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Hadamard Transforms of Pure-Duple Rhythms

Jason Yust[a*],

[a]*School of Music, Boston University, Boston, MA*

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Recent research has demonstrated a number of applications of the discrete Fourier transform to cyclic rhythms, addressing amongst other issues questions about conceptualizations of meter. One can apply a similar operation, the Hadamard transform, when the rhythmic cycle is a power of two. This paper explores some analytical applications of the Hadamard transform to repertoires where pure-duple metrical settings are the norm, such as American ragtime and Balinese gamelan. I compare it to the Fourier transform and highlight special features of the Hadamard transform of particular theoretical value, such as its sorting of rhythmic information into metrical levels, which leads to methods of classifying and quantifying syncopation.

Keywords: Hadamard transform; Walsh matrix; discrete Fourier transform; Rhythm; Syncopation; Ragtime; Balinese gamelan; Clave

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This article introduces the use of Hadamard transforms of rhythms in pure-duple meters. "Pure duple" refers to cycle lengths in powers of two, where each division of the cycle by a power of two has some metrical significance. A substantial proportion of music across a wide range of musical styles and traditions uses pure-duple meters. The Hadamard transform is a lossless transformation of a rhythm into a series of coefficients that partition into distinct metrical levels and can be understood to measure different kinds of syncopation, specific to the given metrical level. Because Hadamard transforms are related to discrete Fourier transforms (DFTs), the established technique of DFT analysis of rhythms can inform the similar use of Hadamard transforms. The two are not equivalent however, making Hadamard transforms useful where considerations of metrical level are more important than generalizing over rotations.

The article begins by introducing the vector representation of rhythms and the Hadamard transform and the musical interpretation of the Hadamard coefficients, and then compares Hadamard transforms and DFTs. The second part of the article presents three theoretical and analytical applications: considering important Afro-diasporic timelines identified by Toussaint (2013), concepts of syncopation in ragtime, and analyzing Balinese melodies.

*Corresponding author. Email: jyust@bu.edu

1. Hadamard transforms, discrete Fourier transforms, syncopation, and rhythmic qualities

1.1. *Hadamard transform of pure duple rhythms*

For the purpose of this article a rhythm is defined as a vector of length 2^n, where each place in the vector is a place in the rhythmic cycle, starting from $0 =$ the downbeat, and the presence of a rhythmic onset is indicated by a 1, the absence by a 0. The Son clave rhythm for instance: ♪. ♪. ♩ ♪ ♩, is represented by (1001001000101000). Alternately, we can allow for other possible values besides 1 and 0, representing for example the number of onsets at a particular location (for a multipart rhythm, for instance), or any sort of weighting of rhythmic values.

The Hadamard transform (AKA the Walsh-Hadamard transform) is the multiplication of this vector by the nth *Hadamard matrix*. This is defined recursively as

$$H_n = H_1 \otimes H_{n-1} \tag{1}$$

where

$$H_1 = \begin{pmatrix} + & + \\ + & - \end{pmatrix} \tag{2}$$

and "$\otimes$" denotes the Kronecker product of matrices (Wang 2007). (I use "+" and "−" here as a shorthand for $+1$ and -1.) In plain language: we duplicate the $(n-1)$th matrix four times in a 2×2 grid, multiplying it by -1 in the lower righthand corner.

The next few Hadamard matrices are

$$\begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{pmatrix} \tag{4}$$

$$\begin{pmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
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+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & +
\end{pmatrix} \tag{5}$$

I will represent the result of multiplying a rhythm vector by this matrix as a Hadamard vector $\vec{h}$, with Hadamard coefficients h_0–h_{2^n-1}.

This article explores possible uses of Hadamard transforms for theory and analysis of rhythms. One important property, which it shares with the discrete Fourier transform, is its invertibility. Applying the Hadamard transform twice returns the original rhythm (times a constant). It is therefore a lossless transformation, a reorganization of rhythmic information into a form that reveals music-theoretically useful properties. Our task here is to specify some of the music-theoretical meaning of the coefficients produced by this transform.

Each coefficient of the Hadamard transform is determined by the corresponding row in the matrix. We can think of these rows as rhythms by replacing the +s and −s with 1s and 0s. I will call these the *Walsh rhythms*. For instance, the Hadamard vector (row vector) for h_7 is $+--+-++-$ and its Walsh rhythm is 10010110.

The relationship of each Hadamard coefficient to duple meter is best understood by numbering them in binary digits. For instance, the binary digit for h_2 is 10 at $2^n = 4$, 010 at $2^n = 8$, 0010 at $2^n = 16$, and so on. The Walsh rhythms for these are 1100, 11001100, and 1100110011001100 respectively. The places of the binary digit represent metrical levels. A zero in the binary digit indicates repetition at that level, and a 1 indicates contrast at that level. The 1s place in the binary digit represents metrical level 1. Since h_2 has a zero in the 1s place, there is repetition at this level. The 2s place represents metrical level 2, which is where h_2 has contrasts. The 4s place of the binary digit is metrical level 3; h_2 has repetition at this and all higher levels. The simplest Walsh rhythms are those indexed in powers of 2. For $2^n = 8$, the rhythm for h_1 is 10101010, the rhythm for h_2 is 11001100, and the rhythm for h_4 is 11110000. Other coefficients mix these, having contrasts at multiple levels. For instance, h_3 has contrasts at levels 1 and 2, with Walsh rhythm 10011001.

The metrical level of a coefficient is that of the leading 1 in its binary digit. So h_2 and h_3 are level 2, h_4–h_7 are level 3, and so on. The concept of metrical level is therefore a grouping of Hadamard coefficients, which can be represented by a binary digit with wildcards ($*$). For $n = 3$, level 1 is 001, level 2 is $01*$ and level 3 is $1**$. Other ways of grouping may sometimes also be of use, as we will see in Section 2.4.

