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Fourier Phase and Pitch-Class Sum

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Summary

- There is a fundamental convergence between two very different music-theoretical worlds, voice-leading space and Fourier space
- This convergence appears only in delicate circumstances.
  - For points, or chords: typically when we limit ourselves to the transpositions of some chord lying in some scale.
  - For voice leadings: typically when our chord divides the octave nearly evenly.
- To explain this correspondence we need to think rigorously about how to represent voice leading in the Fourier perspective:
  - Glide paths vs. crossfade paths

CIRCULAR VOICE-LEADING SPACE

Circular voice-leading space

- Abstract and simplified representations of the higher-dimensional configuration spaces representing *n*-note chords.
  - Points represent *entire chords.*
  - The spaces depict the bijective, strongly crossing-free voice leadings among the transpositions of any *n*-note chord in any *d*-note scale.
- A spiral winding *n* times around an annulus, with *d* chords equally spaced along it.

http://dmitri.mycpanel.princeton.edu/cs.html

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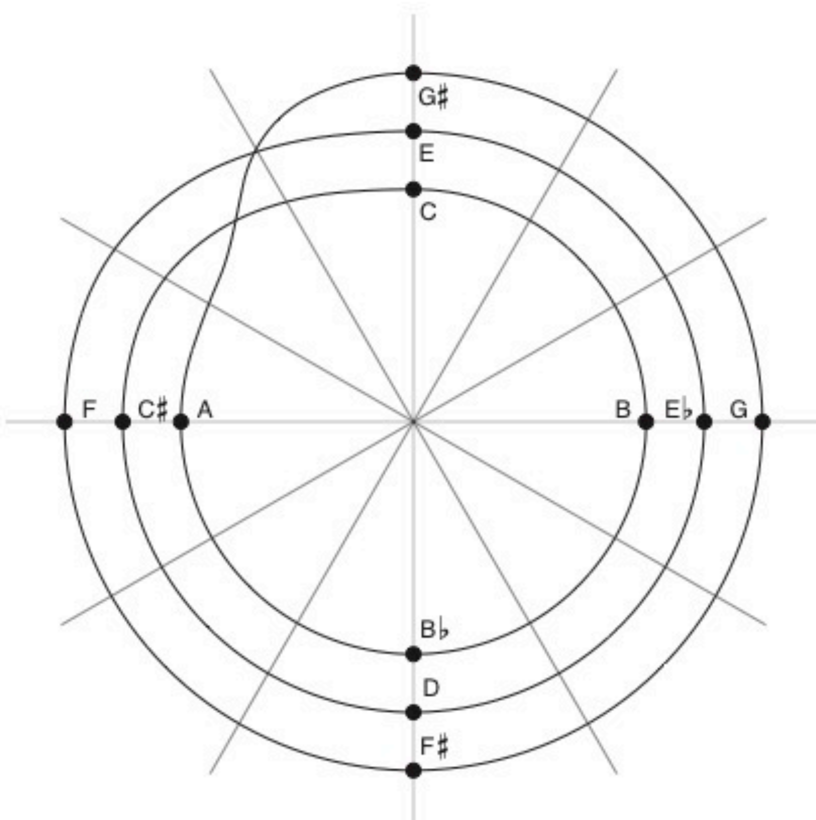
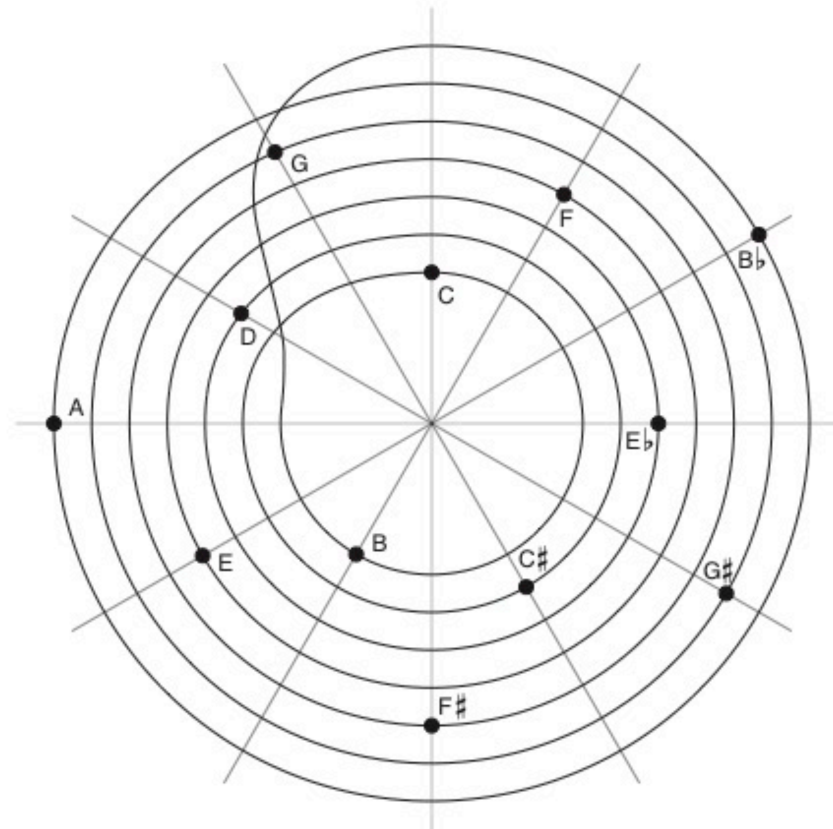
Circular voice-leading space





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Circular voice-leading space

- Moving along the line represents transposition along the scale.
  - Angular position corresponds to pitch-class sum.
- A complete circle, understood as 360° motion along the spiral, followed by radial motion back to the starting point, corresponds to transposition *along the chord*.
  - As if the chord was itself a scale.
- Homotopic paths represent the same voice leading.

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FOURIER SPACE

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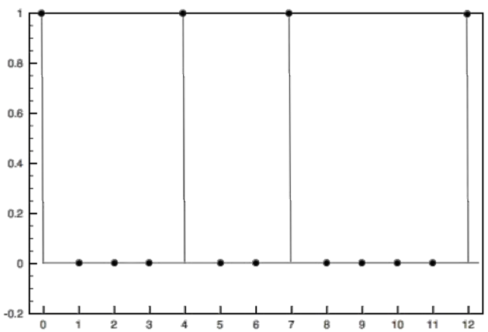
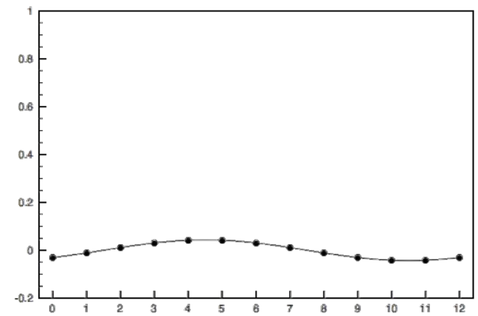
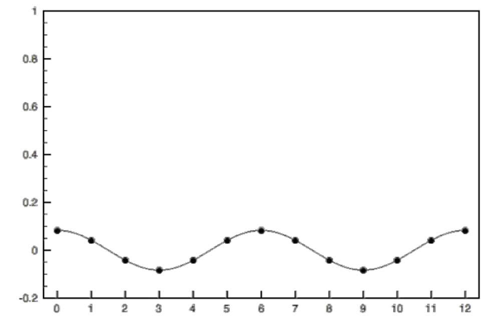
Fourier analysis of a pcset

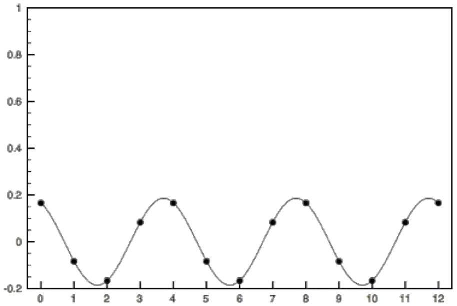
C major triad = F_0 + F_1 + F_2

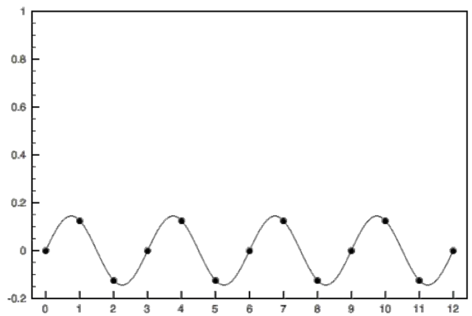


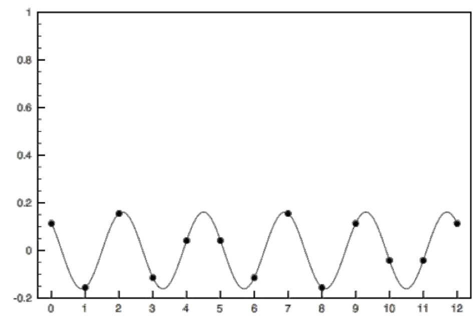


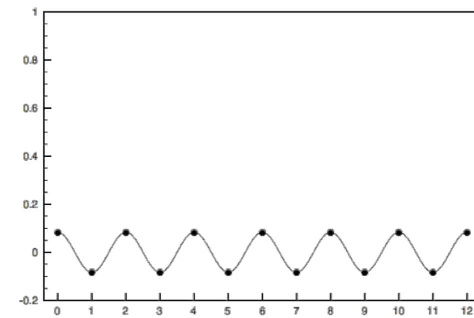


+ F_3 + F_4 + F_5 + F_6 + . . .







