Generalized Trichordal and Tetrachordal Tonnetze: Geometry and Analytical Application

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Outline

(1) Two-Dimensional Generalized Tonnetze

(2) Example: Stravinsky Owl and the Pussycat

(3) Optimizing Spaces

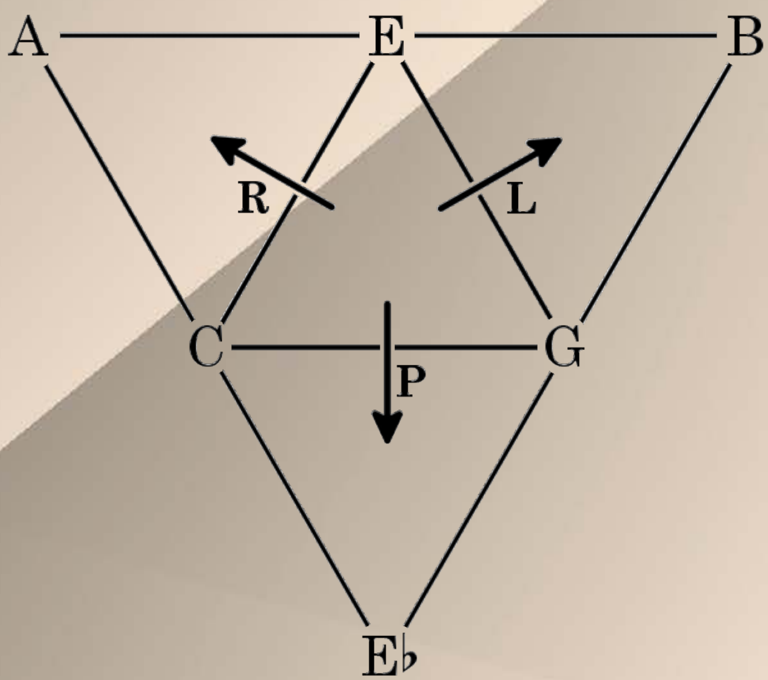
(4) Duplicating Intervals and Foldings

(5) Three-Dimensional (tetrachordal) Tonnetze

(6) Example: Brahms, Sarabande WoO 5

(1) Two-Dimensional Generalized Tonnetze

Triadic Tonnetz



Features of the triadic Tonnetz:

Voice-leading efficiency
Generalized by Tymoczko (*JMT* 2012)

Common-tone retention
Generalized by Cohn (*JMT* 1997)

General 2-dimensional Tonnetz

Yust (*JMT* 2015):

DFT phase spaces $\longrightarrow$ Geometric Tonnetz

Geometric Tonnetz $\longrightarrow$ DFT phase spaces
?

General 2-dimensional Tonnetz

Start from the most general defining features:

- Triangulation:

  Partitions a space into triangular (simplicial) regions

- Vertices of the regions correspond to pitch-classes.
- Transposability:

  Transposition $\rightarrow$ Rigid transformation
of pitch classes of space

General 2-dimensional Tonnetz

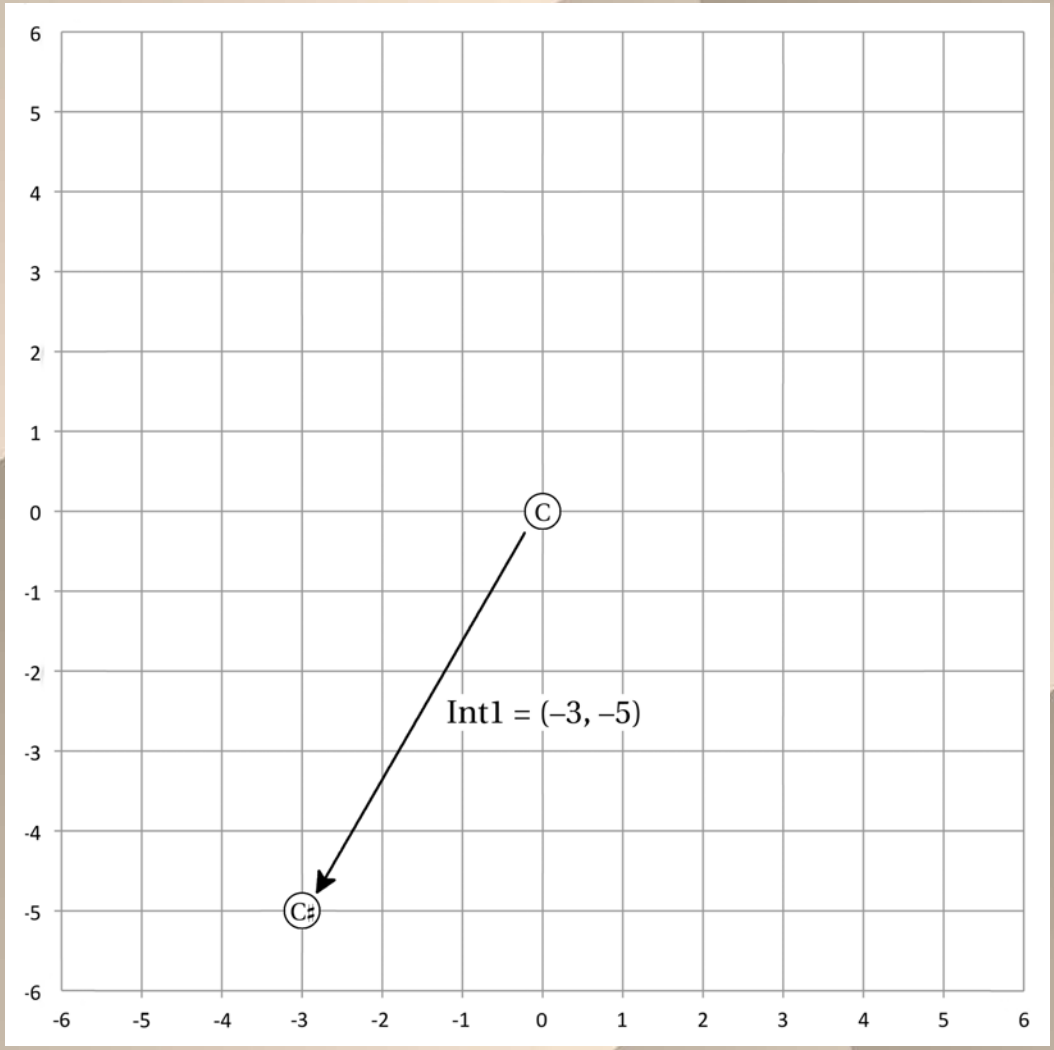
Construction of a two-dimensional Tonnetz:

(1) Place pcs by initial choice of interval

(2) Choose two intervallic axes

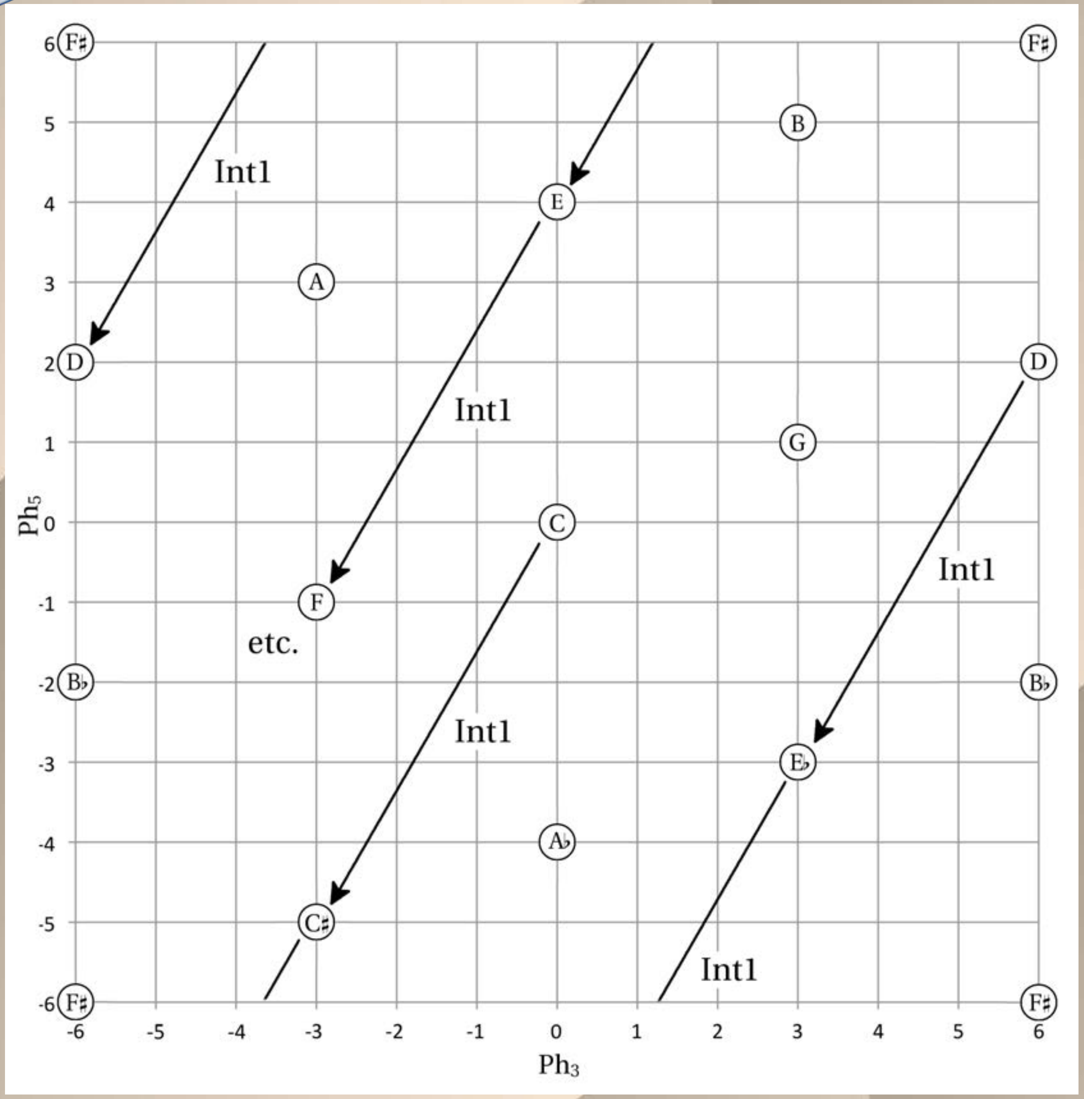
(3) Bisect the resulting quadrilaterals

Example



(1) Define the semitone as the vector (–3, –5)

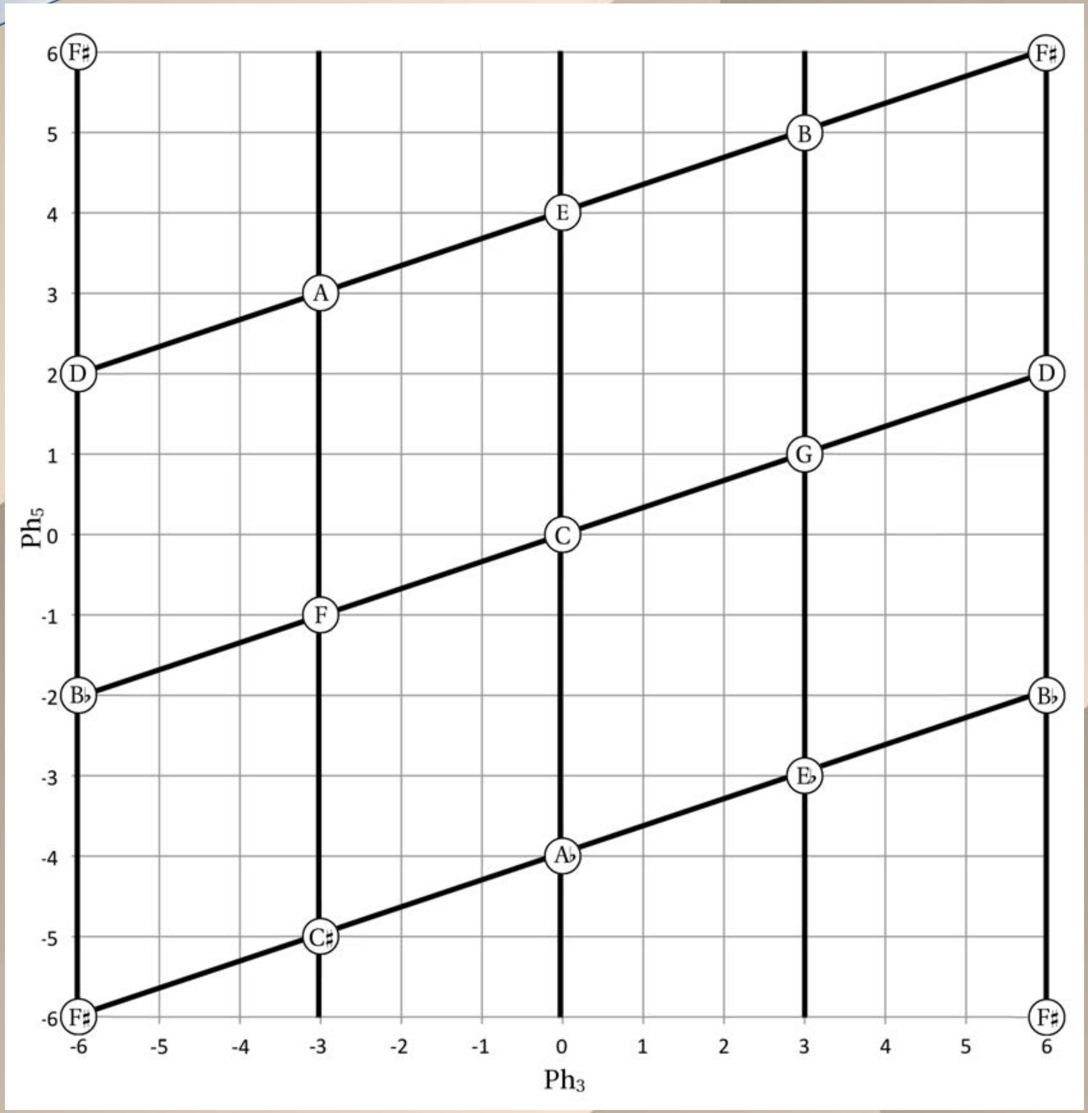
Example



(1) Define the semitone as the vector (–3, –5)

Place all pcs by reiterating this interval.

Example



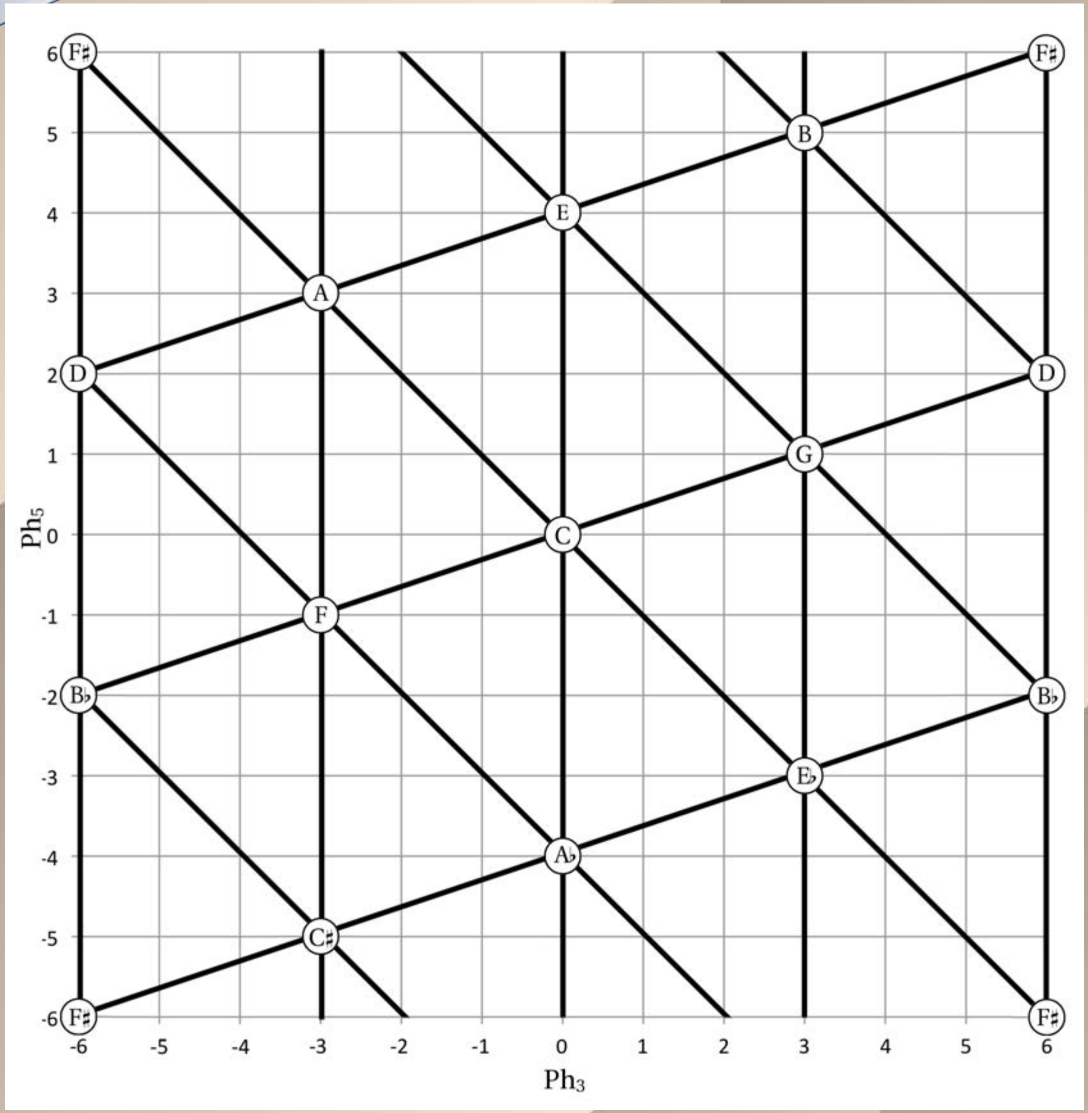
(2) Choose two axes:

One corresponds to ic5.

The other corresponds to ic4.

There are two ways to bisect the resulting quadrilaterals.

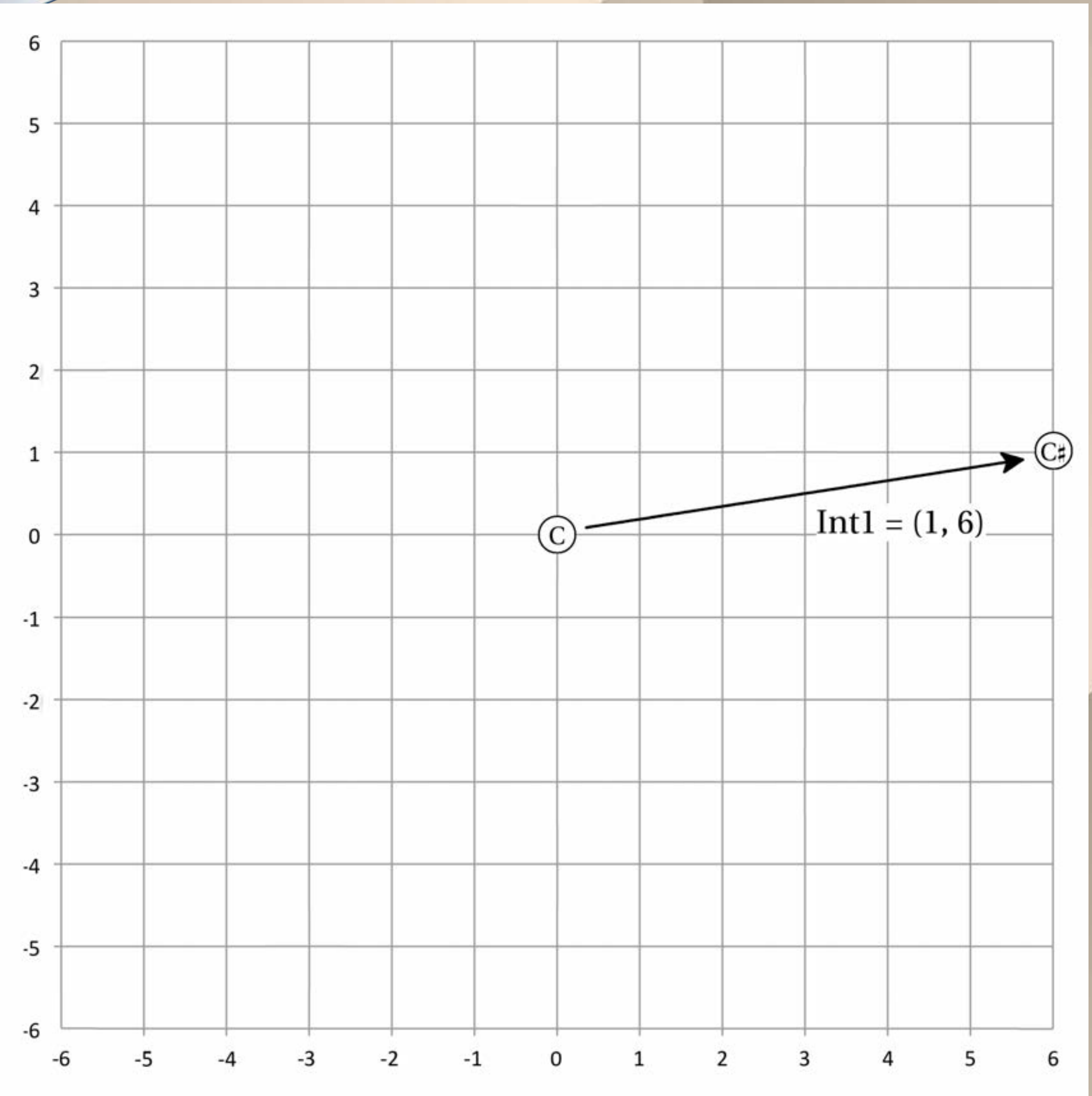
Example



Add a third set of parallel lines bisecting the parallelograms.

The choice of a minor-third axis results in a compact version of the standard triadic Tonnetz

