Non-spectral Transposition-Invariant Information in Pitch-Class Sets and Distributions

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Preliminaries: DFT on PC-vectors

A = {0, 3, 7}

s = $\mathbf{1}_A$ = {1,0,0,1,0,0,0,1,0,0,0,0}

$$\mathcal{F}_A : t \mapsto \widehat{a}_t = \sum_{x \in A} e^{-2i\pi x t/12}$$

The DFT converts pitch-class weights to complex-valued periodic functions (equal divisions of the octave)

$\mathcal{F}_A$(s) =

{3, 0.13 – 0.5*i*, 0.5 – 0.87*i*, 1 + 2*i*, 1.5 – 0.87*i*, 1.87 – 0.5*i*,
–1, 1.87 + 0.5*i*, 1.5 + 0.87*i*, 1 – 2*i*, 0.5 + 0.87*i*, 0.13 + 0.5*i*}

Basic Properties of DFT

Fourier coefficients determine the original distribution.

The <u>magnitude</u> of the Fourier coefficients is invariant by transposition or inversion.
They determine the <u>intervallic content</u> of pc-set *A*.

$$\widehat{a}_{12-t} = \overline{\widehat{a}_t}$$

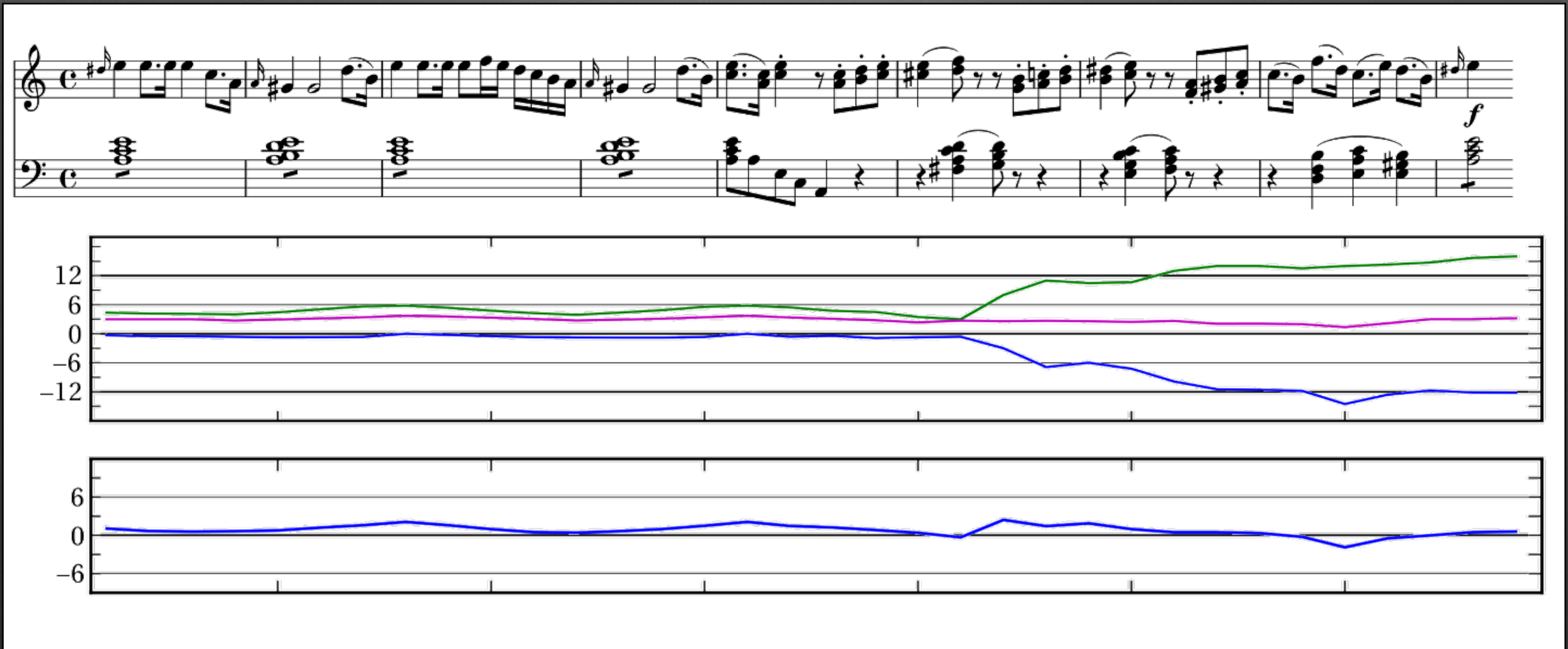
The <u>phase</u> of a Fourier coefficient is defined by

$$\widehat{a}_t = |\widehat{a}_t| e^{i\varphi_t} = |\widehat{a}_t| e^{i\Phi_t \pi/6}$$

(Φ_t is defined modulo 12)

Under transposition of *A*, the phase is translated.

Motivation for studying coefficient products
Example: Mozart K.310 theme (Yust 2016)



Above: ϕ_2, ϕ_3, ϕ_5

Below: $\phi_2 + \phi_3 - \phi_5$
(equivalently $\phi_2 + \phi_3 + \phi_7$)

Motivation for studying coefficient products
Example: Chopin Mazurka Op. 33/2 (Yust 2016)

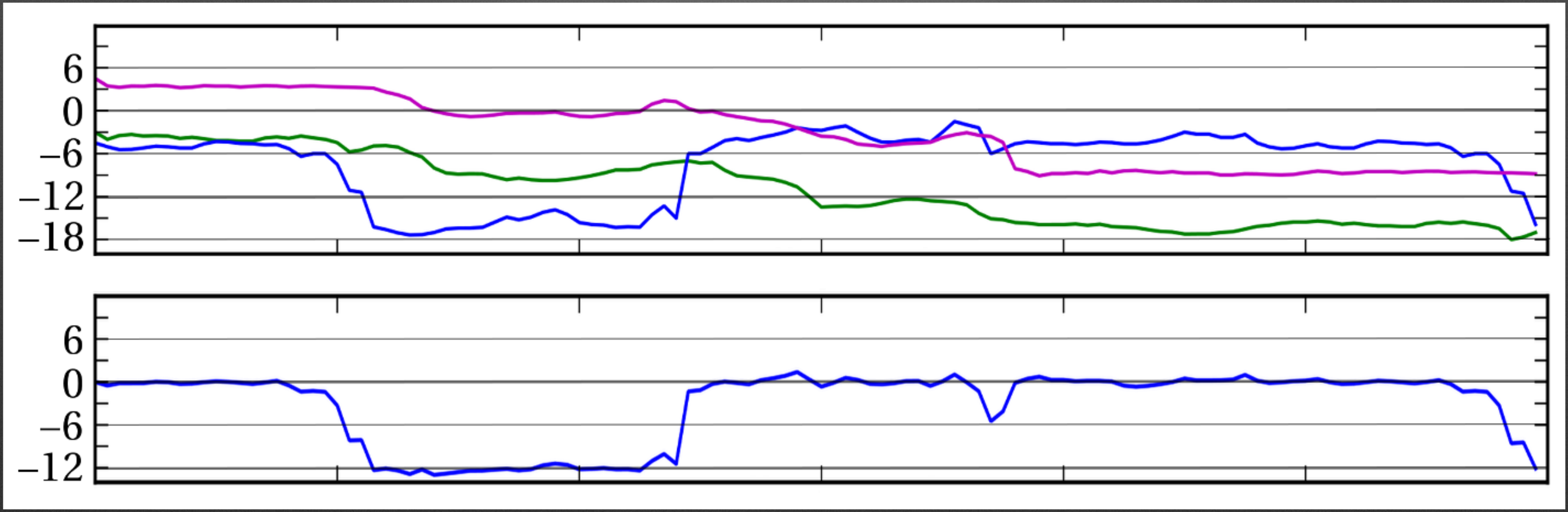
Period 1 in D maj.

Period 2 in B♭ maj.

Period 3: B♭ min. to D♭ maj.

Period 4: F♯ min. to A maj.

Period 5: D maj.



Above: ϕ_2, ϕ_3, ϕ_5

Below: $\phi_2 + \phi_3 - \phi_5$

Outline

- General properties of coefficient products
  - Transpositional invariance
  - Inversion and complementation
  - Sums and generated collection
- Coefficient products in tonal music
  - Corpus data
  - Approximation to clipping function, macroharmony
  - Phase space ($\hat{a}_3$, $\hat{a}_5$) and coefficient products

($\hat{a}_2 \hat{a}_3 \hat{a}_7$ and $\hat{a}_3 \hat{a}_4 \hat{a}_5$)

- Analytical example: Takemitsu *Air*

Fourier coefficients products

We consider regular products of Fourier coefficients, those whose indexes sum up to 12. Ex:

$$\hat{a}_2\hat{a}_3\hat{a}_7, \hat{a}_3\hat{a}_4\hat{a}_5, \hat{a}_4\hat{a}_8 \dots$$

Such a product is real positive when the sum of corresponding phases is zero.

