# Non-Spectral Transposition-Invariant Information in Pitch-Class Sets and Distributions 

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#### Abstract

The spectral information of a pitch-class set or distribution relates to its interval content and what Ian Quinn calls its harmonic qualities, the magnitudes of a discrete Fourier transform of a pitch-class vector. The spectrum is invariant with respect to transposition and inversion, but the existence of Z-related sets, which have equivalent spectra but are not related by transposition or inversion, means that the spectrum is not a complete description of a set class. We show how to isolate transposition-invariant phase information using products of Fourier coefficients. We describe some of the mathematical features of these coefficient products and show how they encode aspects of tonality, and can be useful for analyzing non-tonal music with an example from Takemitsu's "Air" for solo flute.


## 1 Pitch-class set theory and homometry

Allen Forte [7] originally defined set-class equivalence as equivalence of interval vectors, but subsequently reconsidered, using transpositional and inversional equivalence instead [8]. Forte's original definition is known in mathematics as homometry, and, as Amiot [3] has shown, can also be defined as equivalence of spectra. The spectrum is obtained by taking the characteristic function of a pitch class set and considering just the sizes of the coefficients of its discrete Fourier transform (DFT). Ian Quinn [9] refers to the spectrum as a point in quality space. Transpositions and inversions are homometric, but not vice versa. Therefore Forte's original definition of set class was stronger than his later one. The difference between them consists of what he calls "Z-related" sets, sets that are homometric but not related by transposition or inversion. With the exception of hexachords, the Z-relation is somewhat rare for ordinary pitch-class sets, but we can identify many more examples if we consider pitch-class multisets [10] or real-valued characteristic functions, in which case the set of all distributions homometric to a given one is a multi-dimensional torus, the orbit of the so-called spectral units group [3, chapter 4].

While the spectrum therefore provides much of the important information about a pitch-class set, it is not a complete description. Since the DFT is a lossless transformation, that means that there is transposition-invariant information in
the phases of the DFT coefficients. In the first section we show that special coefficient products (specifically with coefficients whose indices sum to twelve) are transposition invariant, and therefore the phases of these include the desired non-spectral information. We then show the importance of these non-spectral set class properties for characterizing tonal sets, and analysis in two non-tonal contexts.

## 2 Products of DFT coefficients

### 2.1 Definitions

Recall that any complex number $z$ can be described by its magnitude $|z| \in \mathbf{R}_{+}$ and its phase $\arg (z) \in \mathbf{R} / 2 \pi \mathbf{Z}$ :

$$
\mathbf{C} \ni z=|z| e^{i \arg (z)}
$$

As mentioned in the preamble, it can be shown that homometry is exactly the equality of all Fourier coefficient magnitudes; Since these are invariant under transposition and inversion, it remains to consider the phases for non-homometry related information. Indeed, phase increases by a constant quantity under transposition and changes signum under inversion.

In the following, we normalize phase modulo 12 (or more generally, $n$, the cardinality of the chromatic aggregate) by setting

$$
\varphi_{k}=\arg \left(\hat{a}_{k}\right) \quad \Phi_{k}=\frac{12}{2 \pi} \varphi_{k}=\frac{6}{\pi} \arg \left(\hat{a}_{k}\right)
$$

where $\hat{a}_{k}=\sum_{x \in X} e^{-2 i \pi k x / 12}$ is the $k$ th Fourier coefficient of pitch-class set $X$.
It was noticed in [15] that many pitch-class sets in tonal context satisfy an improbable equation:

$$
\Phi_{5} \approx \Phi_{3}+\Phi_{2}
$$

This is an exact equality for diatonic scales, fifths, and several other prominent tonal collections. Notice however that the opposite equality ( $\Phi_{5} \approx-\Phi_{3}-\Phi_{2}$ ) yields for a pentatonic scale (thus allowing a way, Fourier-wise, to tell pentatonic and diatonic scales apart, although the magnitude of all their Fourier coefficients except the 0 th are equal). It yields, up to a small error, for major and minor triads.