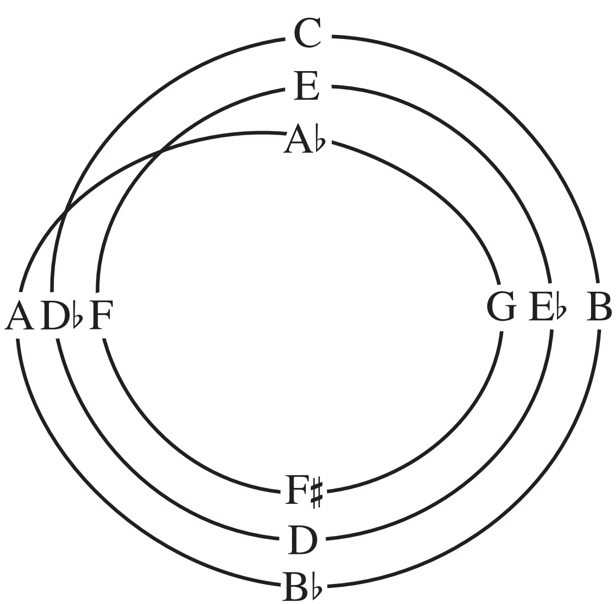


Fourier components as vector sums

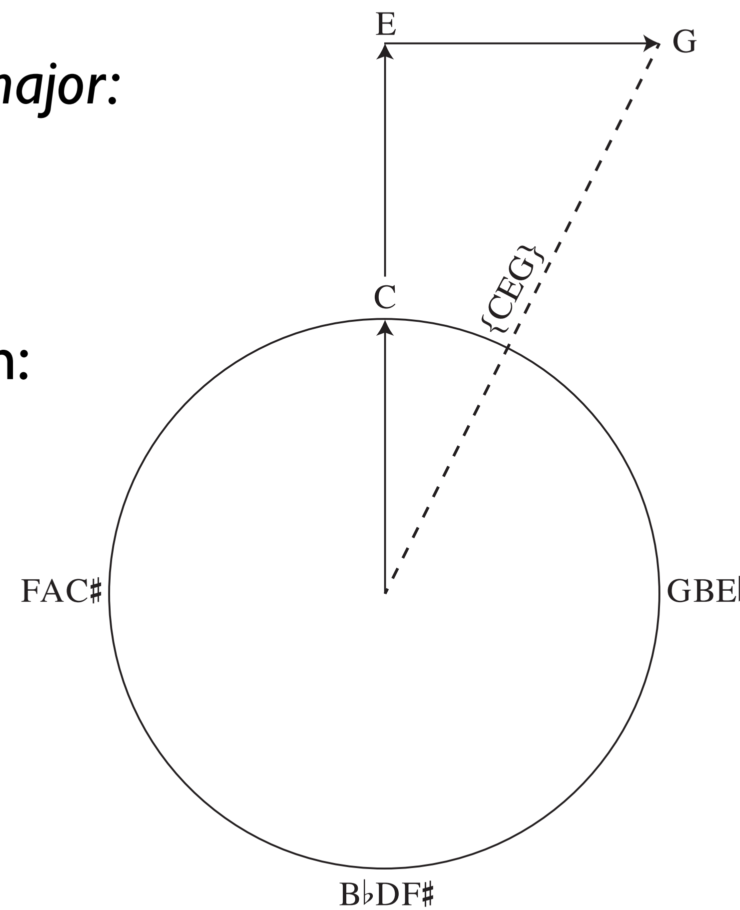
The k^{th} Fourier component, f_k, of a chord is a vector sum of its pitch-classes in complex space.

Example: f_3 of C major:

"Reduced 8ve" of f_3:



Vector sum:

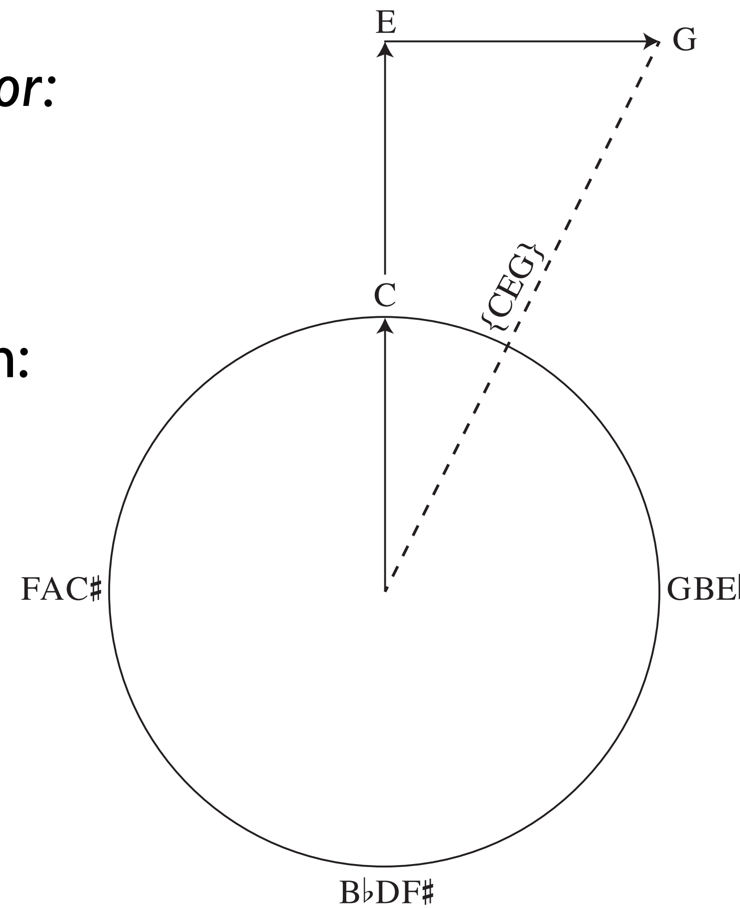


Fourier components as vector sums

The **phase** of the k^{th} Fourier component, φ_k, is the angular component of f_k.

Example: f_3 of C major:

Vector sum:



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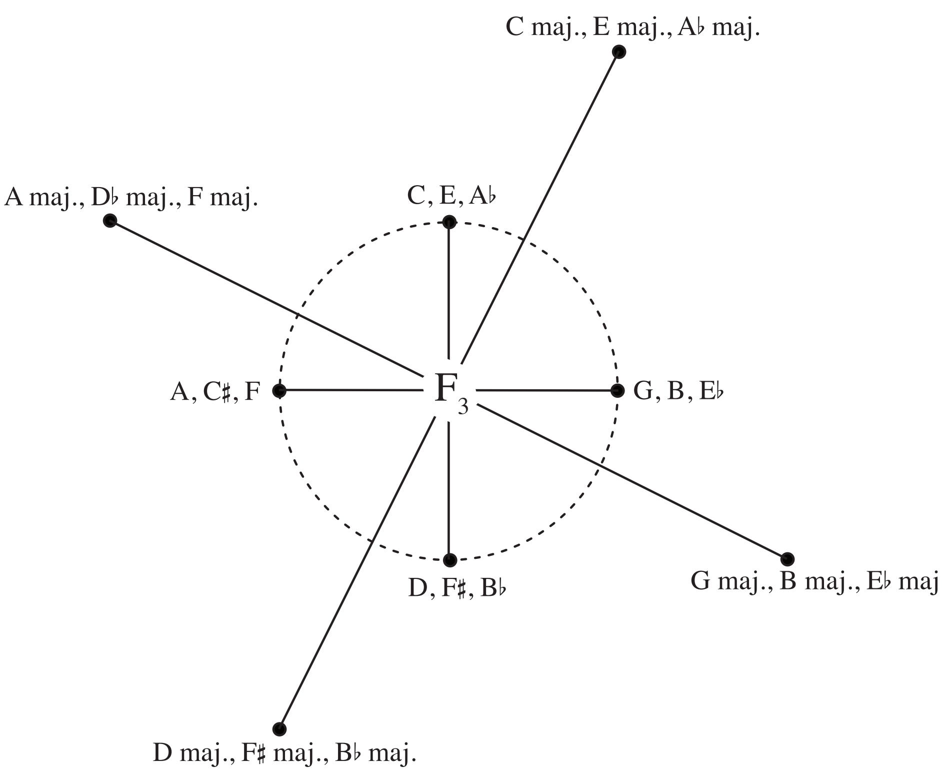
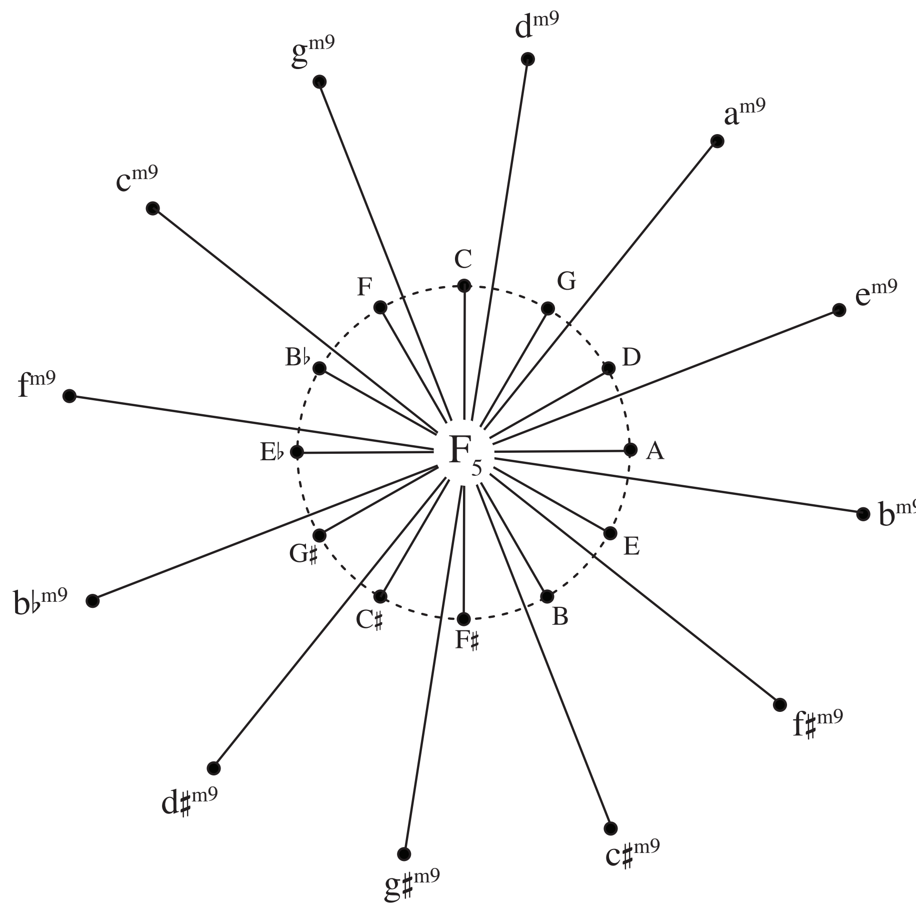
Fourier components as vector sums