This is true inconditionally when *A* is a single pitch-class, or with a regular product of two coefficients.

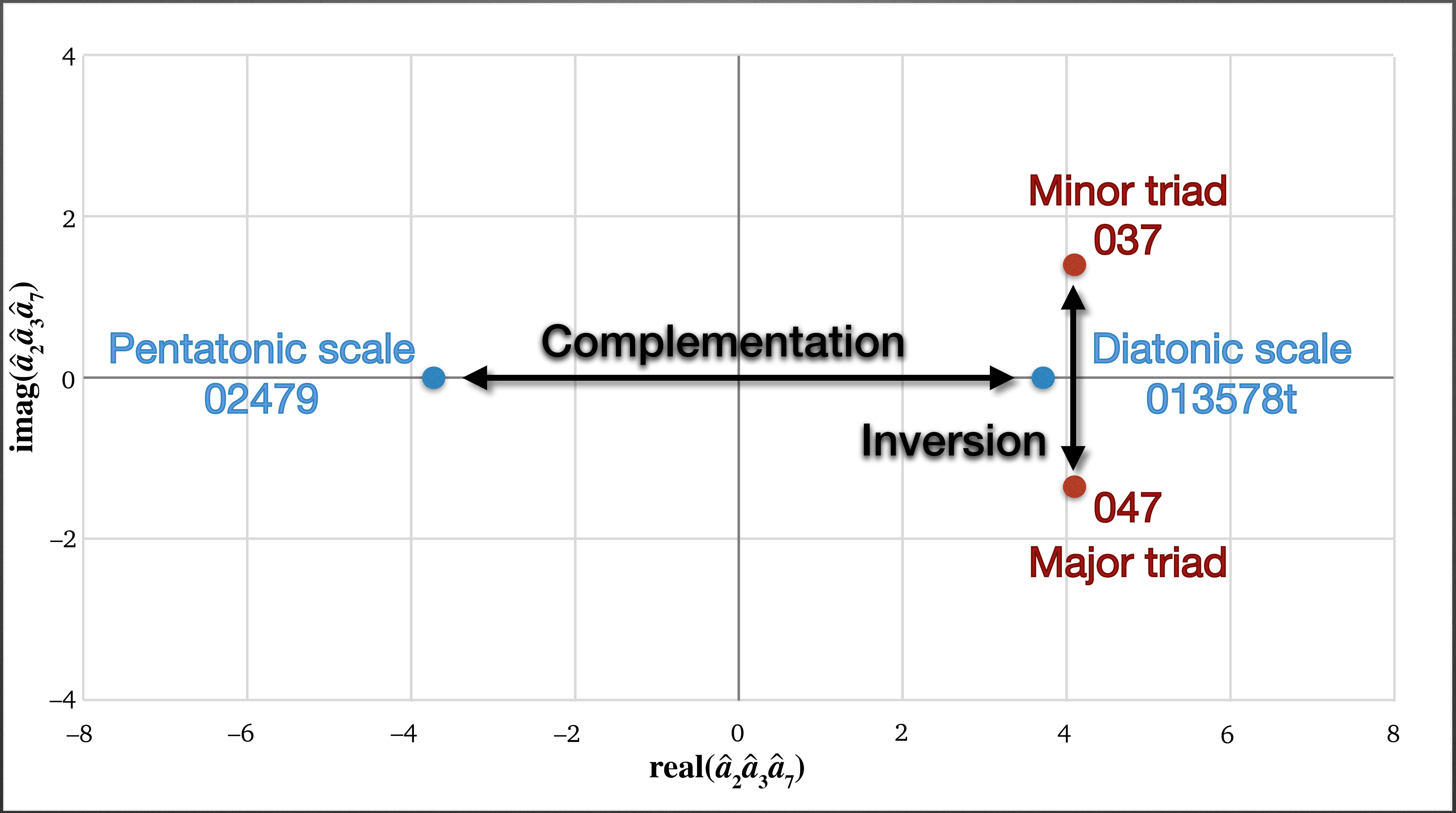
A positive product is called coherent, a real product is called aligned.

General properties of coefficients products

- Invariances
  - ✳ All (regular) coefficient products are invariant under transposition
  - ✳ Under inversion, the real part of a coefficient product is invariant, the imaginary part is negated
- Regular / aligned
  - ✳ If *A, B* are disjoint, aligned, and homometric then their union is aligned. (ex: all dyads)
  - ✳ A *generated* scale is always aligned.

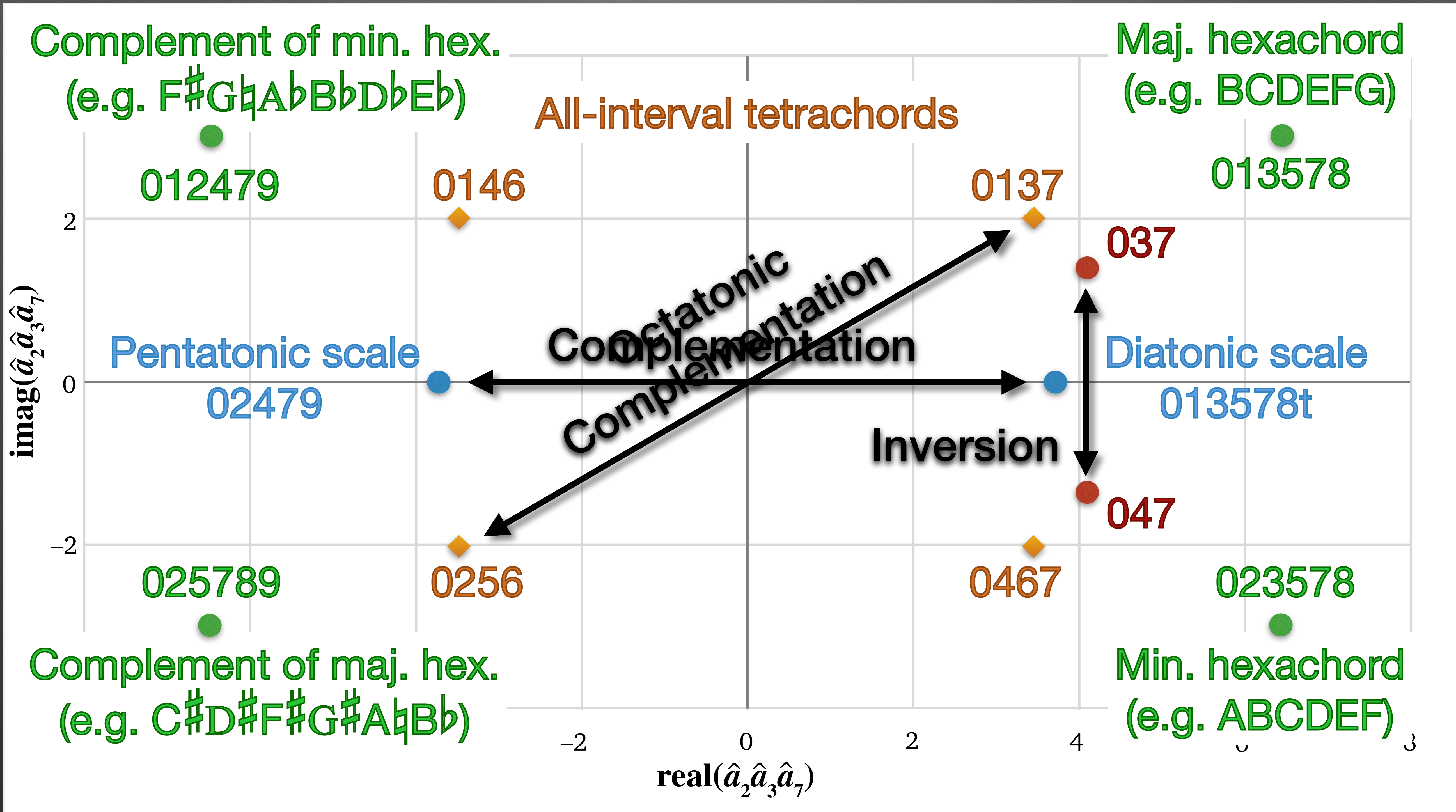
For indexes 2,3, and 7 the diatonic is coherent, the pentatonic aligned.

Example: $\hat{a}_2 \hat{a}_3 \hat{a}_7$ for tonal sets and complements



Complementation reflects over the origin.
Inversion reflects over the real axis.

