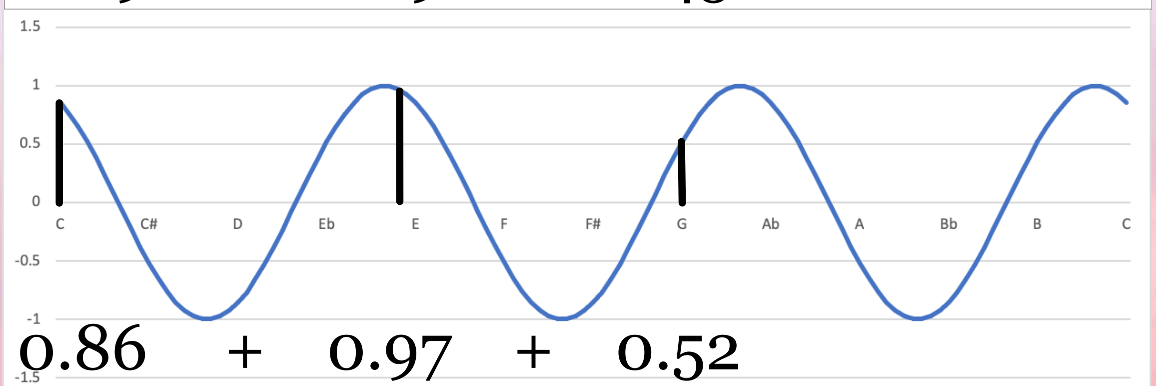
Triadicity

Example: Triadicity of some triads



12-t C major triad

0.89 + 0.89 + 0.45 = 2.23



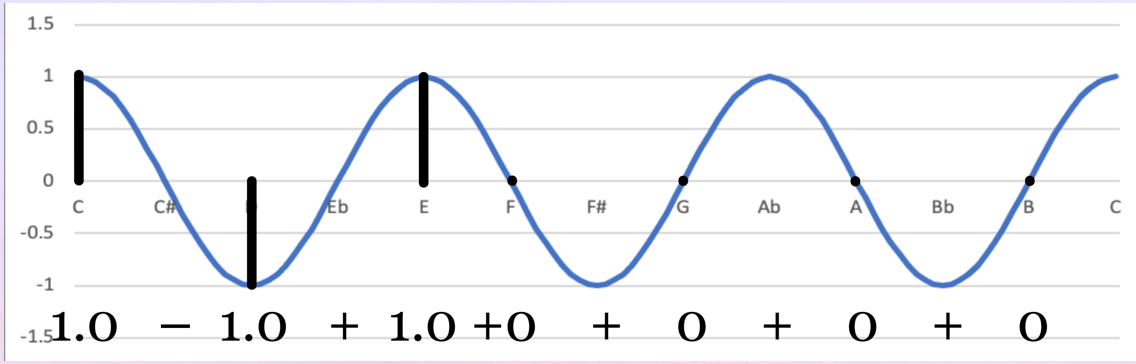
JI C major triad

0.86 + 0.97 + 0.52 = 2.35

No assumption is made of a 12-tone grid

Triadicity

Example: Triadicity of scales



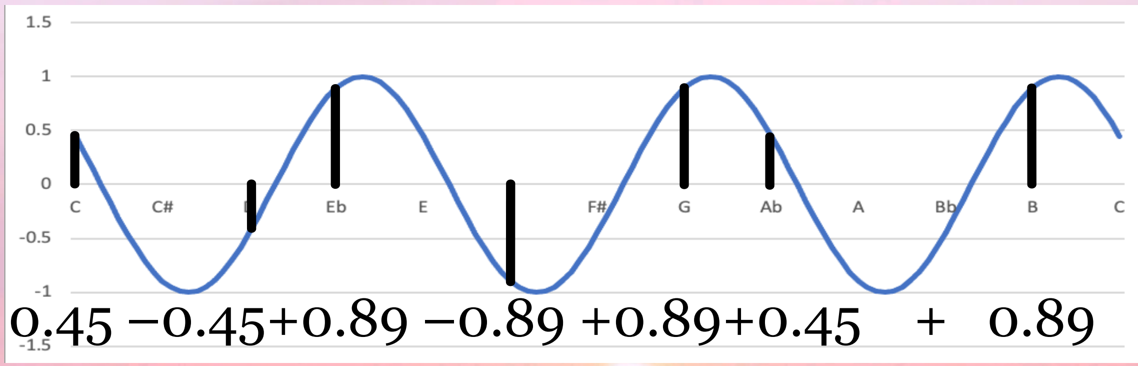
1.0 − 1.0 + 1.0 +0 + 0 + 0 + 0 = 1.0

C diatonic scale

Spans:

C D E F G A B

½ ½ 0 1 0 1 0



0.45 −0.45+0.89 −0.89 +0.89+0.45 + 0.89 = 2.23

C harmonic minor

Spans:

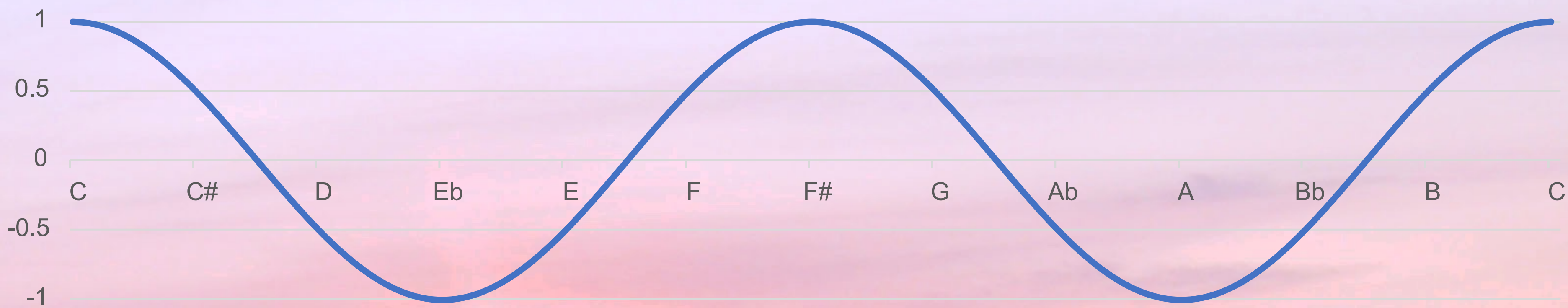
C D E♭ F G A♭ B

1 0 0 1 0 1 0

- Cardinality-flexible: applies to chords of any size
- Not all triad positions need to be represented; multiple notes can represent a single category

Dyadicity

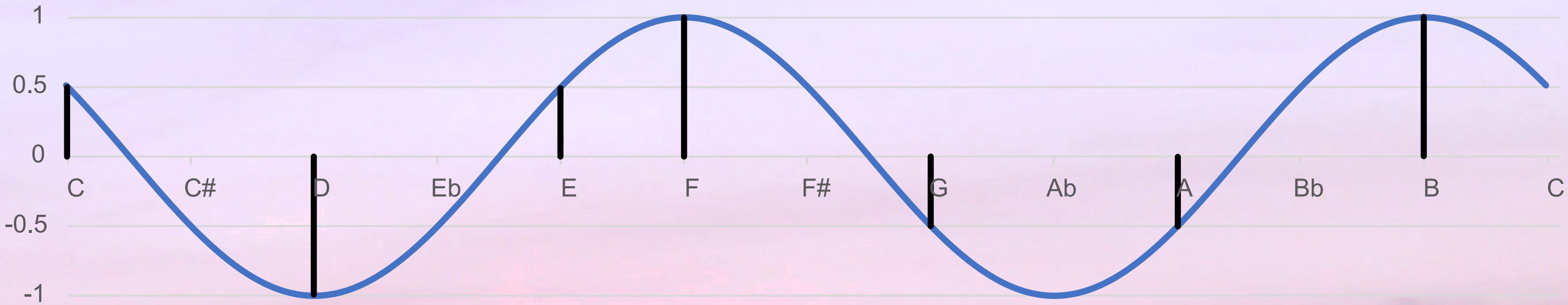
Dyadicity: A cosine function over the pitch-classes with frequency 8ve/2:



- Positive values are good representatives of the dyadicity; negative values are poor representatives.
- The curve can vary in phase (different dyadicities).

Dyadicity

Example: Dyadicity of a diatonic scale



0.5 − 1.0 + 0.5 + 1.0 − 0.5 − 0.5 + 1.0 = 1.0

Spans:

C D E F G A B

½ ½ 0 0 1 0 0

Dyadicity divides the scale into tetrachords ABCD and DEFG

Heptatonicity vs. Diatonicity

The diatonic scale is a ***prototype* of heptatonicity:** it maximizes heptatonicity for a 7-note subset of 12-tET.

Therefore heptatonicity **in a 12-tET context** equates to similarity to characteristic diatonic subsets.

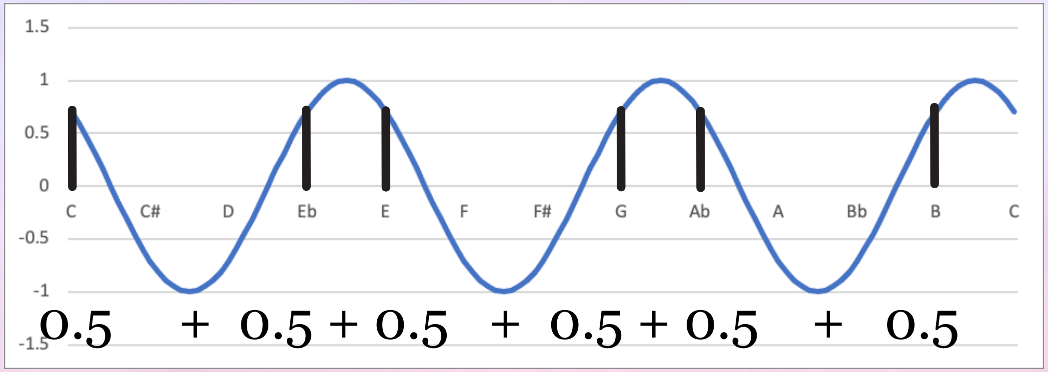
i.e.

Diatonicity = Heptatonicity + chromaticity

The distinction is only relevant in the context of alternate or flexible tuning, but it is also conceptually important.

Hexatonicity and Octatonicity

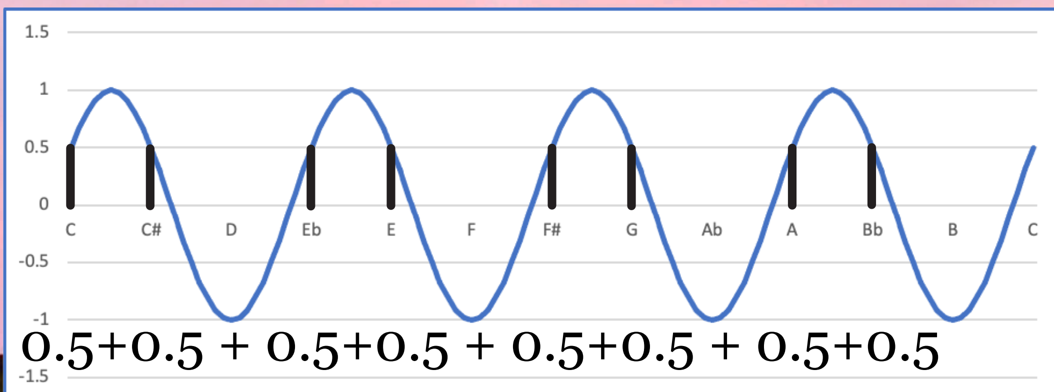
The 12-tET prototype of **triadicity** is a ***hexatonic scale:***



0.5 + 0.5 + 0.5 + 0.5 + 0.5 + 0.5 = 3.0

Hexatonicity = Triadicity + chromaticity

The 12-tET prototype of **tetradicity** is an ***octatonic scale:***



0.5+0.5 + 0.5+0.5 + 0.5+0.5 + 0.5+0.5 = 4.0

Octatonicity = Tetradicity + chromaticity

Fourier Transform

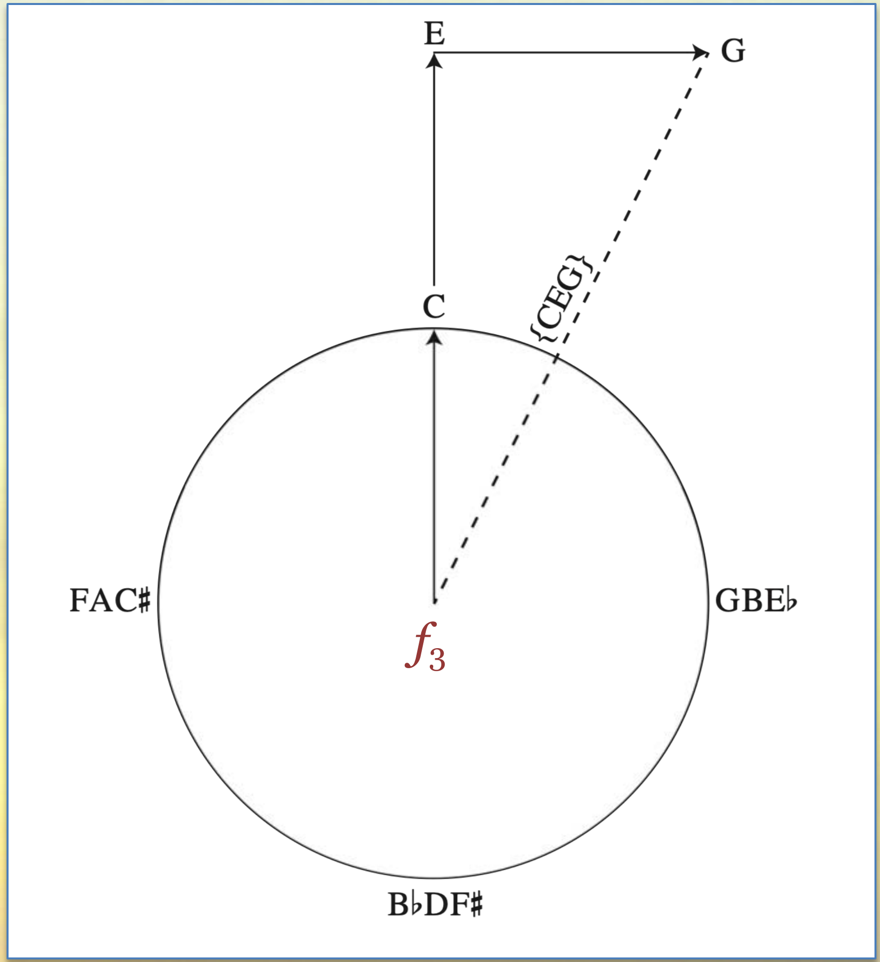
- Coefficient spaces (complex plane)
- Spectra
- Phase spaces
- Coefficient multiplication

Fourier Transform as Vector Sums

Fourier component f_k can be derived as a vector sum with each pitch class as a unit vector, where the unit circle is the 8ve/k.