This is an intriguing feature, since $\Phi$, a complex logarithm, is anything but a linear map; also a comprehensive computation shows that for most pc-sets, equation $(\sharp)$ is quite incorrect. ${ }^{1}$

For the sequel of this paper, we will rephrase it: since $\Phi_{5}=-\Phi_{7}$ (a general feature of Fourier coefficients of characteristic, or in general real-valued, functions) we can state instead

$$
\Phi_{7}+\Phi_{3}+\Phi_{2} \approx 0 \quad(\bmod 12)
$$

[^0]or by exponentiation
$$
\exp \left(i\left(\Phi_{7}+\Phi_{3}+\Phi_{2}\right) \pi / 6\right)=e^{i \Phi_{7} \pi / 6} e^{i \Phi_{3} \pi / 6} e^{i \Phi_{2} \pi / 6} \approx e^{0}=1
$$
or even better, multiplying by the magnitudes of the relevant Fourier coefficients to rebuild them anew, $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7} \approx\left|\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}\right|$, meaning
$$
\hat{a}_{2} \hat{a}_{3} \hat{a}_{7} \text { is (almost) real positive. }
$$

We can then state a general definition, where $2+3+7=12$ is replaced by an integer partition:

Definition 1 Let $n$ be the cardinality of the chromatic aggregate and $k_{1}, k_{2} \ldots k_{r}$ be an integer partition of $n$, i.e. the $k_{i}$ are positive integers ${ }^{2}$ Then $\hat{a}_{k_{1}} \hat{a}_{k_{2}} \ldots \hat{a}_{k_{r}}$ is a (regular) coefficient product. ${ }^{3}$

It is coherent for a given pitch-class if its value for that set is real positive, approximately coherent if it is close to real positive. For short, if the context is clear we will say that a pc-set is coherent if its (regular) product is coherent.

As will be seen later on, the more general case of real-valued regular products (positive or negative) is notable. We will call such a product aligned.

Without spoilers, with partition $2+3+7=12$ we have coherent products for all single notes, dyads, diatonic scales, major sevenths; approximate coherence for major or minor triads; and aligned products for pentatonic scales.

A trivial but illuminating example of coherent product, for which we thank an anonymous reviewer, is any partition of the type $n=k+(n-k)$, since for any pc-set we get $a_{k} a_{n-k}=\left|a_{k}\right|^{2} \geq 0$. In this case, Proposition 3 retrieves that transposed or inverted pc-sets are homometric.

### 2.2 Features

Proposition 1 Singletons are coherent for all regular products.
Proof. Let $a \in \mathbf{Z}_{n}$ be a pitch-class. Then we get for the $k$ th Fourier coefficient of $A=\{a\} \hat{a}_{k}=e^{-2 i k \pi a / n}$ hence

$$
\hat{a}_{k_{1}} \hat{a}_{k_{2}} \cdots=e^{-2 i k_{1} \pi a / n} e^{-2 i k_{2} \pi a / n} \cdots=e^{-2 i\left(k_{1}+k_{2}+\ldots\right) \pi a / n}=e^{-2 i n \pi a / n}=1 .
$$

As we will see this a special case of Prop. 3, $A$ being a transposition of $\{0\}$ for which any product is trivially regular.

Lemma 1 The argument of the sum (or mean) of two complex numbers with equal magnitude is the mean value of their arguments.

[^1]
## Remark.

There is a catch here. Since arguments are defined modulo a whole circle ( $2 \pi, 12$, or $n$, depending on normalization), half-arguments are defined modulo half the circle ( $\pi, 6$, a tritone).

However, out of these two opposite directions, the appropriate one is the mean value which directs the interior of the angles between the two complex numbers, see Fig. 1. In other words, both complex vectors and their sum/mean must lie in the same half-plane.