Hadamard coefficients are closely related with a simple way of defining syncopation, which I will call *onbeatness*. For a given metrical level, we define "on-beat" and "off-beat" positions, and subtract the number of off-beats from the number of on-beats. Higher, positive onbeatness then indicates unsyncopated rhythms, and lower, negative onbeatness, syncopated rhythms. The on-beats at level l are multiples of 2^{l+1}, and the off-beats are halfway between these, the multiples of 2^l that are not on-beats. Level 1 onbeatness is then equivalent to h_1.[1] Level 2 onbeatness is equal to the average of h_2 and h_3, level 3 onbeatness to the average of h_4–h_7, and so on. Because this measurement of syncopation is agnostic to the lower-level off-beats, at higher levels the measurement is divided up between more and more Hadamard coefficients, each of which has a different pattern of $+$s and $-$s on the lower-level off-beats. These are ultimately balanced across all the coefficients at that level.

For example, the rhythm 11100110 has onbeatness 1 at level 1, -1 at level 2, and 1 at level 3. Its full Hadamard transform is $(5, 1, 1, -3, 1, 1, 1, 1)$.

As Toussaint (2013, 67–68) points out, traditional definitions of syncopations are numerous, vague, and conflicting. One element often included in definitions of syncopation, missing from the simple definition above, is that an off-beat note gives a greater sense of syncopation when a subsequent on-beat position is not articulated. Section 2.2 below will consider how such a concept of syncopation may be defined and how it relates to the Hadamard coefficients. Toussaint's (2013, 68–70) own preferred measure of syncopation, *metrical expectedness*, does not have such a sequential element, and therefore is closely related to onbeatness. It can be obtained by multiplying the onbeatness at each level by a constant, and adding them.[2]

The idea of syncopation in the form of onbeatness can also be extended to specific Hadamard coefficients by considering the full binary representation. For instance, h_9 (in binary 1001) involves levels 1 and 4. Its Hadamard vector is $+-+-+-+--+-+-+-+$ which may be understood as subtracting the level-1 onbeatness of the second half of the rhythm from the level-1 onbeatness of the first half. The division of the rhythm into halves is the level-4 aspect of this coefficient.

1.2. *Sequency order and discrete Fourier transform*

Hadamard transforms are similar to discrete Fourier transforms (DFTs), which have been used in a number of studies to analyze rhythms and theorize problems relating to meter and rhythmic similarity (Amiot 2016; Yust 2021b,a). The relationship is made most clear by rearranging the rows of the Hadamard matrix in what is called *sequency order*. For $2^n = 8$ the sequency order is h_0-h_4-h_6-h_2-h_3-h_7-h_5-h_1:

$$\begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \end{pmatrix} \tag{6}$$

[1]This is essentially the same as Toussaint's (2013, 99–106) "off-beatness" measure.

[2]The constants are $(2^{n+1} - 1)/2^n$ at level 0 (cardinality) and $(2^{n-l+1} - 1)/2^{n-l+1}$ at level $1 \leq l \leq n$.

The sequency ordering can be obtained by alternately multiplying the second half of the row by -1 and rotating it, or by changing one digit at a time in the binary numbering of the Hadamard coefficients using the Gray code ordering (Ahmed, Rao, and Abdussattar 1971; Aung, Ng, and Rahardja 2008). Changing the first digit of the binary pattern multiplies the second half of the row by -1, while changing the digit after the first 1 of the binary numbering rotates the pattern (by an amount corresponding to the position of that digit). Sequency arranges the Walsh rhythms by their number of zero crossings, the number of times they switch from $+$ to $-$ or vice versa (Larsen and Crawford 1977; Wang 2007). The odd-numbered rows of the sequency ordering end on a $-$, so if they are conceived as cyclic (as they are in our rhythmic interpretation), we can count one more zero crossing at the wraparound. In this sense the zero crossings group the rows into odd-even pairs, and these are rotations of one another. Each odd-even pair $(a-1)$-a in sequency order $(1 \leq a \leq 2^{n-1}-1)$ corresponds to DFT coefficient $a/2$, while the zeroeth coefficients correspond and the last (h_1) corresponds to DFT coefficient 2^{n-1}.

Consider, for example, h_7 and h_5, with Walsh rhythms 10010110 and 10100101. These are adjacent in sequency order, and the latter is a rotation of the former forward two places. We will find these coefficients especially interesting in the applications below. They are both associated with the 3rd DFT coefficient, and indeed their rhythms have maximum values on this coefficient for four-note rhythms. Therefore rhythms that approximate divisions of the cycle into three or five parts, the tresillo rhythm 10010010 and cinquillo rhythm 10110110, load heavily on h_7 or h_5. Because the two Walsh rhythms are related by rotation, the magnitudes of all DFT coefficients are the same and only the phases differ. The phases are oblique (differing by $\pi/2$) on the 3rd DFT coefficient, so that each is associated with a different axis of the complex plane for this coefficient. (These are not the real and imaginary axes but a rotation that arranges them symmetrically around 0.) These Hadamard coefficients might therefore be loosely understood as a simple re-parametrization of the complex space of the 3rd DFT coefficient, but this is only approximately true, because the Hadamard vectors have non-zero values on the 1st DFT coefficient also. These kinds of associations can be made for all Hadamard coefficients, but the approximation becomes less good as the metrical level of the coefficient gets higher.

Perhaps the most important property in musical applications of the DFT (such as Lewin 2001; Quinn 2006; Yust 2016) is that the *magnitudes* of DFT components are invariant with respect to rotation. In the domain of pitch-class, this means that the DFT isolates properties of pitch-class sets that are invariant with respect to transposition. The DFT magnitudes, which are also invariant with respect to reflection (inversion) and general homometry (Z-relation), might be understood as the purely intervallic properties of a collection. This can be made precise by thinking of the DFT magnitudes as what is preserved in the autocorrelation of the pitch-class or rhythmic vector. By the convolution theorem, autocorrelation corresponds to multiplication of conjugates in the Fourier realm. Amiot's (2016) theorem 1.11 shows that this property (turning convolution into multiplication) is unique to the DFT.

The Hadamard transform therefore does not have this property. Only h_0 is invariant with respect to rotations. This is consistent with the idea that the Hadamard transform as measure types of syncopation in the form of onbeatness: rotation obviously does affect how syncopated a rhythm is. In fact, we can get more specific: rotations by multiples of 2^l do not affect the onbeatness at level l, and rotations of 2^{l-1} (or $k2^l + 2^{l-1}$) precisely reverse the onbeatness. (For a fuller treatment of such properties see Whelchel and Guinn 1968.)