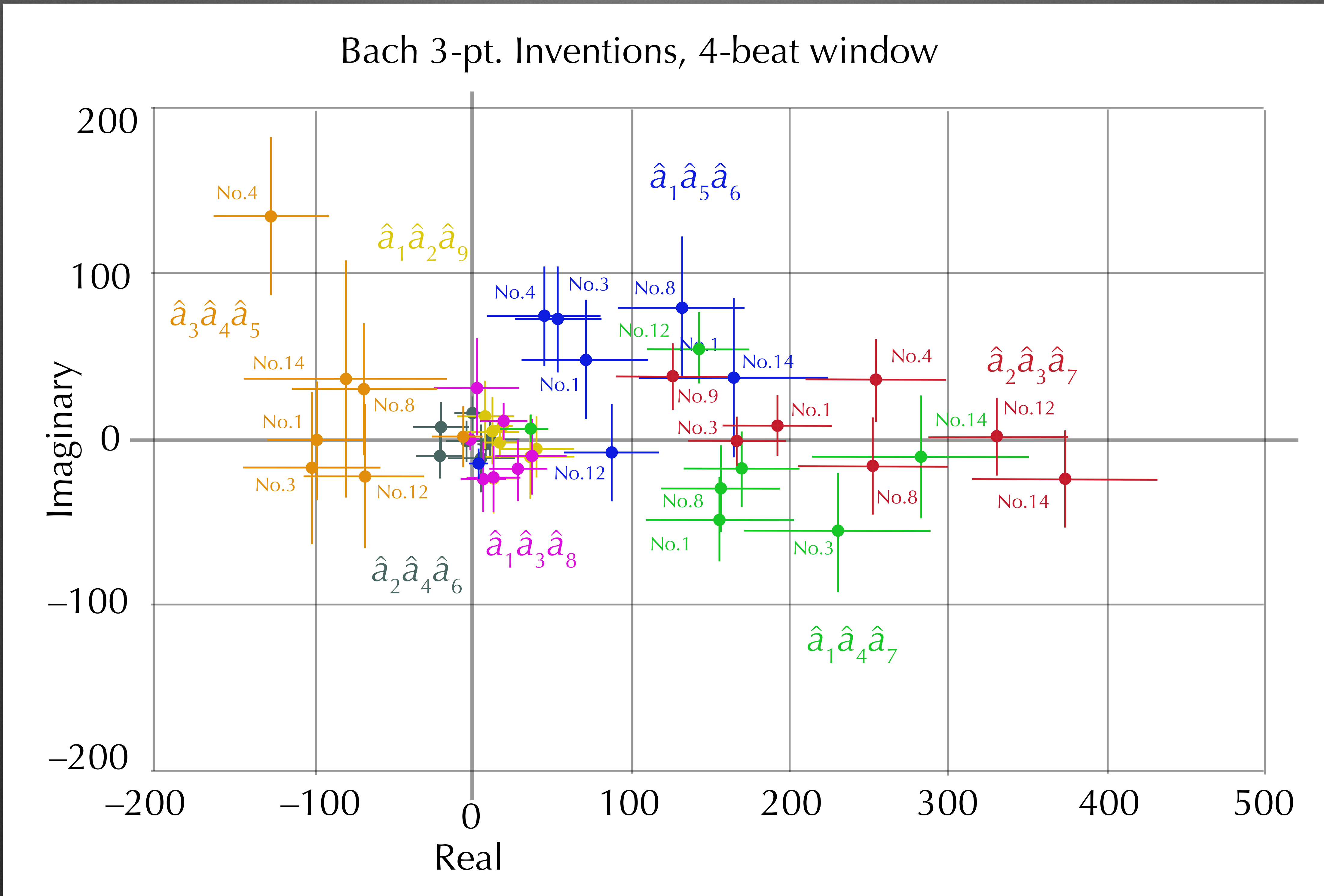
Example: $\hat{a}_2 \hat{a}_3 \hat{a}_7$ for tonal sets and complements



Distance from the origin is spectral (fixed for given interval content)

Phase can vary for homometric sets (inv., Z-rel.) but is still T-invariant

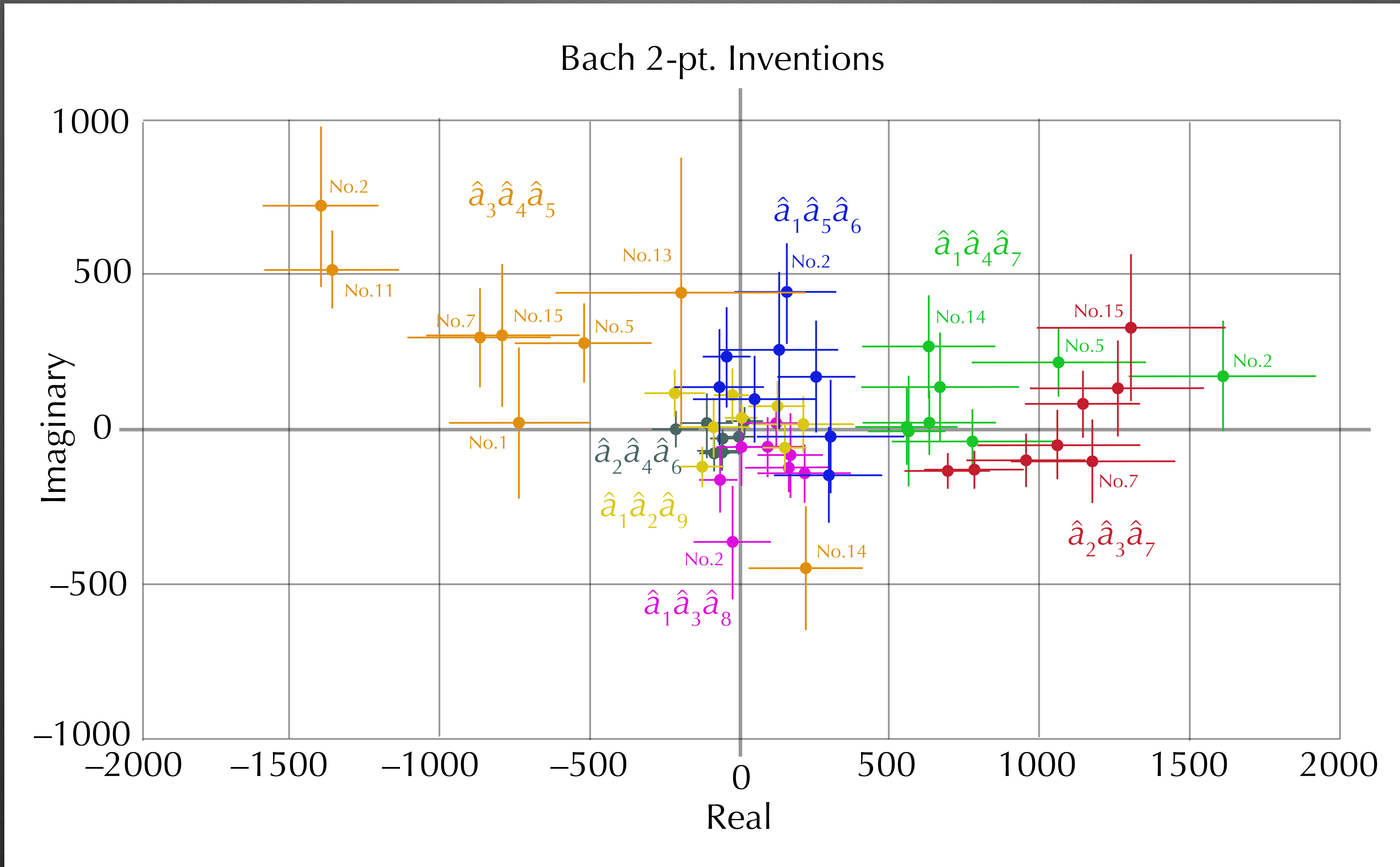
Coefficient products of tonal pitch-class counts



$\hat{a}_2\,\hat{a}_3\,\hat{a}_7$ is consistently real positive,

$\hat{a}_3\,\hat{a}_4\,\hat{a}_5$ is usually negative.