Proposition 2 If $A, B$ are disjoint and homometric, and coherent with respect to some coefficient product, then $A \cup B$ is aligned with respect to that product.

Proof. Let $\hat{a}_{k}=\left|a_{k}\right| e^{i \varphi_{k}}$ be the $k$ th Fourier coefficient for $A$ and similarly $\hat{b}_{k}=$ $\left|\hat{b}_{k}\right| e^{i \psi_{k}}$ for $B$. Then since $A, B$ are homometric, $\left|\hat{a}_{k}\right|=\left|\hat{b}_{k}\right|$. Hence for $C=A \cup B$, one gets

$$
\hat{c}_{k}=\hat{a}_{k}+\hat{b}_{k}=\left|\hat{a}_{k}\right|\left(e^{i \varphi_{k}}+e^{i \psi_{k}}\right)=\left|\hat{a}_{k}\right| \cos \frac{\varphi_{k}-\psi_{k}}{2} e^{i \frac{\varphi_{k}+\psi_{k}}{2}}
$$

hence the sum of the phases (taken modulo $2 \pi$ ) is

$$
\sum \frac{\varphi_{k}+\psi_{k}}{2}=\frac{1}{2}\left(\sum \varphi_{k}+\sum \psi_{k}\right)=0 \quad \bmod \pi
$$



Fig. 1. Sum and phase of two complex numbers with the same length.

For instance for the diatonic partition $2+3+7=12$, reunions of homometric dyads are coherent (though most products are 0). Counter-examples would be for instance the chromatic dyad (01) for partition $3+4+5=12$ : in this case

$$
\hat{a}_{3} \hat{a}_{4} \hat{a}_{5}=1-\sqrt{3}<0 .
$$

and the product is aligned, but not coherent.
Both properties (coherent / aligned) are invariant by transposition and inversion. More precisely,

Proposition 3 Coefficient products are transposition-invariant.
Inversion negates the imaginary part of a coefficient product, while the real part is inversion-invariant.

Proof. For transposition let us have two pc-sets in $\mathbf{Z}_{n}$ such that $B=A+\tau$. Denoting their Fourier coefficients by $\hat{a}_{k}, \hat{b}_{k}$ we derive

$$
\hat{b}_{k}=\hat{a}_{k} e^{-2 i \pi k \tau / n}
$$

and hence

$$
\begin{aligned}
\hat{b}_{k_{1}} \hat{b}_{k_{2}} \ldots \hat{b}_{k_{r}}=\hat{a}_{k_{1}} \hat{a}_{k_{2}} \ldots e^{-2 i \pi k_{1} \tau / n} & e^{-2 i \pi k_{2} \tau / n} \ldots \\
& =\hat{a}_{k_{1}} \hat{a}_{k_{2}} \ldots e^{-2 i \pi\left(k_{1}+k_{2}+\ldots\right) \tau / n} \\
& =\hat{a}_{k_{1}} \hat{a}_{k_{2}} \ldots e^{-2 i \pi n \tau / n}=\hat{a}_{k_{1}} \hat{a}_{k_{2}} \ldots \hat{a}_{k_{r}}
\end{aligned}
$$

whenever $k_{1}+k_{2}+\cdots=n$.
The inversion $A \mapsto-A$ just changes the signs of all phases, conjugating all Fourier coefficients, which leaves the real part invariant and inverts the imaginary part. For other inversions $A \mapsto \tau-A$, notice it is the previous inversion combined with a transposition.

It follows easily that any inversionally symmetric pc-set has aligned product. We can be more specific in the following case:

Proposition 4 For generated scales a regular product is aligned, the sign depending on a product of sines.