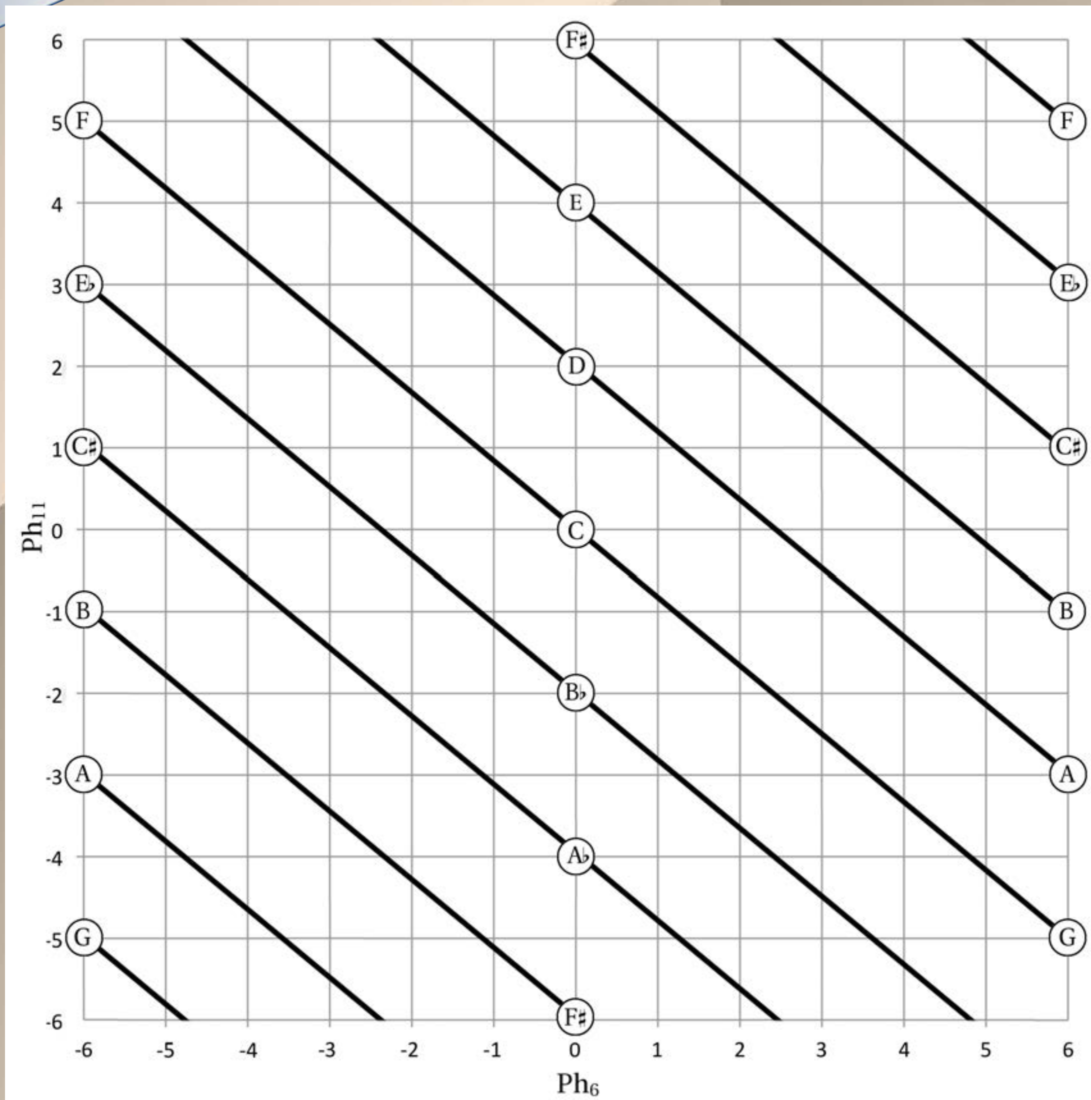
Example



What happens when a different choice is made to place the pcs?

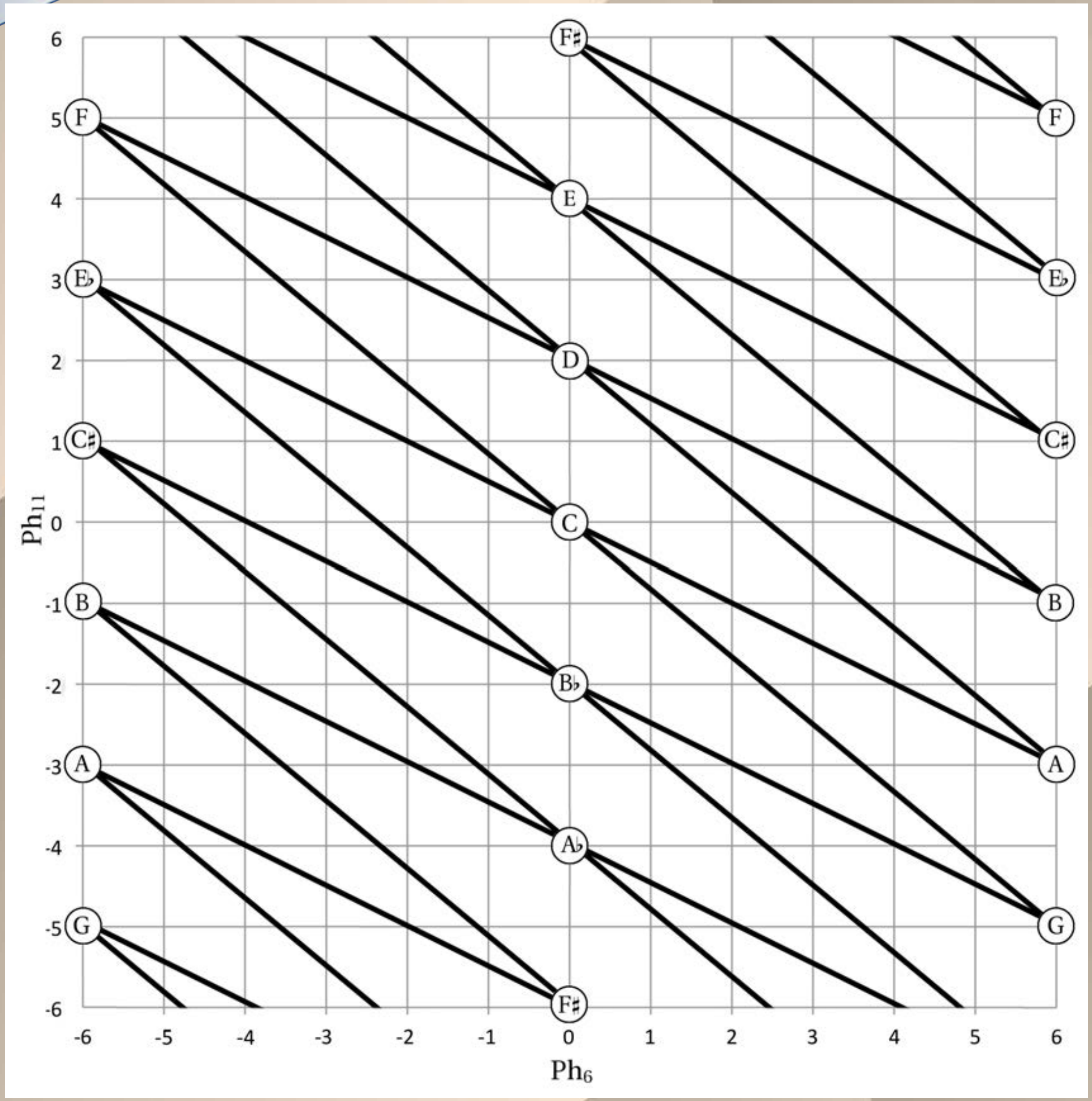
For example,
Let int1 = (1, 6).

Example



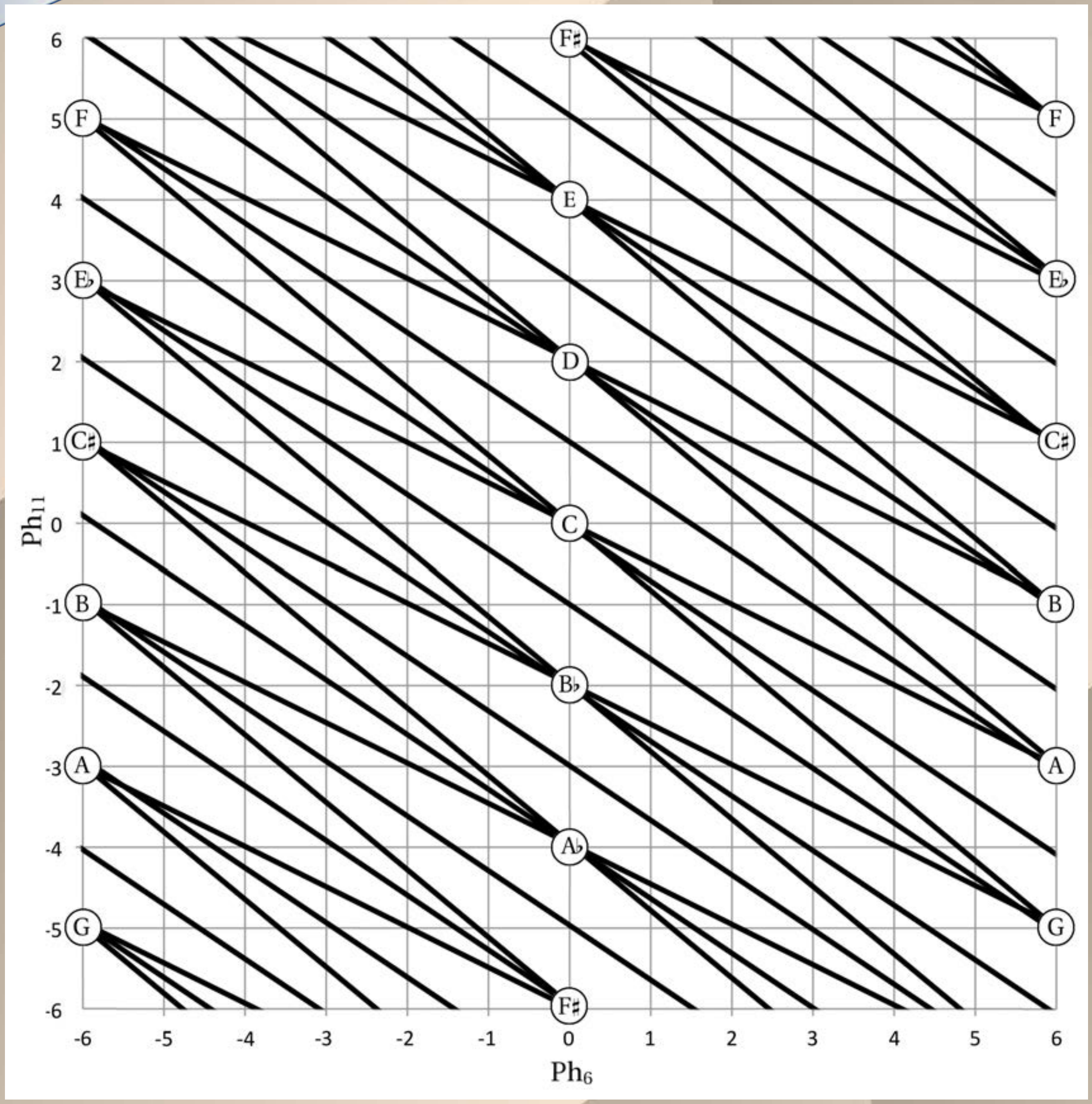
The fifths axis cycles through all pcs, but is also much longer here than in the other space.