Son clave	(1 0 0 1 0 0 1 0 0 0 1 0 1 0 0 0)
Bossa nova	(1 0 0 1 0 0 1 0 0 0 1 0 0 1 0 0)
Shiko	(1 0 0 0 1 0 1 0 0 0 1 0 1 0 0 0)
Rumba clave	(1 0 0 1 0 0 0 1 0 0 1 0 1 0 0 0)
Soukous	(1 0 0 1 0 0 1 0 0 0 1 1 0 0 0 0)
Gahu	(1 0 0 1 0 0 1 0 0 0 1 0 0 0 1 0)

Table 1. Toussaint's (2013) six "good" rhythms (timelines)

2. Applications

2.1. *Toussaint's six clave rhythms*

A central narrative thread of Toussaint's (2013) *The Geometry of Musical Rhythm* are the six clave or timeline rhythms given in Table 1. For Toussaint, these claim special status as representatives of rhythmic "goodness," and he analyzes them with multiple computational tools over the course of the book.

As the shorthand names make clear, they are freely drawn from musical traditions from Africa and Latin America. Toussaint bases his selection of these six rhythms on a mixture of empirical/musicological criteria and formal criteria. The central musicological motivation is to show that the clave rhythm of the Cuban son has special theoretical features that might explain its musicological importance. The other five rhythms are selected on the grounds of features they have in common with Son clave: they all have five onsets in a 16-pulse cycle, and they all share four onsets with Son clave. All but one rhythm (Gahu) differs from Son clave by a minimal "rhythmic voice leading," meaning that we need only move one onset by one unit to relate the rhythms. The remaining case, Gahu, moves one onset by two units (and relates to another, Bossa nova, by a minimal voice leading).

The Hadamard transforms of these rhythms are,

Rhythm	Hadamard transform															
	h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}
Son clave	(5	3	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	3	5)
Bossa nova	(5	1	−1	−1	1	1	−1	3	1	1	−1	3	1	−3	3	3)
Shiko	(5	5	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	3	3)
Rumba clave	(5	1	−1	3	1	1	−1	−1	1	−3	−1	3	1	1	3	3)
Soukous	(5	1	−3	1	3	−1	−1	3	1	1	1	1	−1	−1	3	3)
Gahu	(5	3	−3	−1	1	−1	1	3	1	−1	1	3	1	−1	1	3)

Figure 1 shows these as stacked columns in sequency order. Negative values for Son clave tend to be preserved in other rhythms, and the other rhythms tend to preserve which coefficients have large values (3 or 5) – specifically h_1, h_{14}, and h_{15}. In particular, Son clave has a maximal value of 5 (equal to its cardinality) for h_{15}, and all other rhythms have large values for this coefficient. Two other coefficients, h_{11} and h_7, receive large values for many of the rhythms, and these are precisely the coefficients adjacent to h_{15} in sequency order. All of these facts relate to the minimal voice-leading and maximal intersection relationships of Son clave to the whole set.

Son clave's maximal value of 5 for h_{15} indicates that it is a subset of Walsh rhythm 15. The other large coefficients (h_1 and h_{14}) come from the same four-note subset: (1000 0010 0010 1000), which is a subset of all three Walsh rhythms (1, 14, and 15). It is generally true that the intersection of any two Walsh rhythms, numbered a and b, will also be a subset of a third Walsh rhythm, numbered c, where c is given by the pointwise sum of the binary representations of a and b mod 2.[3] It therefore has a simple Hadamard transform, with a maximum value of 2^{n-2} for h_0, h_a, h_b, and h_c.

The intersection of Walsh rhythms 1 and 14 also has an interpretation as the augmentation of Walsh rhythm number 7. These is because doubling the coefficient number (from 7 to 14) is equivalent to taking the first half of the vector and repeating each element, and taking an intersection with h_1 retains just the even numbered elements of the vector. Walsh rhythm number 7 is significant because of its relationship to the tresillo, (10010010) and cinquillo, (10110110) rhythms, noted in Section 1.2 above. The fact that Son clave contains the augmentation of a tresillo rhythm as a subset is an important feature for its usage in Latin jazz, in which such augmented tresillo rhythms are often explicitly realized and are a useful way to unambiguously indicate the orientation of clave.

The added onset (in position 3) that turns this four-note subset of Walsh rhythms 1, 14, and 15 to the five-note Son clave shifts the weight of the rhythm towards h_{15} and away from h_1 and h_{14}. The particular choice of position 3 draws upon another relationship between Walsh rhythms 7 and 15, which is that they differ just the first (level-4) element binary representations and (as already observed) are adjacent in sequency order. This means that Walsh rhythm 15 is the concatenation of Walsh rhythm 7 and its complement. The added onset in position 3 of Son clave gives it a tresillo rhythm in its first half, which

[3] Each row of the Hadamard matrix is a multiple of the rows given by the 1s in its binary representation, and each of these basic rows is order two. Therefore multiplying rows a and b will give row c. If there is a 1 in the same place in both rows a and b, then, it will also occur in the same place in row c. This is the intersection of Walsh rhythms a and b.

