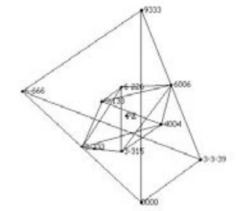
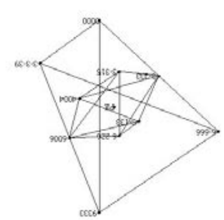


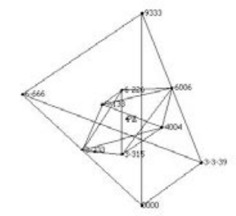
Upright Petrouchka, Proper Scales, and Sideways Neapolitans

Rachel Wells Hall
Department of Mathematics
Saint Joseph's University

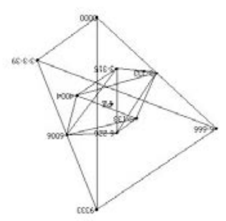
Dmitri Tymoczko
Department of Music
Princeton University

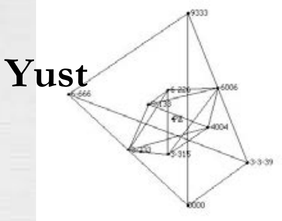
Jason Yust
School of Music
University of Alabama, Tuscaloosa





The "Petrouchka Chord," Rotated Voice Leadings, and Polytonality



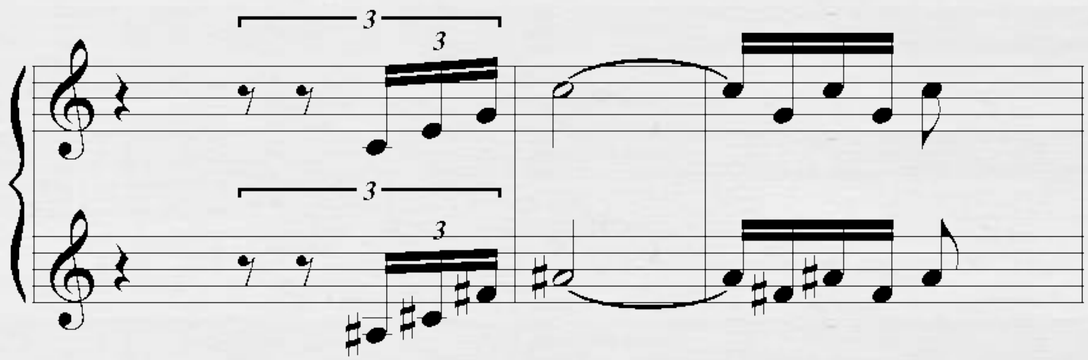


harmonic intervals: 2 3 1 2 1 2 1 2

First appearance of the "Petrouchka chord"

(Second Tableau, r. 49, Clarinets)

The most efficient "neapolitan" voice leadings

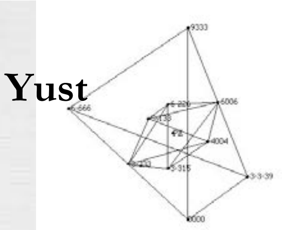


harmonic intervals: 2 3 1 2 1 2 1 2

The Petrouchka chord is a “90º rotation” of a neapolitan voice leading (i.e., melodic intervals become harmonic intervals).

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

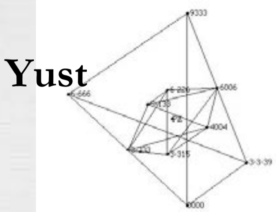
—Rachel Hall, Dmitri Tymoczko, Jason Yust



Vertical Intervals: **6 5 4 6 6 6 6 5 4 6 6 6**

A variation on the "Petrouchka chord" (r.49, mm. 11–12)

In this example, "Petrouchka chord" simultaneous arpeggiations in the clarinets alternate with those in the piano. Stravinsky juxtaposes different major triads in each case, but always maintaining approximately a fourth between the voices.



Another pattern derived from the "Petrouchka chord,"
(Fourth Tableau, r.78: Strings, doubled first by bassoon then clarinet)

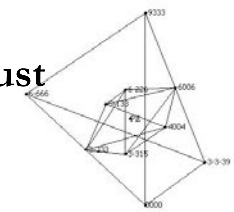
The first part of this example, different major triads are juxtaposed to produce vertical intervals consistently in the vicinity of a major second. The second part of the example juxtaposes the *same* major triads in different ways, so that the intervals swap directions.

harmonic intervals: 12 11 12 11 11 12 11 11 12 11 11

Rite of Spring: *Jeux des Cités Rivales* r.57 m. 3–4, Horns

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust

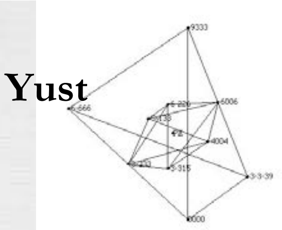


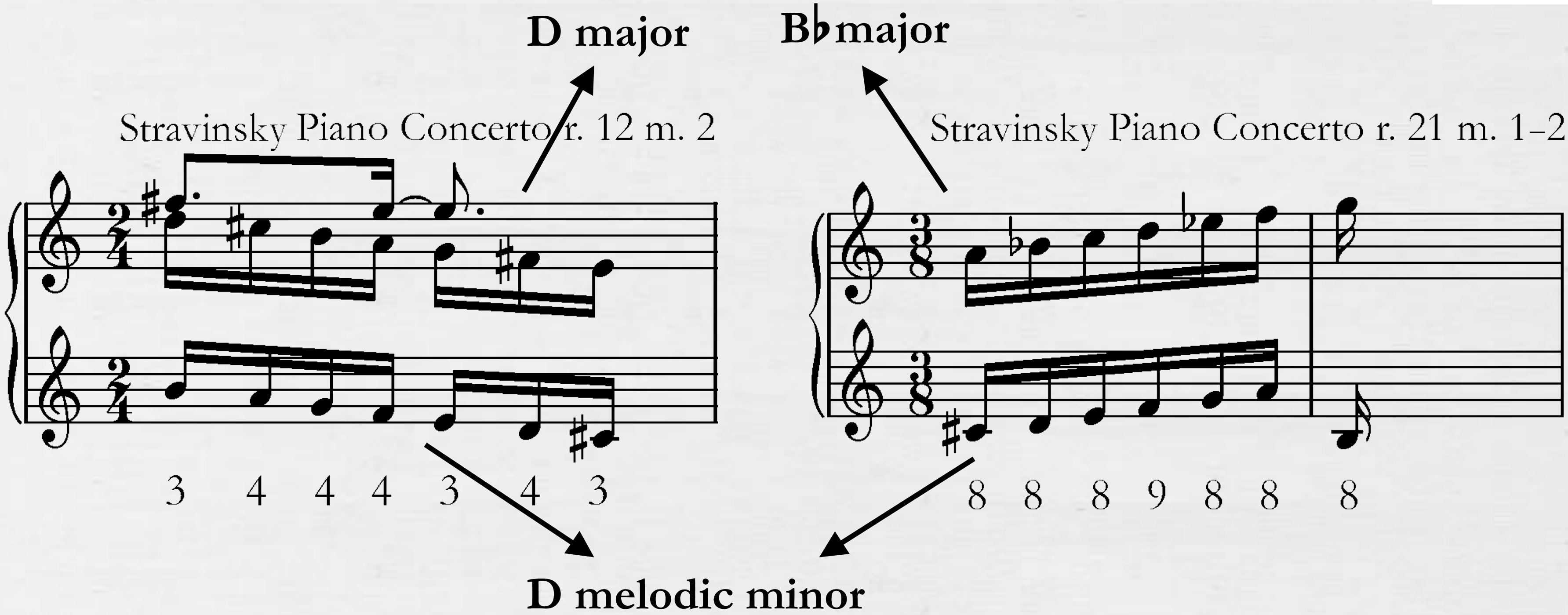
harmonic intervals: 12 11 12 11 11 12 11 11 12 11 11

Rite of Spring: *Jeux des Cités Rivales* r.57 m. 3–4, Horns

D → D♯
F → F♯
G → G♯
C → C♯

This “polytonal” passage can be thought of as a rotated voice leading between diatonic scales.