Coefficient products of tonal pitch-class counts



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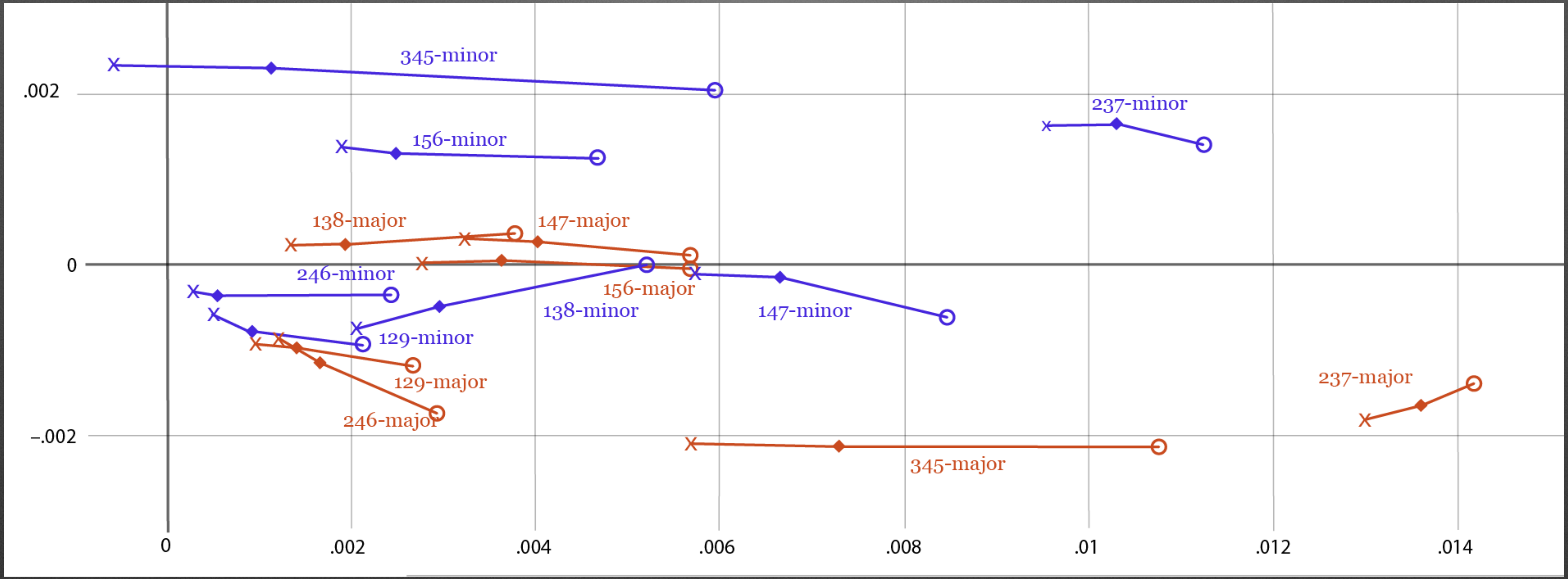
Coefficient products of tonal pitch-class counts

Averages over a large corpus of minuets by Handel, Bach, Haydn, and Mozart (normalized)

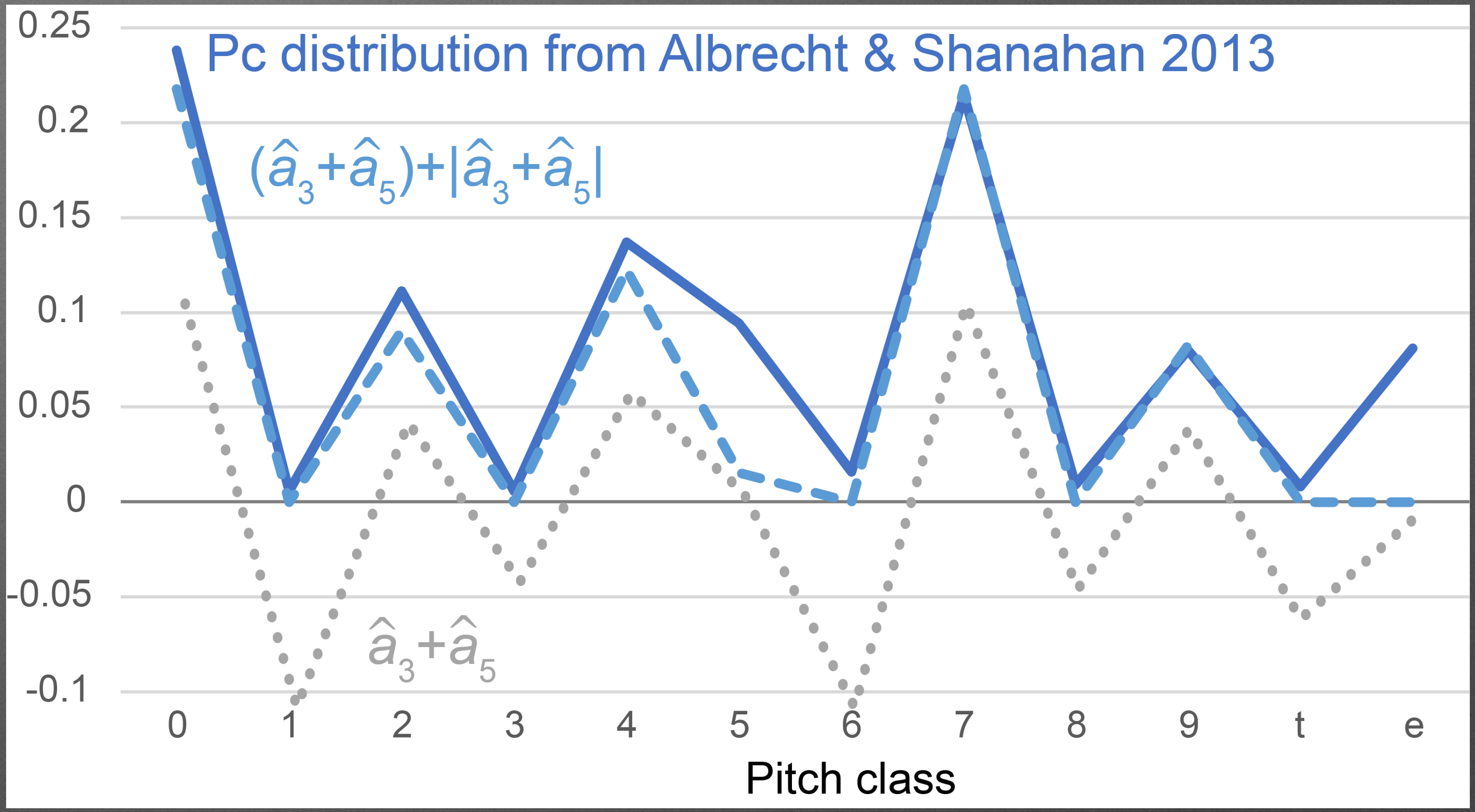


✕: 3-beat window, ♦: 6-beat window, ○: 9-beat window

Negative real values are uncommon, imaginary values are close to 0.