Examples: f_3 of triads, f_5 of minor ninths





THE CORRESPONDENCE (POINTS)

Phase and Voice-Leading Sum

Proposition: For a given *n*-note chord type (transposition type) in a *c*-note scale, phases are equivalent to voice-leading sums, i.e.

$$S(T_x(A)) - S(A) = Ph_n(A) - Ph_n(T_x(A))$$

$$\text{with } Ph_n \cong (c/2\pi)\varphi_n$$

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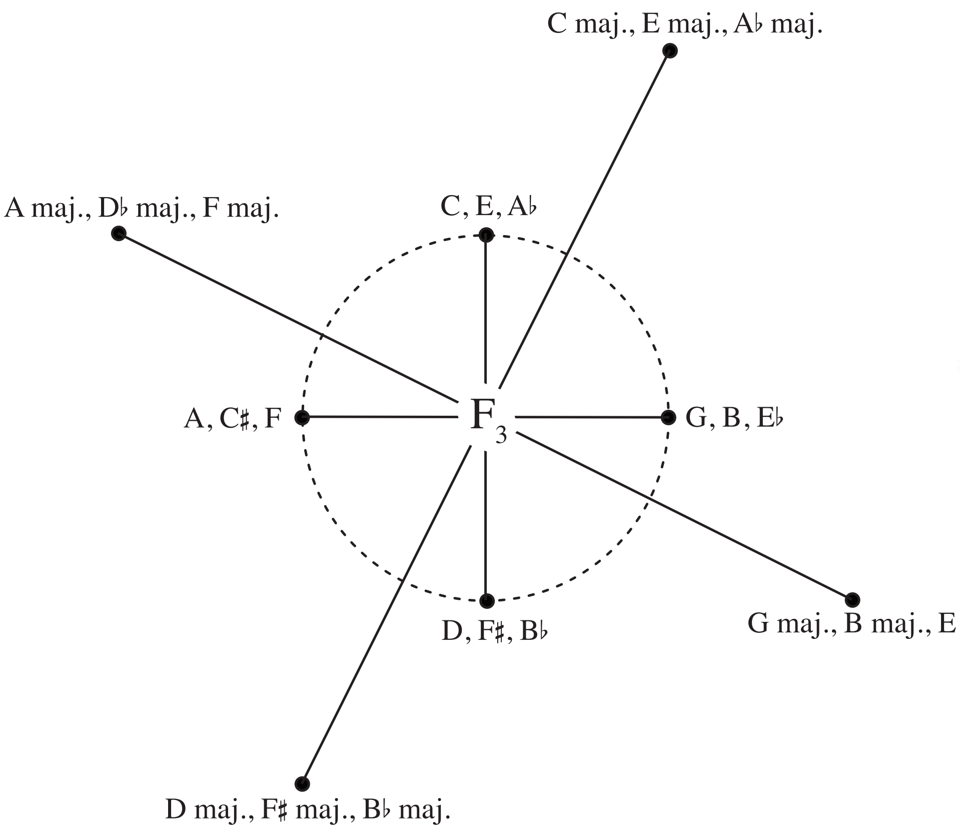
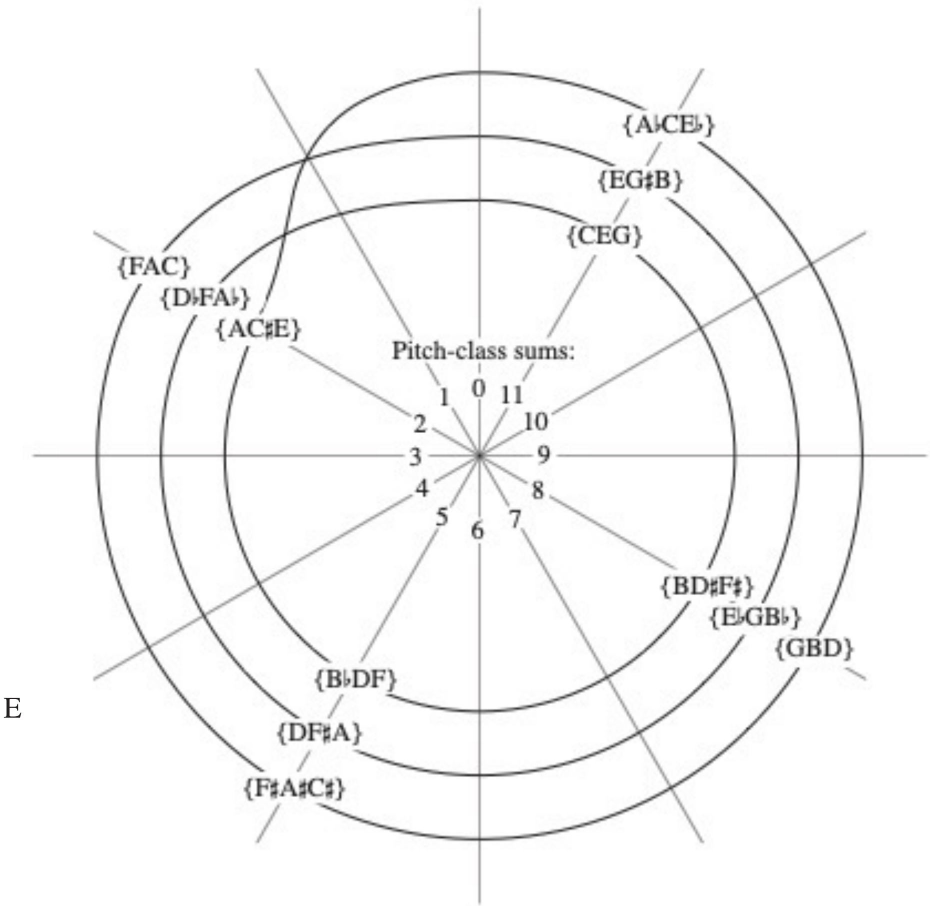
Phase and Voice-Leading Sum

Example:





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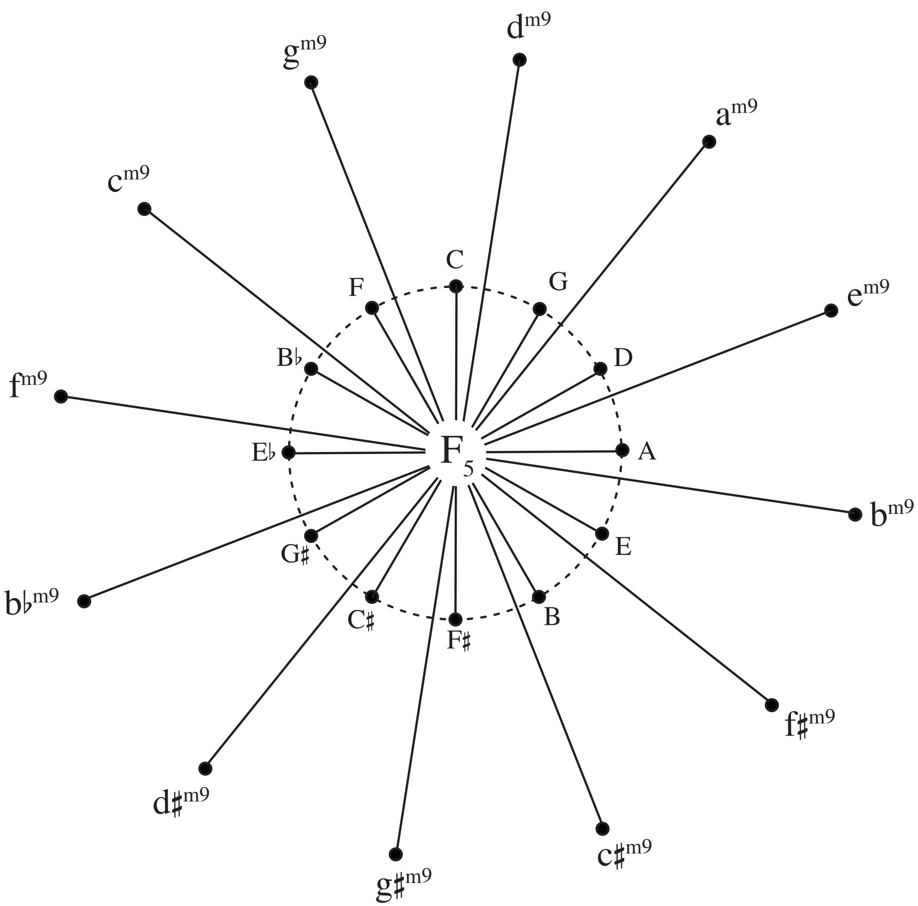
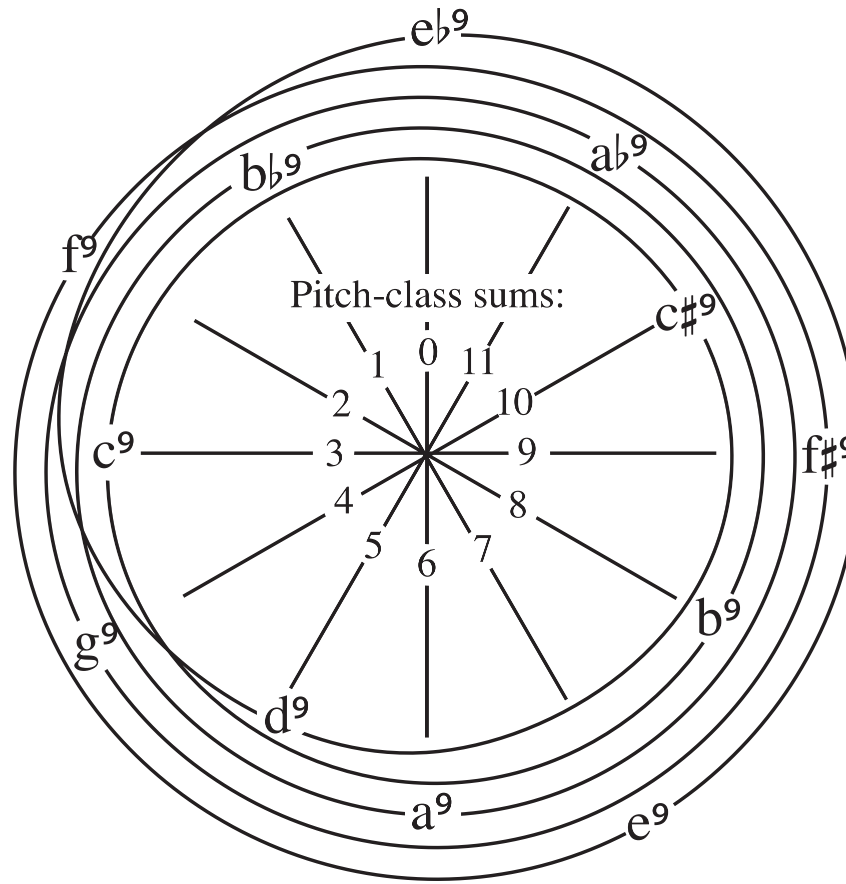
Phase and Voice-Leading Sum

Example:

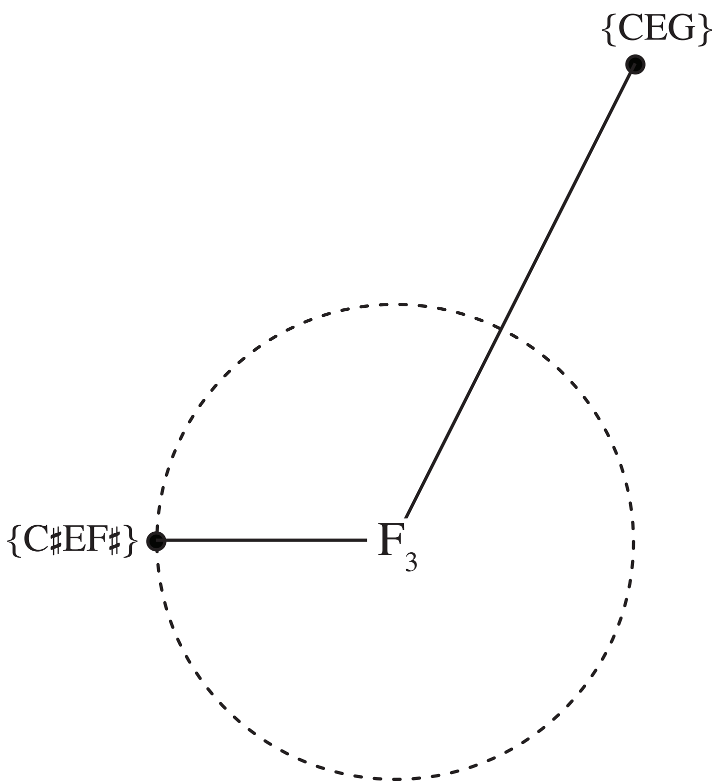


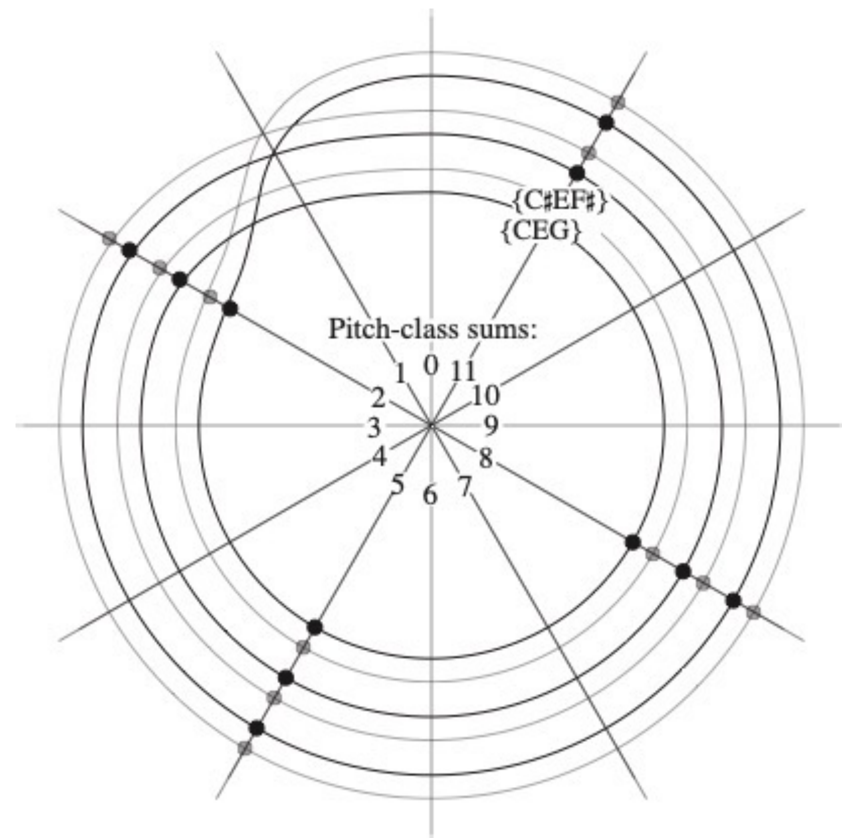


Phase vs. Pitch-class sum: different chord types

Isomorphism of phase and pitch-class sum differences breaks down for a mixture of chord types.

Example: same sum, different phases





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VOICE LEADING IN FOURIER SPACE

Voice leading in Fourier space

- With configuration spaces (voice leading spaces) we typically represent voice leadings as paths:
  - Specifically the path that is taken if each pitch class glides smoothly to its destination at a constant velocity, starting and stopping at the same time.
- How can we represent voice leading in Fourier space?

Voice leading in Fourier space

- Strategy 1: *glide paths*
  - Do exactly what we do in voice-leading space.
  - Let each note glide smoothly to its destination, and take the Fourier transform of every point along this path.
  - Produces smooth paths in Fourier space.
- Strategy 2: *crossfade paths*
  - Linear interpolation between the vectors corresponding to the initial and final chords.

THE CORRESPONDENCE (GLIDE PATHS)

Question:

Are glide paths in Fourier space homotopic with the associated voice-leading paths in circular voice-leading space?

using math we can show this is equivalent to:

Do zero-sum (*balanced*) voice leadings between transpositionally related chords ever produce nontrivial phase space paths?

Answer:

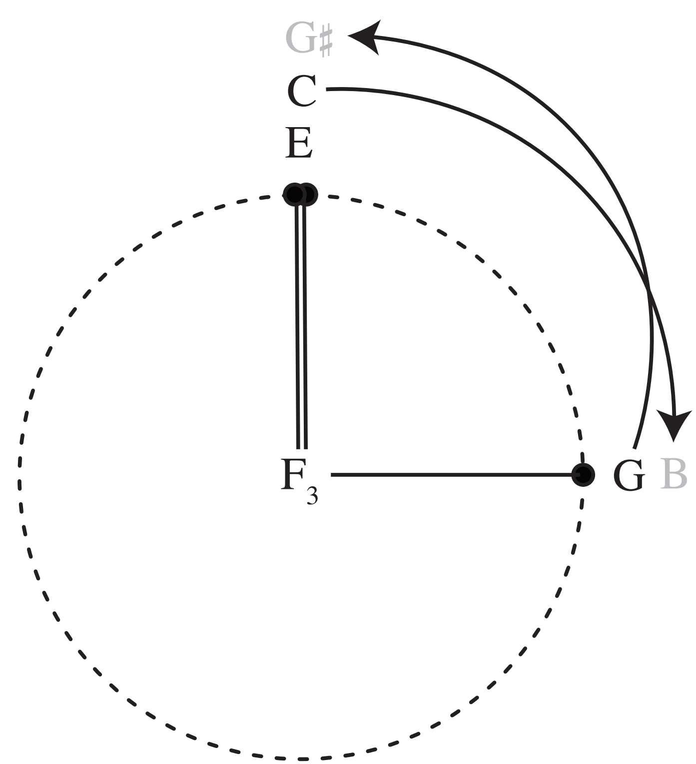
Are the paths homotopic?