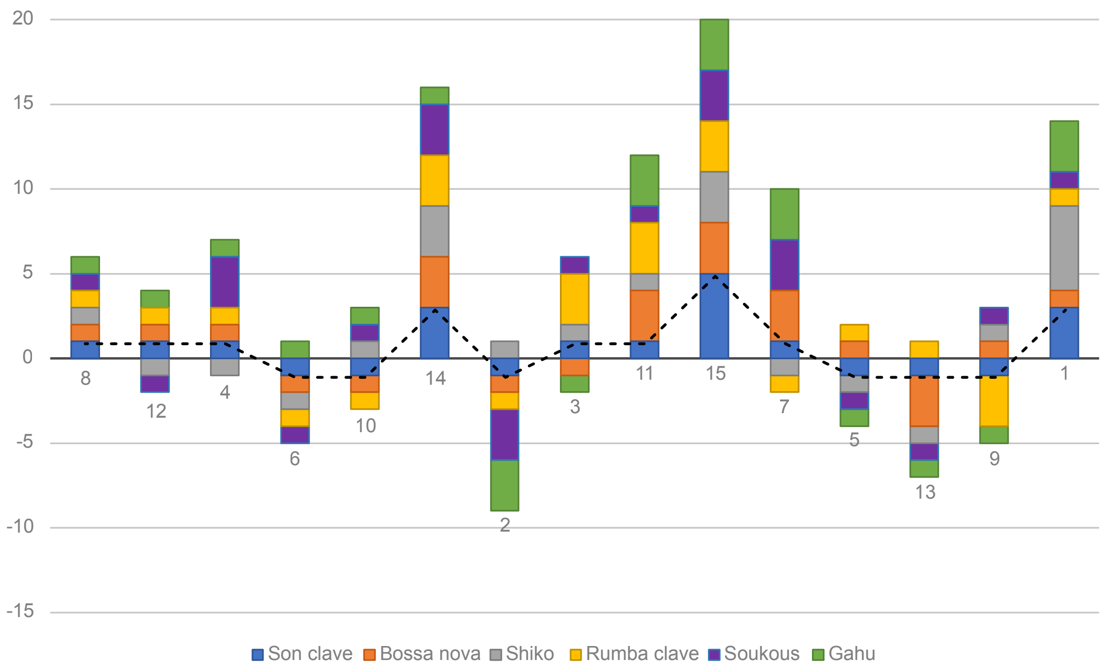

Figure 1. Stacked Hadamard transforms of Toussaint's six timeline rhythms. The dashed line shows the Son clave coefficients.

Latin jazz musicians call the "3-side" (Peñalosa 2009). In other words, Son clave may be understood as a two-leveled tresillo. Its first three onsets are a small tresillo, and every other onset (first, third, and fifth) make an augmented tresillo. The "3-2/2-3 concept" of Latin jazz, the idea that Son clave is made up of two opposed halves, a ying and yang so to speak (Peñalosa 2009), is therefore directly related to basic features of Walsh rhythm 15.

Other rhythms in Toussaint's group share some of these properties of Son clave, and some have their own related special properties. Shiko turns Son clave into an augmented cinquillo, shifting weight from h_{15} to h_1 (all onsets are at even positions). Three of the other rhythms (Bossa nova, Rumba, and Gahu) shift energy to h_{11}, which is a rotation of h_{15}. These are the two coefficients that relate to DFT coefficient 5, so these three rhythms may be understood as preserving or enhancing the approximate evenness of Son clave as a 5-in-16 rhythm. Bossa nova in particular maximizes this property, as a maximally even 5-in-16 rhythm.

2.2. *Unresolved syncopation*

As a theory of syncopation, the onbeatness measure defined above simply counts the notes occurring in metrically strong and weak positions at different levels. This is different than the conventional idea that a syncopation involves not only the appearance of a note in a weak beat, but also the absence of a note in a subsequent strong position. For instance, the rhythms 10010000 and 10000001 have the same level-1 onbeatness of 0, but the first would conventionally be regarded as syncopated while the second would not. This is because the strongly positioned note directly follows the weak one in the second rhythm, but not the first. Let us use the term *unresolved syncopations* for instances where an articulated off-beat at some level is followed by an unarticulated on-beat at the same level. The idea is that an on-beat note directly following an off-beat one "resolves" it, making it feel less like a syncopation.

A level-1 unresolved syncopation can be detected with an indicator vector like $(0,0,0,-,+,0,0,0)$. Multiplying by this vector will give -1 for an unresolved syncopation at position 3, and 1 or 0 otherwise. We can create a complete $8{\times}8$ invertible matrix that contains all such vectors, here in rows 1–4:

$$\begin{pmatrix}
+ & + & + & + & + & + & + & + \\
+ & 0 & 0 & 0 & 0 & 0 & 0 & - \\
0 & 0 & 0 & - & + & 0 & 0 & 0 \\
0 & - & + & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & - & + & 0 \\
+ & 0 & 0 & 0 & 0 & - & - & + \\
0 & - & - & + & + & 0 & 0 & 0 \\
+ & - & - & - & - & + & + & +
\end{pmatrix} \tag{7}$$

This is, in fact, a non-normalized Haar matrix up to rotation of the columns and rearrangement of the rows, suggesting that a more general approach to unresolved syncopation might apply discrete wavelet transforms (Fino 1972; Ahmed, Rao, and Abdussattar 1973; Wang 2007). Let us refer to this matrix as the "basic syncopation transform" with coefficients s_1–s_7. In addition to the trivial s_0 and the indicators for level-1 syncopation, s_1–s_4, the matrix has two coefficients, s_5 and s_6, for level-2 syncopation and one, s_7, for level-3 syncopation. These level-2 and level-3 coefficients only give intuitively satisfying results for rhythms limited to the beats of the given level. A better measure of level-2 syncopations is given by the sums of coefficients for syncopations over the same strong beat: $s_1 + s_5$ and $s_2 + s_6$. Similarly, level-3 syncopation is best measured by $2s_1 + s_5 + s_7$.

How then does the basic syncopation transform relate to the Hadamard transform and the concept of onbeatness? We can address this by multiplying it by the Hadamard matrix, giving the result

$$2\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 2 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2 & 2 & 2 & 2 \end{pmatrix}. \tag{8}$$

Each row of the above matrix gives the loadings of the Hadamard coefficients, in order, on the syncopation coefficients. Notice that all the level-1 syncopations involve a positive loading of h_1. This means that a decrease of level-1 onbeatness makes all forms of level-1 unresolved syncopation more likely. Other Hadamard coefficients are balanced across the level-1 syncopations, so they create syncopations only in certain contexts. When $h_2 + h_1 < 0$, one of the stronger syncopations of rows 1 and 2 will occur. When $h_1 - h_2 < 0$, one of the weaker level-1 syncopations of rows 3 and 4 will occur. The other relevant parameters for level-1 syncopations are $h_4 + h_7$ and $h_5 - h_6$, which generate syncopations when their absolute values are larger than $h_1 + h_2$ or $h_1 - h_2$ respectively.