Reconstruction of a tonal pc distribution

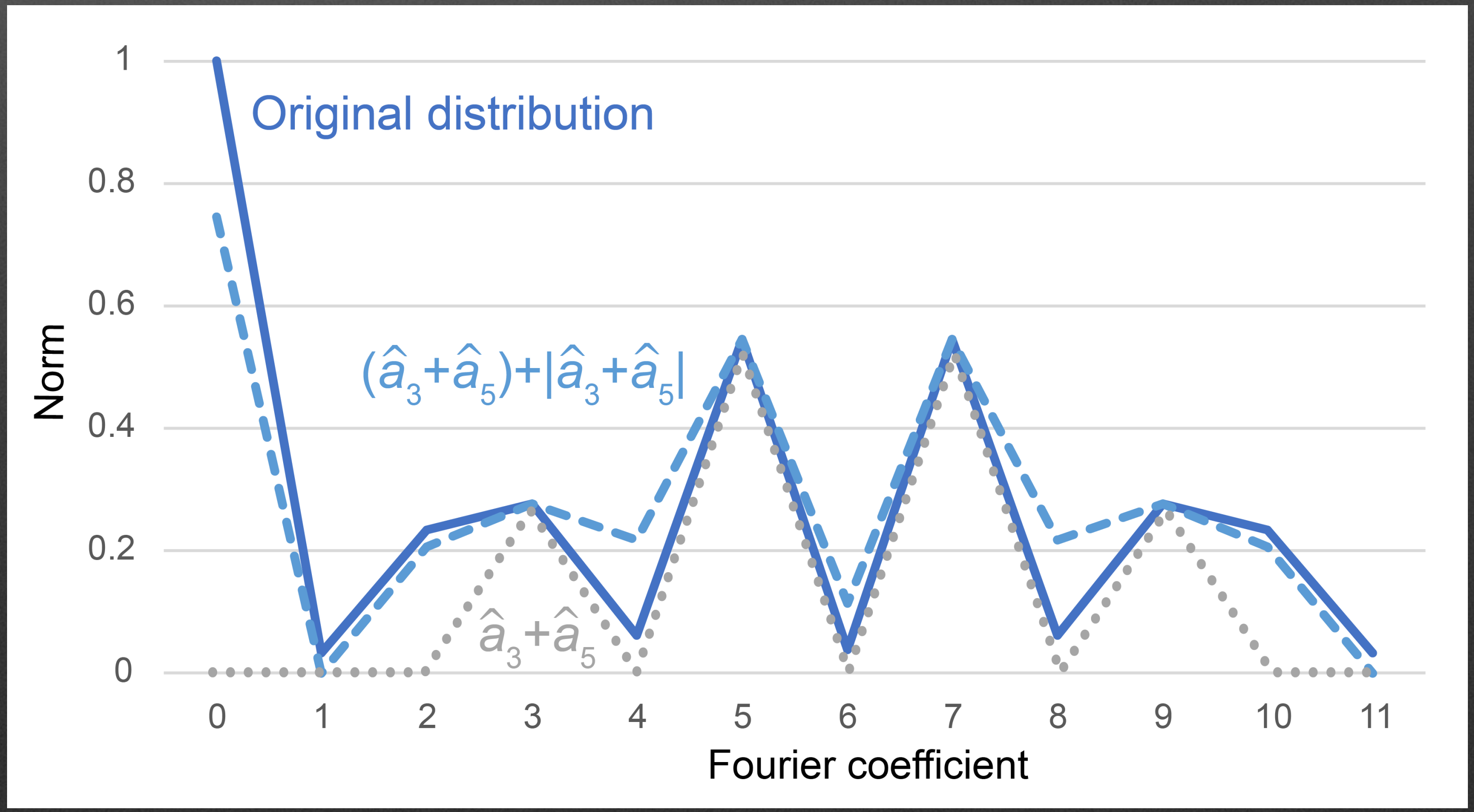
Pc vector



Spectrum

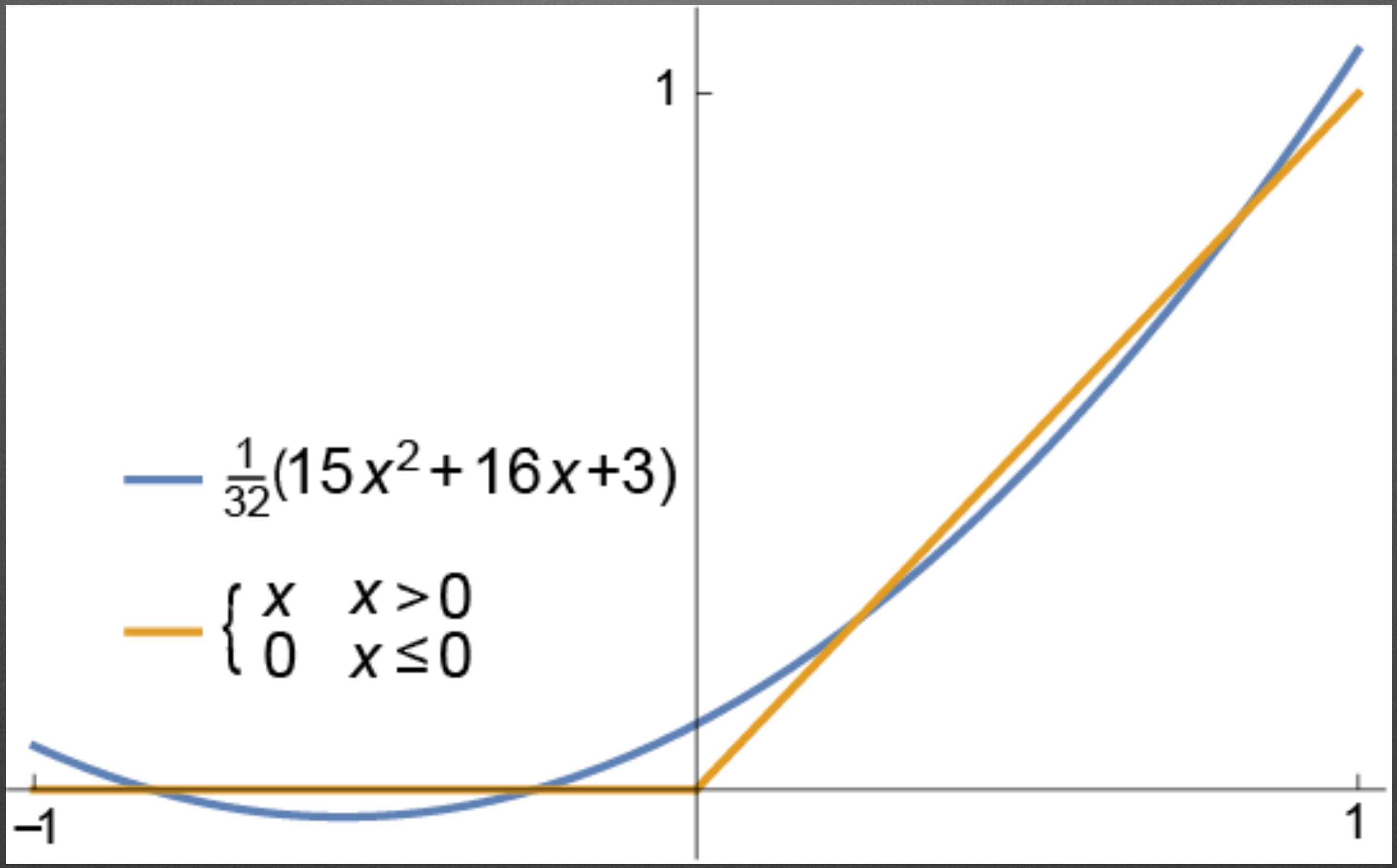
(1) Isolate the two largest coefficients ($\hat{a}_3$ and $\hat{a}_5$)

(2) Clip



The clipping function mimics a *limited macroharmony* property.

There is a fairly good quadratic approximation to this:

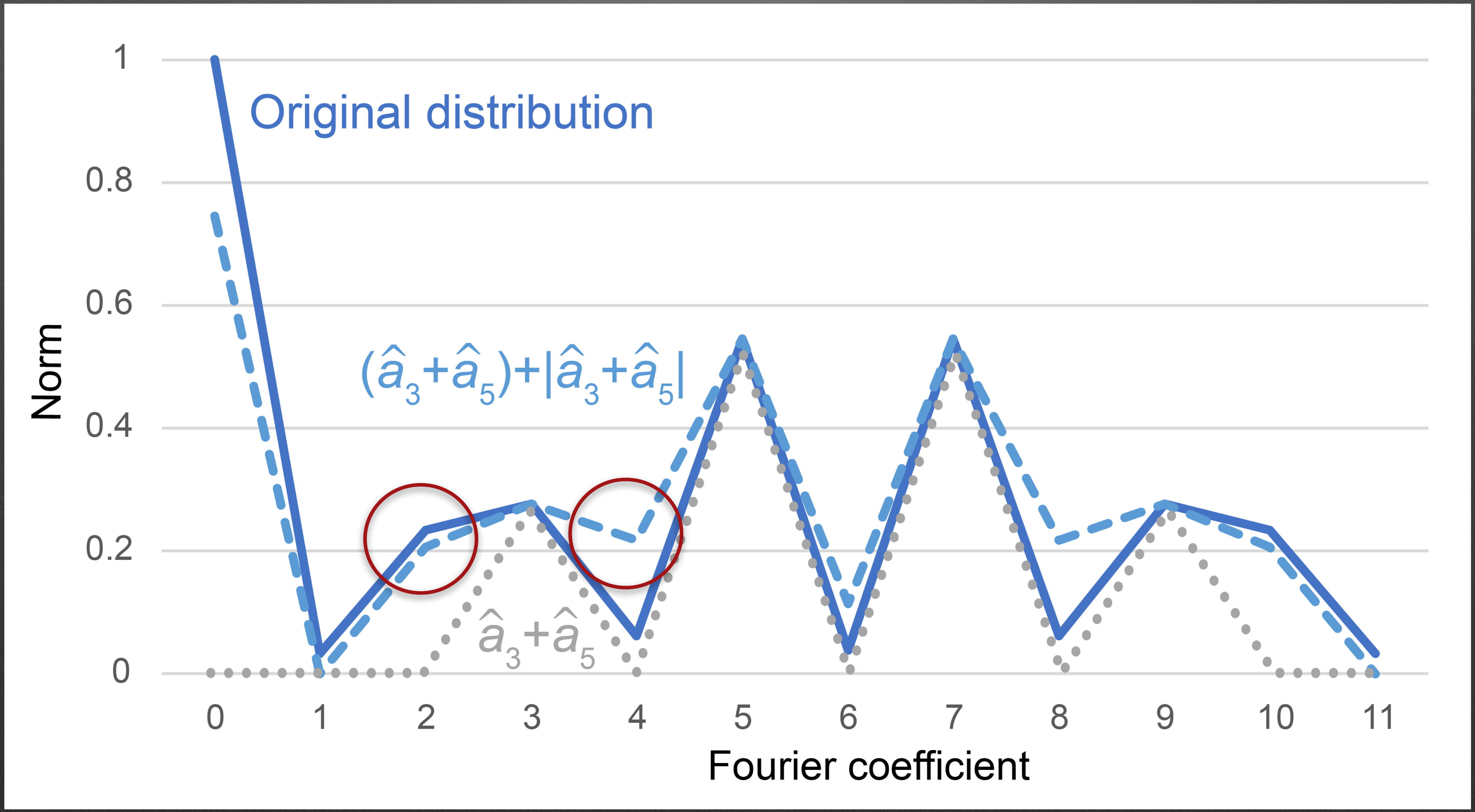


The quadratic term contributes products of coefficients of the original DFT (replace *x* by $\hat{a}_1$, $\hat{a}_2$ . . . .)