Proof. According to the last proposition we can assume that the generated scale begins on 0 :

$$
A=\{0, f, 2 f, 3 f, \ldots(d-1) f\} \quad \text { if the generator is } f \text { and the cardinality } d \text {. }
$$

Then we compute $\hat{a}_{k}=\sum_{j=0}^{d-1} e^{-2 i \pi k j f} / n$, a geometric sum:
$\hat{a}_{k}=\frac{e^{-2 i \pi d k f / n}-1}{e^{-2 i \pi k f / n}-1}=\frac{e^{-i \pi d k f / n}\left(e^{-i \pi d k f / n}-e^{+i \pi d k f / n}\right)}{e^{-i \pi k f / n}\left(e^{-i \pi k f / n}-e^{+i \pi k f / n}\right)}=e^{-i(d-1) k f \pi / n} \frac{\sin (d k f \pi / n)}{\sin (k f \pi / n)}$.
Hence, depending on the sign of the sines quotient, the phase is either $\varphi_{k}=-(d-1) f k \pi / n$ or $\varphi_{k}=-(d-1) f k \pi / n+\pi$, or in normalized format

$$
\Phi_{k}=-\frac{(d-1) f k}{2} \text { or }-\frac{(d-1) f k}{2}+\frac{n}{2} .
$$

Then for any partition $n=k_{1}+k_{2}+\ldots$, the sum of the $-\frac{(d-1) f k_{i}}{2}$ is a multiple of $n / 2$ and so is the sum of all phases, meaning that the coefficient product is real.

These two cases are exemplified by the diatonic and pentatonic scales in 12 TeT , which are fifth- (or fourth-) generated: $n=12, f=5$ (or 7 ), and $d=5$ or 7 . The sine in the denominator is $\sin (k f \pi / n)=\sin (5 k \pi / 12)$, which is positive for $k=2$, negative for $k=3,7$; and the numerator $\sin (d k f \pi / n)=\sin \left(5^{2} k \pi / 12\right)=$ $\sin (\pi / 12)$ or $\sin (5 \times 7 k \pi / 12)=-\sin (\pi / 12)$, hence the result.

NB: generally, albeit random pc-sets usually do not satisfy coherence, the previous propositions help us understand informally why man-made music may: it is not uncommon to compose using pc-sets built up from small units or bricks, like dyads, symmetric tetrachords, bits of generated scales, or pc-sets close to these, etc.

## 3 Example: Tonal pitch-class distributions

Pitch-class distributions of tonal music have a number of regular features observable through the DFT, in particular high magnitudes of the fifth and third coefficients [16]. Phases of the fifth and third coefficients can be used to estimate the key of a passage [15]. Tonal distributions also have a clearly observable regularity in one of the coefficient products, $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$. This means that the pitch-class watermarks of tonal music include not only the spectral features relating to intervallic content (diatonicity and triadicity) but also at least this one non-spectral feature, determined by the phases of $\hat{a}_{2}, \hat{a}_{3}$, and $\hat{a}_{5}$.

Figure 2 shows all of the coefficient products for a windowed analysis of Bach's 3-part inventions, excluding those with duplicated coefficients. ${ }^{4}$ Only the inventions in $4 / 4$ are included, and the distributions are taken over all four-beat windows in the piece. In addition to always being the largest coefficient product in all but one case (no. 3), the $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ values also are most consistent in phase, staying close to the positive real axis. Despite being small, the imaginary values also reliably distinguish mode, with the two minor mode pieces (numbers 4 and 9) having the only consistently positive imaginary values.

In Figure 2 we observe that when the average coefficient product is reliably distinct from the origin, it is usually also approximately coherent, with the main exceptions being in one product, $\hat{a}_{3} \hat{a}_{4} \hat{a}_{5}$. This might be explained by the limited macroharmony tonal music, where macroharmony is Tymoczko's term for "the total collection of notes used over moderate spans of musical time" [11, p.4]. Specifically, tonal music usually deals with a limited number of pitches in circulation at a time, so it is impossible to differentially weight all twelve pitch classes. Rather, the composer chooses a limited set of pitches (the macroharmony) and differentially weights these according to their status in the key and chord, and omits the rest. We therefore expect to see a pitch-class distribution with a "floor." We can model such distributions by imagining starting with the