Another method for classifying syncopation is to focus on the level of the missing strong beat, what Huron and Ommen (2006) call the "lacuna," instead of the syncopated note itself ("pre-lacuna"). An advantage of this method is that it is possible to isolate the syncopation indicators in 2^{n-1} linearly independent factors. For instance, at $2^n = 8$ we have

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 7 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & -2 & -2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & -6 \end{pmatrix} \tag{9}$$

Let us refer to this as a "rest-based" syncopation transform with coefficients r_0–r_7. Coefficients 1–4 indicate different kinds of syncopation: r_1 for syncopation across the downbeat, r_2 for syncopation across position 4, r_3 across position 2 (equivalent to s_3) and r_4 across position 6 (equivalent to s_4). The other three coefficients are a parametrization of the remainder of rhythmic information not related to syncopation in this sense.

As with the basic syncopation matrix, we can multiply by the Hadamard matrix to see the relationship of onbeatness to rest-based syncopation, which yields

	Rhythm	Hadamard	Simple syncopation	Rest-based syncopation
1.	10000001	$(2, 0, 0, 2, 0, 2, 2, 0)$	$(2, 0, 0, 0, 0, 2, 0, 2)$	$(2, 6, 0, 0, 0, 0, 0, -6)$
2.	10010000	$(2, 0, 0, 2, 2, 0, 0, 2)$	$(2, 1, -1, 0, 0, 1, 1, 0)$	$(2, 6, -1, 0, 0, -2, 1, 1)$
3.	10000011	$(3, 1, -1, 1, -1, 1, 3, 1)$	$(3, 0, 0, 0, 1, 1, 0, 3)$	$(3, 5, 0, 0, 1, 0, -2, -5)$
4.	10010010	$(3, 1, -1, 1, 1, -1, 1, 3)$	$(3, 1, -1, 0, 1, 0, 1, 1)$	$(3, 5, -1, 0, 1, -2, 1, 2)$
5.	10010100	$(3, -1, 1, 1, 1, 1, -1, 3)$	$(3, 1, -1, 0, -1, 0, 1, 1)$	$(3, 5, -1, 0, -1, -2, -1, 2)$
6.	10100011	$(4, 2, -2, 0, 0, 2, 2, 0)$	$(4, 0, 0, 1, 1, 1, -1, 2)$	$(4, 4, -1, 1, 1, 1, -1, -4)$

Table 2. Examples of Hadamard, basic syncopation, and rest-based syncopation transformations of 8-cycle rhythms

$$2\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 0 & 2 & 2 & 2 & -3 & -1 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 2 & 3 & -1 & -1 & -3 \\ 0 & 3 & 3 & -4 & 3 & -4 & -4 & 3 \end{pmatrix}. \tag{10}$$

Syncopation across the downbeat, r_1, is straightforward: it is given by a sum across all Hadamard coefficients except h_0. Level-2 rest-based syncopation, r_2, is positively correlated with level-1 and level-2 onbeatness and negatively with level 3. Also h_4 plays a special role because a level-2 syncopation must occur in the first half of the cycle. Syncopation types r_3 and r_4 are already familiar from the basic syncopation matrix.

A difference between these three ways of quantifying syncopation (onbeatness, simple syncopation, and rest-based syncopation) and many of the measures applied in computational and corpus-analysis studies, such as those discussed by Toussaint (2013), is that it breaks syncopation up in to a number of types differing in level, rather than producing a single quantity. Of course, it is possible to produce a single quantity by weighting and summing the different types, but for many applications it is useful to be able to keep the different types separate.

Table 2 provides some examples of how the two syncopation transformations work. Examples 1 and 2 are syncopated and unsyncopated rhythms, respectively, with the same onbeatness values. The syncopation is evident in the negative values of s_2 and r_2. Working from the Hadamard coefficients, we can associate the syncopation in rhythm 2 with the positive h_4+h_7. Rhythm 1 has a zero value for both h_4+h_7 and h_5-h_6. Similar points can be made about the unsyncopated rhythm 3 and the tresillo rhythm, number 4, which is syncopated. Notice how the presence of h_7 creates syncopations of type s_2 and r_2 in the absence of level-1 or level-2 onbeatness (either positive or negative).

Rhythm 5 of Table 2 is a rotated tresillo rhythm that is more syncopated than rhythm 4 because it has a negative level-1 onbeatness. This adds syncopations of type s_4 $(= r_4)$ and preserves the one of type s_2 and r_2. Rhythm 6 is an example with level 2 syncopation only. This is indicated by a negative level-2 onbeatness in the Hadamard transform, a negative $s_2 + s_6$, and a negative r_2. The basic syncopation matrix has the benefit of distinguishing the type of syncopation in rhythm 6 from those in rhythms 2, 4, and 5, whereas the rest-based syncopation matrix has the benefit of a simple representation of level-2 syncopation, involving a single coefficient instead of a sum of two.

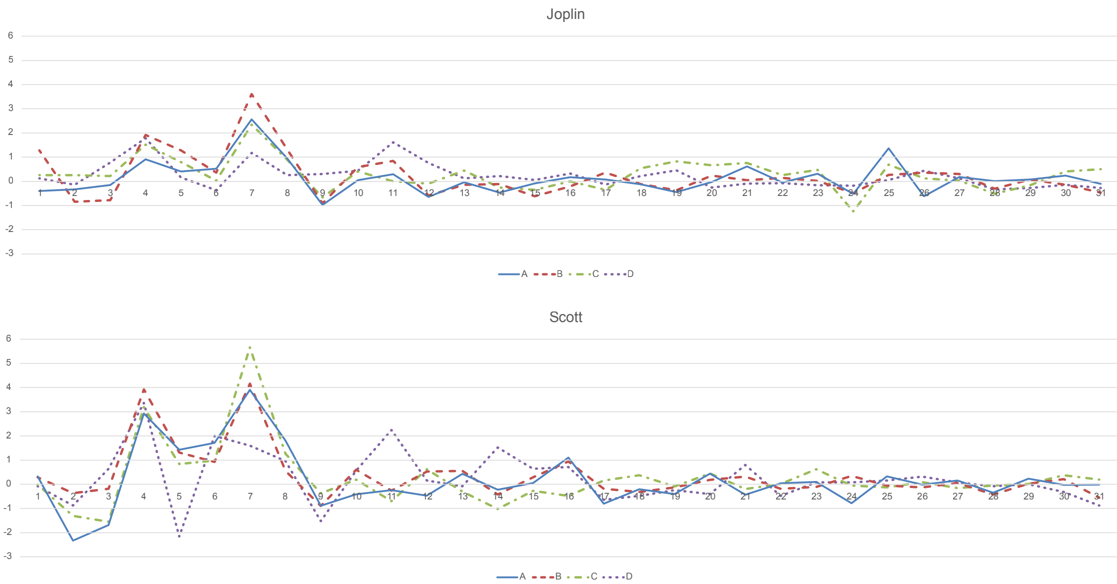

Figure 2. Average Hadamard coefficients of first and third phrases from a corpus of rags by Scott Joplin and James Scott, divided by strain order.