The length of the resulting vector is the **magnitude** of the component, and the angle is its **phase.**

Example: 12-tET C maj. triad, $k = 3$

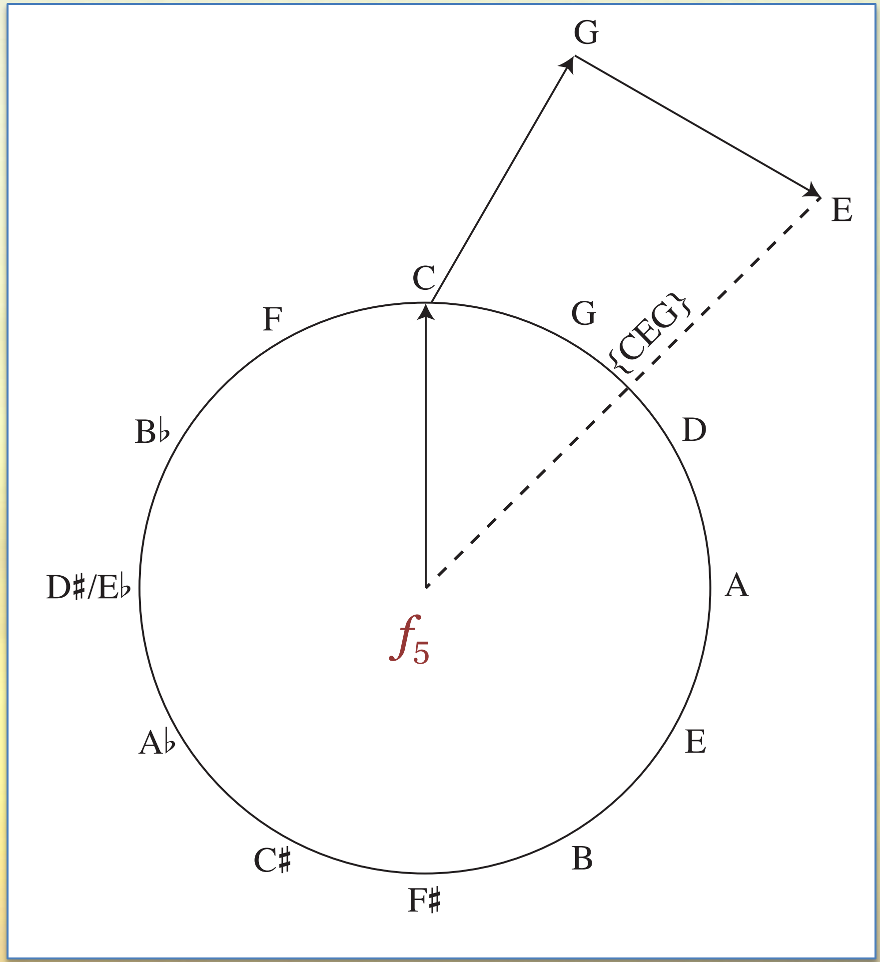


Fourier Transform as Vector Sums

Fourier component f_k can be derived as a vector sum with each pitch class as a unit vector, where the unit circle is the 8ve/k.

The length of the resulting vector is the **magnitude** of the component, and the angle is its **phase.**

Example: 12-tET C maj. triad, $k = 5$

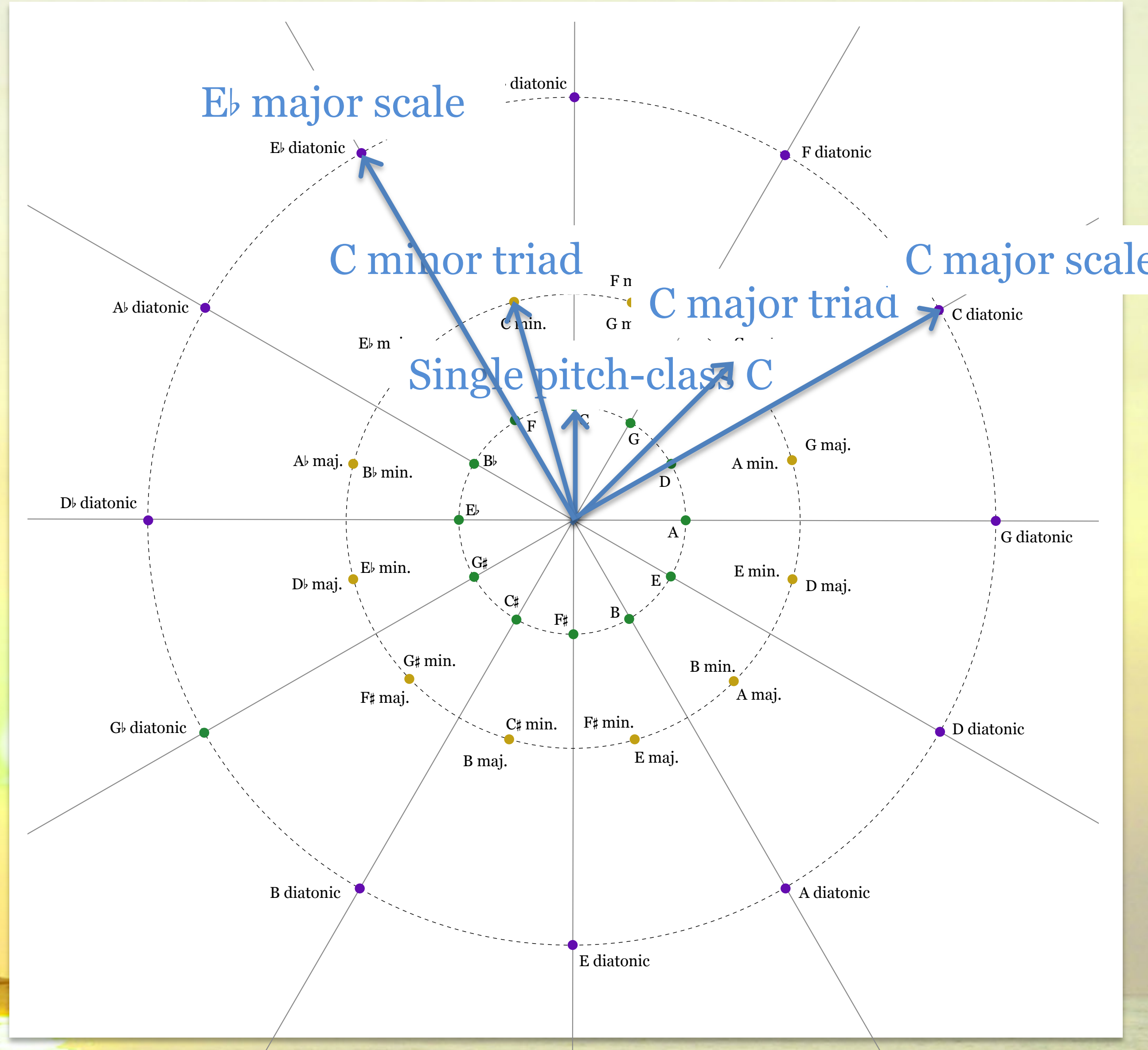


Single component spaces (complex plane)

Example:
12t-ET sets in
f_5 space

Distance from the center is the ***magnitude*** of f_5

Angle is the ***phase*** of f_5



Spectra

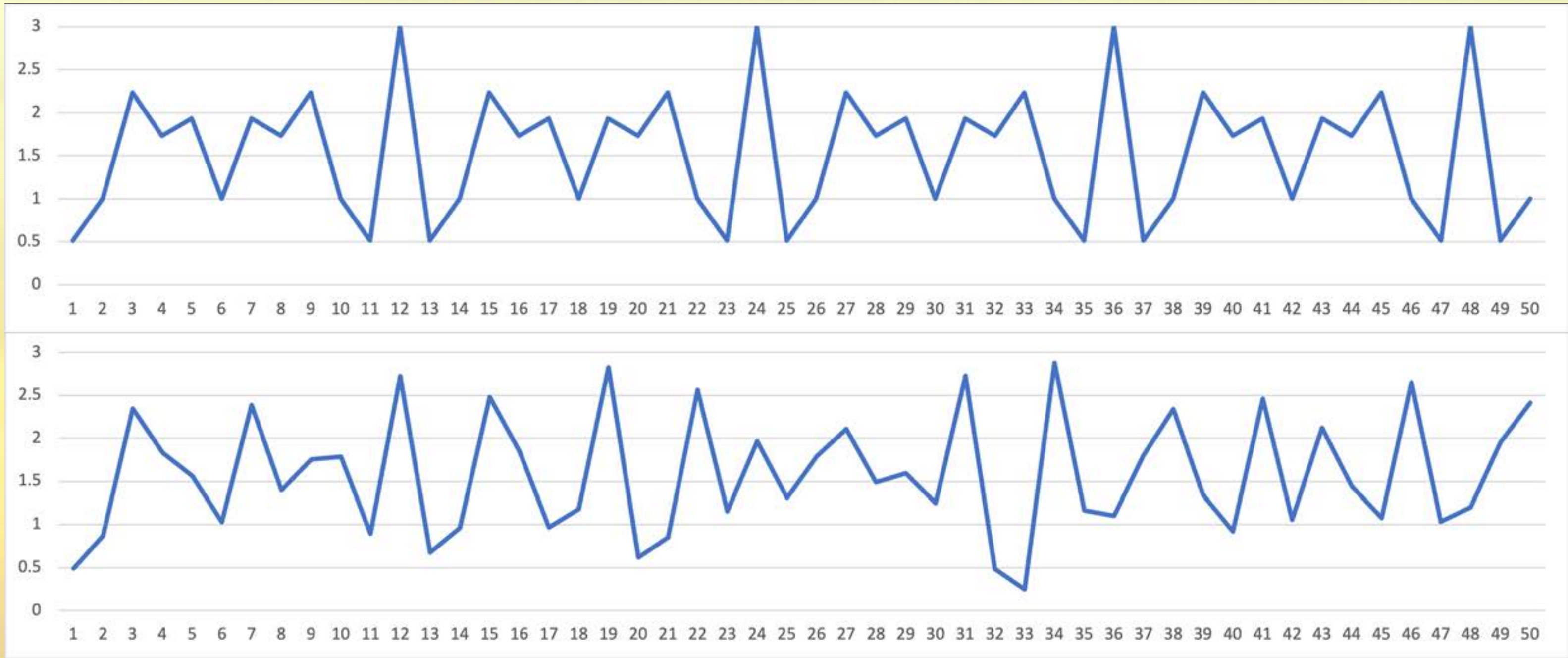
The ***spectrum*** of a pitch-class vector shows the magnitudes of all its Fourier coefficients (ignoring phases)

The spectrum is **invariant with respect to transposition and inversion** (i.e. it is a *set class* property)

Examples:

12tET major triad

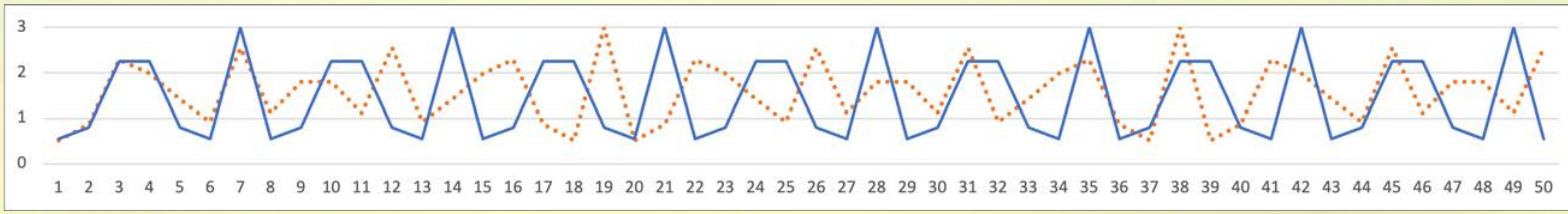
Just major triad



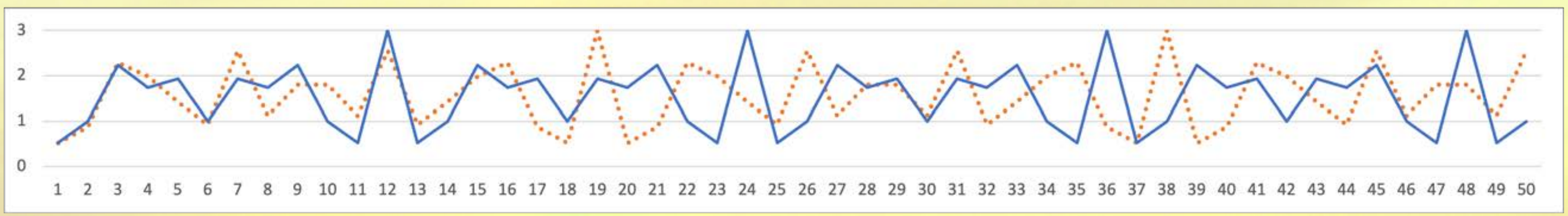
Spectra

The ***spectrum*** of a pitch-class vector shows the magnitudes of all its Fourier coefficients (ignoring phases)

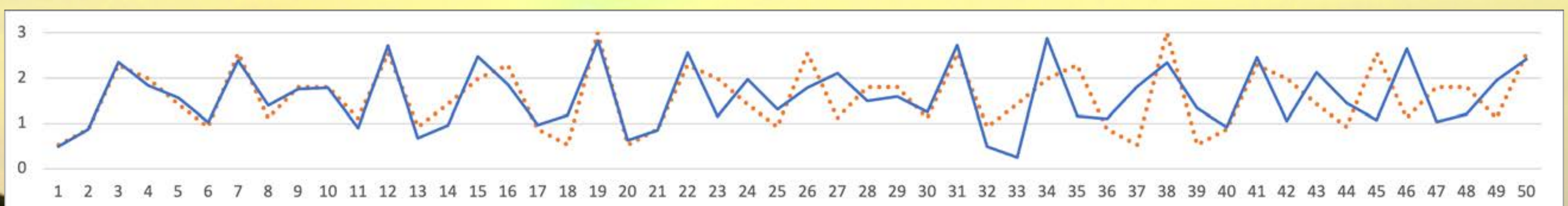
More examples: Just triad (dotted) compared to . . .



7tET triad



12tET triad

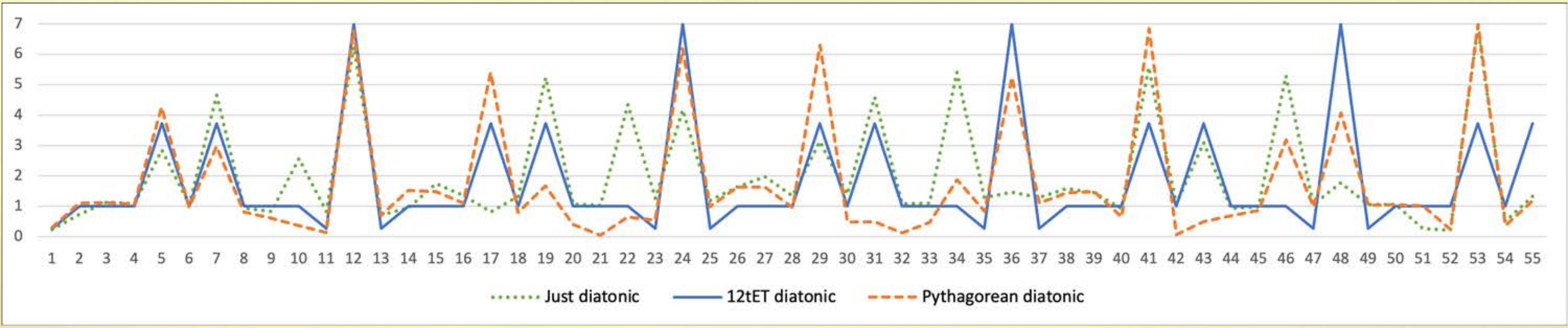


19tET triad

Spectra

The ***spectrum*** of a pitch-class vector shows the magnitudes of all its Fourier coefficients (ignoring phases)

Example: Diatonic scales in 12-tET, Pythagorean, just tuning

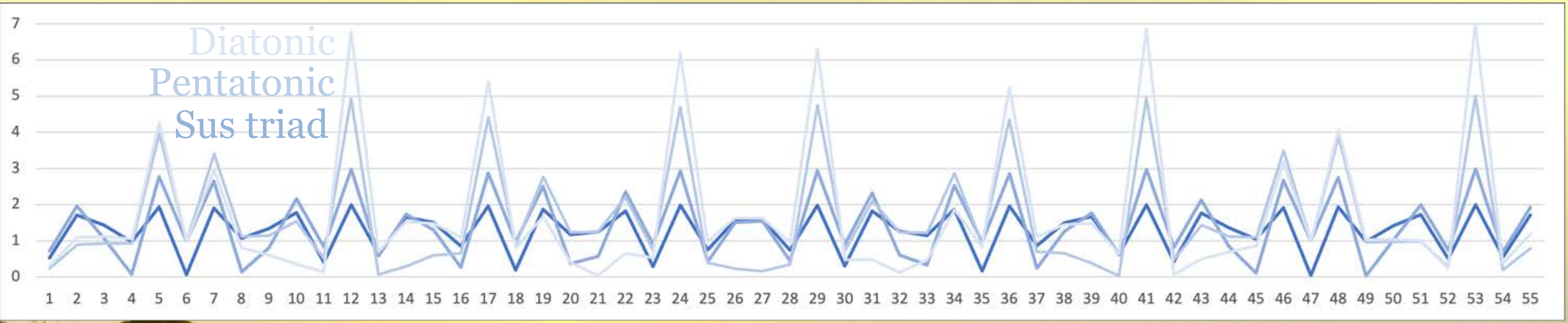


Spectra

The spectrum of an interval gives the ET approximations

A ***generated*** set intensifies the spectrum of the generating interval

Spectrum of the just perfect fifth

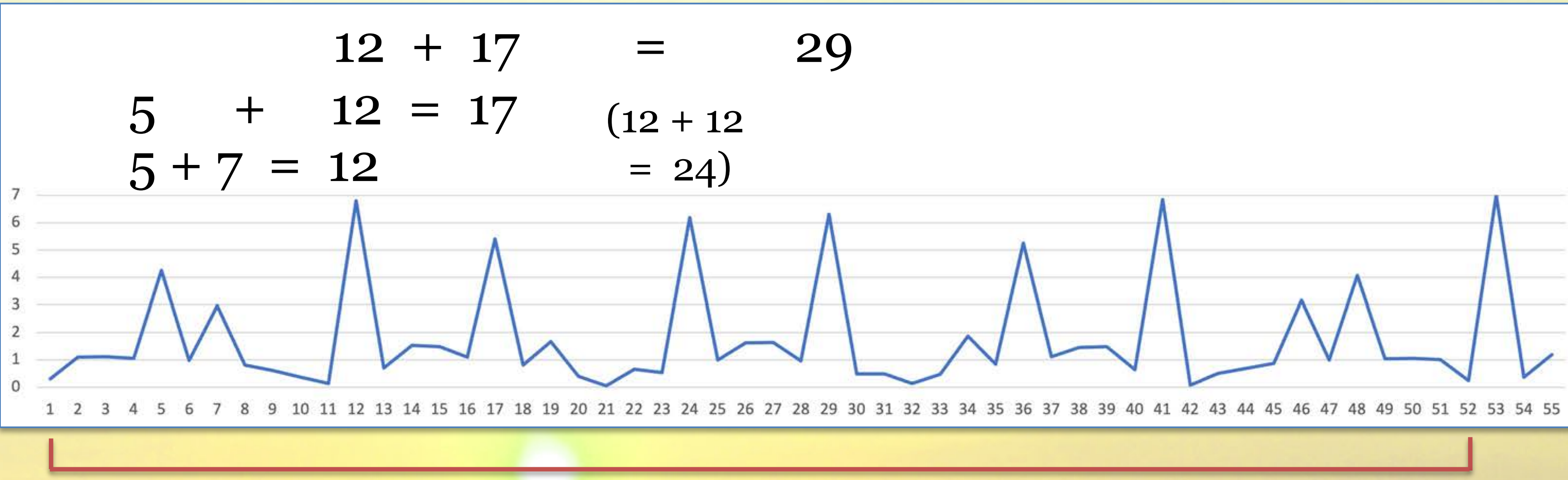


Spectra of fifth-generated collections

Spectra

The spectrum tends to have peaks at ***sums*** and ***differences*** of other peaks, especially for ***generated collections***

Pythagorean diatonic:



Approximate symmetry (5^{th} in 53tET is very close)

Phase Spaces

A ***phase space*** uses the phases of two (or more) coefficients as coordinates.

Phases are **cyclic**, so phase spaces are **toroidal.**

Transpositions correspond to translations (rotations) of the phase space. Inversions correspond to reflections.

Coefficient Products

Spectral ⇒ Transposition invariant

but

Transposition invariant ⇏ Spectral

For any $a + b = c$,

$$f_a f_b \overline{f_c}$$

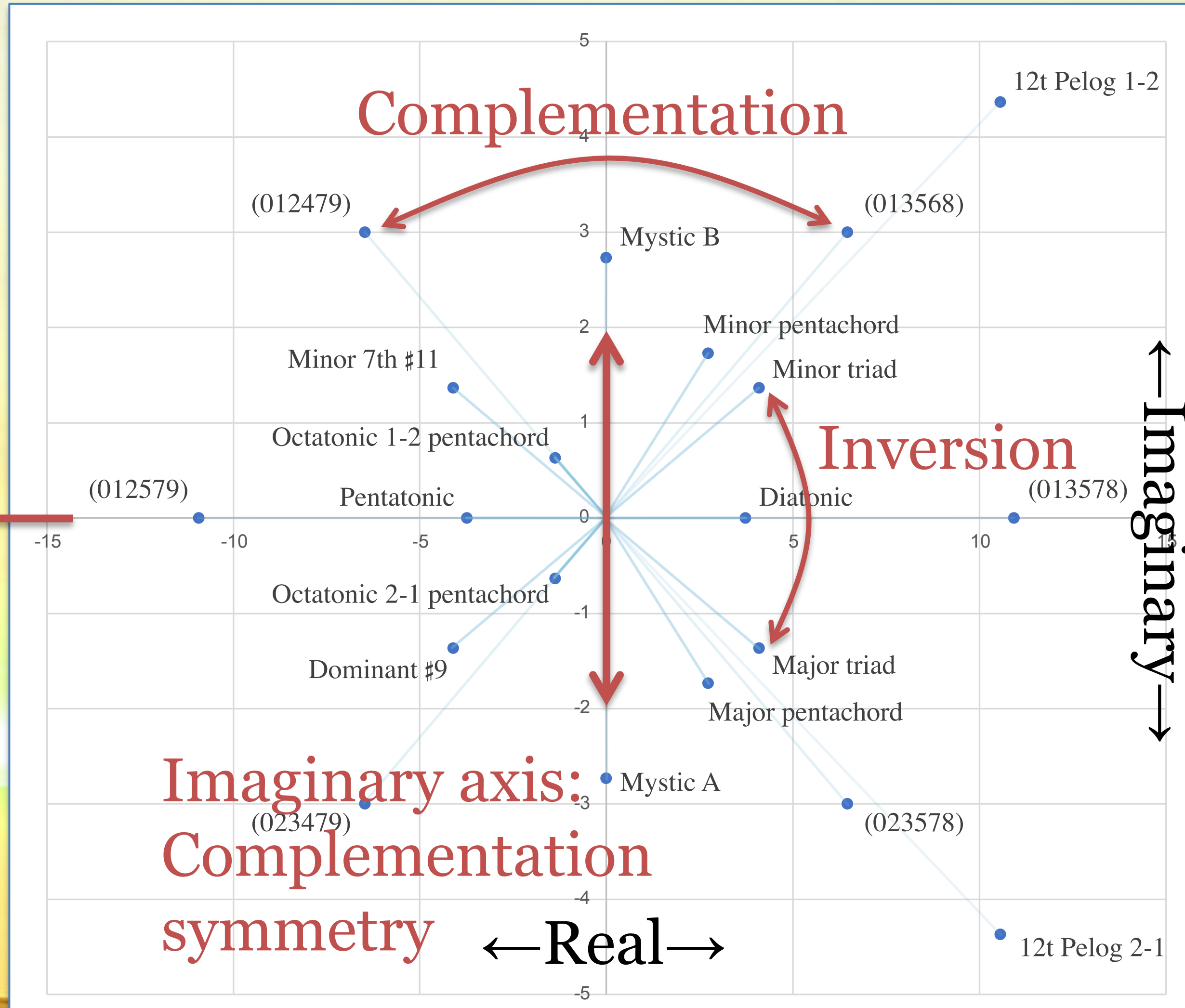
is a transposition-invariant complex number, with phase $\varphi_a + \varphi_b - \varphi_c$

Thanks to Emmanuel Amiot

Coefficient Products

Example: $f_2 f_3 \overline{f_5}$ of 12tET sets

Real axis: Inversional symmetry



Coefficient Products

Example: $f_2 f_3 \overline{f_5}$ of 12tET sets

Pentatonic:

	C	D	E	G	A	
f_2 spans:	0	0	1	0	1	+
f_3 spans:	½	½	1	0	1	≠
f_5 spans:	1	1	1	1	1	

Negative $f_2 f_3 \overline{f_5}$

Diatonic:

	C	D	E	F	G	A	B	
f_2 spans:	½	½	0	0	1	0	0	+
f_3 spans:	½	½	0	1	0	1	0	=
f_5 spans:	1	1	0	1	1	1	0	

Positive $f_2 f_3 \overline{f_5}$

Coefficient Products

Example: $f_2 f_3 \overline{f_5}$ of 12tET sets

(013568):								(023578):							
	B	C	D	E	F	G			A	B	C	D	E	F	
f_2 spans:	0	0	1	0	0	1	+	f_2 spans:	0	0	1	0	0	1	+
f_3 spans:	0	1	0	0	1	1	=	f_3 spans:	1	0	0	1	0	1	=
f_5 spans:	0	1	1	0	1	2		f_5 spans:	1	0	1	1	0	2	

Positive real $f_2 f_3 \overline{f_5}$ *and positive imaginary*

Positive real $f_2 f_3 \overline{f_5}$ *and negative imaginary*

Coefficient Products

Example: $f_2 f_3 \overline{f_5}$ of 12tET sets

(012479):

C C♯ D E G A

f_2 spans:	0	0	0	1	0	1	+
f_3 spans:	0	0	1	1	0	1	≠
f_5 spans:	1	0	1	1	1	1	

Negative real $f_2 f_3 \overline{f_5}$
and positive imaginary

(023479):

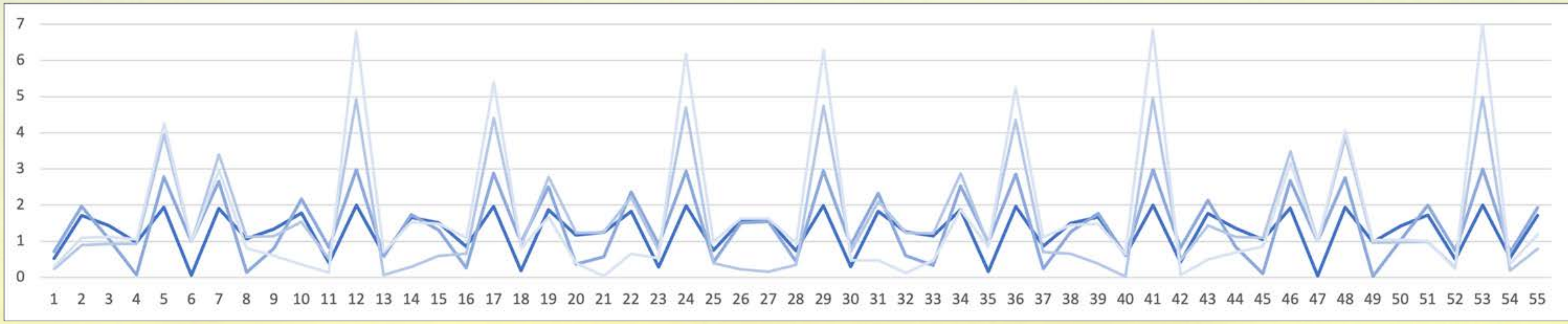
C D E♭ E G A

f_2 spans:	0	0	0	1	0	1	+
f_3 spans:	1	0	0	1	0	1	≠
f_5 spans:	1	0	1	1	1	1	

Negative real $f_2 f_3 \overline{f_5}$
and negative imaginary

Coefficient Products

Spectra of fifth-generated collections



Generated collections are inversionally symmetrical ⇒ real only

Well-formedness rule for coefficient product of spectral peaks
cardinality < large coefficient ⇒ positive
cardinality = large coefficient ⇒ negative

Maximal evenness: cardinality = smaller coefficient,
Use ET of large coefficient

Balinese Pelog

Andrew Toth’s measurements

Pelog spectra

Begbeg–Sedeng–Tirus models

$f_2 f_7 \overline{f}_9$ space

Andrew Toth's measurements

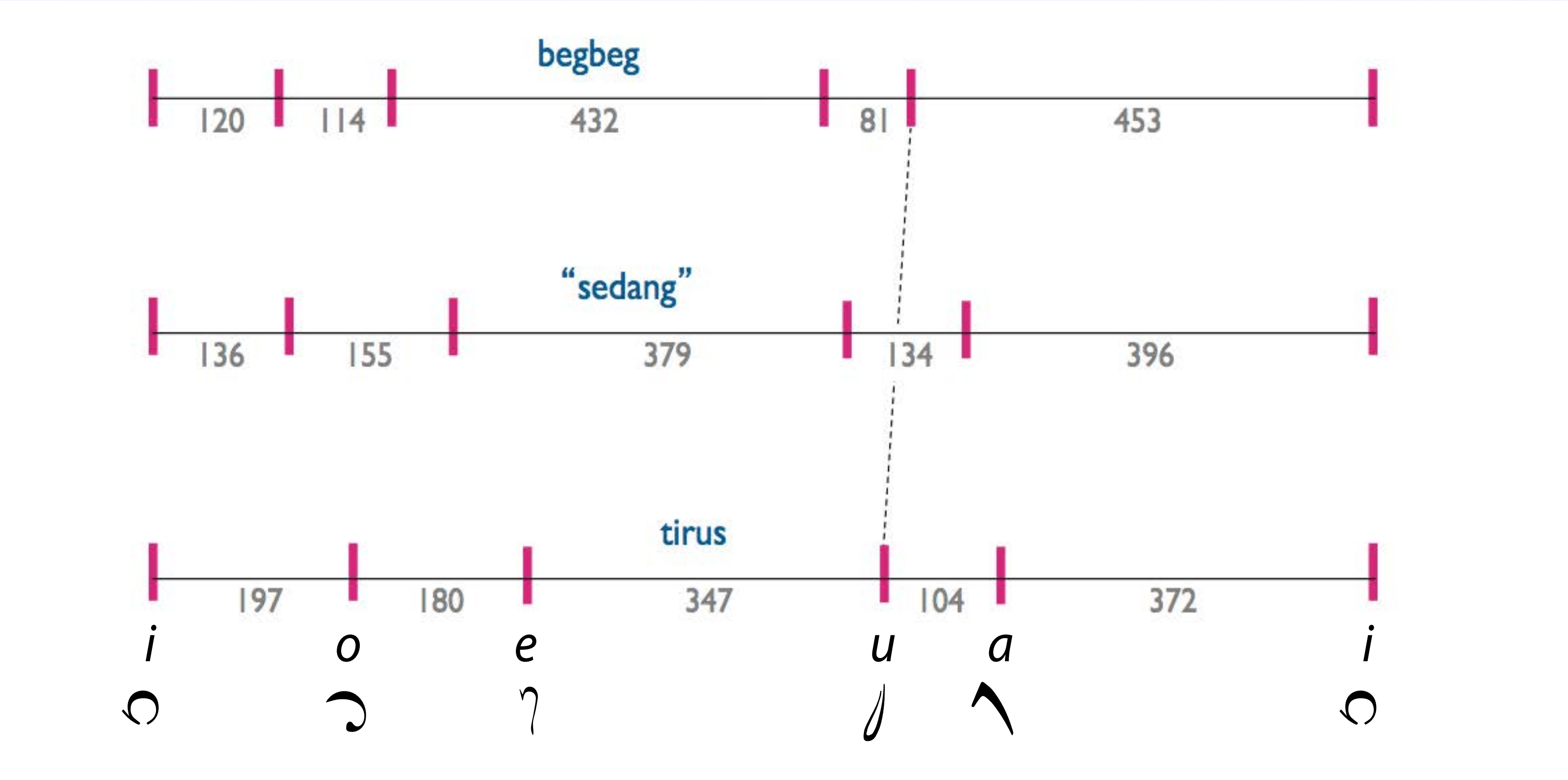
Toth measured 50 gamelans across all regions of Bali

Thanks to Wayne Vitale and Bill Sethares for data.
("Balinese Gamelan Tuning: The Toth Archives" forthcoming in *Analytical Approaches to World Music*)

Processing:

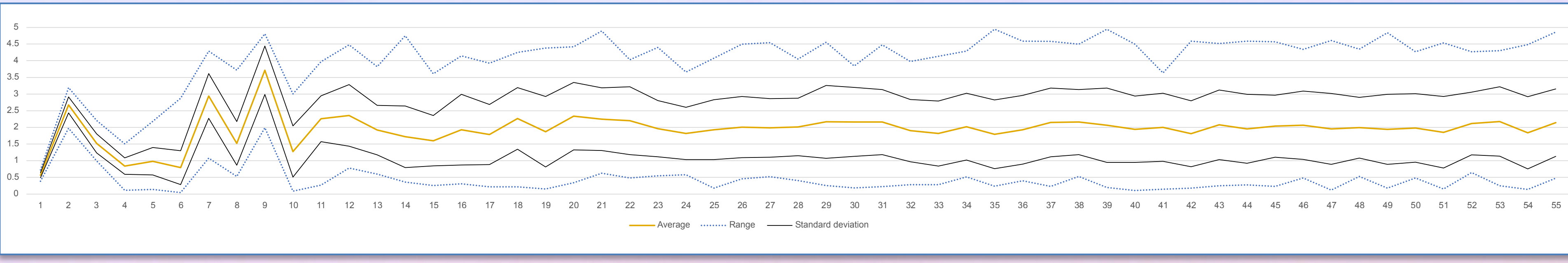
- Average across instruments.
- Average step sizes between second and third octave.
- Stretch/compress to a 1200¢ octave.

Models: Begbeg–Sedang–Tirus



Toth's idealized models of pelog tuning varieties (from testimony of master tuners)

Pelog Spectra



Peaks at f_2, f_7, and f_9 and troughs in between are consistent.

Above f_9, little discernable consistency.

Pelog Spans

	i	*o*	*a*	*e*	*u*		*i*	*o*	*a*	*e*	*u*		*i*	*o*	*a*	*e*	*u*	
f_2:	0	0	1	0	1	(all the same)												
f_3:	0	1	1	0	1	*or*	1	0	1	0	1							
f_5:	0	1	2	0	2	*or*	0	1	1	1	2	*or*	1	0	2	0	2	*or* . . .
f_7:	1	1	2	1	2	(all the same)												
f_9:	1	1	3	1	3	(all but one the same)												
f_{12}:	1	2	4	1	4	*or*	2	1	4	1	4	*or* . . .						

Tuning Variants: Spectral

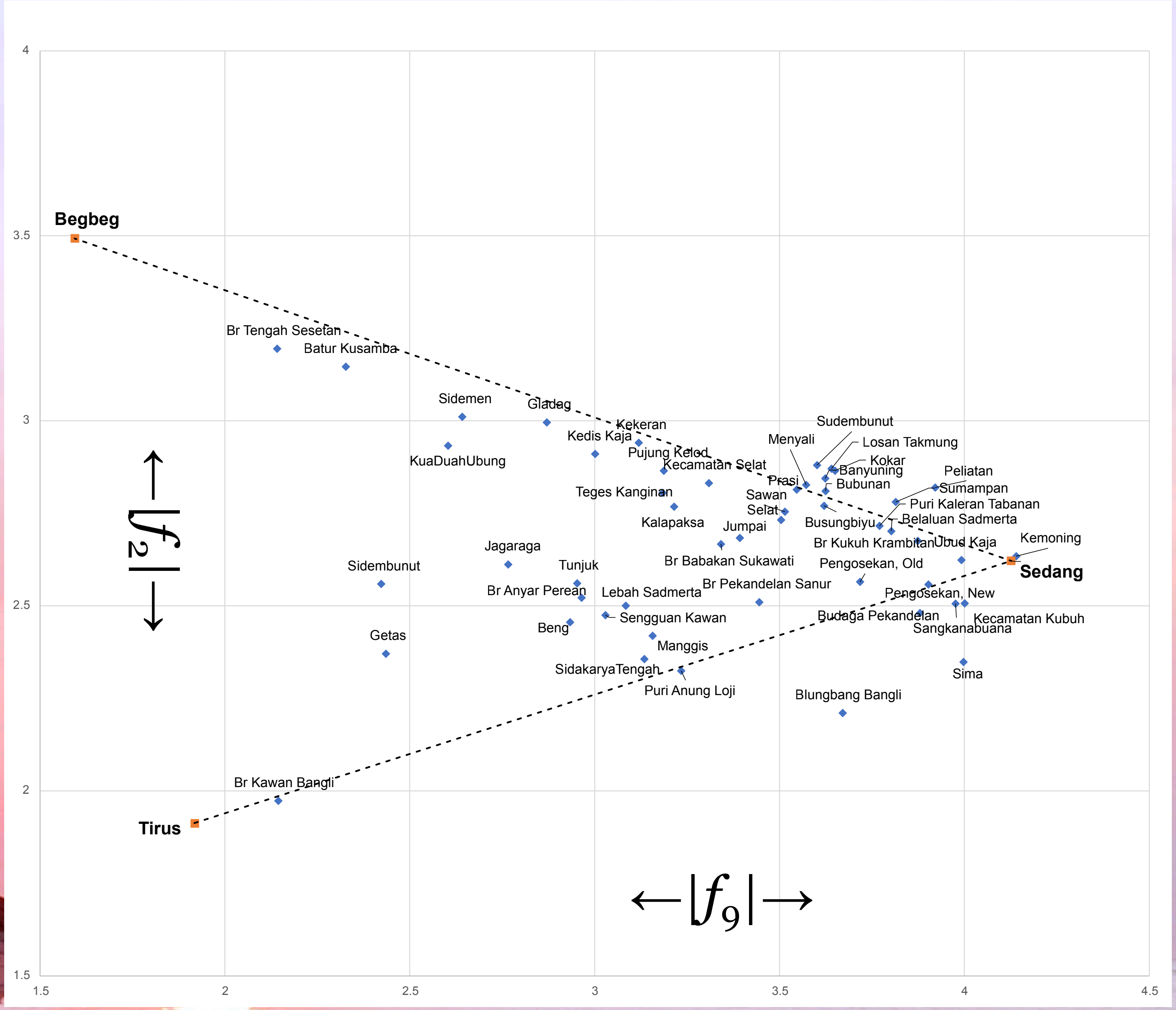
$|f_9|$ is highly correlated with $|f_7|$, $|f_{11}|$, $|f_{13}|$,
anti-correlated with $|f_5|$ and $|f_{12}|$
but relatively uncorrelated with $|f_2|$