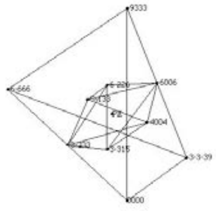
These passages from Stravinsky's Concerto show the same kind of harmonic consistency as the example from the Rite of Spring, but juxtapose two *different* scale types (diatonic and acoustic).

In the foregoing examples:

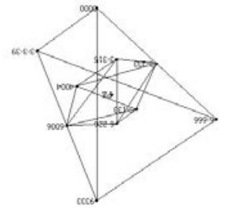
- **Stravinsky "rotates" familiar voice leadings so that melodic intervals appear vertically and harmonic intervals appear horizontally.**

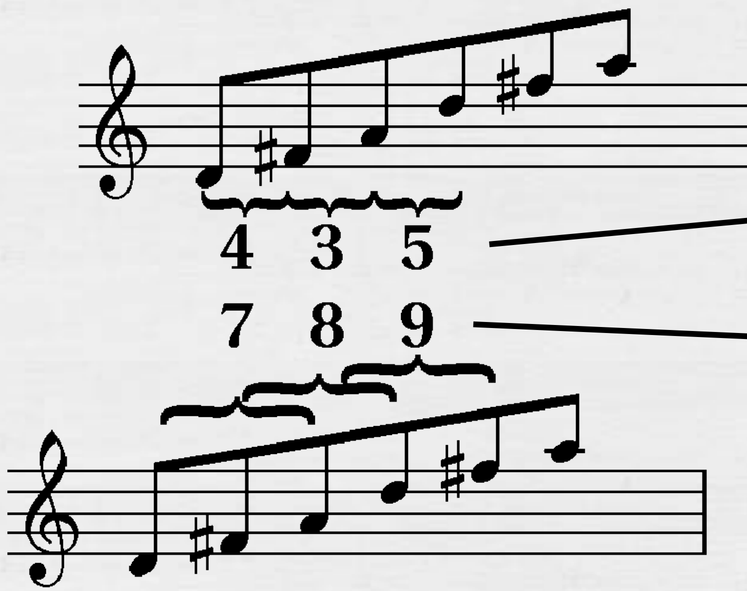
- **The vertical intervals are all similar in size giving the passages a palpable sense of consistency that is difficult to explain in traditional theoretical terms.**

How can we understand this process?



Scalar Interval Matrices

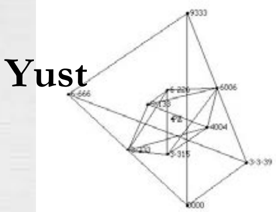


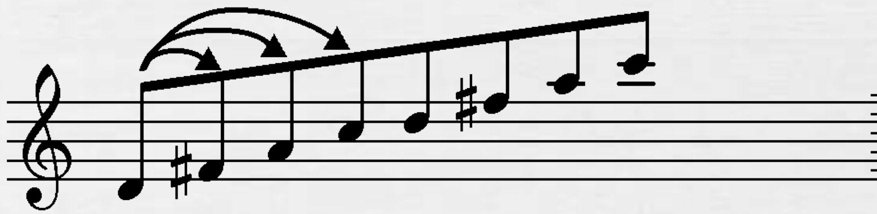


Intervals between *adjacent* notes in the triad.

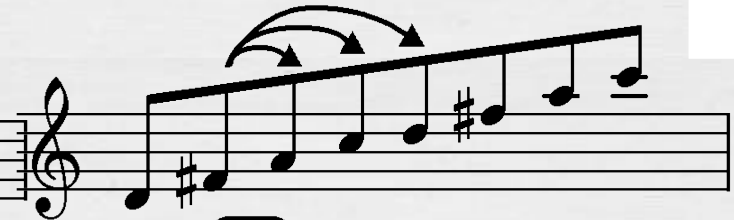
Intervals between *non-adjacent* notes in the triad.

Scalar interval matrix for a major triad

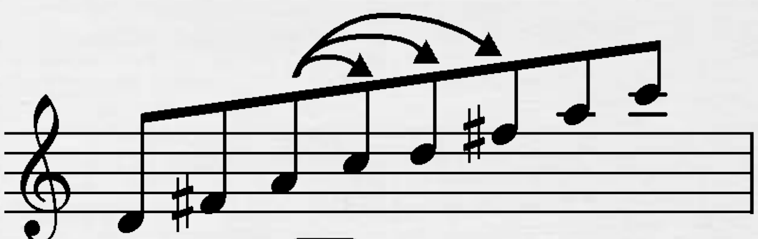




4	3	3	2
7	6	5	6
10	8	9	9



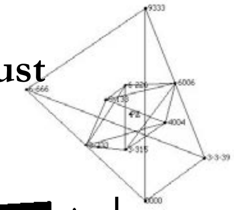
4	3	3	2
7	6	5	6
10	8	9	9



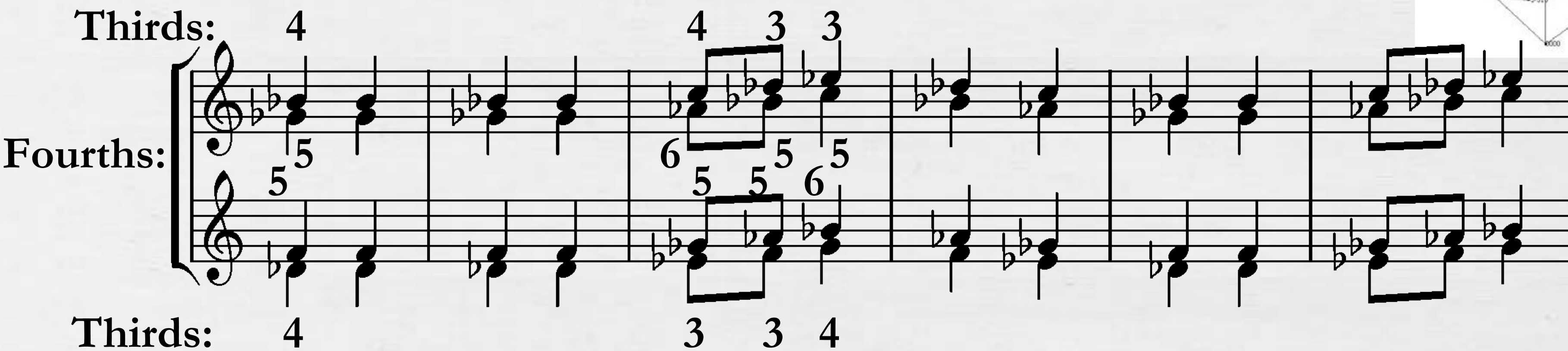
4	3	3	2
7	6	5	6
10	8	9	9

4	3	3	2
7	6	5	6
10	8	9	9

Scalar interval matrix for the dominant seventh



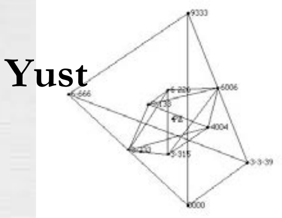
Rite of Spring: *Augures Printaniers* r.28 m. 5–10, Trumpets