Example



Add the minor-third axes.

Example



Add the other major-third axes.

This Tonnetz is the same as the previous one skewed. The fit of space to Tonnetz is poor, resulting in non-compact regions.

General 2-dimensional *Tonnetz*

Resulting space:

Equivalent to some Fourier phase space:

See Amiot MCM 2013, *Music through Fourier Space* (2016); Yust *JMT* 2015, *JMT* 2016, *Organized Time* (Forthcoming, Oxford University Press), ch. 10

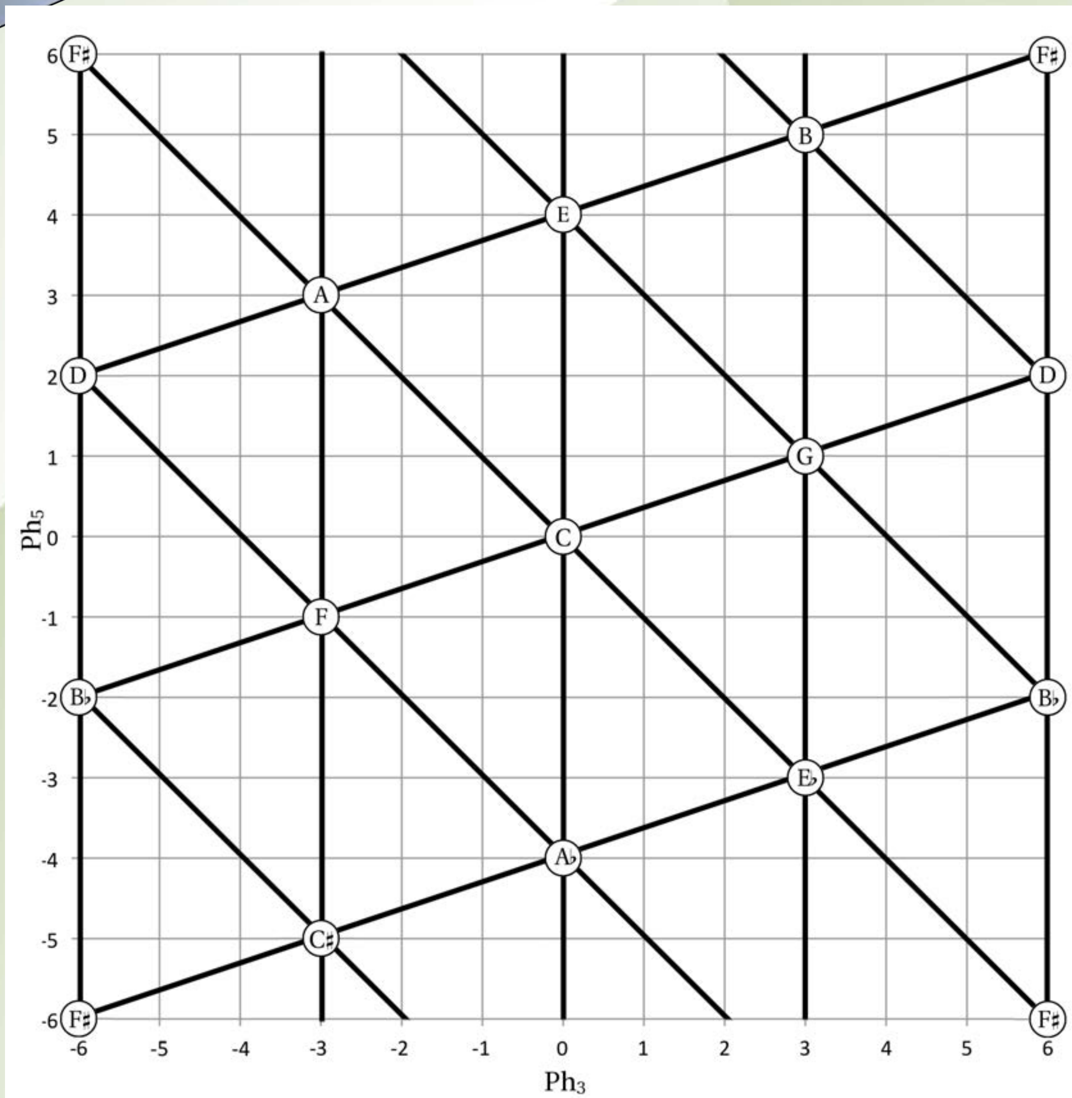
I.e., each dimension represents one of the interval cycles.

Triangulation into 24 (or 2*u*) regions.

Any set class is possible (as long as Z_{12} is its minimal embedding universe).