$\hookrightarrow$ $|f_9| \times |f_2|$ is a good space to distinguish pelog spectra

Tuning Variants: Spectral

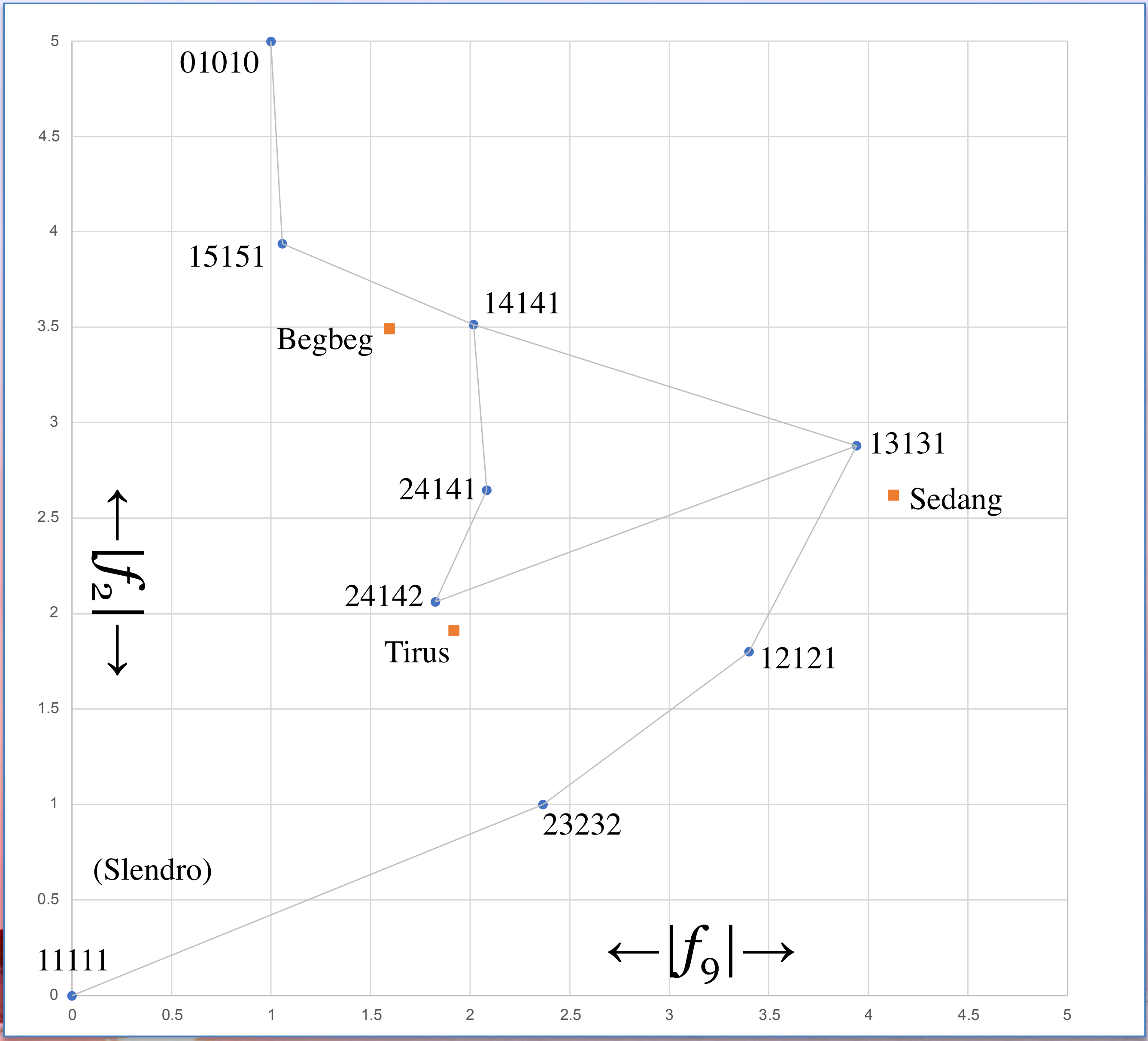
$|f_2|$ is a good model of "Begbeg–Tirus axis,"

but Sedang also differs in $|f_9|$.



Tuning Variants: Spectral

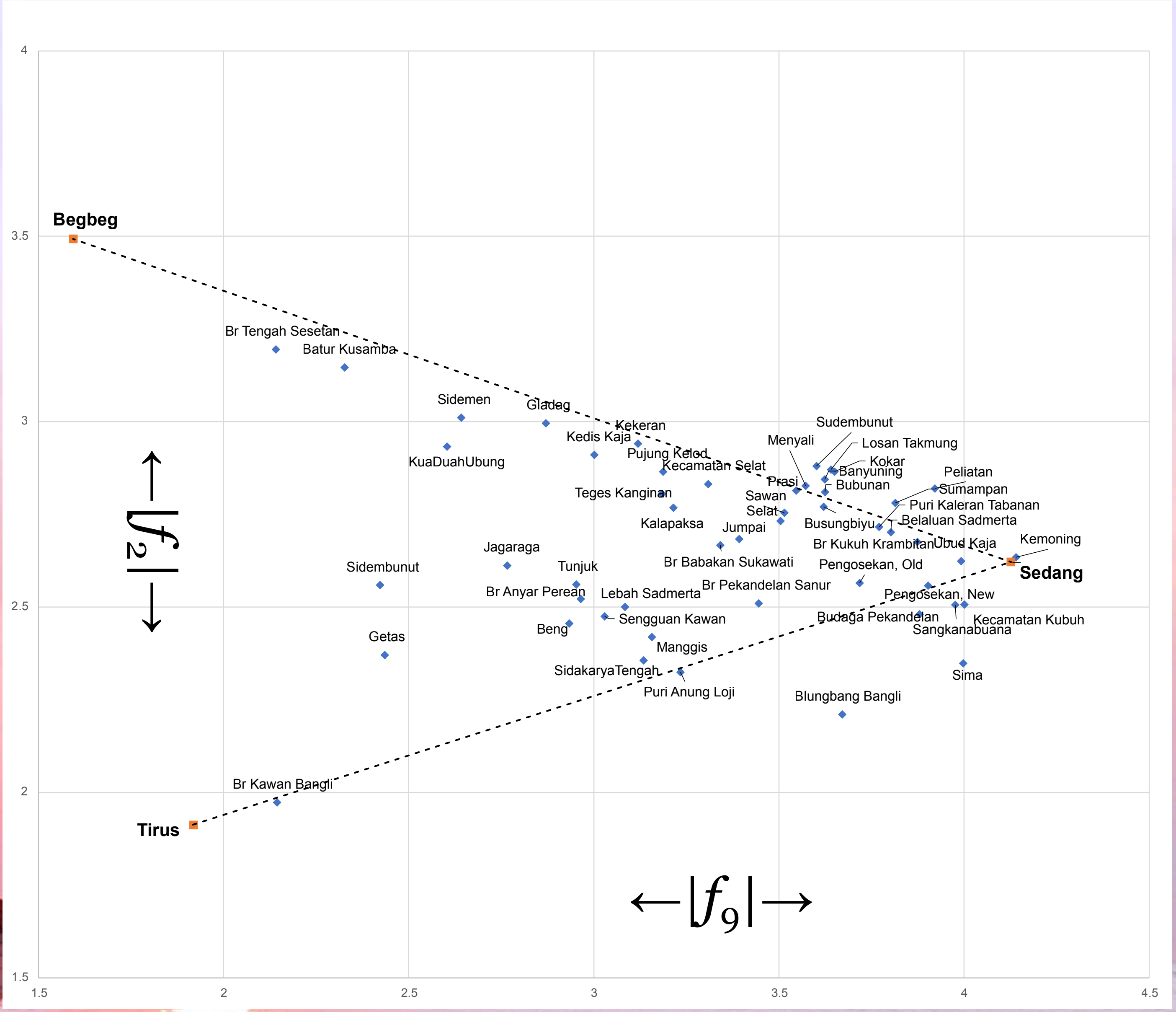
ET models in $|f_2|$-$|f_9|$ space



Tuning Variants: Spectral

Most tunings are close to Sedang.

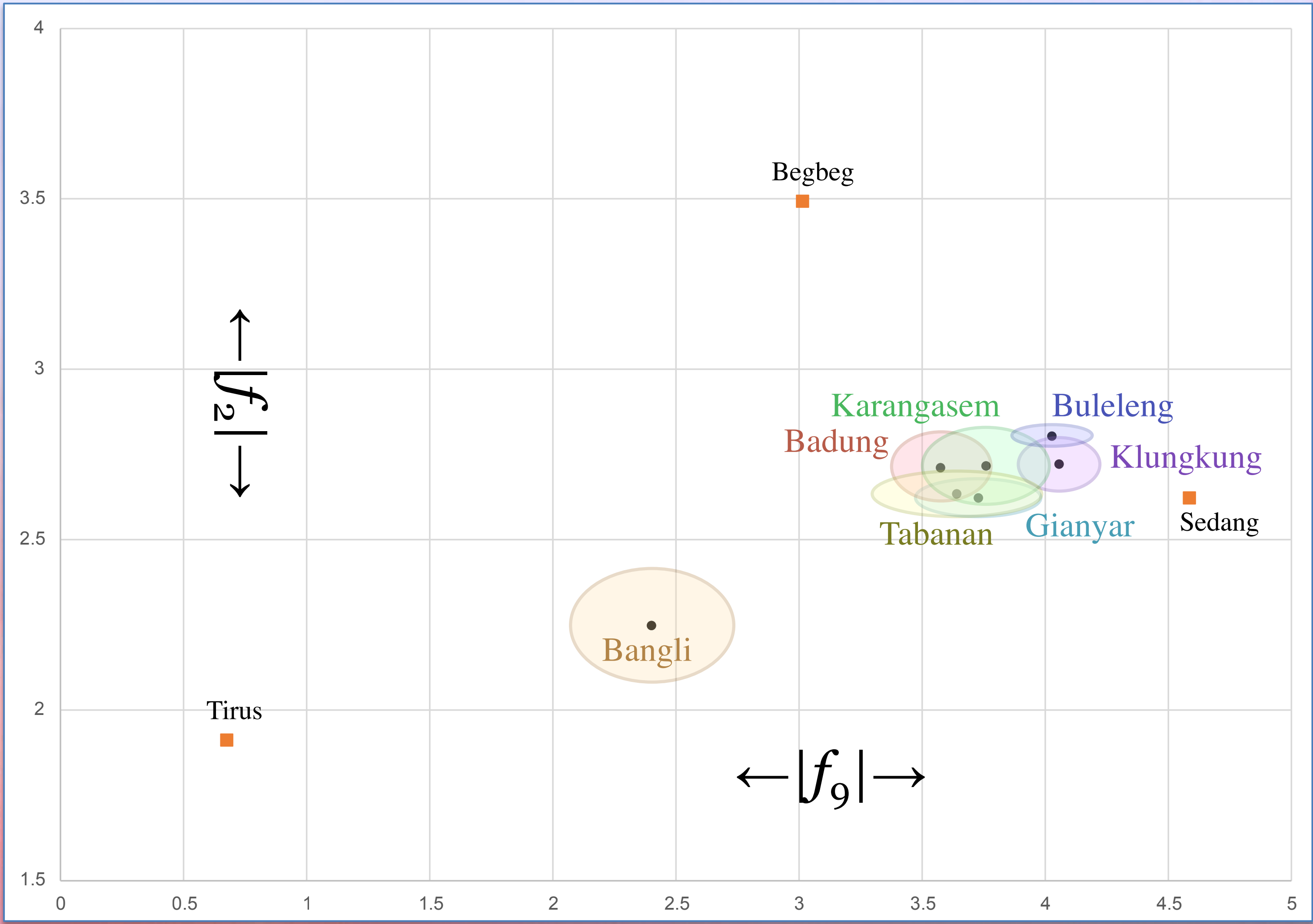
Begbeg and Tirus are outside the range of observed tunings.



Spectral Tuning Variation by Region

Only Bangli region (central highlands) is reliably distinct (include all most Tirus tunings).