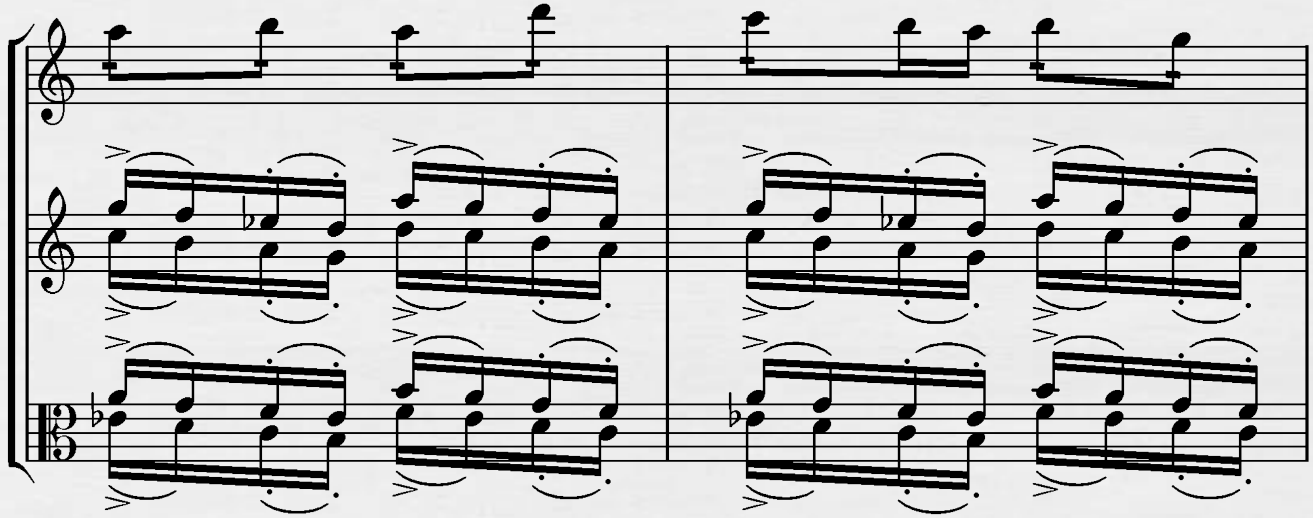


Rite of Spring: *Augures Printaniers* r.28 m. 5–10, Trumpets

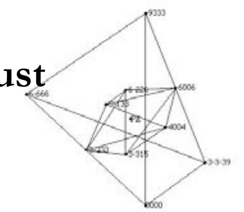
2	2	1	2	2	2	1
4	3	3	4	4	3	3
5	5	5	6	5	5	5
7	7	7	7	7	7	6
9	9	8	9	9	8	8
11	10	10	11	10	10	10
12	12	12	12	12	12	12

Scalar interval matrix for the diatonic

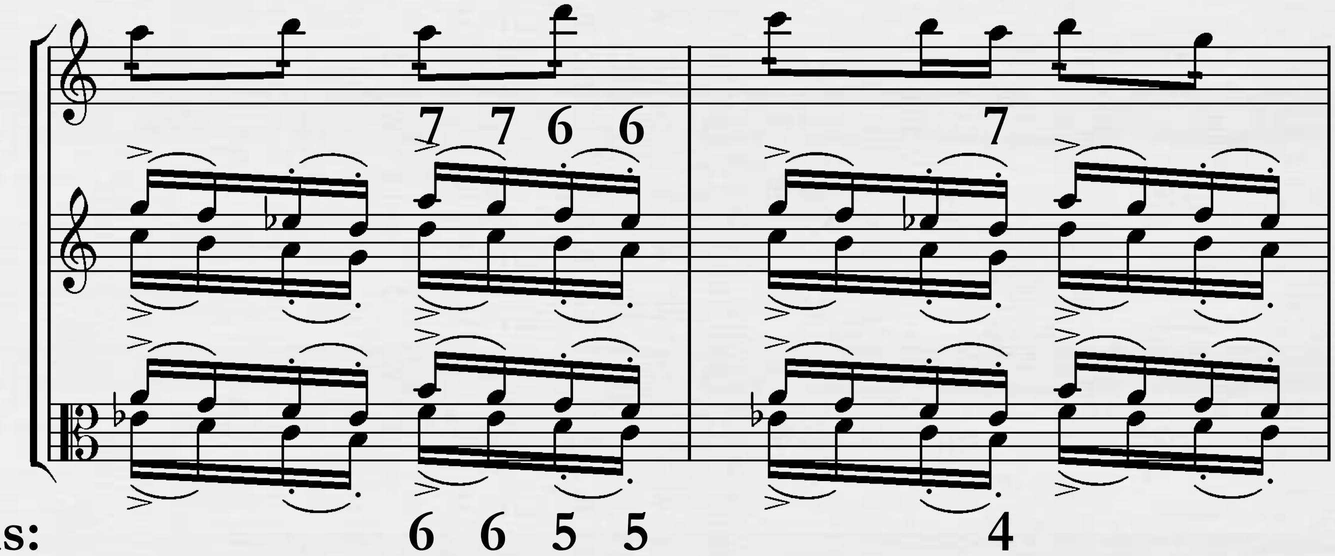




Rite of Spring: *Augures Printaniers* r.31 m. 17–18, Strings



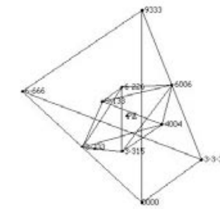
Fifths:



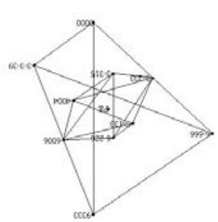
Fourths:

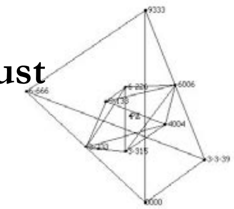
2	2	2	1	2	1	2
4	4	3	3	3	3	4
6	5	5	4	5	5	6
7	7	6	6	7	7	8
8	8	8	8	9	9	9
10	10	10	10	11	10	10
12	12	12	12	12	12	12

Scalar interval matrix for the acoustic scale (melodic minor)



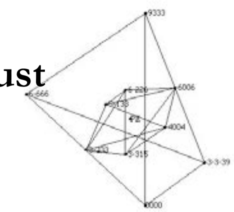
Rotational arrays
and scalar interval matrices
(a fortuitous connection)





0	1	7	5	6	11
0	6	4	5	10	11
0	10	11	4	5	6
0	1	6	7	8	2
0	5	6	7	1	11
0	1	2	8	6	7

Rotational array for Stravinsky's *Movements*



All Combinatorial Hexachord

0	1	7	5	6	11
0	6	4	5	10	11
0	10	11	4	5	6
0	1	6	7	8	2
0	5	6	7	1	11
0	1	2	8	6	7

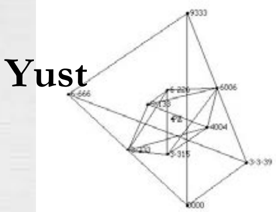
Rotational array for Stravinsky's *Movements*

0	1	7	5	6	11
0	6	4	5	10	11
0	10	11	4	5	6
0	1	6	7	8	2
0	5	6	7	1	11
0	1	2	8	6	7

All Combinatorial Hexachord

Intervals between successive elements of the hexachord

Rotational array for Stravinsky's *Movements*

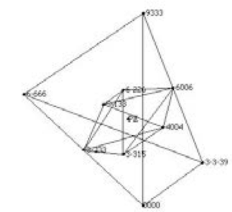


0	1	7	5	6	11
0	6	4	5	10	11
0	10	11	4	5	6
0	1	6	7	8	2
0	5	6	7	1	11
0	1	2	8	6	7

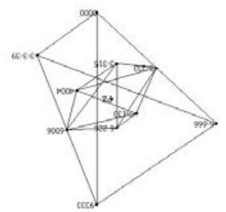
All Combinatorial Hexachord

Intervals between elements two places apart in the hexachord

Rotational array for Stravinsky's *Movements*



Proper Scales



Scalar interval matrix for Dom. 7th

4	**3**	**3**	**2**	*Row 1: Range = 2*
7	**6**	**5**	**6**	*Row 2: Range = 2*
10	**8**	**9**	**9**	*Row 3: Range = 2*
12	**12**	**12**	**12**	*Row 4: Range = 0*

Row 1–2: Gap = 1

Row 2–3: Gap = 1

Row 3–4: Gap = 2

Row 4–5: Gap = 2
(= Row 0–1)

The dominant seventh chord is a *proper* scale, because there is no overlap in interval sizes between the rows of the interval matrix.

Scalar interval matrix for Dom. 7th

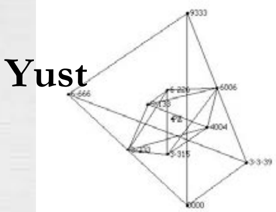
4	3	3	2
7	6	5	6
10	8	9	9
12	12	12	12

$\xrightarrow{-3}$

Interscalar interval matrix for Dom. 7ths

1	0	0	–1
4	3	2	3
7	5	6	6
9	9	9	9

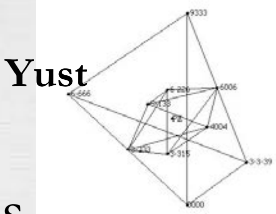
The interscalar matrix for any transposition of a scale inherits the propriety property, since adding a constant does not change the ranges of interval sizes within rows or the gaps between rows.