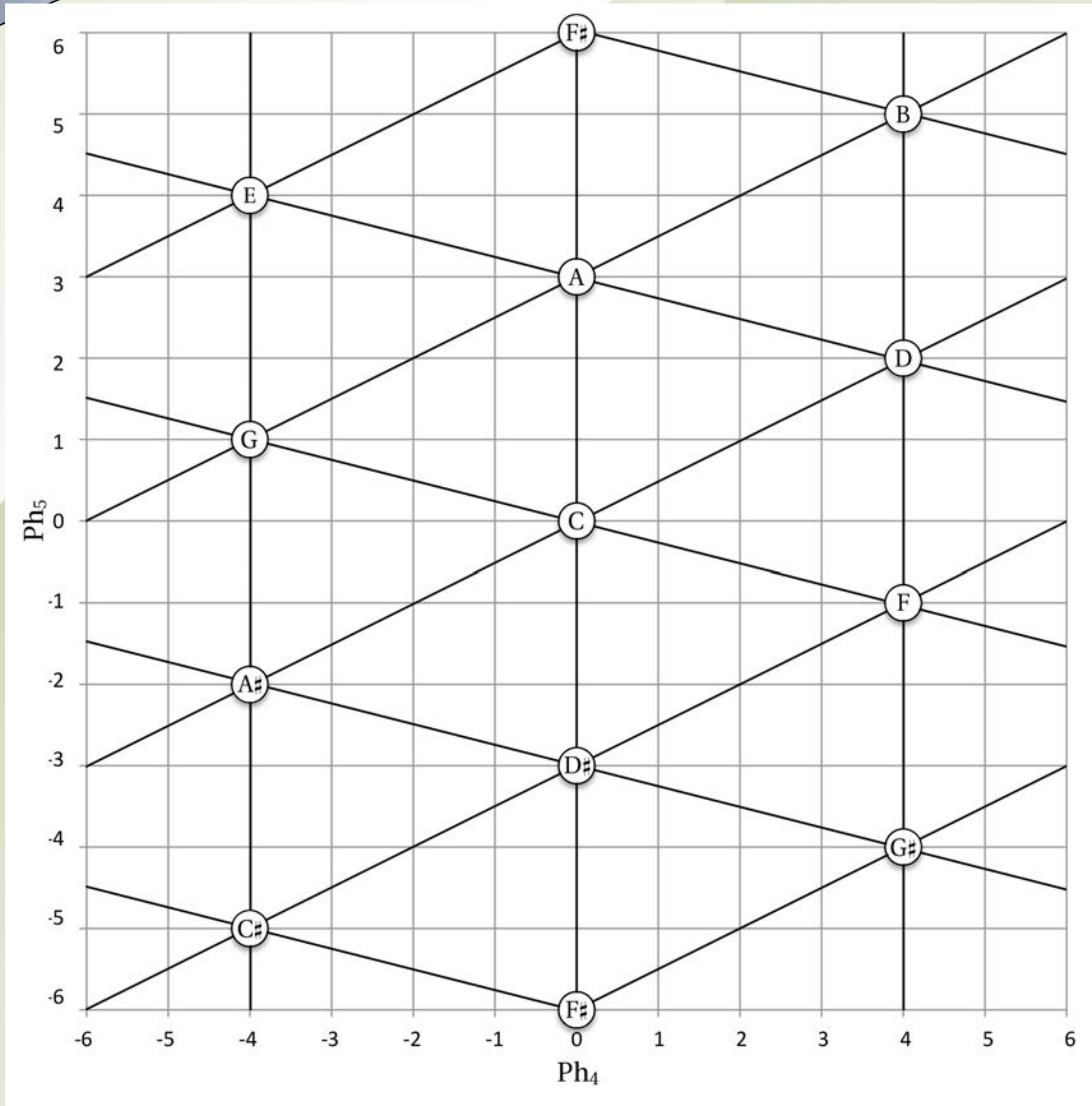
Examples

Triadic Tonnetz



Standard triadic Tonnetz
in triadic-diatonic
(Ph_3/Ph_5) space

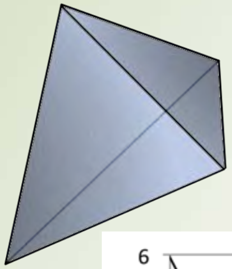
(025) Tonnetz

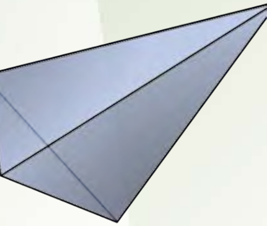


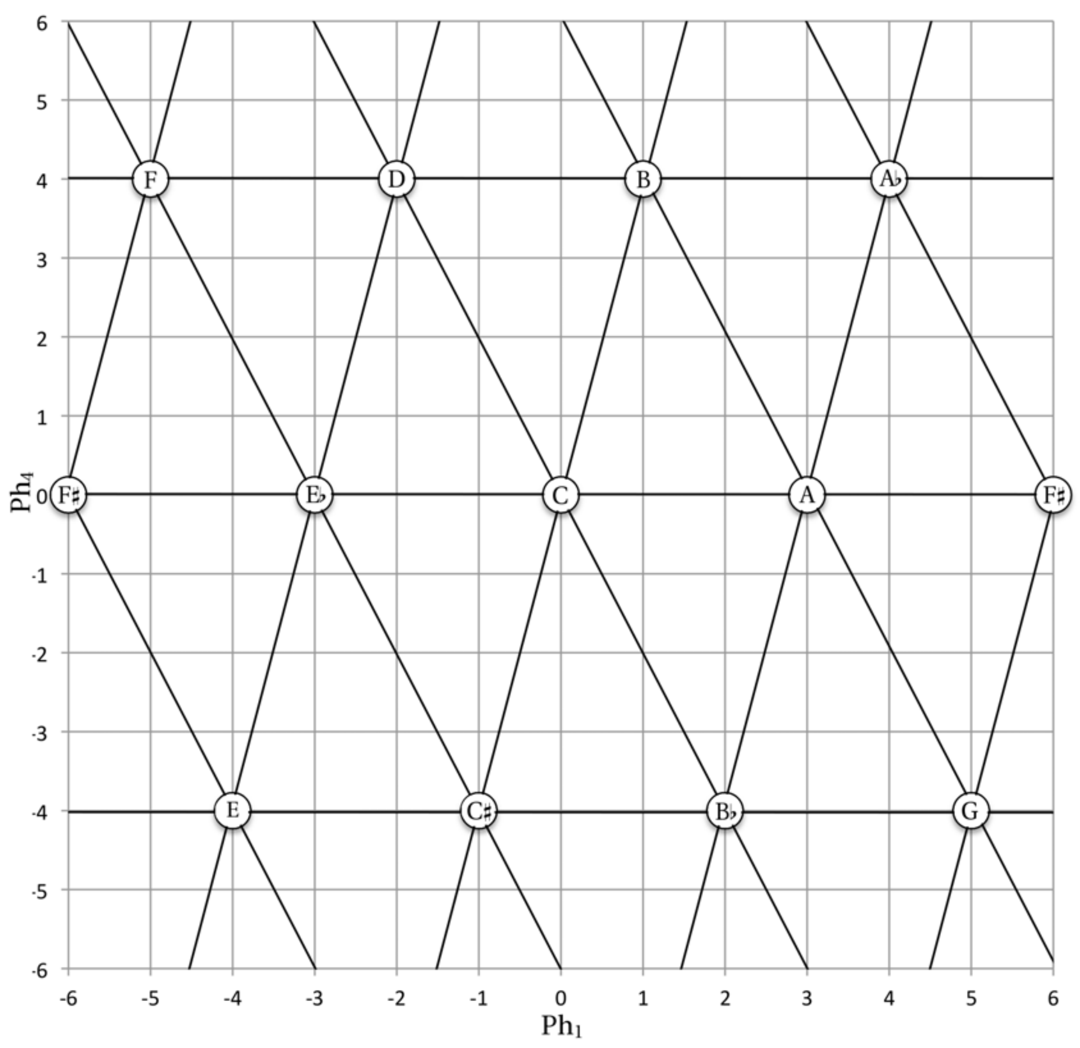
(025) Tonnetz

in octatonic-diatonic (Ph_4/Ph_5) space

(013) Tonnetz