Buleleng, Klungkung more Sedang



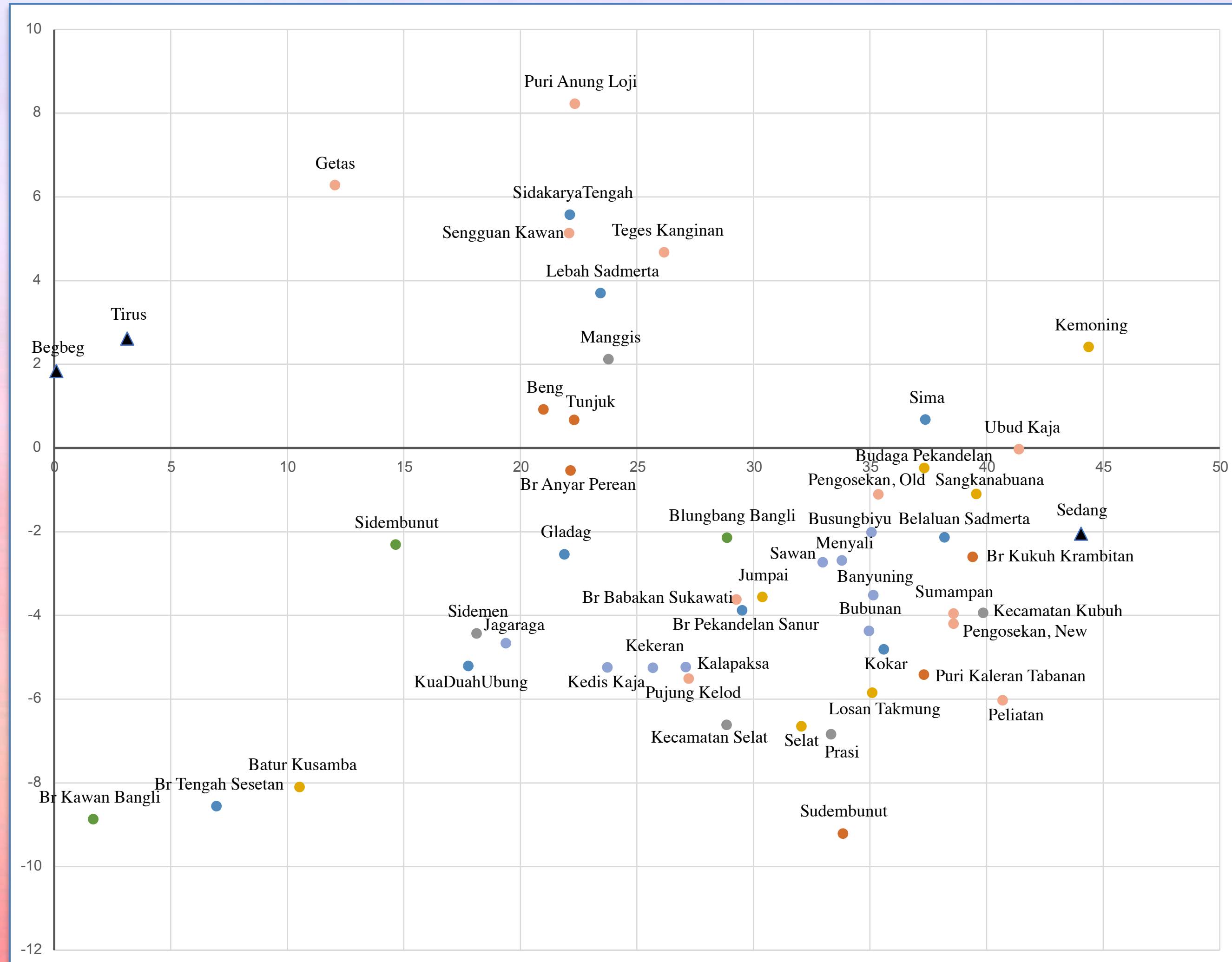
Ellipses show standard error

Tuning Variants: $f_2 f_7 \overline{f_9}$ space

All tunings have positive real values.

Consistency of spans:

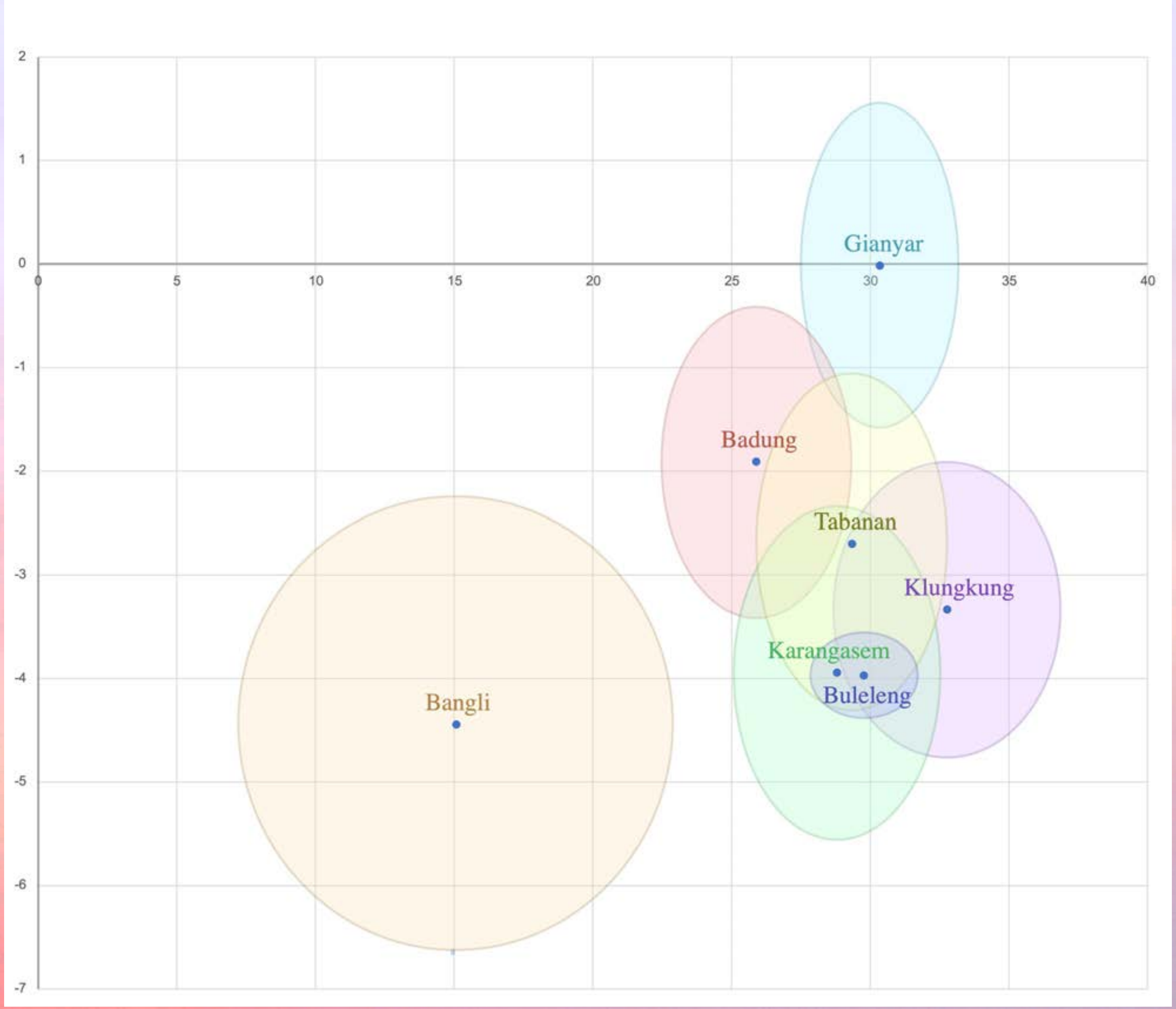
i	o	e	u	a
0	0	1	0	1
1	1	2	1	2
1	1	3	1	3



Tuning Variants: $f_2 f_7 \overline{f_9}$ space

Regions vary consistently on imaginary axis:
  Only Gianyar is balanced around zero
  Other regions consistently negative.

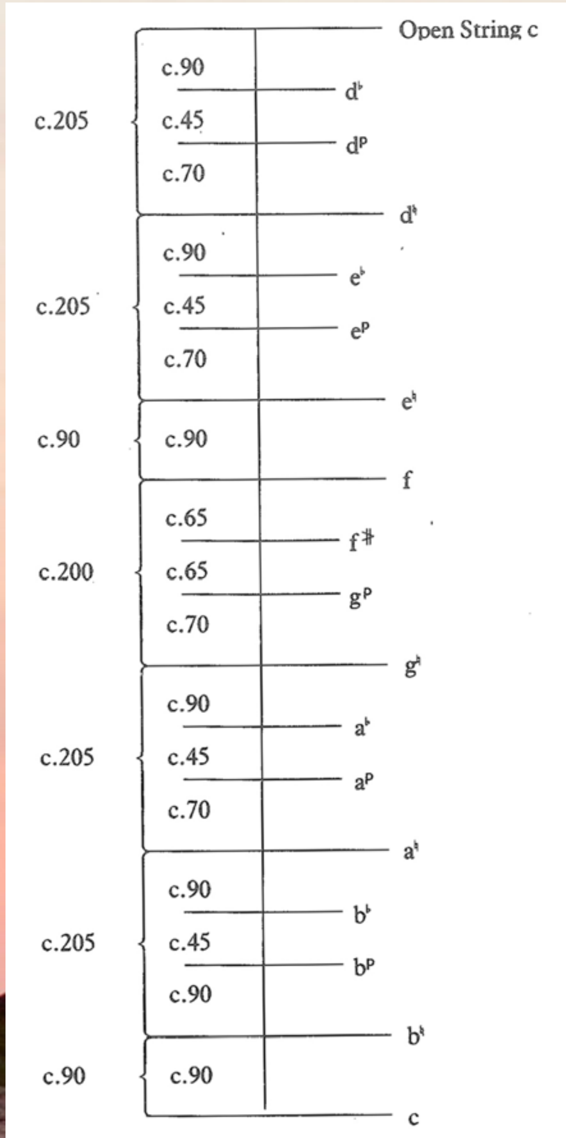
Ellipses show standard error.



Persian *Dastgah* Tuning

- Farhat's tuning and the *Dastgah* system
- Spectral analysis of scales and tetrachords
- Coefficient-product spaces

Farhat's Tuning

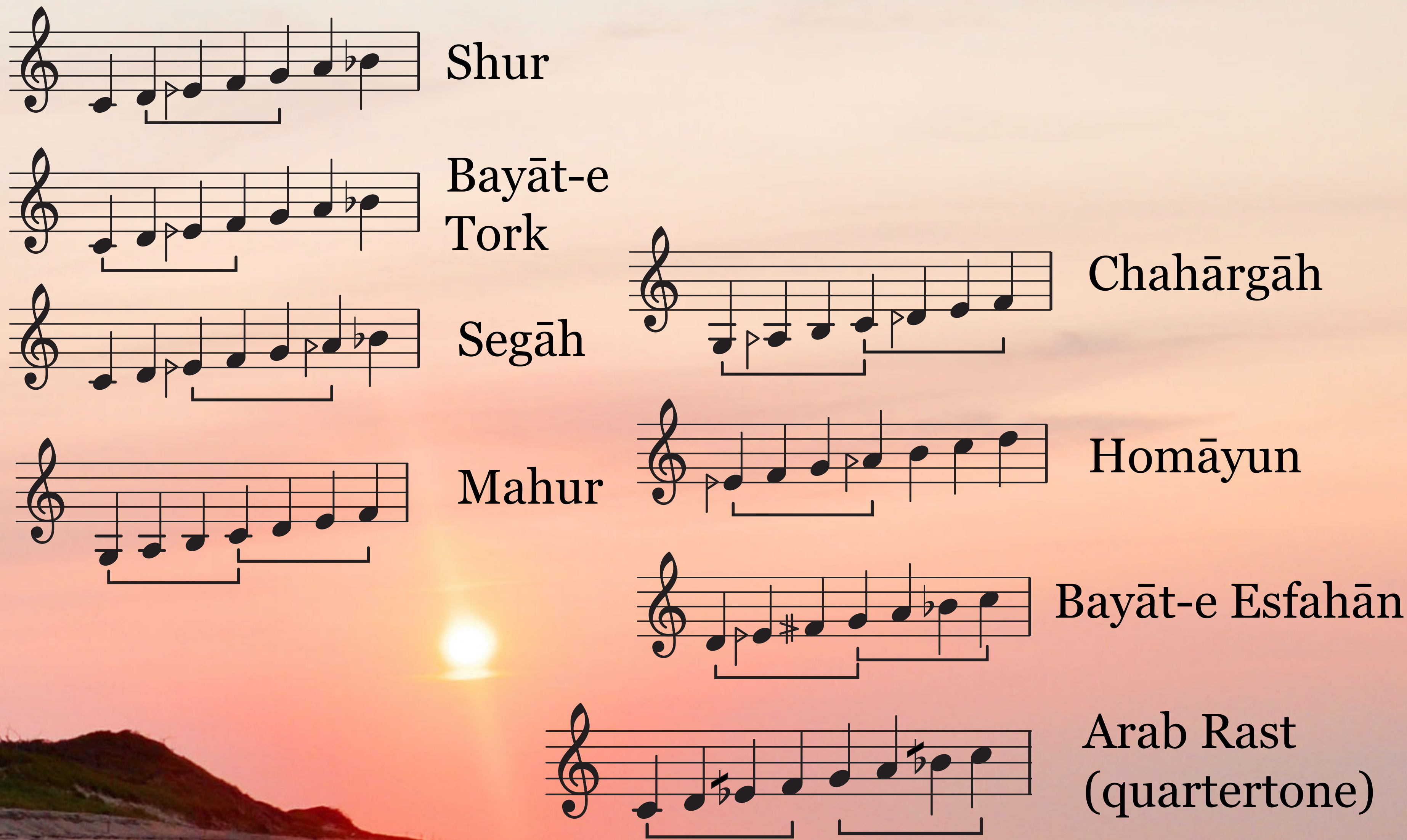


– Loosely empirical (based on measurements but no data reported)