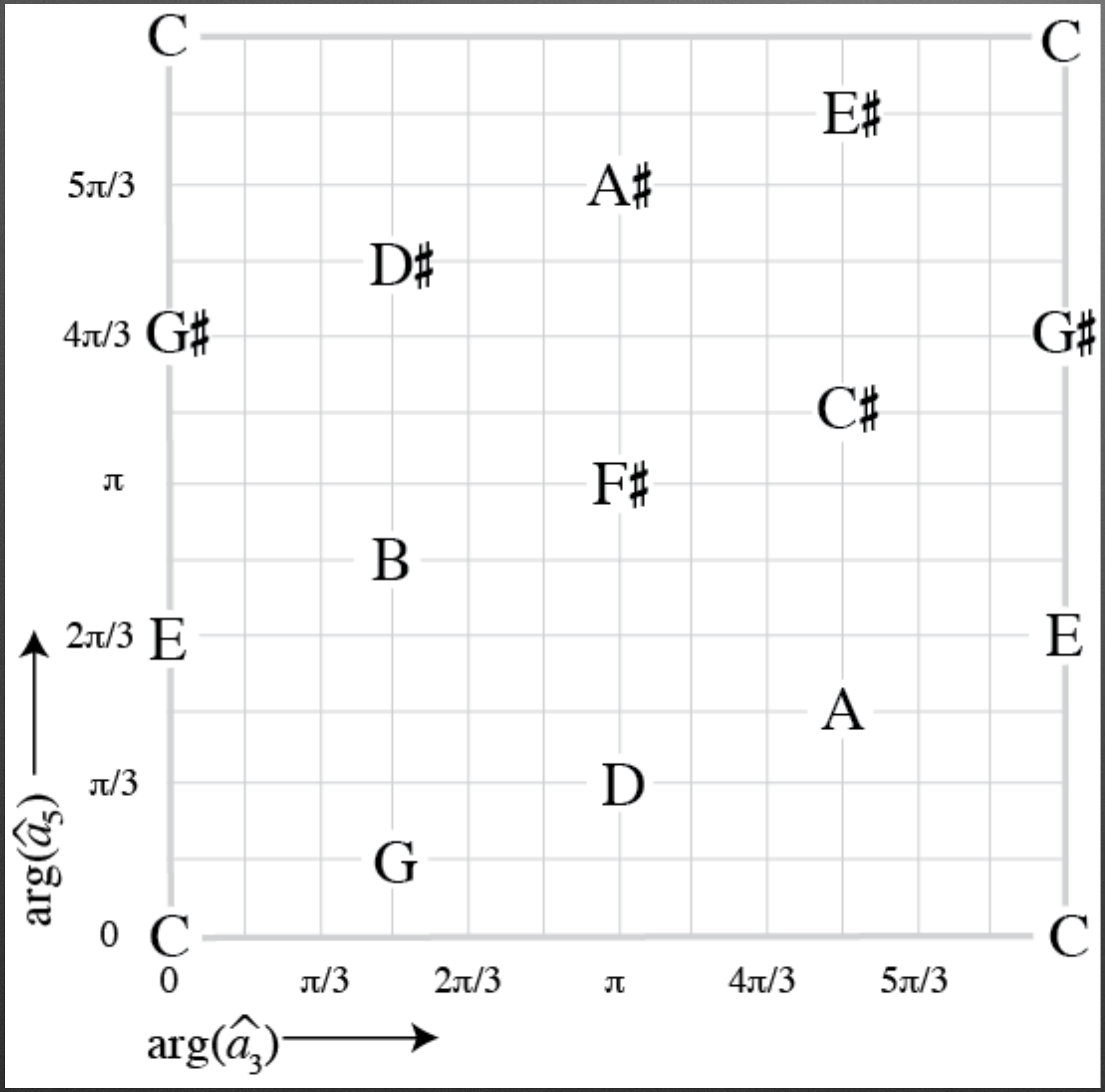
This leads to coherent coefficient products in the resulting pc vector.

Starting from a simple spectrum (built from two coefficients) the clipping function adds coefficients that make regular products with the original two:

5 – 3 = 2, 7 – 3 = 4, 7 – 5 = 2



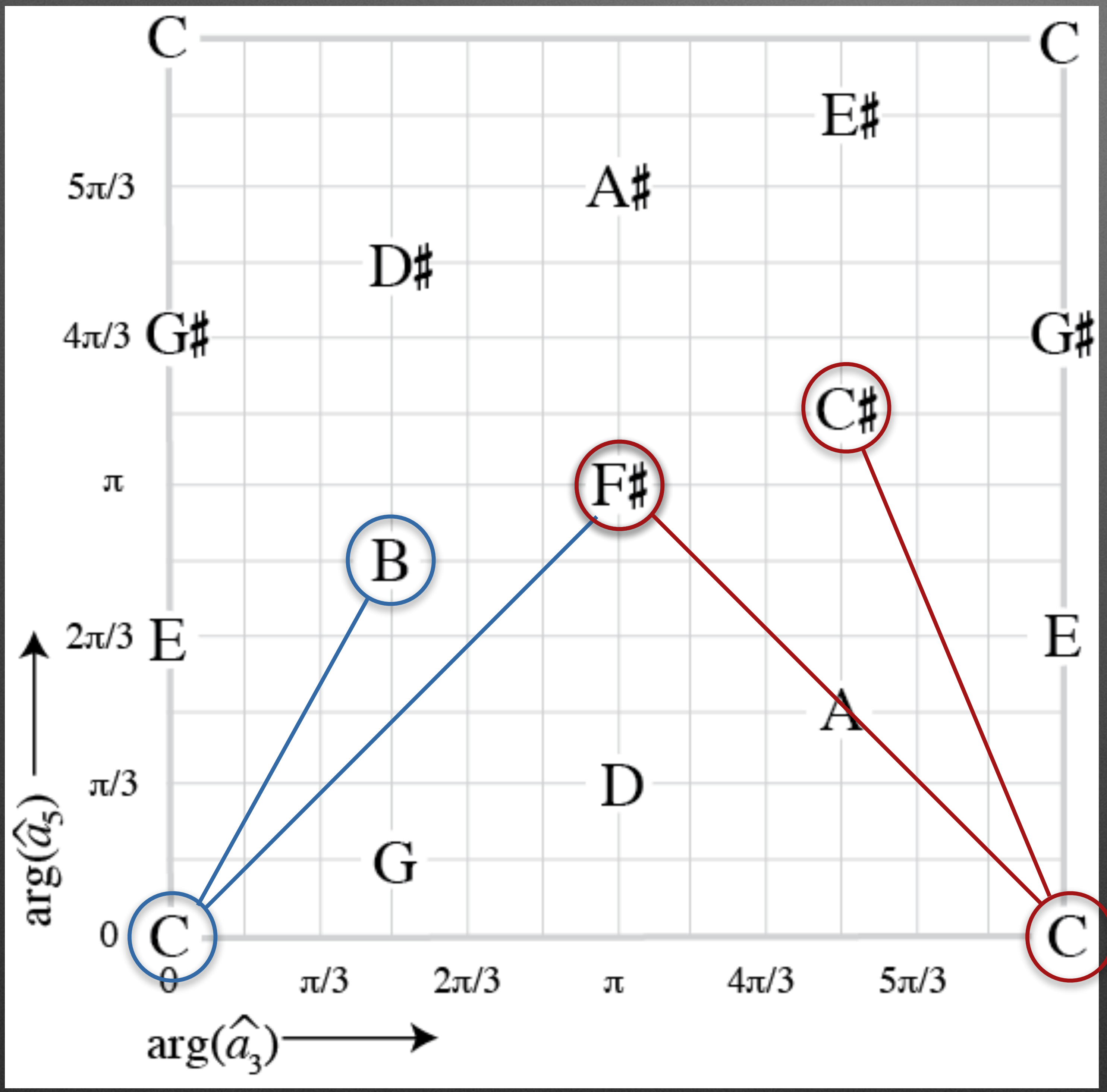
Coefficient products and phase space orientation



A toroidal space on ϕ_3 and ϕ_5 is a good model for tonal keys and harmony (triadic/ diatonic space)