[^2]

Fig. 2. Average values of coefficient products from common-time Bach three-part inventions. We take averages over all four-beat windows. Standard errors, shown with bars, are corrected for overlap.
full space of distributions with values balanced around zero and normalized to lie between -1 and 1 , then applying a clipping filter to eliminate negative values:

$$
x \mapsto \begin{cases}x & \text { for } x>0 \\ 0 & \text { otherwise }\end{cases}
$$

This is the product of identity by a step function, and can also be expressed as $(x+|x|) / 2$.

In the space of continuous maps on (say) $[-1,1]$, with hermitian norm $f \mapsto$ $\sqrt{\int_{-1}^{1} f^{2}}$, we get a decent quadratic approximation to this map with $x \mapsto(3+$ $\left.16 x+15 x^{2}\right) / 32$, as can be seen in Fig. 3.

Given some distribution in the full space, then, the corresponding clipped distribution will differ primarily by the addition of a positive quadratic term. The DFT of this quadratic term will consist of the products of coefficients in the original distribution, and adding these to the coefficients of the original distribution will push all coefficient products in the direction of coherence.

For instance, if we take the sum of pitch classes from a large number of majorkey pieces transposed to C major we get a pitch-class distribution like the one in Figure 4 (here we use the distribution obtained in [1], but very similar ones could be taken from many other studies). We can approximately resynthesize this from just its two largest Fourier coefficients, $\hat{a}_{3}$ and $\hat{a}_{5}$, by taking a sum of these and applying the clipping filter. The result is similar to the original distribution, in


Fig. 3. The clipping function and its best quadratic approximation.
particular recovering an $\hat{a}_{2}$ similar to the one in the original distribution. The main difference is that the derived distribution has an $\hat{a}_{4}$ which is suppressed in the original distribution. These $\hat{a}_{2}$ and $\hat{a}_{4}$ components of the derived distribution are attributable to the squared term in the quadratic approximation of the clipping function. ${ }^{5}$

We might interpret the data in Figure 2, then, with the claim that the entire pitch-class distribution is determined roughly by $\hat{a}_{3}, \hat{a}_{4}$, and $\hat{a}_{5}$, plus the assumption of limited macroharmony. The limited macroharmony (clipping) filter accounts for the observed values of $\hat{a}_{1}$ and $\hat{a}_{2}$, which make coherent products with the other coefficients $\left(\hat{a}_{1} \hat{a}_{4} \hat{a}_{7}\right.$ and $\left.\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}\right)$.


Fig. 4. On the left, a major-key pitch-class distribution from [1], the sum of its 3rd and 5th Fourier coefficients, and the clipping filter applied to this. On the right, the spectra of these.

The coefficient products $\hat{a}_{3} \hat{a}_{4} \hat{a}_{5}$ and $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ both involve the third and fifth coefficients. We might contrast coherence/incoherence in these two products by considering how intervals that are farther apart in $\hat{a}_{3}$ and $\hat{a}_{5}$, tritones and semitones, appear in sets that are otherwise relatively concentrated in these dimensions. Figure 5 shows the $\hat{a}_{3} / \hat{a}_{5}$ phase space, a toroidal space where the coordinates are phases of different DFT coefficients [2,12]. There are two relatively

[^3]parsimonious ways to connect semitones and tritones, along SW-NE or NW-SE diagonals. The former is associated with diatonic semitones and tritones, the latter with chromatic semitones and tritones along the minor-thirds axis. ([12] refers to these as "intervallic axes.") We suggest the term "blues tritone" for NW-SE orientation because it could result from adding blue notes to a pentatonic scale. If pitch classes tend to cluster around a diatonic SW-NE diagonal, it will have a positive $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ and negative $\hat{a}_{3} \hat{a}_{4} \hat{a}_{5}$, which is what we observe in the Bach inventions. If, on the other hand, they cluster around a chromatic/octatonic NW-SE diagonal, we will see the opposite pattern, a negative $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ and positive $\hat{a}_{3} \hat{a}_{4} \hat{a}_{5}$. This pattern, although it can involve equally large values of $\left|\hat{a}_{3}\right|$ and $\left|\hat{a}_{5}\right|$, would be rarely observed in eighteenth-century tonal pitch-class distributions. We will use the term "diatonic" to refer to the first type of set, and "anti-diatonic" the latter type.