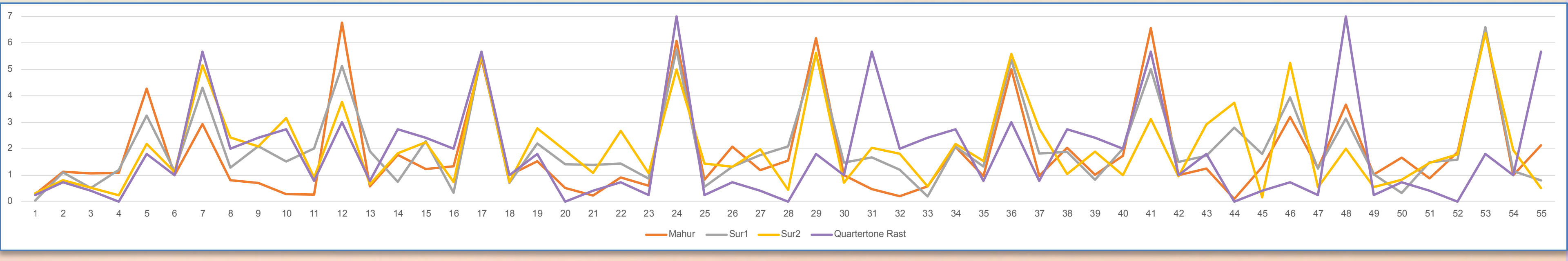
– Generated by two basic intervals:
  - Perfect fifth (two Pythagorean scales of 11 and 6 notes each) and
  - Neutral step, which Farhat estimates at 135¢ (Pythagorean second – koron 205¢ – 70¢)

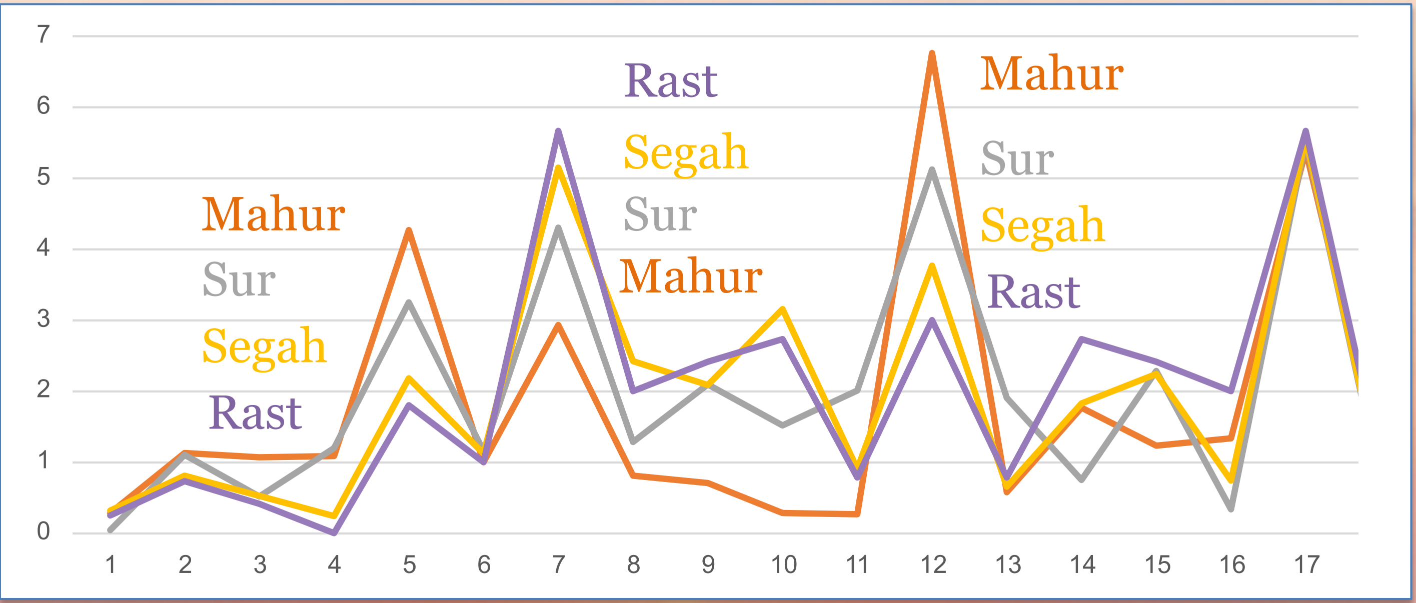
  (Large neutral step is semitone + koron, 90¢ + 70¢ = 160)