Rite of Spring: *Augures Printaniers* r.28 m. 5–10, Trumpets

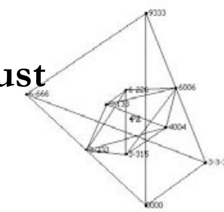
Thirds: 4 3 3 4 (4 3 3)
Fourths: 5 5 5 6 (5 5 5)

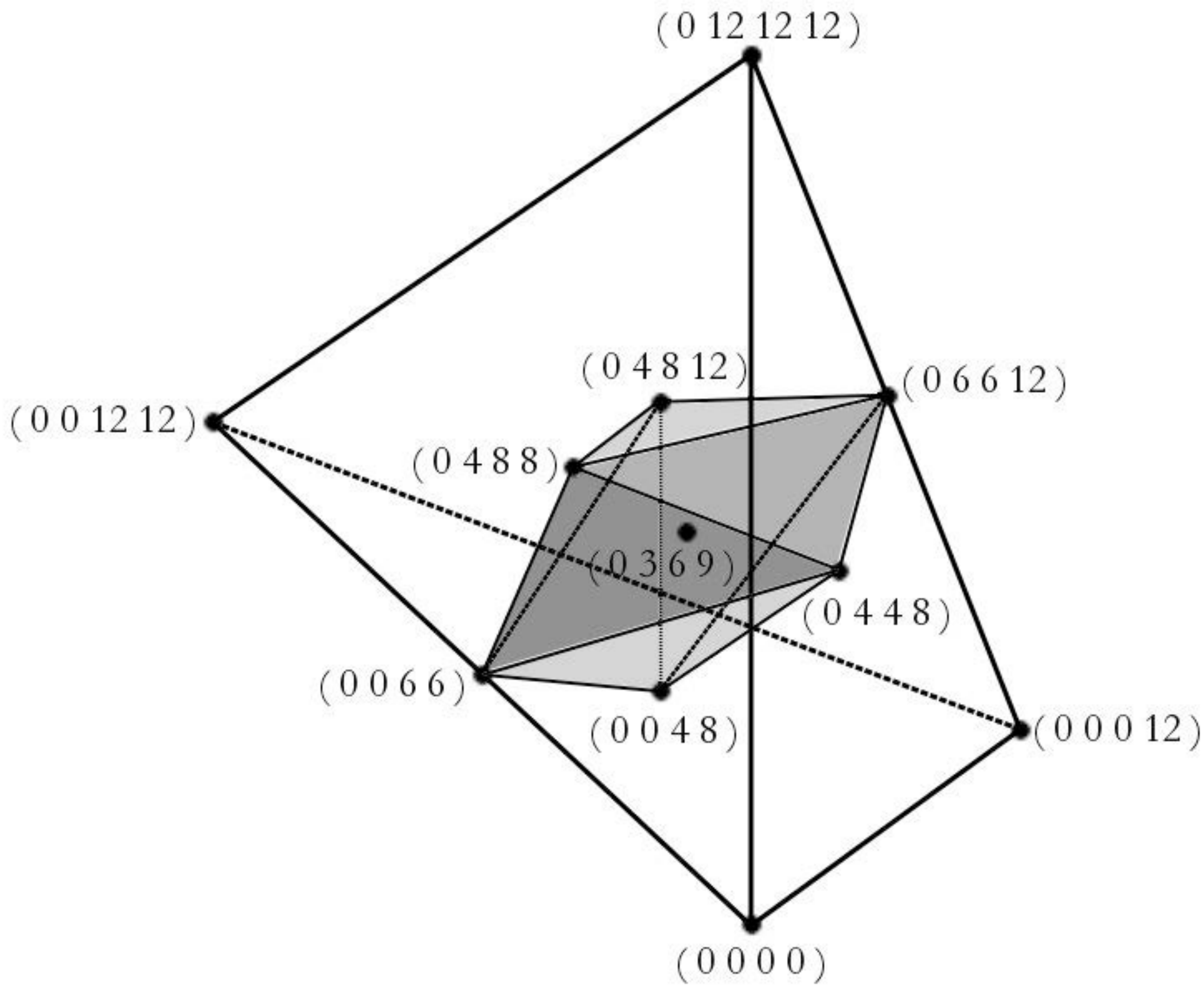
In the diatonic scale, thirds are always smaller than fourths



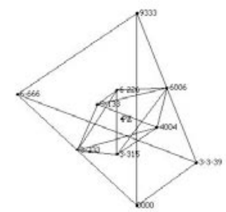
Rite of Spring: *Jeux des Cités Rivales* r.57 m. 3–4, Horns

Different juxtapositions of the diatonic scales in *Jeux des Cités Rivales* would produce vertical intervals consistently larger than an octave or consistently smaller than a major seventh, because the diatonic scale is proper.

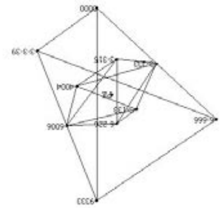


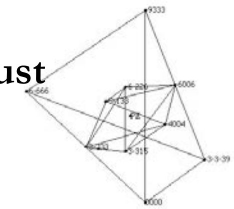


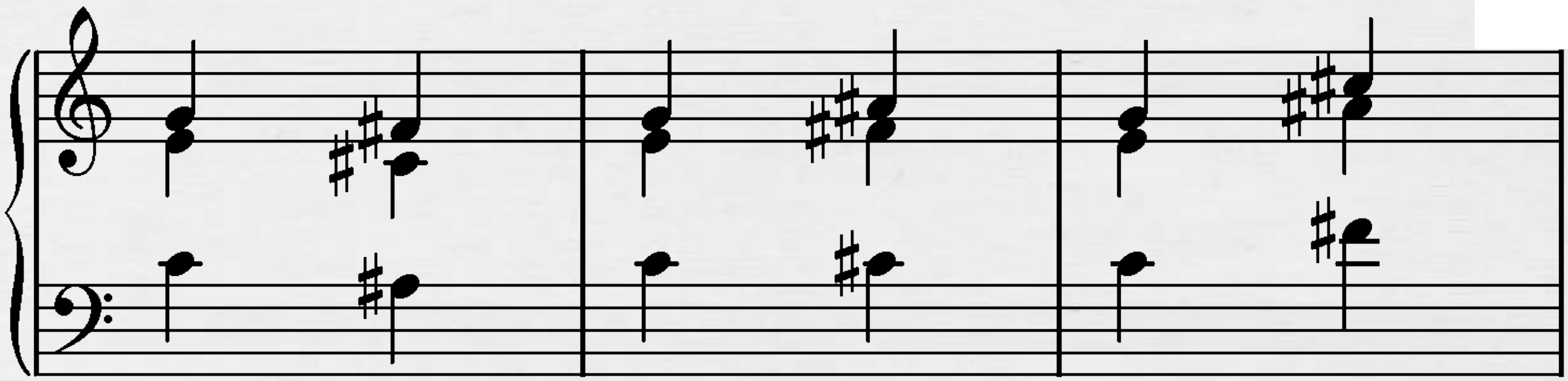
The proper scales form a *compact*, *convex* region at the center of *n*-note chord space



Interscalar Interval Matrices
(for transpositionally related scales)