2.3. *Rhythm and form in ragtime*

A recent study (Yust and Kirlin 2021) uses the Hadamard transform to explore statistical trends in the "big three" ragtime composers, Scott Joplin, James Scott, and Joseph Lamb. The typical rag has four 16-measure strains, each individually repeated, which are essentially independent. Although the first strain sometimes acts as a refrain, returning in between other strains, in general the form is often simply progressive, moving from one strain to another, and not infrequently later strains are in a different key than the initial strain, giving the impression of a simple patchwork formal procedure. This is a false impression, however, because it misses important ways that composers express progress through the form using rhythm.

The data in Yust and Kirlin 2021 show how composers modify a typical rhythmic profile over the course of a complete rag. We analyzed the rhythm over four-measure phrases, and because the minimal rhythmic unit is a sixteenth-note in 2/4, this results in rhythmic vectors of 32 elements. Figure 2 shows average Hadamard coefficients across a number of rags for two of the composers, Joplin and Scott, divided between strains, labeled A, B, C, and D according to their order in the piece. Only the first and third phrases of each strain are included, to exclude the cadential rhythms that typically occur in second and fourth phrases. The Hadamard transform immediately shows that there is little to no systematic contrast between the two halves of the four-measure phrase (once cadential phrases are eliminated), because h_{16}–h_{31} are essentially flat. The only exceptions are Joplin's A- and C-strain rhythms, where h_{24} and h_{25} show a tendency to weight the interior (second and third) measures with eighth-note syncopations. Similarly, contrasts between individual measures (first-second or third-fourth), h_8–h_{15}, are less important than the distinctive rhythms within measures, as is evident in the strong profile of h_1–h_7.

One overall trend is immediately evident: large values for h_7 and h_4, and negative values for h_2 and h_3. Drawing upon the last section, we can say that syncopation in ragtime tends to originate in negative level-2 onbeatness $(h_2 + h_3)$ and large values for

$h_4 + h_7$, which are associated with level-1 syncopations crossing level-2 beats. These are what Temperley (2021) calls “fourth-position syncopations,” which he shows occur in ragtime, and not in earlier British and African American published songs. Berlin (1980) and Volk and de Haas (2013) also show that fourth-position syncopation becomes more common in post-1900 ragtime music (which includes most of the Yust and Kirlin 2021 corpus discussed here). The later is particularly notable because it provides a method to create syncopation without negative onbeatness, and because of the relationship of h_7 to the tresillo rhythm, which, as Cohn (2016) and others have pointed out, is common in ragtime.

For both composers, however, the profile of D-strain rhythms is distinctly different. In these concluding strains h_7 becomes less prominent, and h_{11} more so. The binary representations of these coefficients are 00111 and 01011 respectively. This means they share the combined level-1 and level-2 contrasts, but h_7 further contrasts these at level 3 and h_{11} at level 4. In other words, there is a tendency towards rhythmic broadening from 1-measure to 2-measure patterns for both composers, which evidently has a special concluding function, giving a satisfying form to the whole rag.

In Scott’s rags, an even more distinct pattern is evident in D strains involving negative h_5 and positive h_{14}. This pattern focuses the syncopation specifically on the downbeats of even measures. Note also that Walsh rhythm 5 is a rotation of Walsh rhythm 7, and 14 is an augmentation of it. Therefore we can say that Scott achieves this characteristically conclusive broadening of syncopations while retaining the tresillo-like quality (h_5) of his rhythms, and adding an augmented-tresillo quality (h_{14}).

Scott’s “Ragtime Oriole” illustrates this typical pattern. Figure 3 shows the melody of the first phrase of each strain, and Table 3 gives the first half of the Hadamard transform (excluding level-5 coefficients). The A strain is dominated by a common high-saturation rhythm whose distinctive property is eighth-note syncopation across second beats. The B strain starts with completely flat rhythms, and then has tresillo-like rhythms that closely resemble Walsh rhythm 7. The C strain is where the true commitment to Walsh rhythm 7 materializes – the melody almost perfectly matches it almost all the way through. While the A and B strains have modestly high values for h_7, the C strain has a value almost equal to the cardinality (the maximum possible). The juxtaposition of this with strain D highlights the change of character. Strain D has a similar quality to C in the sense of tresillo-like repetition at the dotted eighth, but this pattern syncopates across the downbeats of the even measures rather than second beats of all mesures. The “tresillo-like” quality, which we might associate with DFT coefficient 12 (in the 32-cycle, equivalent to coefficient 3 of the 8-cycle), loads negatively on h_5 rather than positively on h_7, its sequency partner. In the DFT, this would appear as a high value for coefficient 12 in all strains, but a change of phase by $\approx \pi/2$ for this coefficient in the D strain. Scott marks C and D as a Trio in this rag, but the same rhythmic progression occurs in other rags without Trio markings or da capo returns. Such use of rhythm to shape the form is underappreciated by traditional form theory with its primary emphasis on tonality.