Yes, when we look at the *n*th Fourier component of a **relatively even** n-note chord.

Example:

{C, E, G} → {B, E, G♯}

Trivial in both voice-leading and fourier space



Answer:

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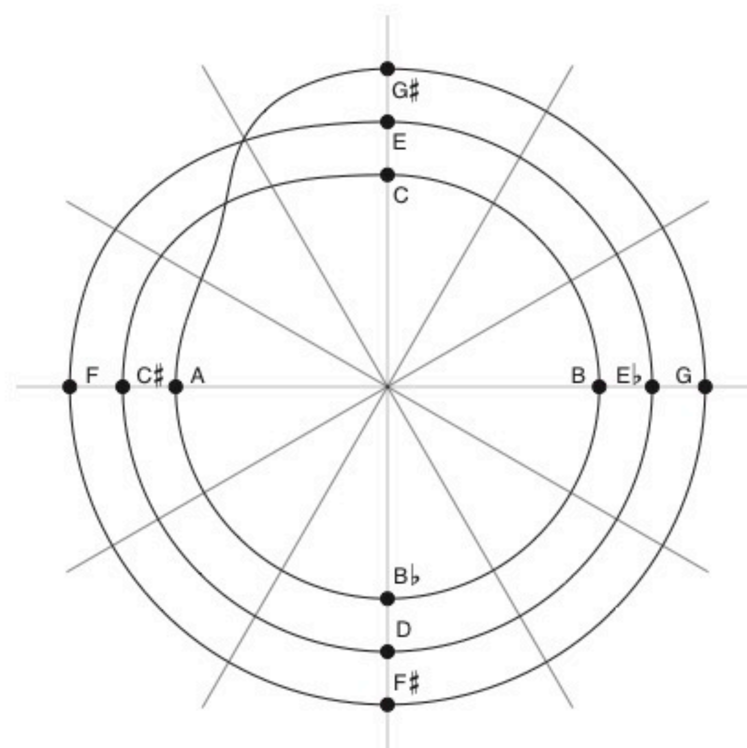


Answer:

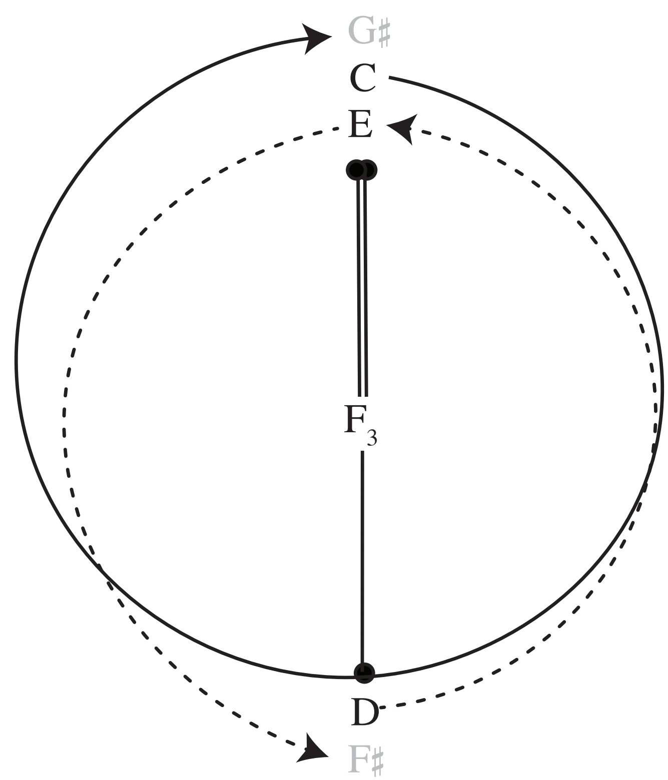
Are the paths homotopic?

No, when we look at the *n*th Fourier component of a **not very even** n-note chord.

Example:

{C, D, E} → {G♯, E, F♯}

Trivial in voice-leading space, **nontrivial** in fourier space



An open question:

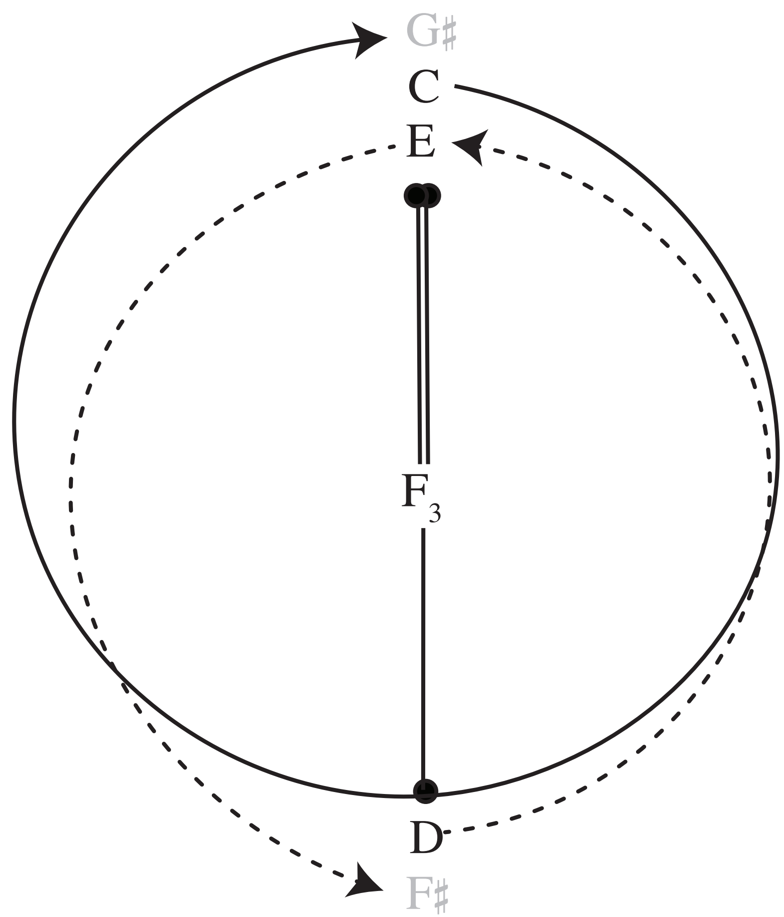
How uneven does a chord have to be for a balanced voice leading to cycle phase space?

We don't know.

Example:

{C, D, E} → {G♯, E, F♯}

In this example, one voice travels by a full 8ve/*n*



THE CORRESPONDENCE (CROSSFADE PATHS)

Crossfade paths

- Represent voice leading as a linear interpolation between the vectors corresponding to the initial and final chords.
  - "fade out" the initial chord while "fading in" the second chord.
- Always take the *shortest* path along circular Fourier phase space.
  - Cannot represent the different paths between antipodal points.
- To distinguish sharpward from flatward requires intermediary points.

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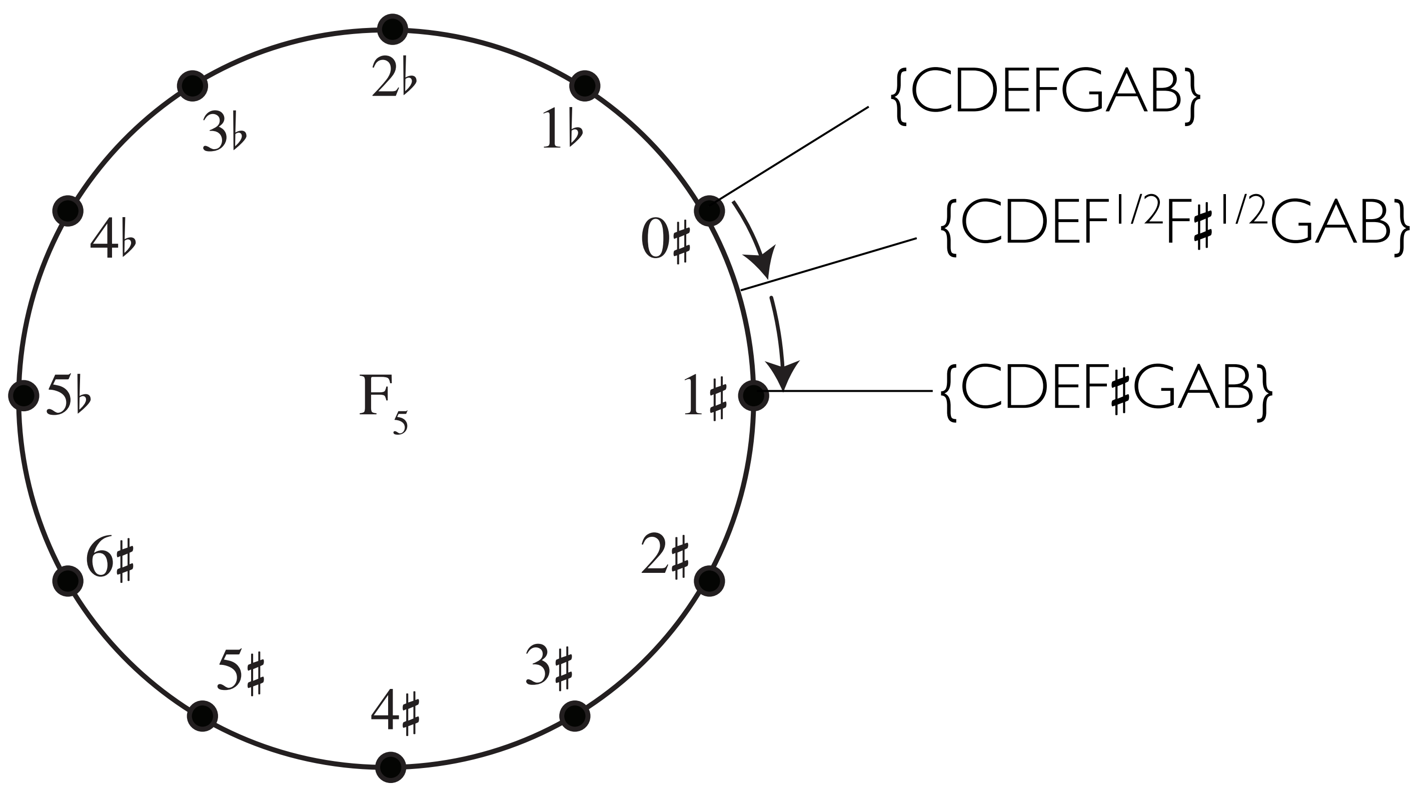
Cross-Fade vs. Glide Paths