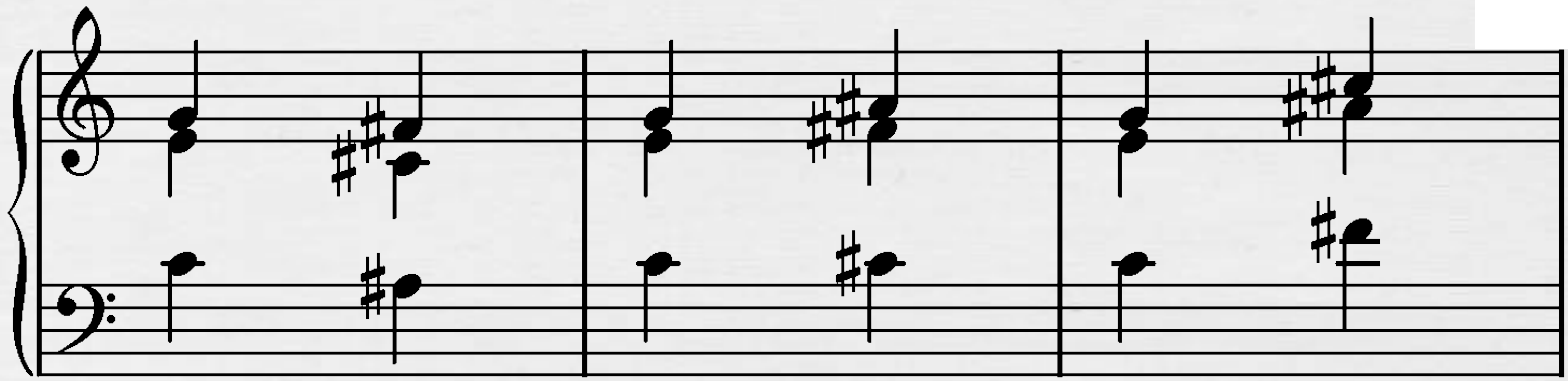


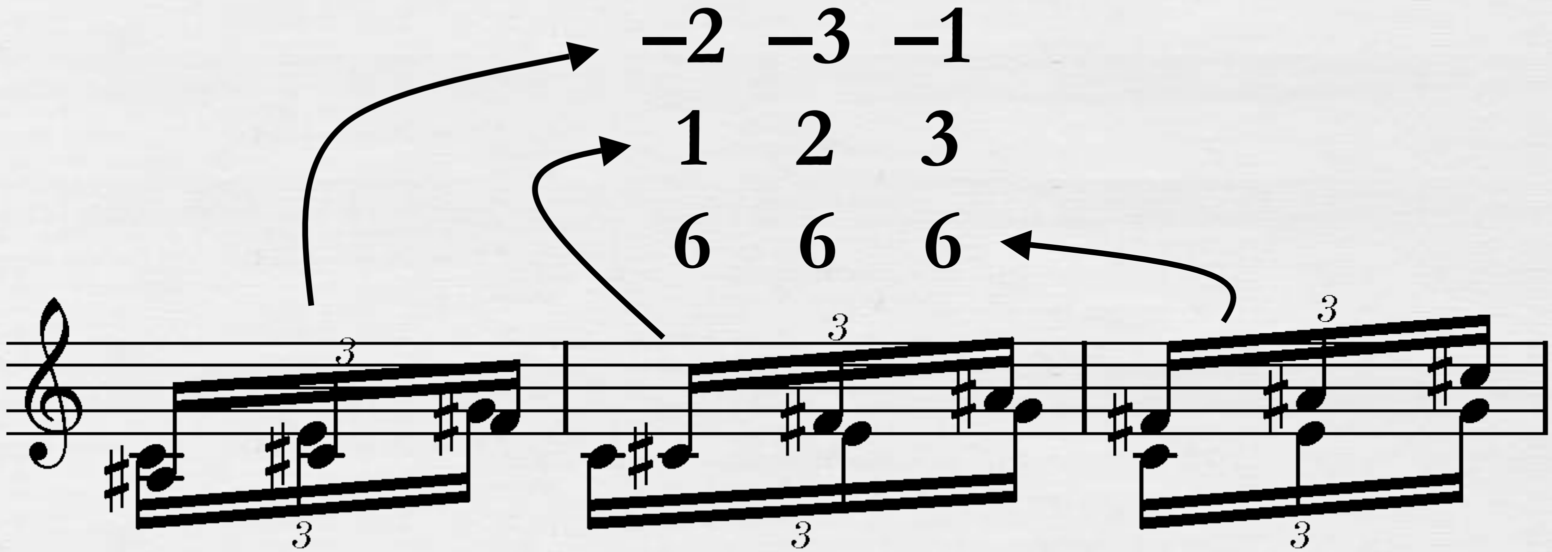




$$\begin{matrix} -2 & -3 & -1 \\ 1 & 2 & 3 \\ 6 & 6 & 6 \end{matrix}$$

The *interscalar interval matrix* for major triads a tritone apart represents all one-to-one crossing-free voice leadings between the chords

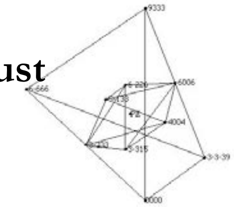




The interscalar interval matrix can also represent the *harmonic* intervals resulting from simultaneous arpeggiations in different positions

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

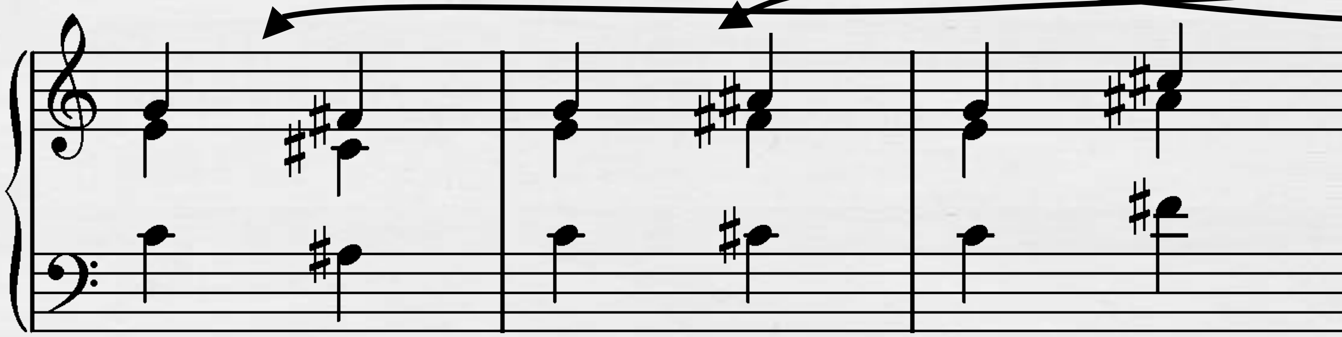
—**Rachel Hall, Dmitri Tymoczko, Jason Yust**



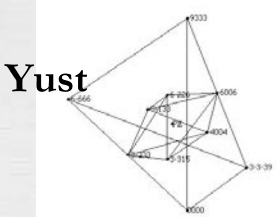
$$\begin{matrix} 4 & 3 & 5 \\ 7 & 8 & 9 \\ 12 & 12 & 12 \end{matrix} \xrightarrow{-6} \begin{matrix} -2 & -3 & -1 \\ 1 & 2 & 3 \\ 6 & 6 & 6 \end{matrix}$$