2.4. *Balinese gamelan gong cycles*

The music of Balinese Gamelan Gong Kebyar centers around gong cycles, referred to as *tabuh*, which are a fixed length in number of beats. All of the traditional *tabuh* have lengths in powers of two. A piece will usually contain a number of these gong cycles, and the basic melodic content of each is largely derived from a skeletal melody called the *pokok*. Tenzer (2000) quotes a number of *neliti* from the standard Kebyar repertoire,

	h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}
A	27	−3	−5	−3	3	5	3	5	1	−1	1	−1	1	−1	1	−1
B	25	3	−3	−1	3	1	−1	5	1	−1	1	−1	−1	1	−1	1
C	17	−1	−1	1	−1	1	1	15	−1	1	1	−1	1	−1	−1	1
D	26	−2	−2	2	−2	−6	2	−2	−2	2	2	6	2	−2	6	2

Table 3. Hadamard coefficients 0–15 (excluding 16–31) for the initial phrases of the four strains of "Ragtime Oriole," shown in Figure 3.

which are simple even-rhythm elaborations of the *pokok*. Each *neliti* expresses different kinds of motion at different levels, with each level reducing the previous one by a factor of two by taking every other note, until a single note, the gong tone, remains. Different instrumental layers of the gamelan engage with the *neliti* at different levels: if we assign the *neliti* itself to level 0, the *calung* plays the *neliti* at level 1 (reducing by a factor of 2), the elaborating parts usually engage with the *neliti* at levels 1 or 2, and so on.

This multi-leveled strictly duple rhythmic design suggests that the Hadamard transform might be a good tool for analyzing these *neliti*, but in a literal sense they are rhythmically undifferentiated, articulating every beat. However, they can be differentiated by the simple procedure of partitioning the total, undifferentiated, rhythm into five rhythms, one for each tone in the Balinese *pelog* scale, named ding, dong, deng, dung, and dang. (The larger intervals, approximately a major third or less, are from deng to dung and dang to ding. The other intervals, ranging from about a semitone to a narrow whole tone, are from ding to dong, dong to deng, and dung to dang.)

Tenzer (2000) chooses eight *neliti* of length 16 as representative of the style, with an emphasis on illustrating a range of types. One way to analyze these melodies is to sum Hadamard coefficients across each level. Figure 4 shows three of Tenzer's examples, with the melody written in a novel notation using the vowels (i, e, u, o, a) to indicate each note, and with a single staff line between dang and ding to show octaves. The dashed line shows the *pokok*, played by the *calung*, and the solid line a reduction of that played

Figure 3. The first four measures of each strain of James Scott's "Ragtime Oriole," right hand part, and vector notation for the rhythm.

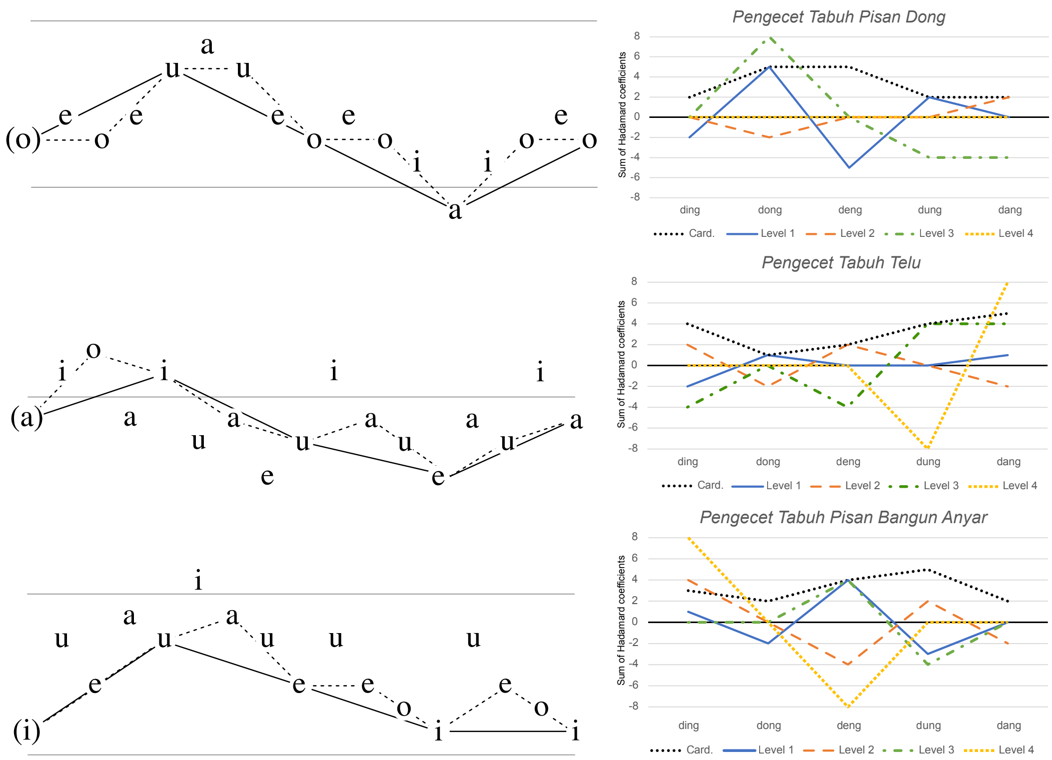

Figure 4. Three *neliti* from Tenzer (2000), *Pengecet Tabuh Pisan Dong*, *Pengecet Tabuh Telu*, *Pengecet Tabuh Pisan Bangun Anyar*, and their Hadamard profiles summed over four metrical levels.

by *jegogan*. The graphs on the right produce an analysis similar to Tenzer's, presenting motion and stasis at different metric levels. The first has motion at levels 1 and 3, and relative stasis at levels 2 and 4. Positive values at each level (such as dong at level 3) indicate strong metrical positions and negative values (such as dung and dang at level 3) weak positions at the given level. Note that the values for each tone must sum to 0, except the gong tone, which sums to 16.

The second melody contrasts with the first, having more motion at levels 3 and 4, and less at level 1. The level-4 motion simply indicates that what Tenzer calls the "axis" note (the midpoint) is different than the gong tone. The flat profile of level 1 does not precisely correspond to Tenzer's idea of stasis, but instead shows that the tones are relatively evenly distributed between *pokok* tones and intermediate tones. The melody largely achieves this by patterning in threes, so that subsequent repetitions of a given tone usually alternate strong-weak.