Paths in Fourier spaces can be constructed as **cross-fades** or as glide paths

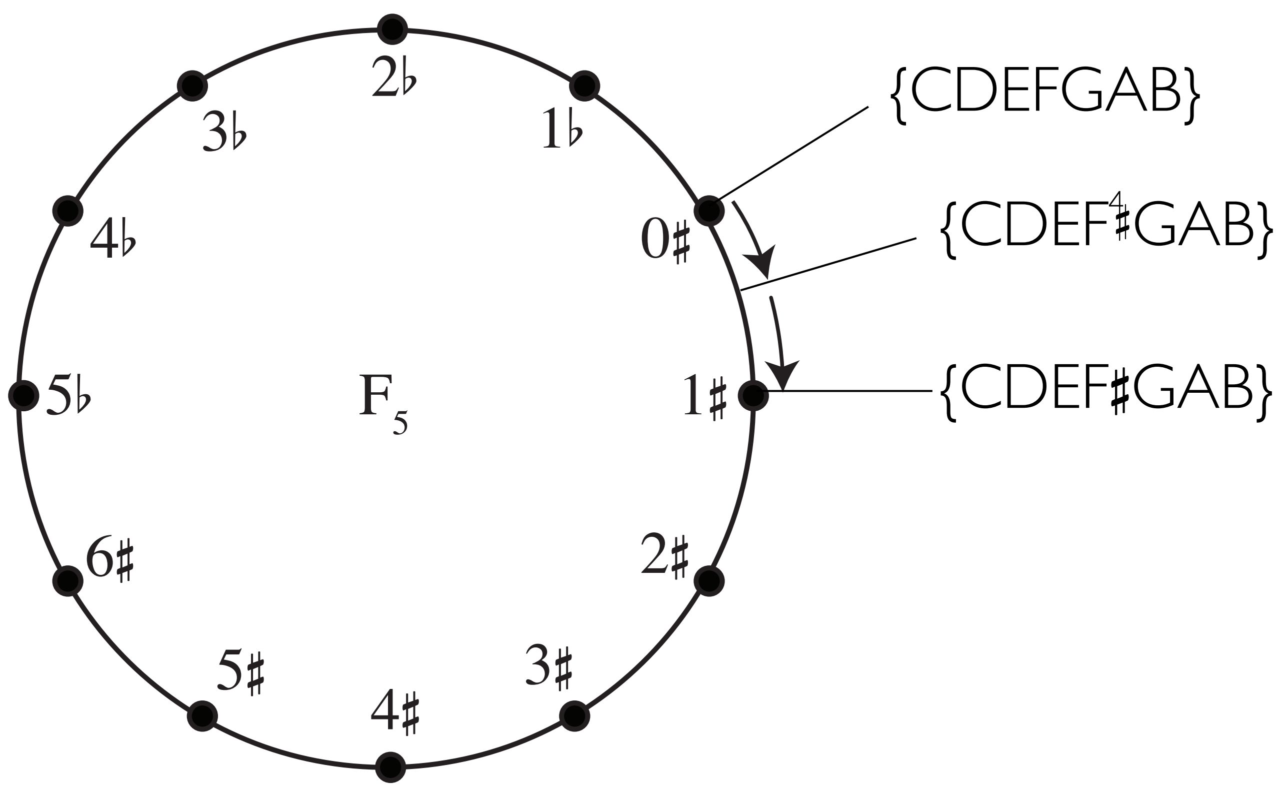
Cross-fade:



Cross-Fade vs. Glide Paths

Paths in Fourier spaces can be constructed as **cross-fades** or as glide paths

Glide:



Cross-Fade vs. Glide Paths

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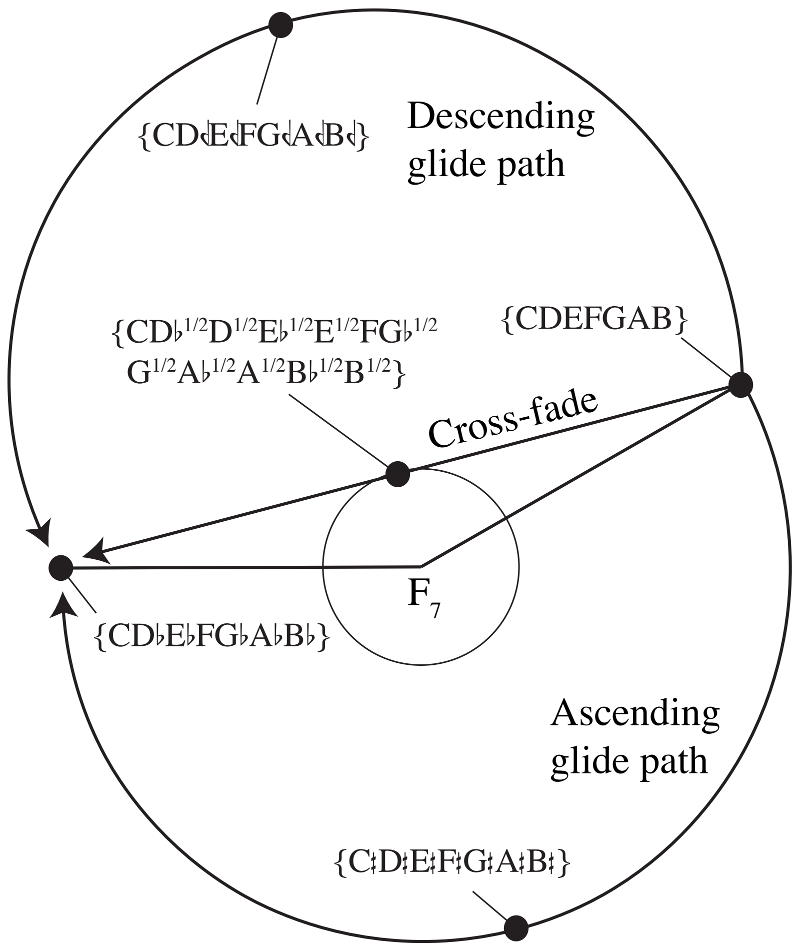
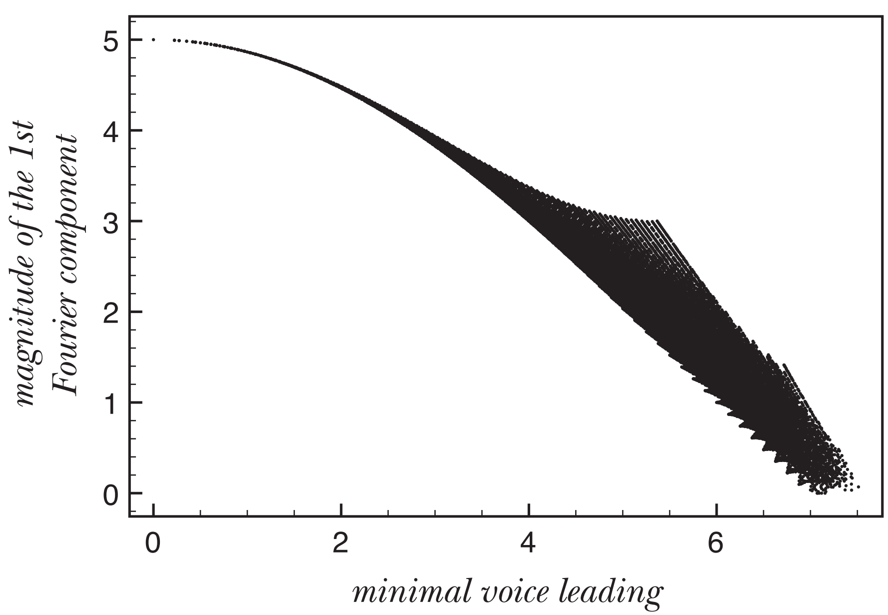


A MORE GENERAL POINT OF VIEW

Voice-Leading Approximation of Fourier Magnitude

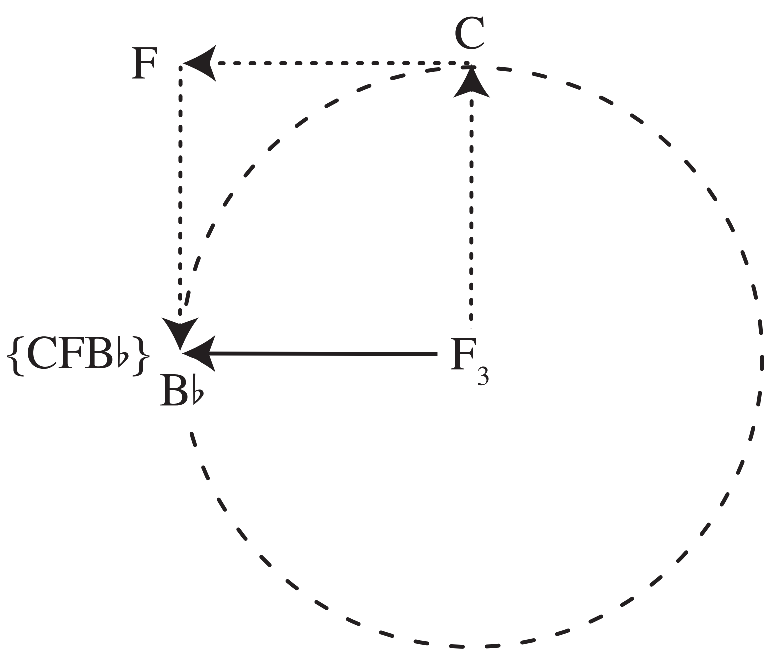
Tymoczko (2008) argued that Fourier magnitudes closely approximate the proximity to the nearest doubled subset of a perfectly even chord.