Scalar interval matrix
for a major triad

Interscalar interval matrix
for tritone-related major triads



The interscalar interval matrix is derived from a *scalar interval matrix*

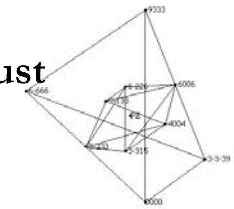


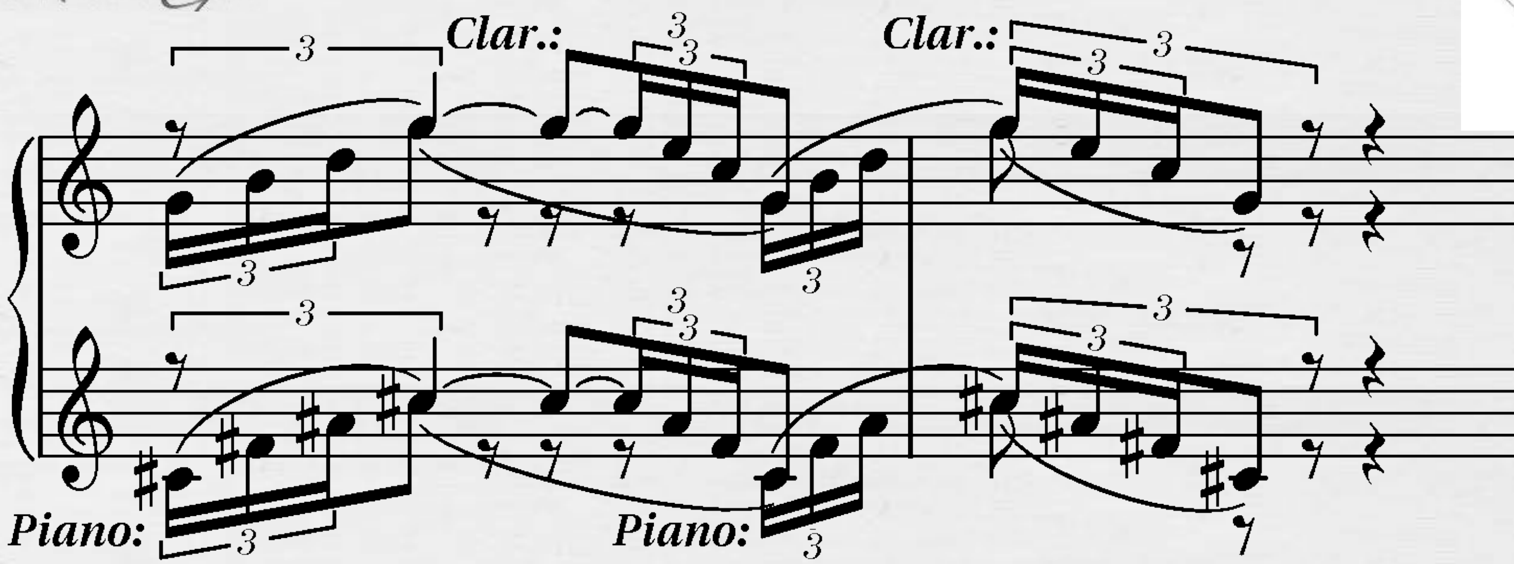
harmonic intervals: 2 3 1 2 1 2 1 2

$$\begin{matrix} -2 & -3 & -1 \\ 1 & 2 & 3 \\ 6 & 6 & 6 \end{matrix}$$

Interscalar interval matrix for major triads a tritone apart

The harmonic intervals of the Petrouchka chord represent a row of the interscalar interval matrix.





Vertical Intervals: 6 5 4 6 6 6 6 5 4 6 6 6

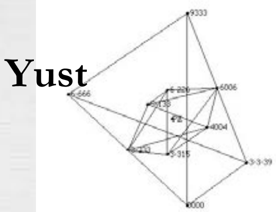
ISI matrix for major triads at T_1

$$\begin{matrix} 1 & 1 & 1 \\ 6 & 5 & 4 \\ 10 & 8 & 9 \end{matrix}$$

ISI matrix for major triads at T_6

$$\begin{matrix} 1 & 2 & 3 \\ 6 & 6 & 6 \\ 10 & 9 & 11 \end{matrix}$$

When Stravinsky contrasts forms of the Petrouchka chord juxtaposing different major triads, the vertical intervals come from different interscalar matrices for the major triad. They articulate rows of these matrices that that have similar interval sizes.



1 1 1 1 3 2 1 1 1 1 3 2 etc. . . .

. . . 1 1 1 –1 –3 –2 2 1 3 –1 –3 –2 2 1 3 etc. . . .

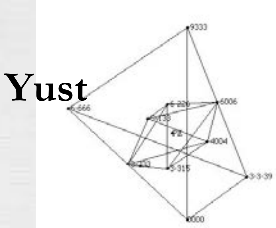
1	**1**	**1**	**1**	**2**	**3**
6	**5**	**4**	**6**	**6**	**6**
10	**8**	**9**	**10**	**9**	**11**

–1	**–2**	**–3**
3	**1**	**2**
6	**6**	**6**

The first part of this example articulates different rows of the same matrices as the previous example. The latter part of the example articulates different rows of a single matrix (the one for the original Petrouchka chord)

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust

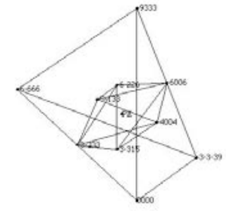


1 1 1 1 3 2 1 1 1 1 3 2 etc. . . .

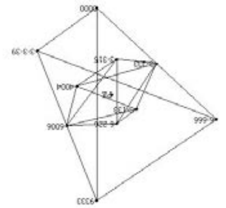
. . . 1 1 1 –1 –3 –2 2 1 3 –1 –3 –2 2 1 3 etc. . . .

–1	**–2**	**–3**
3	**1**	**2**
6	**6**	**6**

Because the major triad is proper, there is no overlap in interval sizes between the two rows of the ISI matrix realized in this example. This makes Stravinsky's "direction flipping" effect possible.

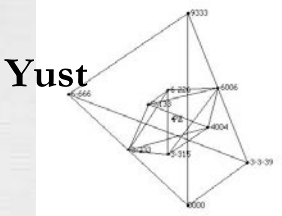


Interscalar Interval Matrices
(for scales not related by transposition)



Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust



harmonic intervals: 12 11 12 11 11 12 11 11 12 11 11

0	0	0	0	0	0	0
2	2	1	2	2	2	1
4	3	3	4	4	3	3
5	5	5	6	5	5	5
7	7	7	7	7	7	6
9	9	8	9	9	8	8
11	10	10	11	10	10	10

Scalar interval matrix for the diatonic

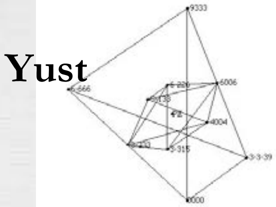
$\xrightarrow{+\,8}$

8	8	8	8	8	8	8
10	10	9	10	10	10	9
12	11	11	12	12	11	11
13	13	13	14	13	13	13
. . .						

Interscalar interval matrix for diatonic scales a major third apart

A variant of the Petrouchka chord
(Second tableau, r.60, ostinato in piano and strings)

Stravinsky's procedure of harmonic juxtaposition is not limited to transpositionally related chords or scales . . .



A variant of the Petrouchka chord
(Second tableau, r.60, ostinato in piano and strings)

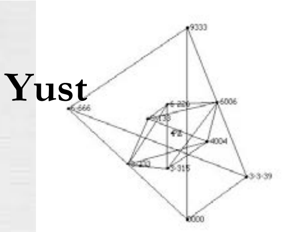
1	**1**	**–1**
4	**5**	**4**
8	**8**	**9**

Interscalar interval matrix
for D minor and F♯ major triads

. . . but neither are interscalar interval matrices.

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust

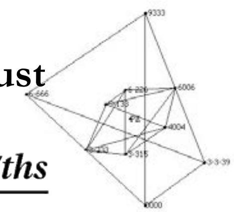


1	**1**	**–1**
4	**5**	**4**
8	**8**	**9**

Interscalar interval matrix
for D minor and F♯ major triads

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust



Scalar interval matrix for Dom. 7th

4	3	3	2
7	6	5	6
10	8	9	9
12	12	12	12

$\xrightarrow{-3}$

Interscalar interval matrix for Dom. 7ths

1	0	0	–1
4	3	2	3
7	5	6	6
9	9	9	9

$\downarrow +1$

1	1	0	–1
4	4	2	3
7	6	6	6
9	10	9	9

Interscalar interval matrix for Min. 7th → Dom. 7th

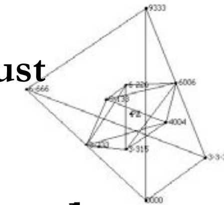
0 1 0 0

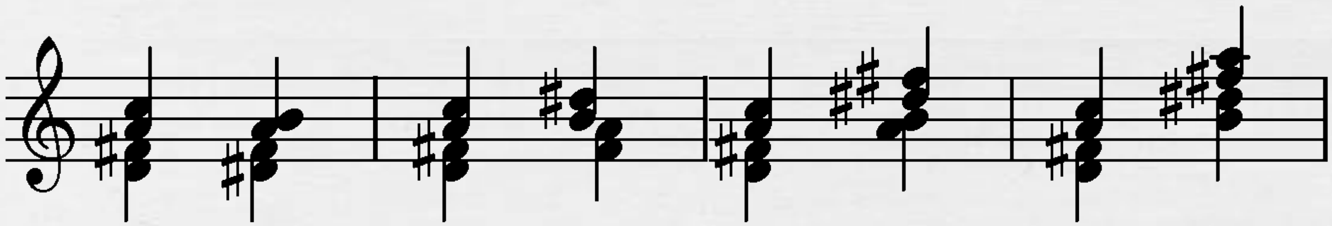
A voice leading from a min. 7th to a dom. 7th

An interscalar interval matrix for different set types can be constructed from a scalar interval matrix and a voice leading between the set types