Fig. 5. Different kinds of semitones and tritone in $\hat{a}_{3} / \hat{a}_{5}$ phase space associated with positive real $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ (diatonic semitone and tritone) and positive real $\hat{a}_{3} \hat{a}_{4} \hat{a}_{5}$ (chromatic semitone and blues tritone).

The following example shows that these non-spectral distinctions between diatonic and anti-diatonic material remain salient for twentieth composers in non-tonal contexts.

## 4 Example: All-interval tetrachords and Takemitsu

The all-interval tetrachords (AITs), set classes (0146) and (0137), are a unique example of small-cardinality Z-related sets, and as such are of particular interest for musically exploring non-spectral properties of set types. The four AIT set types occupy unique locations in $\hat{a}_{1} \hat{a}_{2} \hat{a}_{9}$ and $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ spaces. These are products
that do not involve $\hat{a}_{4}$ or $\hat{a}_{8}$, which means that the eight-note chromatic complement of a tetrachord is equal to its four-note octatonic complement (because the octatonic is nil on all coefficients). Therefore the AIT pairs (0137)-(0256) and (0146)-(0467), which are octatonic complements, behave like ordinary complements in these spaces, with equal magnitude and opposite phases. This special relationship between AITs and the octatonic relates to the CUP property explored by Childs [6] and Capuzzo's Q-operations [5].

The differences in $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ explain how the AITs differ in quality despite their equivalent intervallic content. The imaginary dimension is associated with the major/minor contrast. Inversion reverses the sign of the imaginary part, so (0137) and (0467) have the same real part in $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$, but opposite imaginary part. The "major" (0467) and (0256) have negative imaginary values. The real dimension distinguishes whether the thirds and fifth are arranged to imply a diatonic semitone and tritone (positive) or a chromatic semitone and blues tritone (negative). The (0137) tetrachords are distinguished by their triadic subset and the (0146)s by the distinctive non-diatonic subset, (014).

Takemitsu's solo flute piece, "Air," uses an all-interval tetrachord as its principal motive, and the major/minor and diatonic/anti-diatonic contrasts of $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ space are important to the harmonic language of the piece. Figure 6 shows a parsing of the first 14 measures, and Figure 7 plots these in $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ space. The central thematic role of (0467) is immediately apparent in its prominent statement in the opening and in m .6 . The opening gesture also defines a larger set, (014578), which is similar to (0467) in $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ : it is close in phase, and slightly farther from the origin. This larger set returns in m. 9. Altogether, this establishes a departure-return script in which the principal motive alternates with harmonically contrasting material.

Octatonic and whole-tone material are essential to Takemitsu's harmonic methods even though complete octatonic and whole-tone collections never appear. These collections are special in that they have $\hat{a}_{2}=\hat{a}_{3}=\hat{a}_{5}=0$. This makes them useful to create a kind of negative space: while adding a complete octatonic or whole-tone collection has no effect on $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$, adding an incomplete collection has the effect of negating the missing pitch class(es), which may be understood as a kind of partial complementation [13]. The set (024689), for example, is a whole-tone collection plus an "anti-semitone" (the added note and omitted note are a semitone apart): (02468t) \t $\cup 9$. The set type (023468), directly following it, similarly, is a whole-tone collection plus "anti-fifth": (02468t) \t $\cup$ 3. The next set, (0346), is a diminished seventh plus an anti-fifth, and is therefore has the same $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ value. Whereas dyads are always on the real axis in a coefficient product space, anti-dyads are always on the imaginary axis.