Some scales and tetrachords

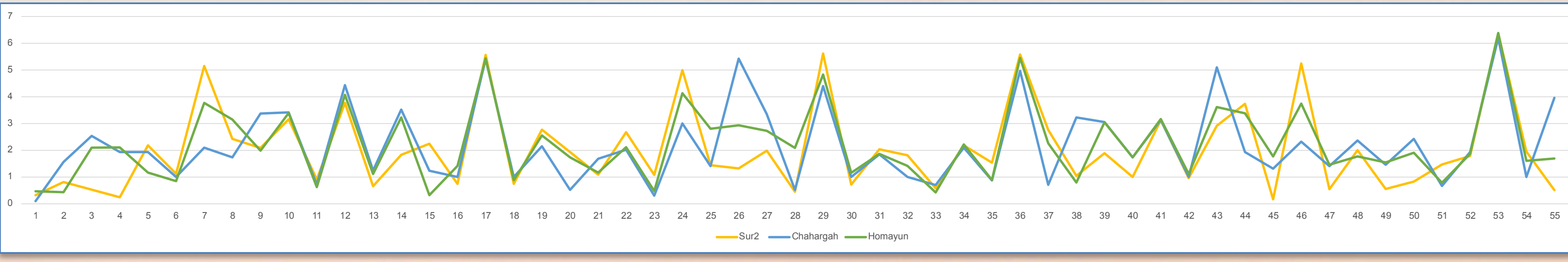


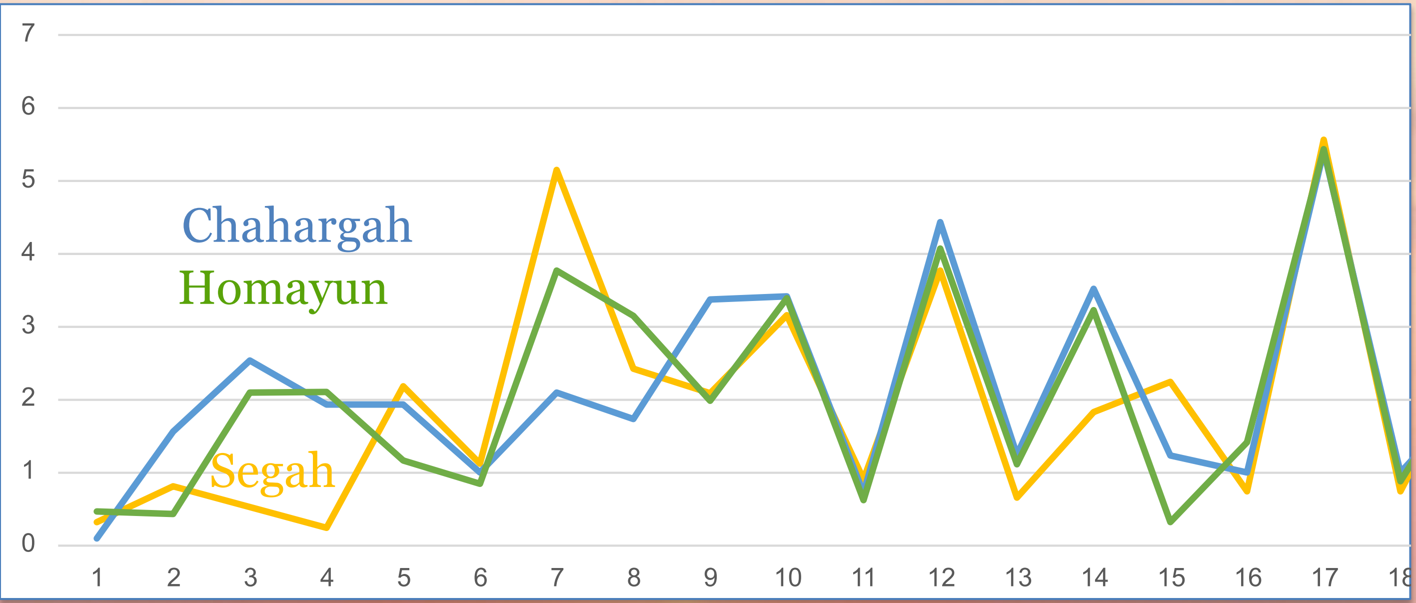
Spectra for scales





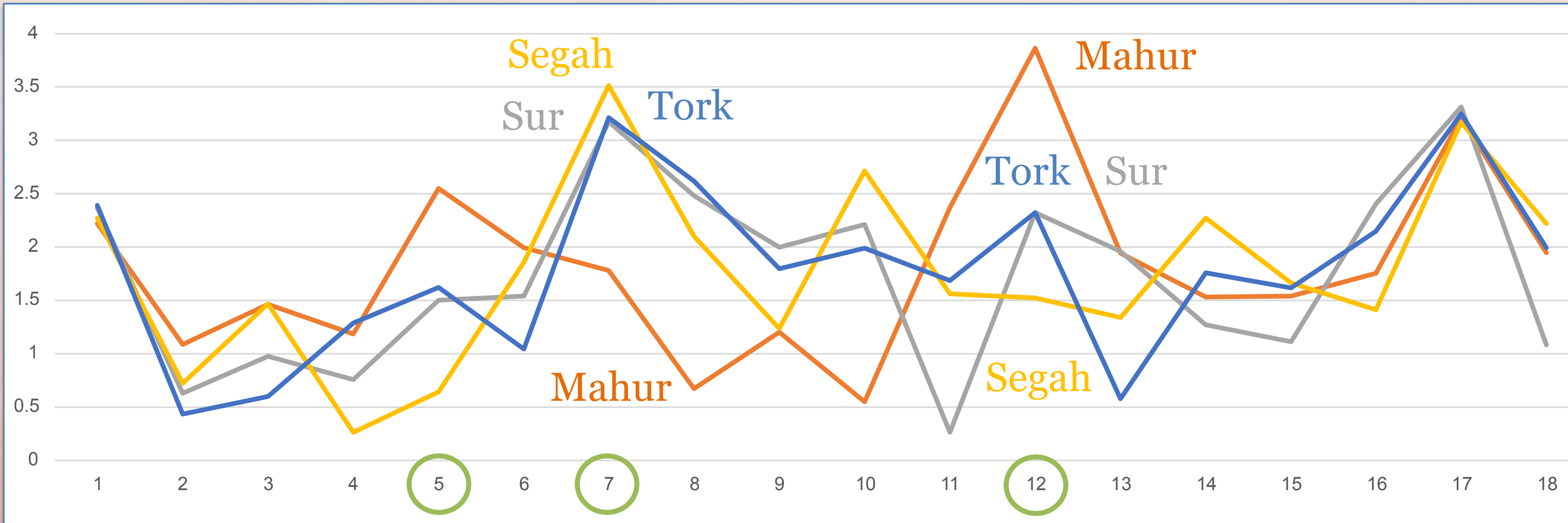
Spectra for scales





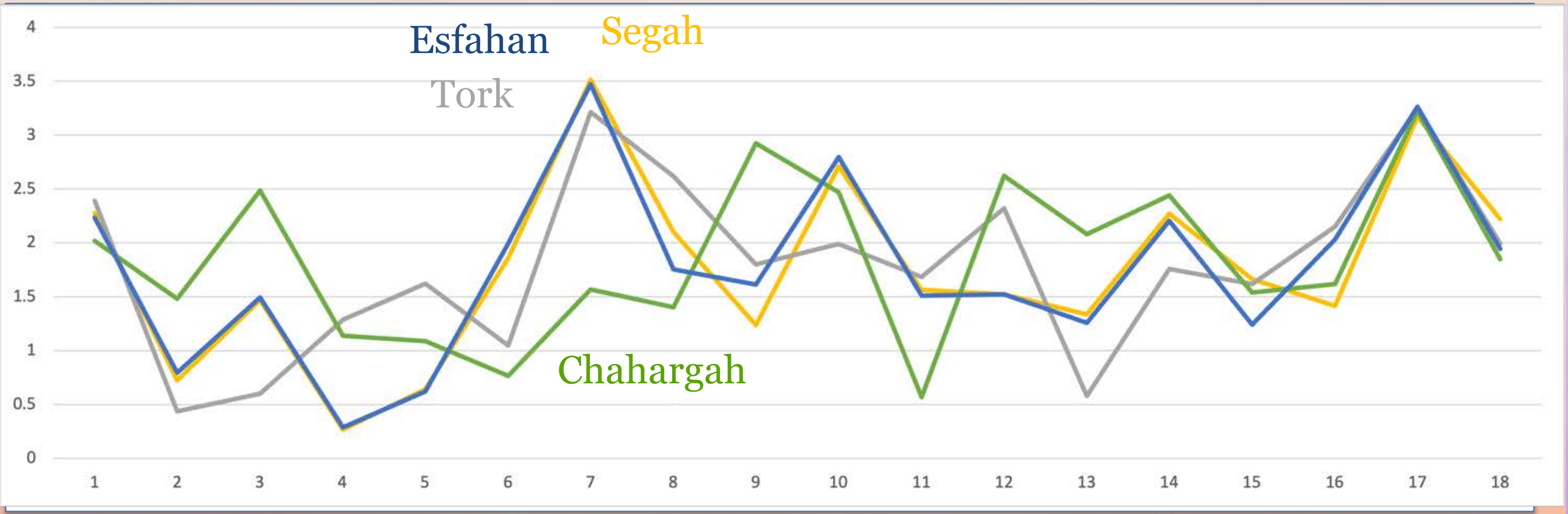
Spectra for tetrachords

Tetrachords show similar pattern in $|f_5|$, $|f_7|$, and $|f_{12}|$

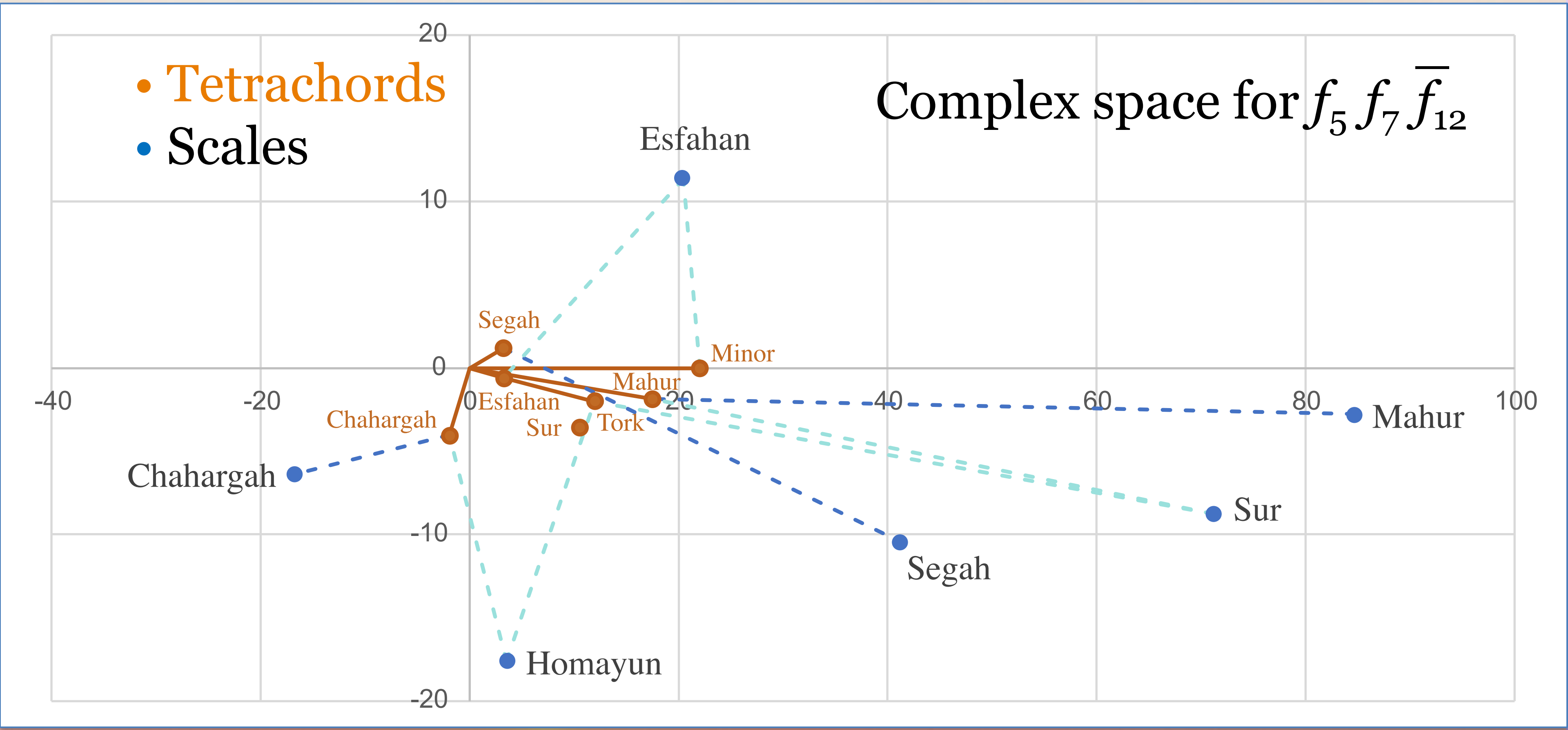


Spectra for tetrachords

Tetrachords show similar pattern in $|f_5|$, $|f_7|$, and $|f_{12}|$

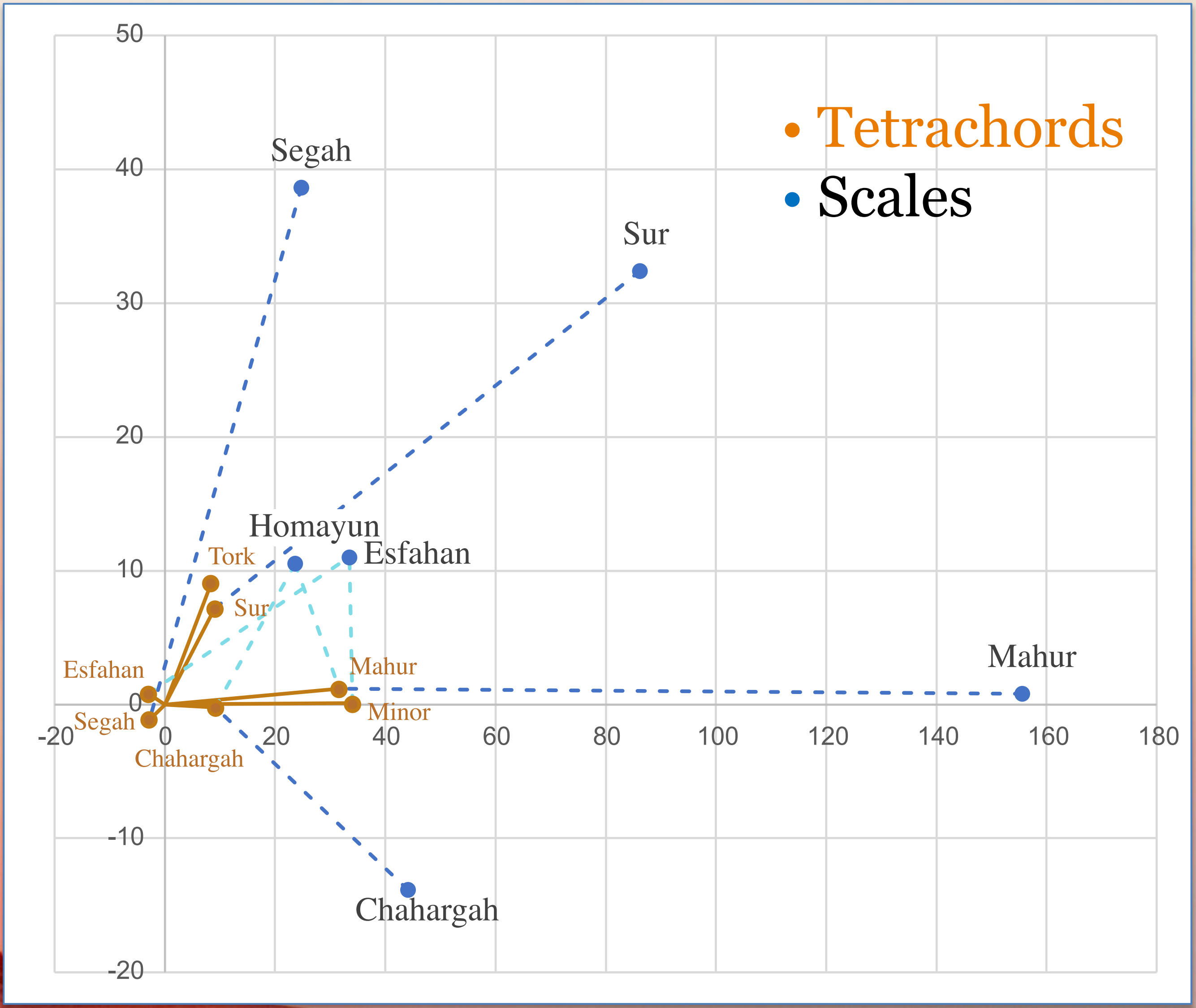


Coefficient Product Space

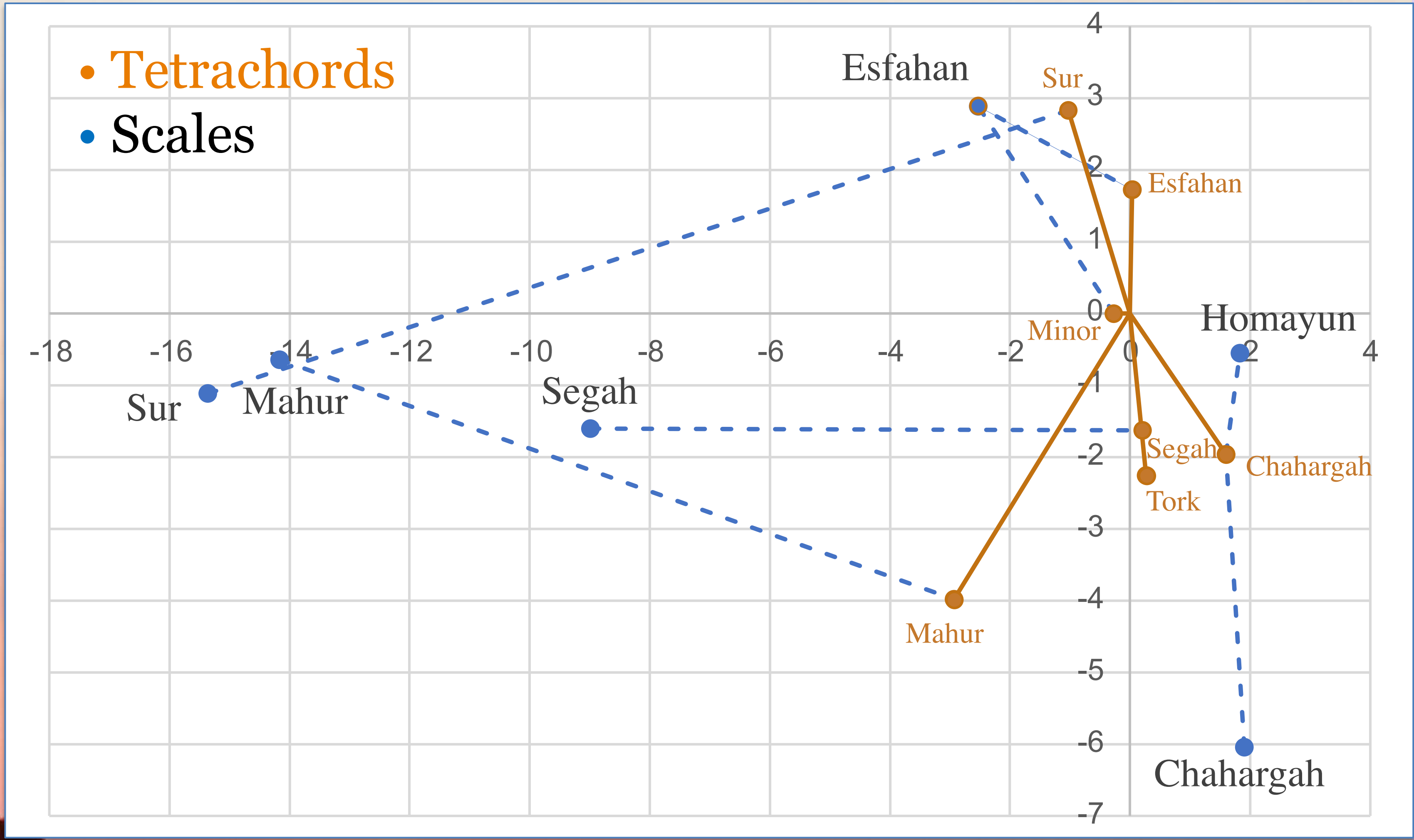


Coefficient Product Space

Complex space for $f_5 f_{12} \bar{f}_{17}$



Coefficient Product Space



Complex space for $f_2 f_5 \overline{f_7}$

Conclusions

- Interval categories defined by even divisions (heptatonicity, triadicity, chromaticity) have many theoretical uses.
- Tuning systems can be defined by multiple interval categorizations.
  —These have the advantage of setting limits of intonation flexibility, rather than idealized "correct" interval sizes.
- Real tuning systems relate to the WF sequence of the acoustical fifth.
- Two kinds of transposition-invariant features of collections:
  —Spectrum
  —Coefficient product phases

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