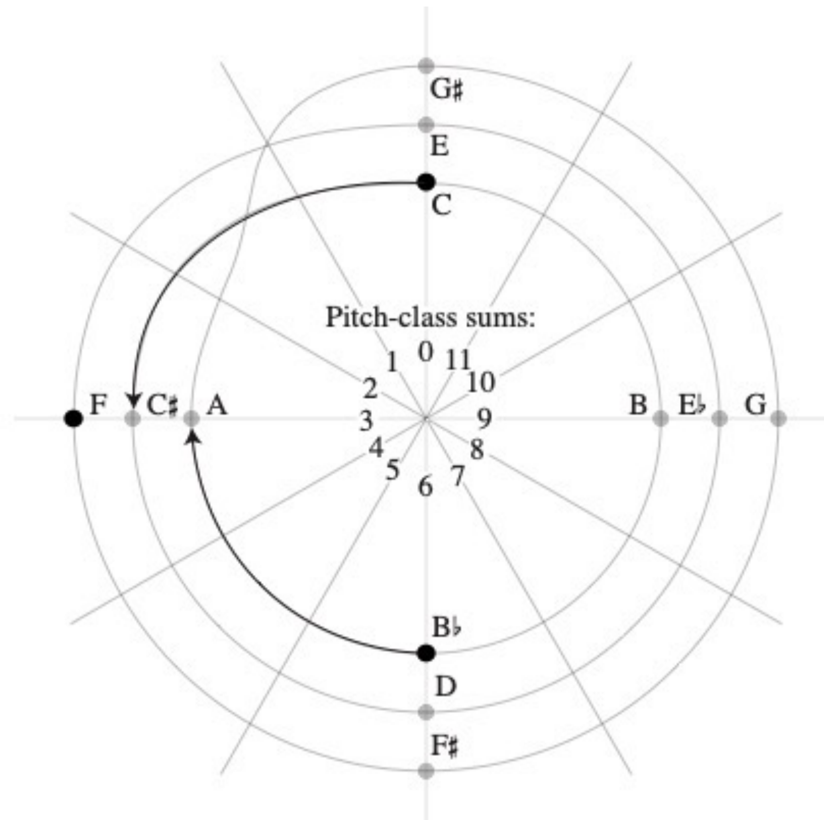


Voice-Leading Approximation of Fourier Phase

Fourier *phases* closely approximate the *transpositional level* of the nearest doubled subset of an *n*-note perfectly even chord

Example:





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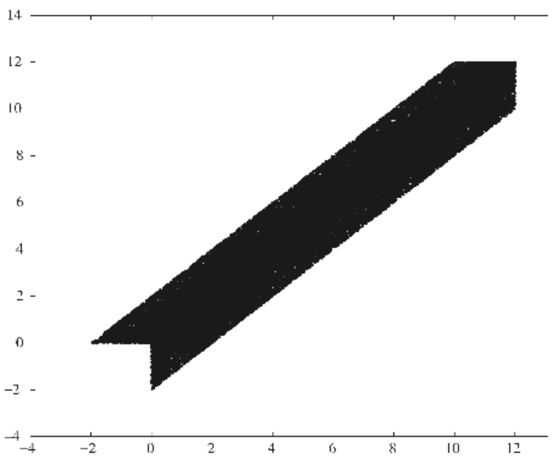
Voice-Leading Approximation of Fourier Phase

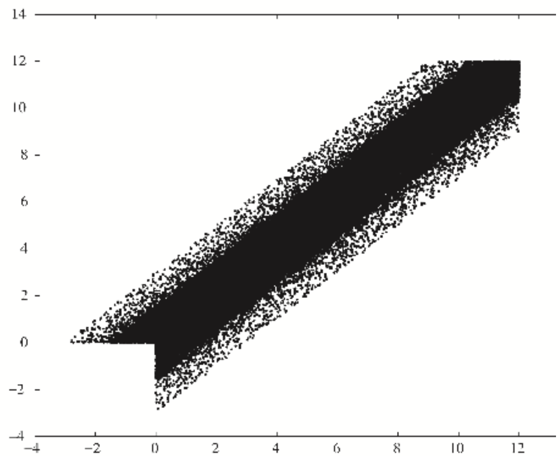
Fourier phases closely approximate the transposition of the nearest doubled subset of an *n*-note perfectly even chord

Phase by transposition of the nearest doubled subset of PE for

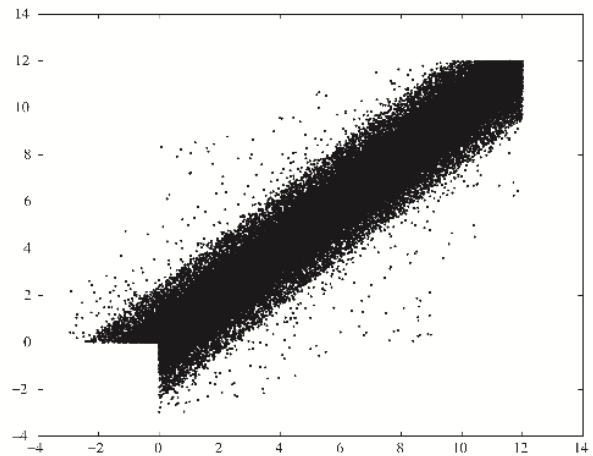
Trichords:



Tetrachords:



Pentachords:



Voice-Leading Approximation of Fourier Analysis

Using these relationships we can approximate many features of the Fourier perspective using voice leading tools.

Question: what features of Fourier analysis resist such translation?

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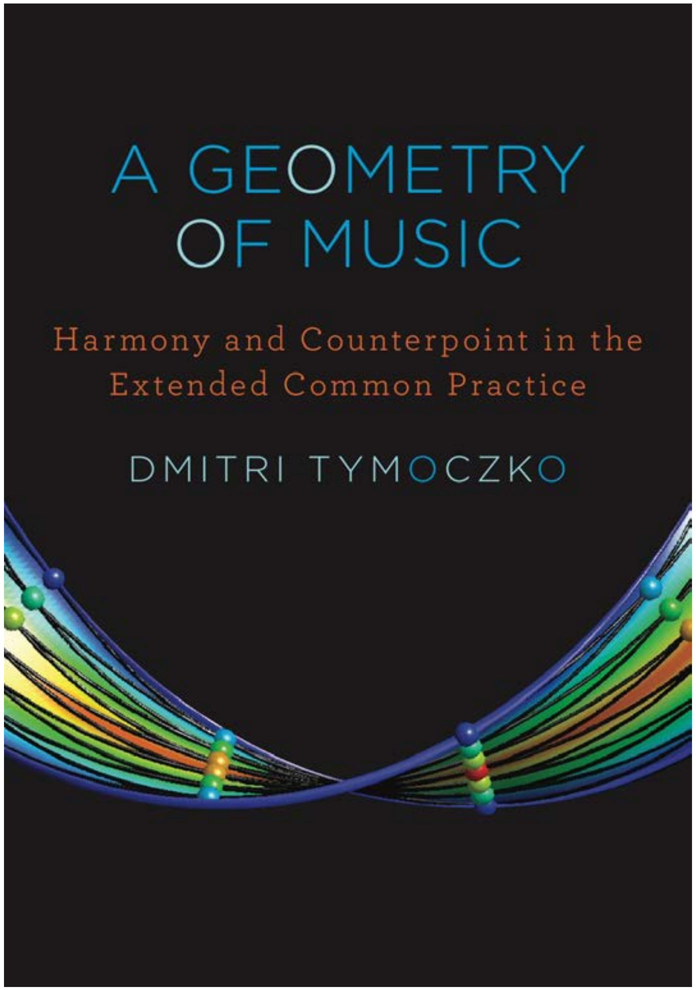
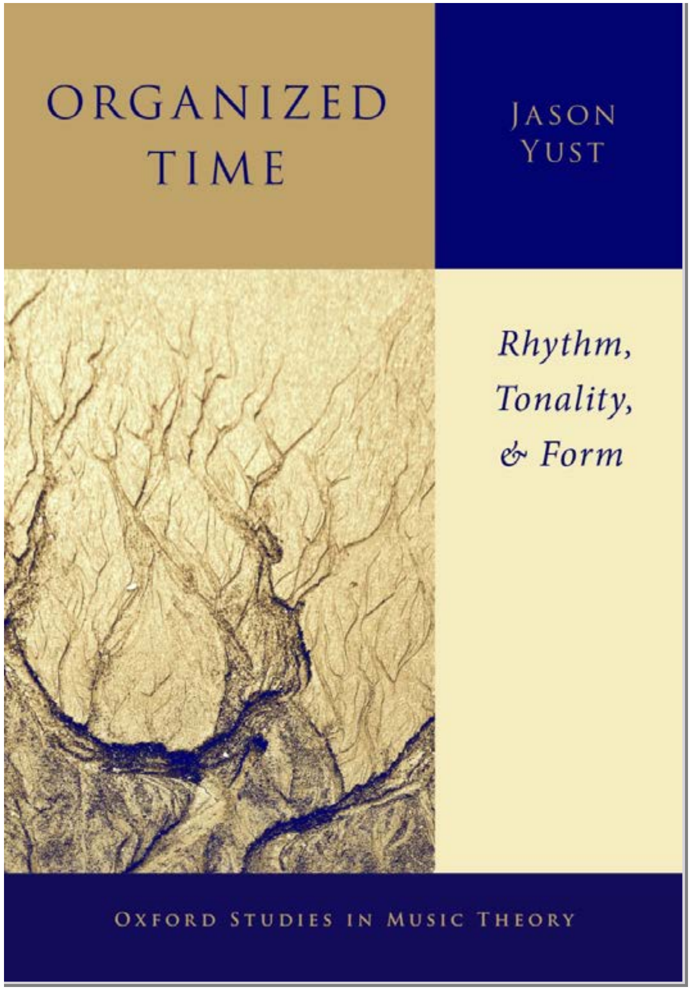


Thank you!