These coefficients are involved in two products,
$\hat{a}_2\,\hat{a}_3\,\hat{a}_7$ and $\hat{a}_3\,\hat{a}_4\,\hat{a}_5$

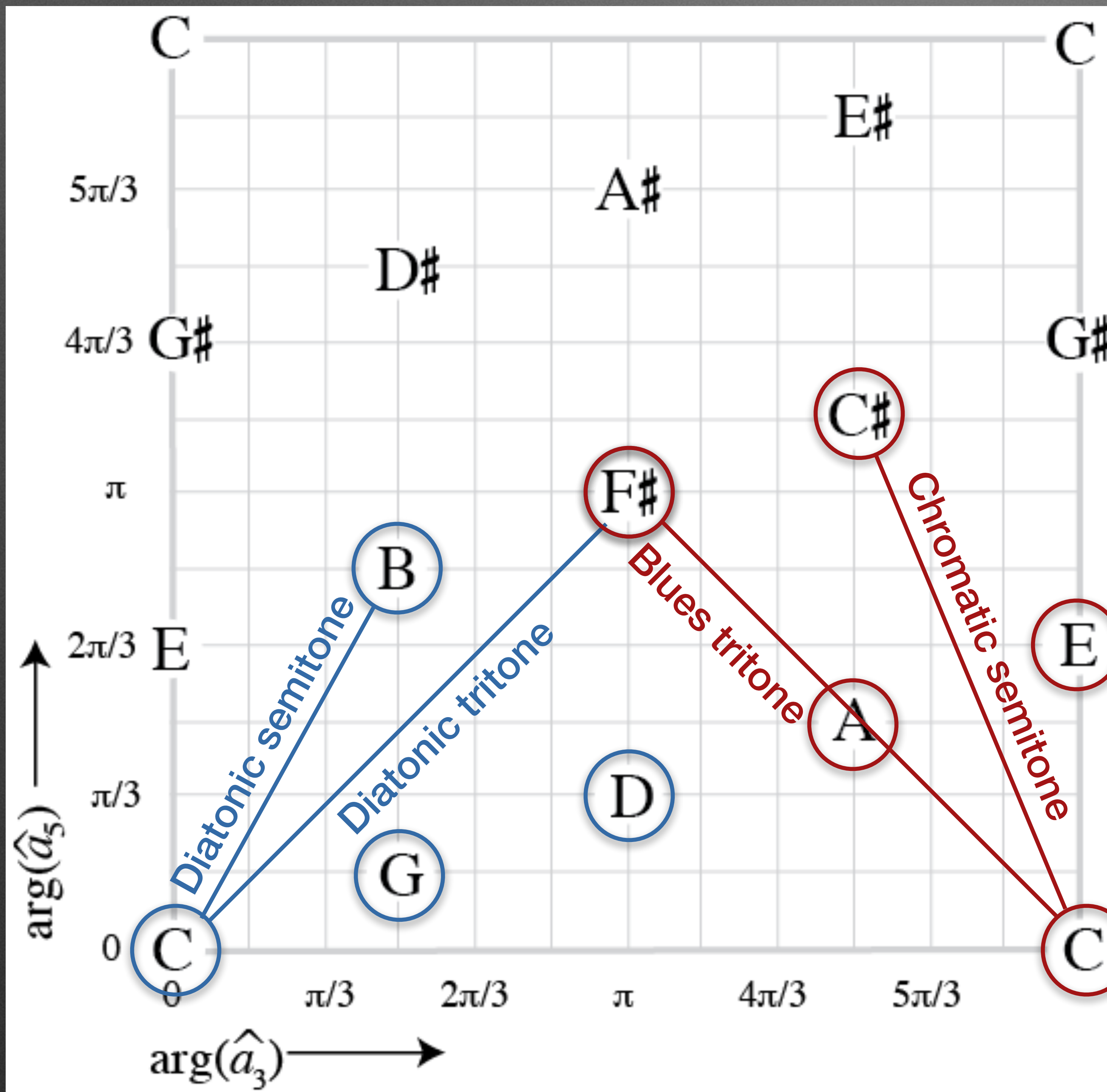
Coefficient products and phase space orientation



Orientation in the space relates to the real parts of $\hat{a}_2\,\hat{a}_3\,\hat{a}_7$ and $\hat{a}_3\,\hat{a}_4\,\hat{a}_5$

The same intervals can be represented by differently oriented vectors

Coefficient products and phase space orientation



Positive orientation corresponds to diatonic intervals,

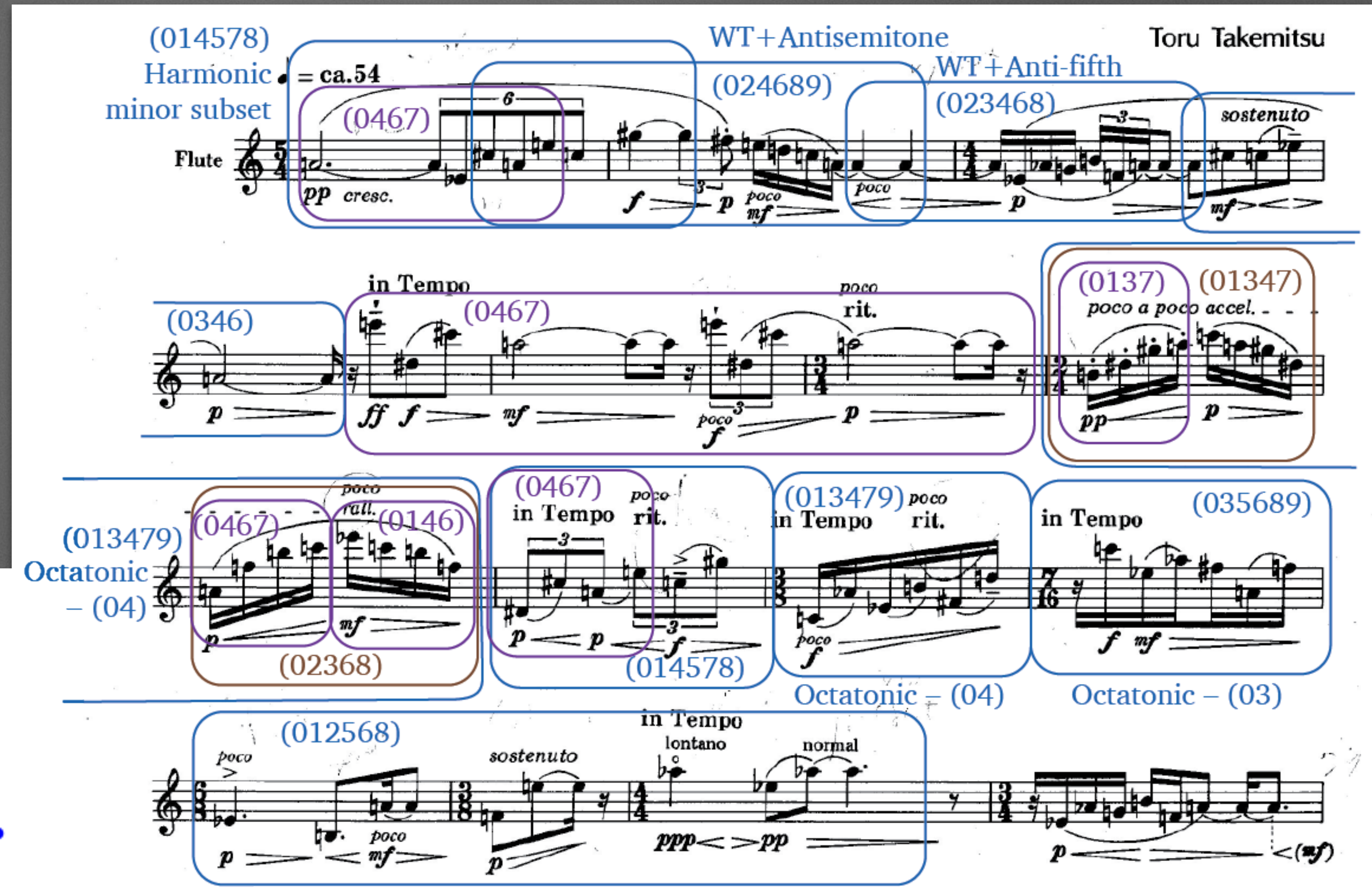
negative orientation to anti-diatonic intervals.