Takemitsu's first contrast juxtaposes the principal motive with these two whole-tone-plus-anti-dyad collections, which neutralize the diatonic element, and highlight major/minor contrasts on the imaginary axis. In particular, the large minor value of (024689) contrasts with the large major value of the principal motive.


Fig. 6. Meas. 1-14 of Takemitsu's Air for solo flute, with important 4-6 note sets identified.


Fig. 7. Pitch-class sets from Air in $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ space. Diamonds show the AITs. Dots and squares show set types that appear in the passage.

The second contrast takes us into the anti-diatonic region through the use of octatonic collections. This is the first place where Takemitsu uses the contrasting AITs, as well as a transposition of the initial (0467). These combine into larger octatonic sets, the octatonic complement of (034) in m. 7 and the octatonic complement of (046) in m. 8. The entire two measures constitute a single
hexachord, which is the octatonic complement of (04). The two pentachords are equivalent in $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ phase to two sets that we have already heard (the principal motivic hexachord in m .1 , and the first contrasting hexachord in m. 2). Their combination, however, introduces a new anti-diatonic element. As if to underscore the point, Takemitsu restates this anti-diatonic octatonic hexachord more compactly in m .10 , immediately after repeating the head motive in m .9 .

The last gesture uses a pitch-class set that is harder to easily characterize, yet Takemitsu communicates a sense of return with the rhythmic broadening, the clear phrase break, and the return to the high $\mathrm{G} \sharp / \mathrm{Ab}$ that marked the registral goal of the basic idea in mm. 1 and 9 . The $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ value is consistent with this: it is large and close in phase to (0467) and (014578).

## 5 Conclusion

Coefficient products were first discovered empirically in $[14,15]$, in the form of coherent $\hat{a}_{2} \hat{a}_{3} \hat{a}_{7}$ in tonal distributions, which were at first difficult to explain. The present study reveals some of the general properties of coefficient products and why we might observe coherent products in distributions from real music. While Fourier coefficients are mathematically independent in principle, the constraints on real distributions mean that they are not always independent in practice. In tonal distributions, the presence of significant $\hat{a}_{2}$ may actually be a mathematical artifact of $\hat{a}_{5}, \hat{a}_{3}$, and limited-macroharmony constraints. At the same time, a significant $\hat{a}_{4}$ coefficient may actually be concealed by similar artifacts.

We have also revealed a potential wealth of other applications of coefficient products, including analysis of Z-related sets, post-tonal music, and distinguishing anti-diatonic sets, particularly those like the pentatonic that share a large $\left|\hat{a}_{5}\right|$ with diatonic sets.

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[^0]:    ${ }^{1}$ For $80 \%$ of pc-sets, the error is larger than $10 \%$.

[^1]:    ${ }^{2}$ Not necessarily distinct.
    ${ }^{3}$ The qualification "regular" distinguishes these from an arbitrary product, but since non-regular products are of no evident interest, we will typically omit the qualifier.

[^2]:    ${ }^{4}$ For example, $\hat{a}_{5} \hat{a}_{5} \hat{a}_{2}$ or $\hat{a}_{4} \hat{a}_{4} \hat{a}_{4}$. These are also regular coefficient products and can have interesting applications, but we focus instead on products of three unique coefficients here.

[^3]:    ${ }^{5}$ A complication here is that there are two contributors to $\hat{a}_{2}$ in the quadratic term, $\hat{a}_{3} \hat{a}_{7}$ and $\hat{a}_{5} \hat{a}_{5}$, and the latter is larger in the distribution derived using the clipping filter. In the original distribution, the phase of $\hat{a}_{2}$ is closer to that of $\hat{a}_{3} \hat{a}_{7}$ than $\hat{a}_{5} \hat{a}_{5}$.