The third melody has a relatively strong profile at all levels. At levels 2 and 4, the gong tone ding alternates with deng, while at levels 1 and 3, deng tends to occupy the strong positions alternating especially with dung.

Summing across levels, however, removes some potentially useful information from the rhythmic profiles of the melodies. To see which coefficients are most active overall, we can sum squared values of each coefficient across the five scale tones. Figure 5 gives the averages of these sums of squares across the eight sixteen-beat *neliti*s. The four lines split the results up based on the first and last value of the binary representation of the coefficients, to show the general trend of high values for coefficients of the form $**\,*1$ and lower-level coefficients of the form $0***$. The other trend occurs within the

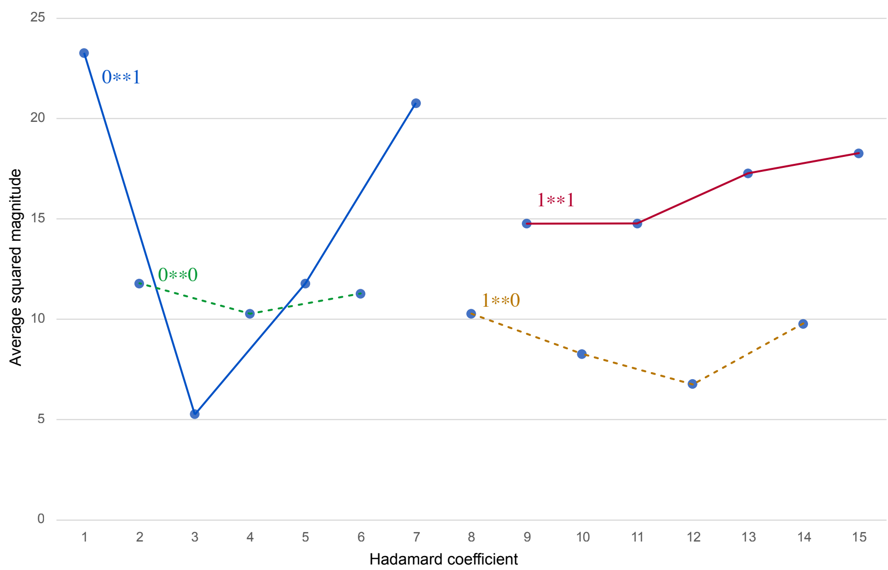

Figure 5. Average squared Hadamard coefficients of Tenzer's eight sixteen-note *neliti*.

coefficients of the form $0{*}{*}1$ where h_1 (0001) and h_7 (0111) have particularly high levels of activity. The first of these shows a tendency for scale tones to be differentiated at level 1 (*pokok* vs. intermediate). The high activity in h_7 is particularly interesting: this is the coefficient with the tresillo-like (3-generated) Walsh rhythm that we found to be significant in ragtime as well as Toussaint's clave rhythms. Appearing in the the Balinese *neliti*s, it indicates a tendency for notes and patterns to repeat at intervals of three beats. The melodies' quality of motion ("*majalan* quality") at level 3 is usually produced by such patterning in threes. This is particularly noticeable in the second *neliti* of Figure 4, for instance. Similar patterns of repetitions in threes over the duple framework occur at 4× diminution in conventional *ubit-ubitan kotekan* patterns (see Tenzer 2000 pp. 226–231), suggesting that this might represent a stylistic trait that transfers from this conventionalized lower-level manifestation to the more structural melodic constructions of *neliti*.

Figure 5 could lead us to guess that other ways of profiling *neliti*, besides simply by level, may also be of use. Figure 6 shows profiles of h_1, h_7, h_9, and h_{15} for the second melody of Figure 4. This selection includes the important lower-level coefficients (1 and 7), and their level-4 partners – in binary notation $abb1$, with $a, b = 0$ or 1. The cardinality line is reproduced on both positive and negative sides to show the maximum and minimum possibles value of each coefficient for each scale tone. This profile reveals that the level-3 motion of the *Tabuh Telu* melody is attributable to h_7, as the alternate, contra-metrical, tresillo-based parsing of the melody (the solid and dashed lines on the left of Figure 6) shows.

3. Conclusion

Unlike its cousin the DFT, the Hadamard transform only applies to certain types of rhythmic cycle, those whose length is a power of two. These types of rhythmic cycle are exceedingly common, however, in a wide variety of musical styles. They have the

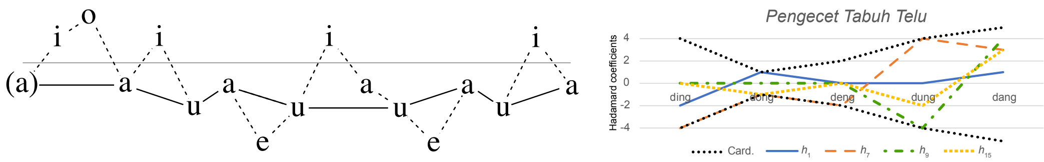

Figure 6. An alternate Hadamard profile for *Pengecet Tabuh Telu*.

special property that they admit of a single dense metrical representation, with a large number of metrical levels relative to the number of elements in the cycle. That is, there are no "hemiolas" in the pure-duple situation, where different ways of arranging factors give rise to different metrical interpretations. These special features are related to the one exploited in this paper, the existence of a Hadamard transform. The property that distinguishes the Hadamard transform from the DFT, and makes it sometimes more useful, is its relationship to the duple metrical interpretation. Even where the Hadamard transform can play a valuable role in representing types of syncopation with respect to such a metrical framework, however, some of the findings of this paper are best interpreted through the relationship of Hadamard coefficients to DFT coefficients through the sequency ordering. Prominent among them is the repeated finding of a special role for h_7 in a variety of circumstances – clave rhythms, ragtime, and Balinese melodies. Two seemingly quite different explanations for this finding are in fact logically equivalent: first, that h_7 can produce strong level-1 syncopations without requiring negative level-1 onbeatness (section 2.2), and second, that h_7 approximates the DFT coefficients that divide measures by three, and points to their corresponding maximally even tresillo and cinquillo rhythms (section 1.2). The Hadamard transform is therefore useful less as a replacement for than as a counterpart to the DFT and traditional concepts of syncopation.

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