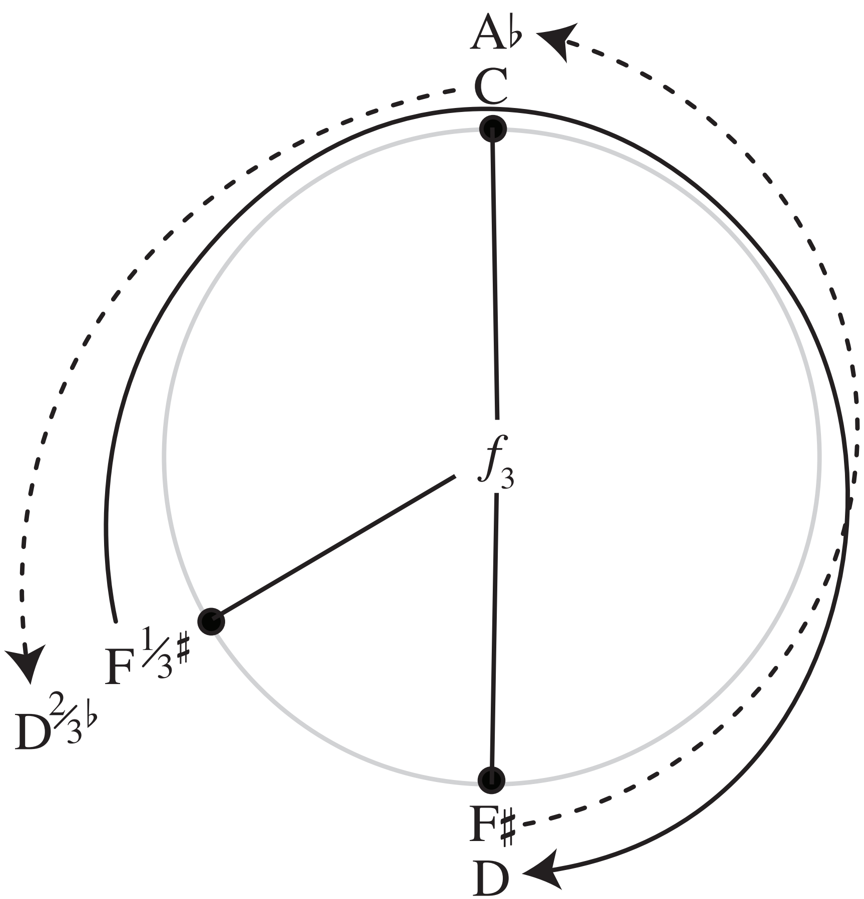
Another question:

How uneven does a chord have to be for a balanced voice leading to cycle phase space?

Example:

{C, F+1/3, F♯} → {C♯+1/3, D, G♯}

In this example, Ph_3 makes a full cycle, but all individual voices travel less than 8ve/*n*



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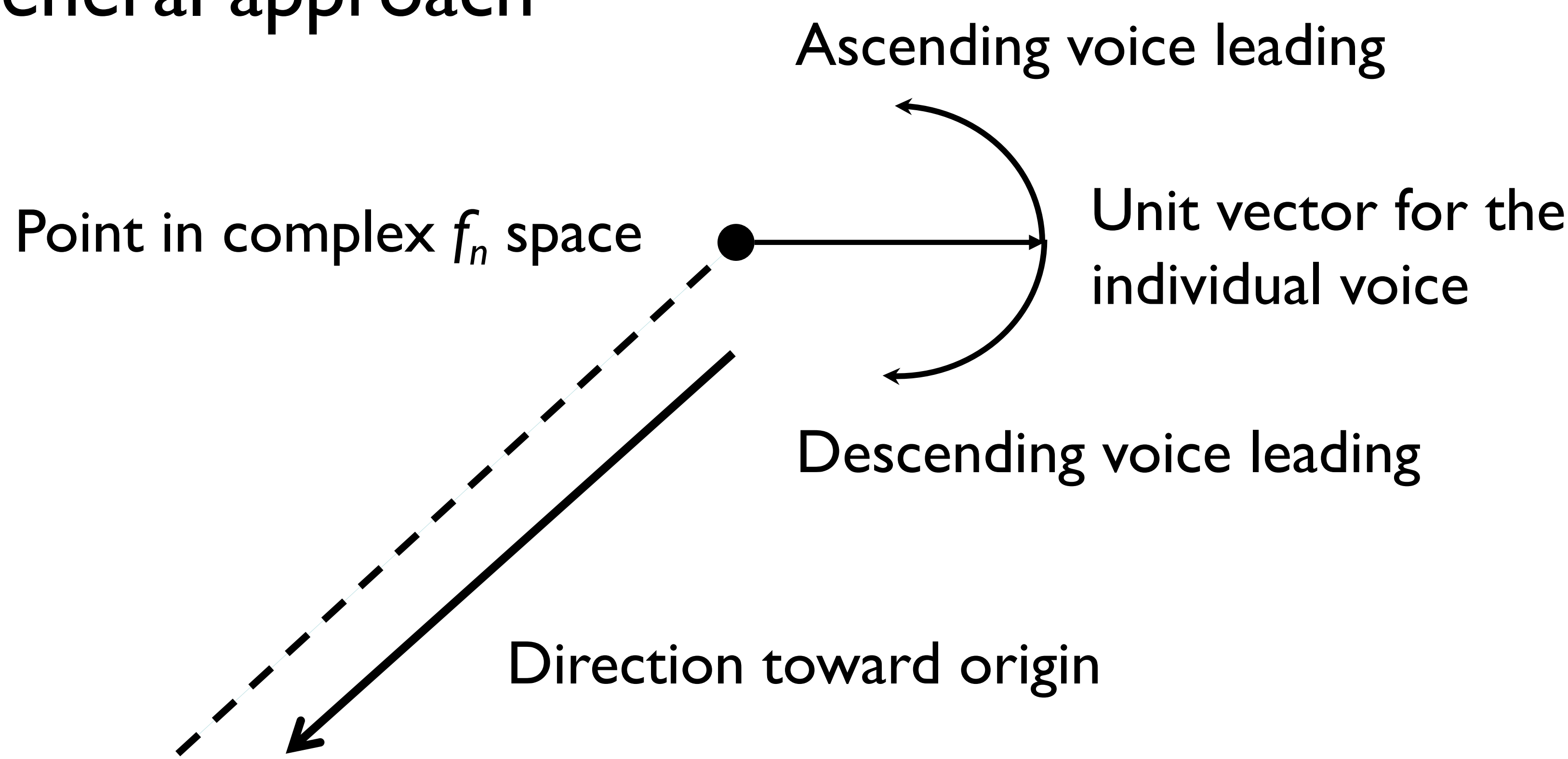
Voice-Leading Consistency

Phase/Voice-leading consistency may offer a general approach



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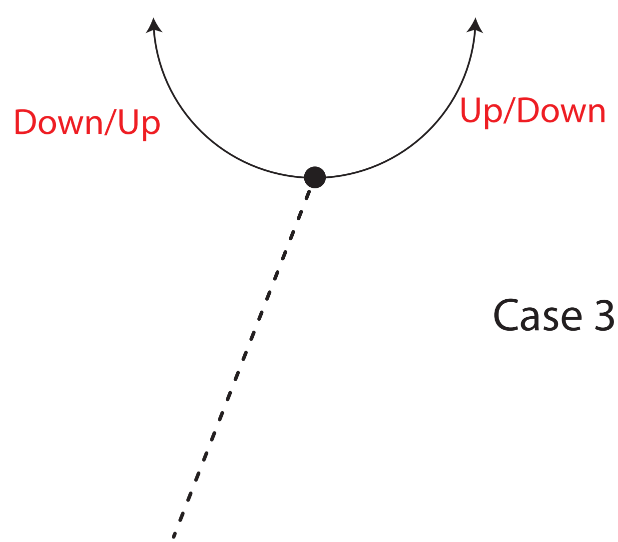
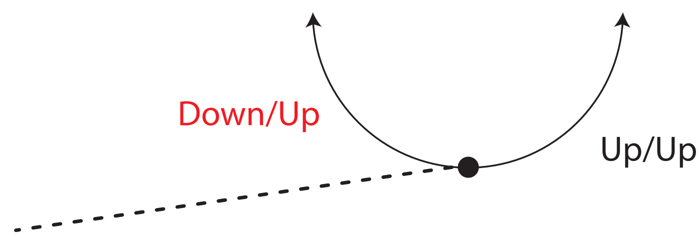
Voice-Leading Consistency

Case 1: Voice distant in phase,
both voice leadings contradictory



Case 2: Voice oblique in phase,
towards center is consistent,
away from center is contradictory



Case 3: Voice close in phase,
both voice leadings consistent,