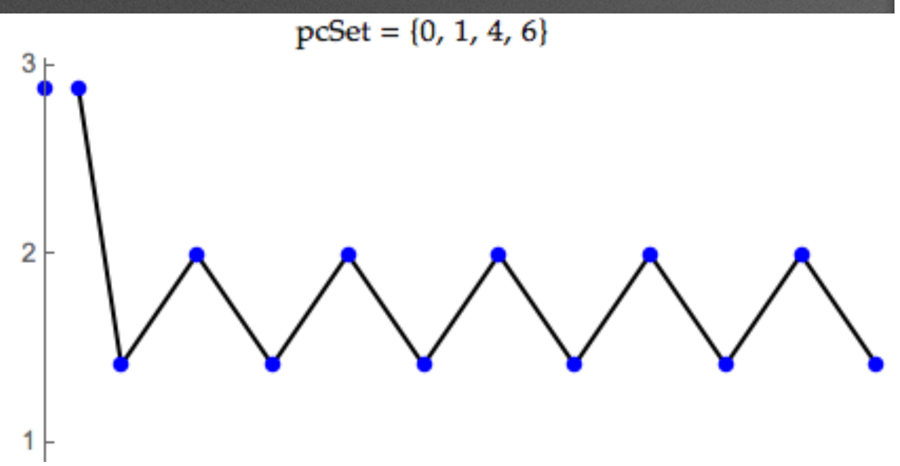
Context determines orientation.

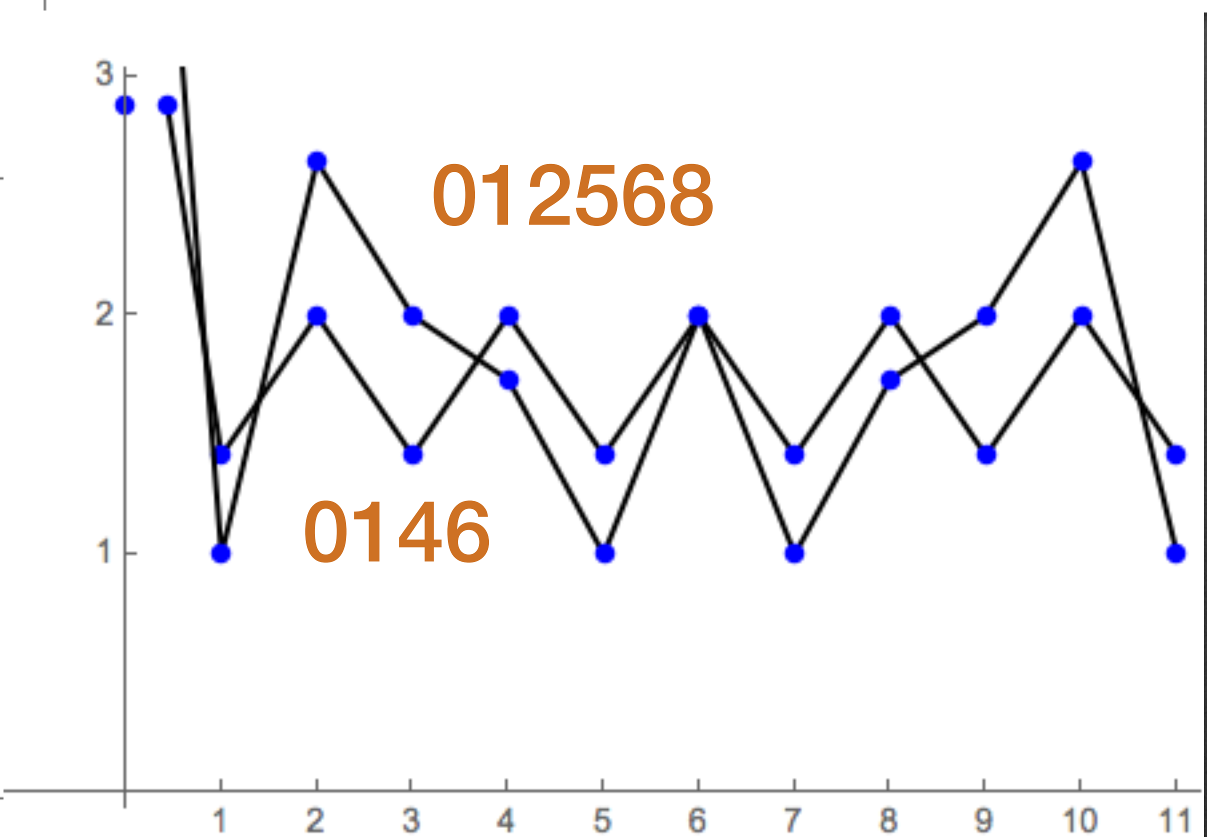
BCDF♯G has a positive real $\hat{a}_2 \hat{a}_3 \hat{a}_7$ and negative real $\hat{a}_3 \hat{a}_4 \hat{a}_5$.

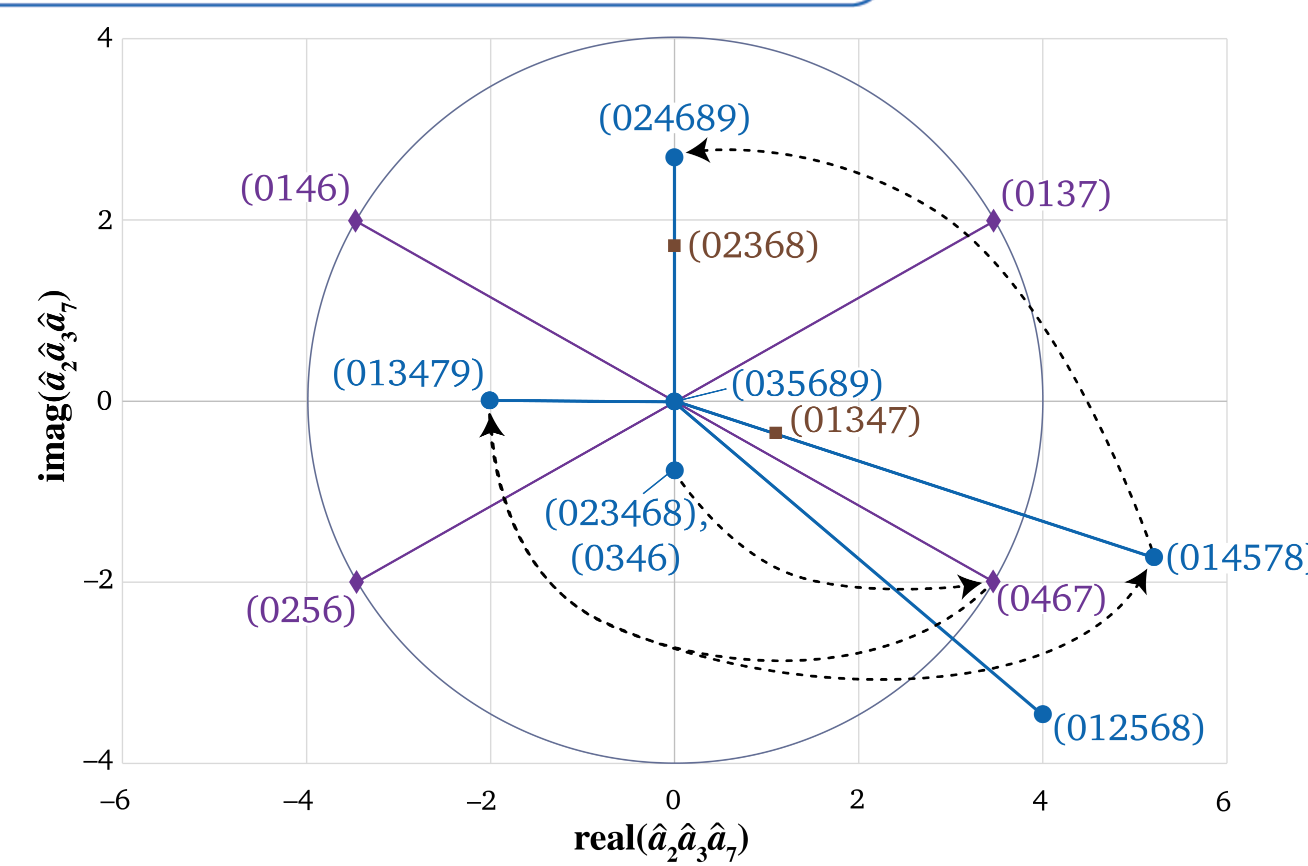
ACC♯EF♯ has a negative real $\hat{a}_2 \hat{a}_3 \hat{a}_7$ and positive real $\hat{a}_3 \hat{a}_4 \hat{a}_5$.

Example Analysis: Takemitsu *Air* and $\hat{a}_2 \hat{a}_3 \hat{a}_7$









Conclusions

- Regular coefficient products are *transposition invariant* but include non-spectral phase information. Therefore they distinguish complements, inversions, and Z-related sets.
- Coherent products are predicted by limited macroharmony, explaining the approximate linear dependence of ϕ_2 on ϕ_3 and ϕ_5 (but not the *in*dependence of ϕ_4!).
- $\phi_2 + \phi_3 - \phi_5$ captures an non-spectral aspect of the tonalness of pc sets.