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust

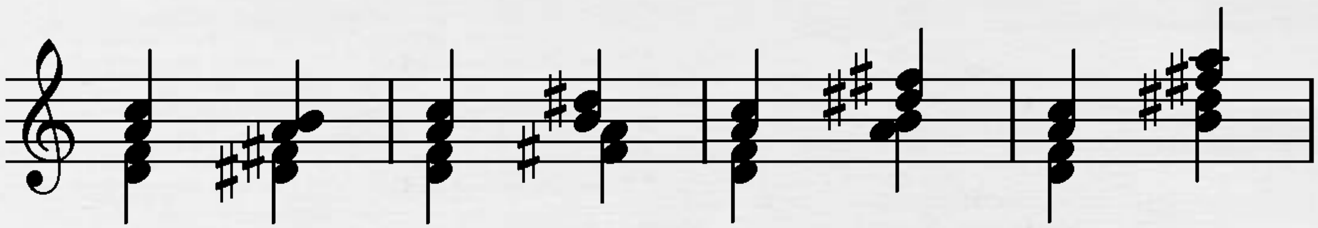




(1 0 0 –1) (4 3 2 3) (7 5 6 6) (9 9 9 9)

Interscalar Interval Matrix for T_3-related Dom. 7ths

1	0	0	–1
4	3	2	3
7	5	6	6
9	9	9	9

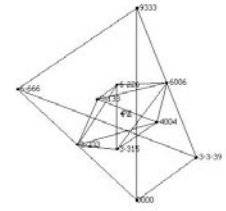


(1 1 0 –1) (4 4 2 3) (7 6 6 6) (9 10 9 9)

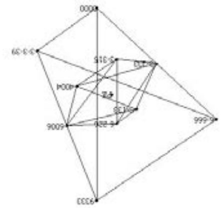
Interscalar Interval Matrix for Min. 7th and Dom. 7th

1	1	0	–1
4	4	2	3
7	6	6	6
9	10	9	9

Interscalar interval matrices (whether for the same set type or different set types) catalogue the possible voice-leadings between chords, or the vertical intervals that result from juxtaposing different rotations of them.

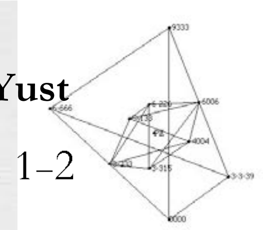


Co-Proper Scales



Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust



Stravinsky Piano Concerto r. 12 m. 2

3 4 4 4 3 4 3

Voice leadings from
D melodic minor to D major

Stravinsky Piano Concerto r. 21 m. 1–2

8 8 8 9 8 8 8

Voice leadings from
D melodic minor to B♭ major

0	0	1	0	0	0	0
2	2	2	2	2	2	1
4	3	4	4	4	3	3
5	5	6	6	5	5	5
7	7	8	7	7	7	6
9	9	9	9	9	8	8
11	10	11	11	10	10	10
12	12	13	12	12	12	12

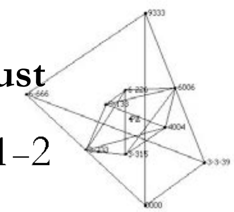
T_4

0	–1	0	0	0	–1	–1
1	1	2	2	1	1	1
3	3	4	3	3	3	2
5	5	5	5	5	4	4
7	6	7	7	6	6	6
8	8	9	8	8	8	8
10	10	10	10	10	10	9
12	11	12	12	12	11	11

Vertical intervals in the polytonal scalar passages from Stravinsky's piano concerto come from transpositionally related ISI matrices.

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust



Stravinsky Piano Concerto r. 12 m. 2

3 4 4 4 3 4 3

Stravinsky Piano Concerto r. 21 m. 1–2

8 8 8 9 8 8 8

0	**0**	**1**	**0**	**0**	**0**	**0**
2	**2**	**2**	**2**	**2**	**2**	**1**
4	**3**	**4**	**4**	**4**	**3**	**3**
5	**5**	**6**	**6**	**5**	**5**	**5**
7	**7**	**8**	**7**	**7**	**7**	**6**
9	**9**	**9**	**9**	**9**	**8**	**8**
11	**10**	**11**	**11**	**10**	**10**	**10**
12	**12**	**13**	**12**	**12**	**12**	**12**

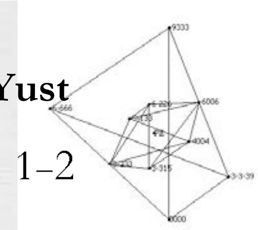
} Gap = 0
} Gap = 1
} Gap = 1
} Gap = 0
} Gap = 0
} Gap = 1
} Gap = 0

(Gaps between the range of interval sizes in adjacent rows)

When interscalar interval matrices for distinct transpositional set classes are proper, the set classes are *co-proper*. Diatonic and acoustic scales are co-proper.

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust



Stravinsky Piano Concerto r. 12 m. 2

3 4 4 4 3 4 3

0	**0**	**1**	**0**	**0**	**0**	**0**	} Gap = 0
2	**2**	**2**	**2**	**2**	**2**	**1**	} Gap = 1
4	**3**	**4**	**4**	**4**	**3**	**3**	} Gap = 1
5	**5**	**6**	**6**	**5**	**5**	**5**	} Gap = 0
7	**7**	**8**	**7**	**7**	**7**	**6**	} Gap = 0
9	**9**	**9**	**9**	**9**	**8**	**8**	} Gap = 1
11	**10**	**11**	**11**	**10**	**10**	**10**	} Gap = 0
12	**12**	**13**	**12**	**12**	**12**	**12**	

Stravinsky Piano Concerto r. 21 m. 1–2

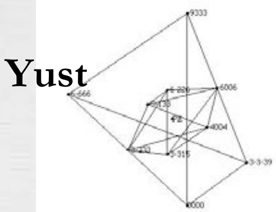
8 8 8 9 8 8 8

0	**–1**	**0**	**0**	**0**	**–1**	**–1**	} Gap = 1
1	**1**	**2**	**2**	**1**	**1**	**1**	} Gap = 0
3	**3**	**4**	**3**	**3**	**3**	**2**	} Gap = 0
5	**5**	**5**	**5**	**5**	**4**	**4**	} Gap = 1
7	**6**	**7**	**7**	**6**	**6**	**6**	} Gap = 1
8	**8**	**9**	**8**	**8**	**8**	**8**	} Gap = 0
10	**10**	**10**	**10**	**10**	**10**	**9**	} Gap = 1
12	**11**	**12**	**12**	**12**	**11**	**11**	

Adding a constant or rotating rows obviously does not effect the ranges within or between rows, so co-proper scales are co-proper regardless of the transpositions of the individual scales.

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust



1	**3**	**4**	**3**	**2**
4	**6**	**7**	**5**	**3**
7	**9**	**9**	**6**	**6**
10	**11**	**10**	**9**	**10**
12	**12**	**13**	**12**	**12**

Gap = 0
Gap = –1
Gap = –1
Gap = 0
Gap = 1

The minor minor ninth is not co-proper with the dominant minor ninth

2	**4**	**4**	**3**	**2**
5	**6**	**7**	**5**	**4**
7	**9**	**9**	**7**	**7**
10	**11**	**11**	**10**	**9**
12	**13**	**14**	**12**	**12**

Gap = 0
Gap = 0
Gap = 0
Gap = 0
Gap = 1

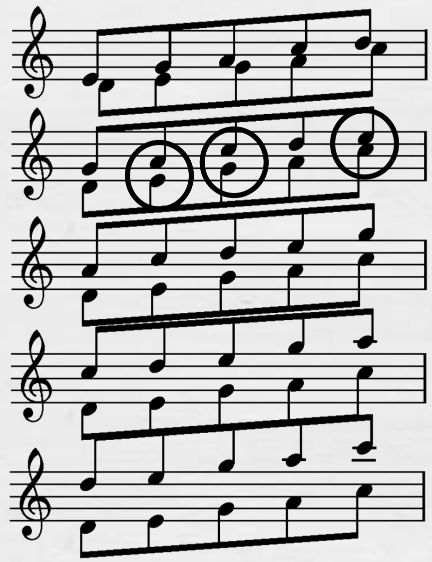
The minor minor ninth *is* co-proper with the pentatonic scale.

A scale can be co-proper with some relatively even scales, but not co-proper with other (less even) scales, even if it itself is not proper (as is the case with the minor minor ninth)

Scalar interval matrix for minor minor ninth

1	2	4	3	2
3	6	7	5	3
7	9	9	6	5
10	11	10	8	9
12	12	12	12	12

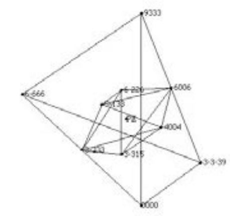
Scalar interval matrix for pentatonic scale



2	3	2	3	2
5	5	5	5	4
7	8	7	7	7
10	10	9	10	9
12	12	12	12	12

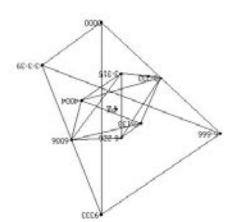
Co-propriety can be determined from scalar interval matices alone.

For two scales to be co-proper it is necessary and sufficient that there is no overlap between adjacent rows of the two different scalar interval matrices. (This is a non-trivial result).

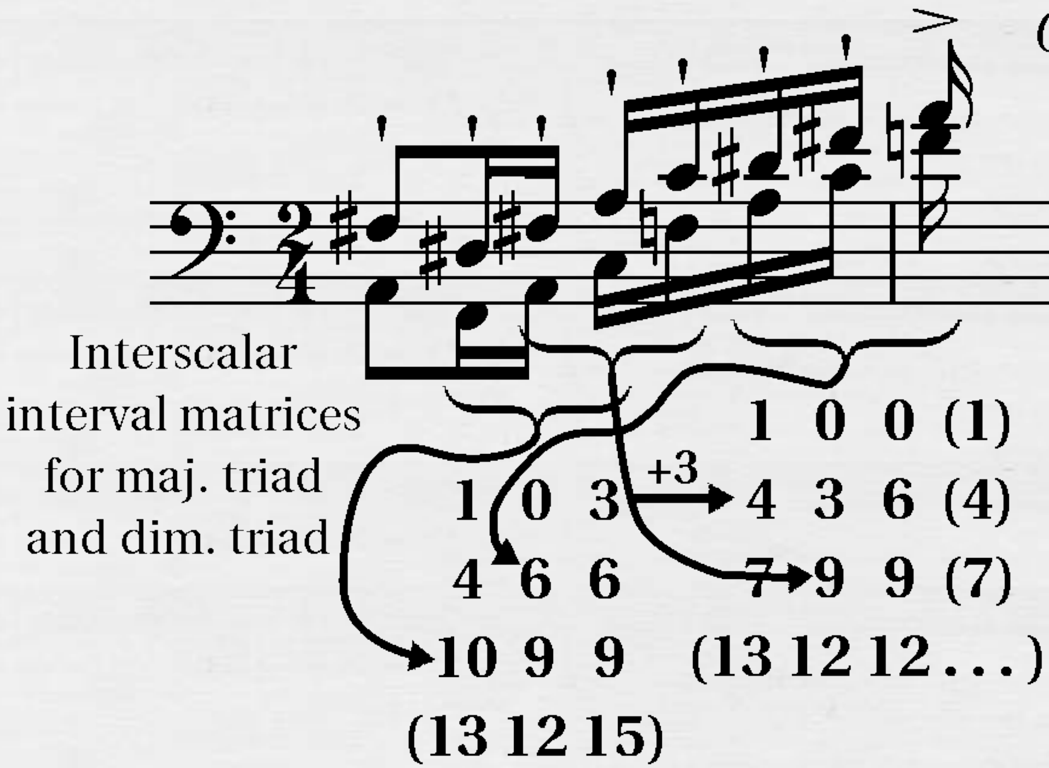


Stravinsky's T-chaining Technique,

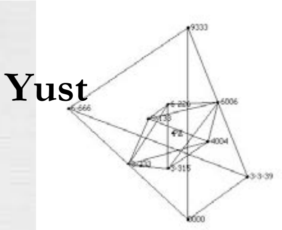
exploiting copropriety and the interscalar interval matrix

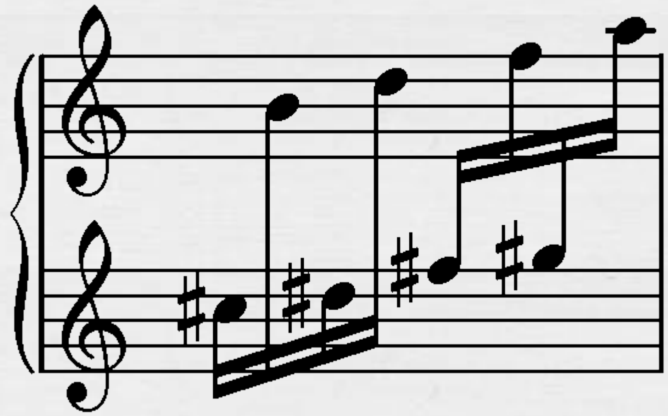


Octet r.60, m.6, bassoons



Another, more sophisticated, procedure we find in Stravinsky explores multiple rows of interscalar interval matrices at different transpositional levels by T-chaining two forms of one set against a single form of another. In this passage from the *Octet*, Stravinsky does this with major and diminished triads, which are co-proper.





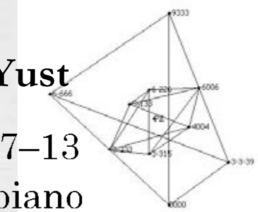
–2 –1 –1 1

One more variant of the Petrouchka chord
Second tableau, r. 59 m.7 , Piano

This variant on the Petrouchka chord juxtaposes
two different *four-note* sets:
A half-dim. seventh (0 2 5 8) and a fifths-generated chord, (0 2 5 7).
These sets are co-proper.

Upright Petrouchka, Proper Scales, and Sideways Neapolitans

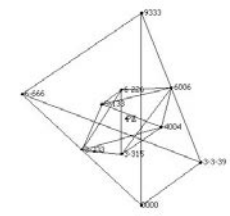
—Rachel Hall, Dmitri Tymoczko, Jason Yust



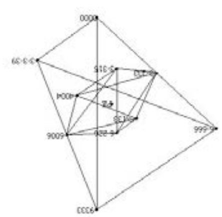
Petrushka r.59, m.7–13
(Second Tableau), piano

					1	1	1	2
−2	−1	−1	1	+5 →	3	4	4	6
1	2	3	3		6	7	8	8
4	6	5	6		9	11	10	11
8	8	8	9		(13	13	13	14)
(10	11	11	13)					

Stravinsky uses the T-chaining procedure in this example, T-chaining transpositions of (0 2 5 7) to present three rows of the interscalar matrix.

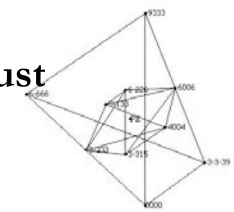


Conclusions



Upright Petrouchka, Proper Scales, and Sideways Neapolitans

—Rachel Hall, Dmitri Tymoczko, Jason Yust



• Voice leading is a phenomenon that can occur in many guises: ordinary chordal voice leading, modulations between scales, and also vertically in the kind of polytonal passages we have been considering.

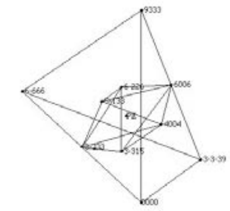
• Scalar and interscalar interval matrices are useful for understanding voice leadings between scales.

• Scalar interval matrices are closely related to the rotational arrays that appear in Stravinsky's music.

• The concept of "scalar propriety" was originally introduced by David Rothenberg to describe the gap between the rows of a scalar interval matrix. This concept can be extended to interscalar interval matrices.

• Co-proper scales provide a wealth of opportunities for the sorts of simultaneous polytonal arpeggios and scales found in Stravinsky's music.

•This fact might be useful to contemporary composers, many of whom are returning to the polytonal ideas found in Stravinsky's early music.



Upright Petrouchka, Proper Scales, and Sideways Neapolitans

Rachel Wells Hall — Department of Mathematics, Saint Joseph's University

Dmitri Tymoczko — Department of Music, Princeton University

Jason Yust — School of Music, University of Alabama, Tuscaloosa