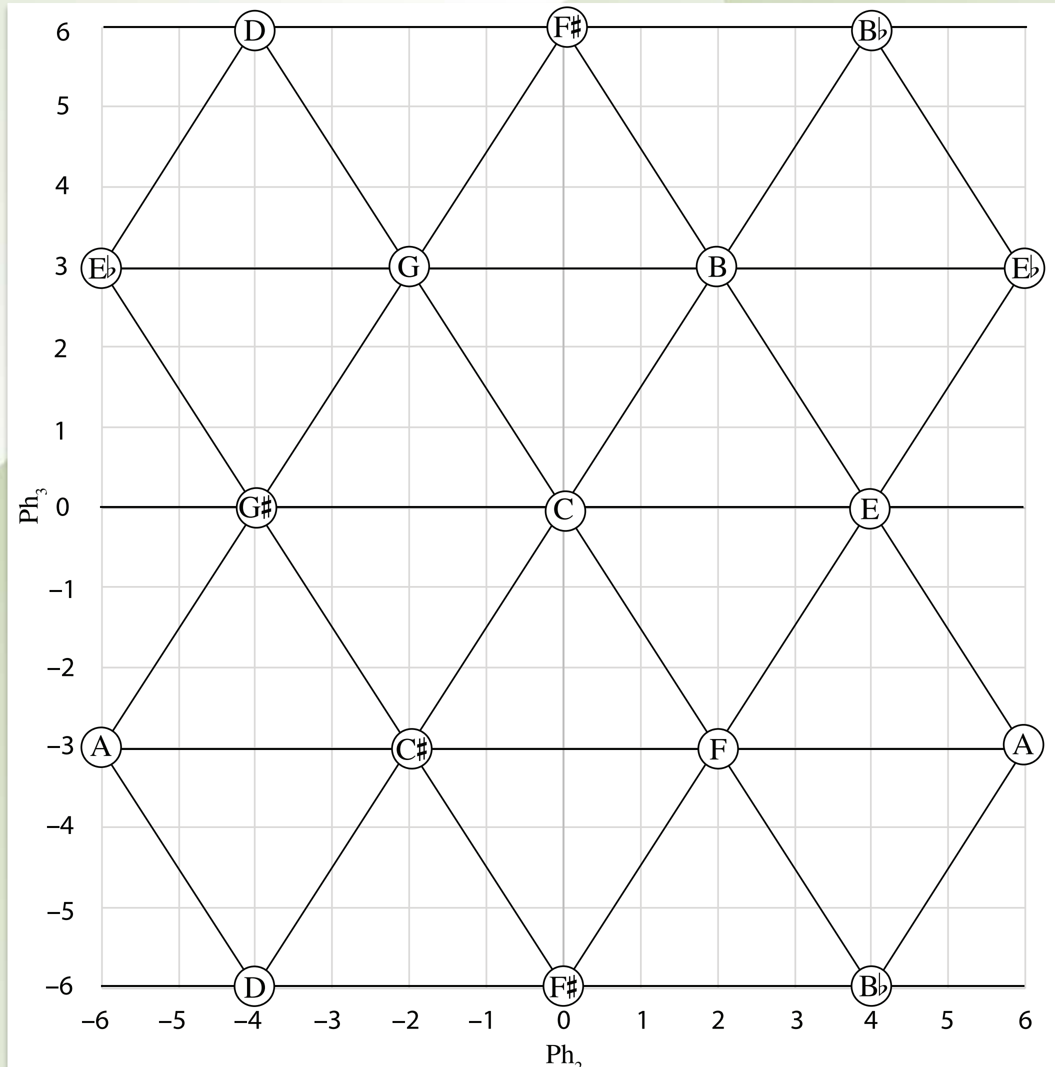




(013) Tonnetz

in chromatic-octatonic (Ph_1/Ph_4) space

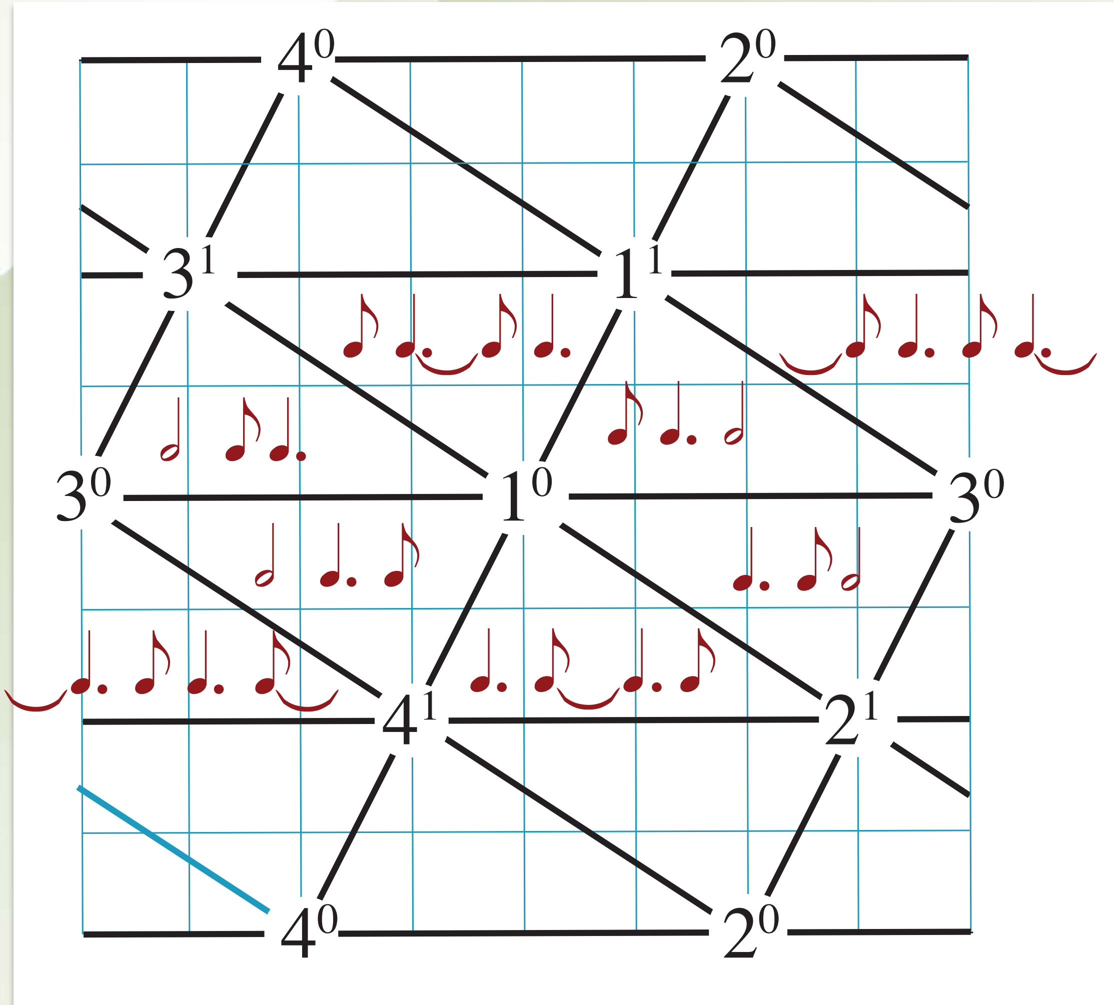
(015) Tonnetz



(015) Tonnetz

in dyadic-triadic (Ph_2/Ph_3) space

Rhythmic *Tonnetz*

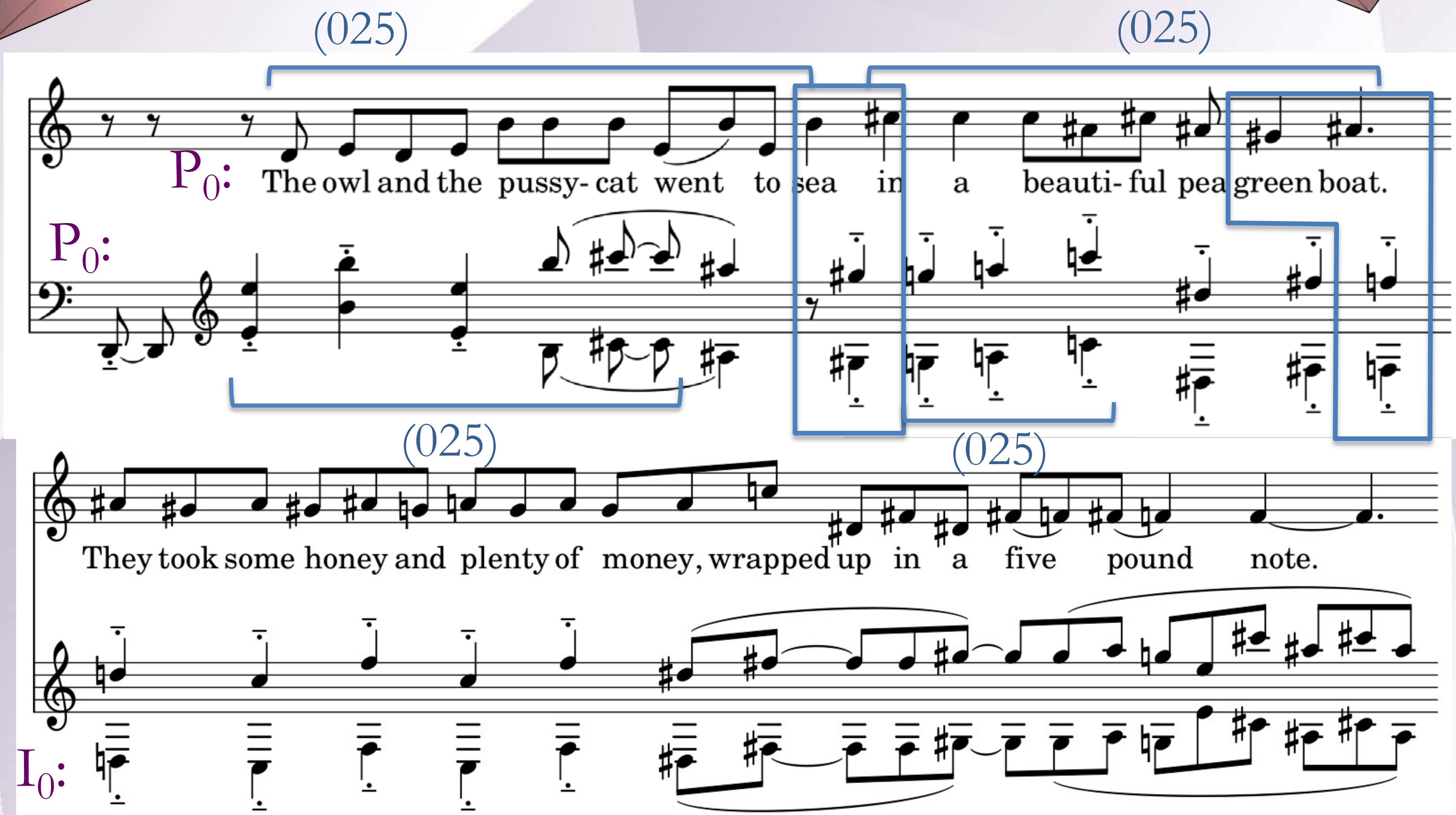


(2) Music Analysis with Non-Triadic *Tonnetze*

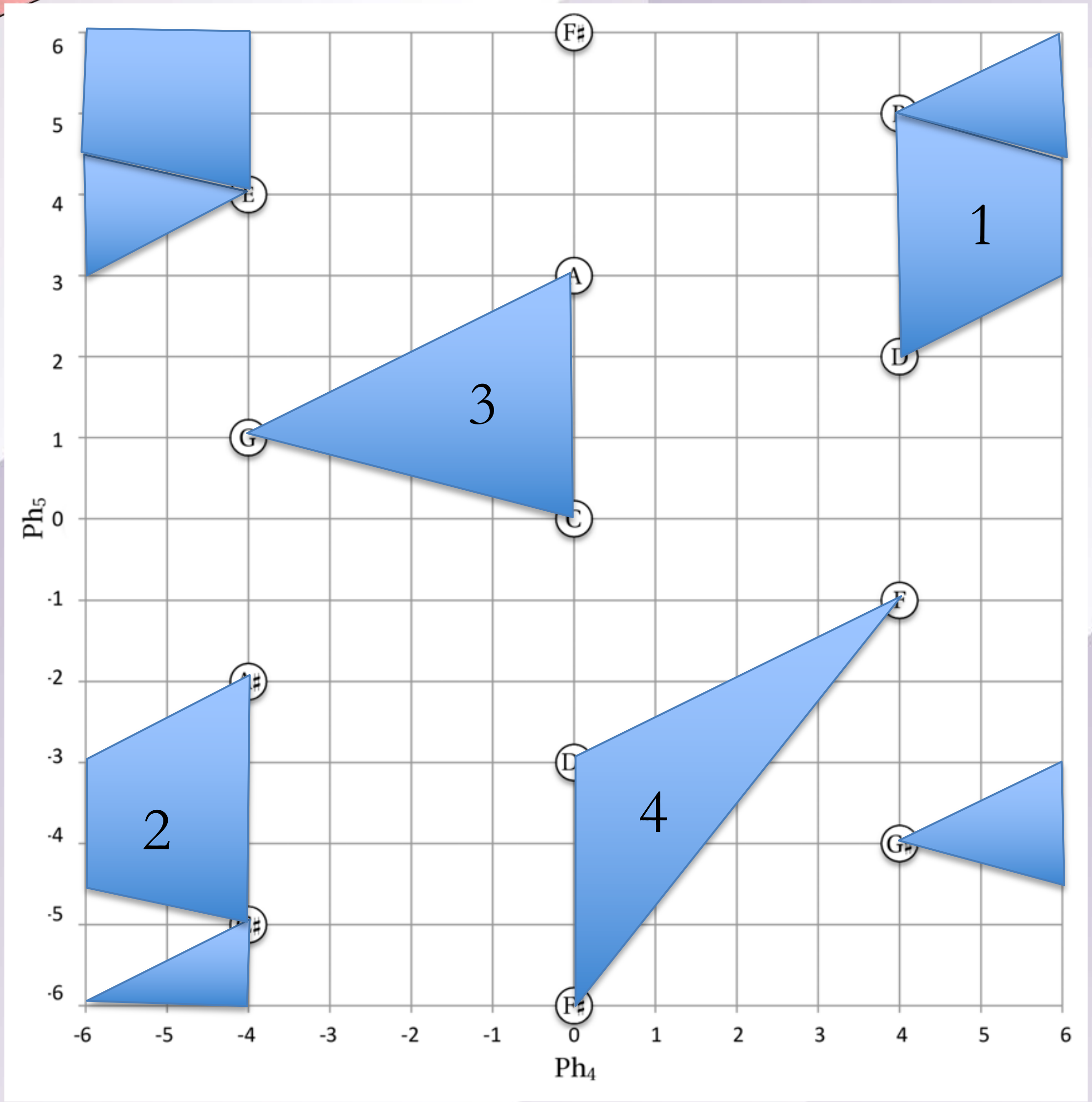
Example:

(025) *Tonnetz*, Stravinsky "Owl and the Pussycat"

Stravinsky: "Owl and the Pussycat"



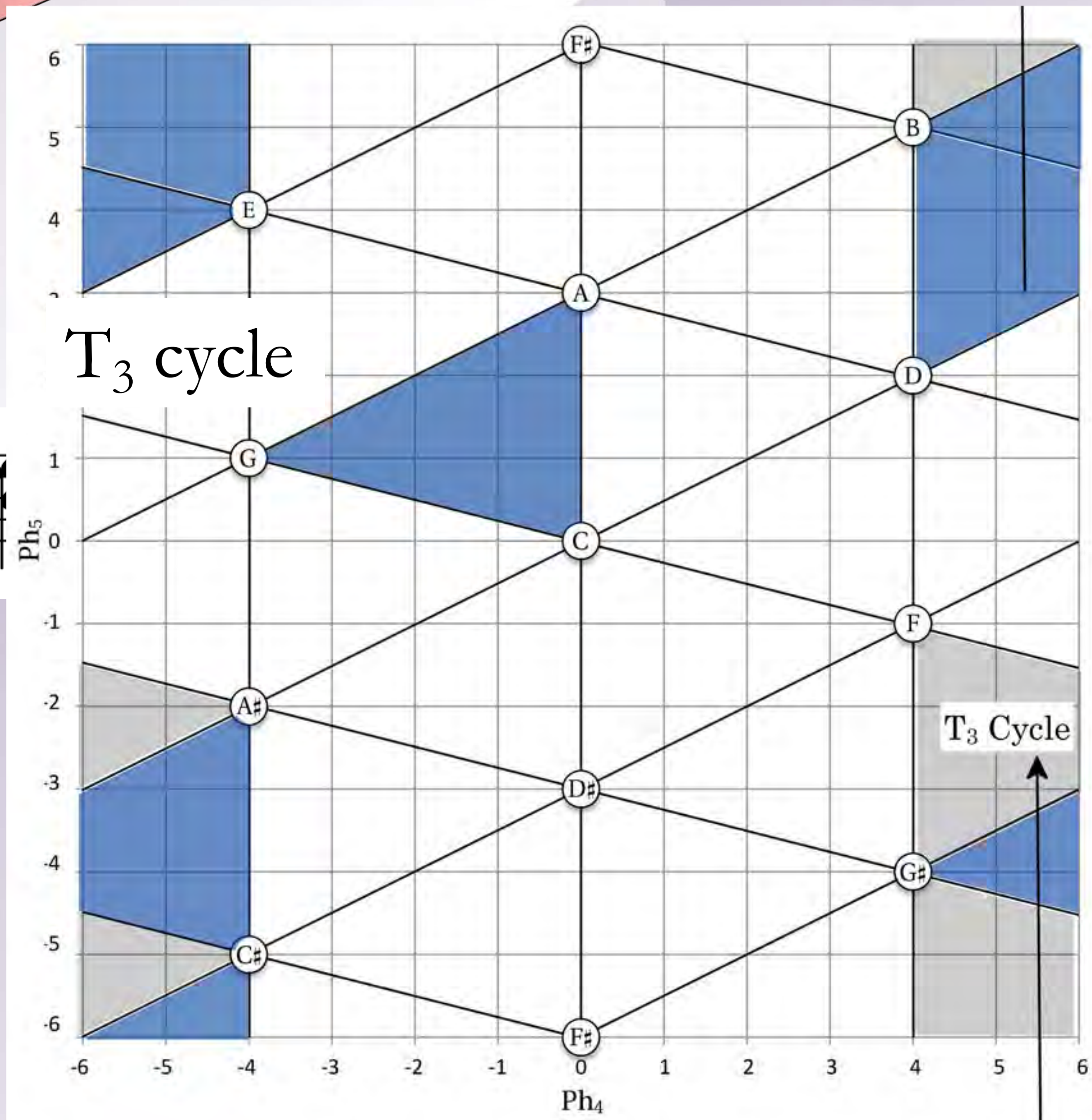
Stravinsky: "Owl and the Pussycat"



Trichords of P_0

Most are (025)s, but last trichord is (013)—not in Tonnetz, compact in Ph_4, not in Ph_5

Stravinsky: "Owl and the Pussycat"



(025)-Tonnetz in octatonic-diatonic (Ph$_4$/Ph$_5$) space

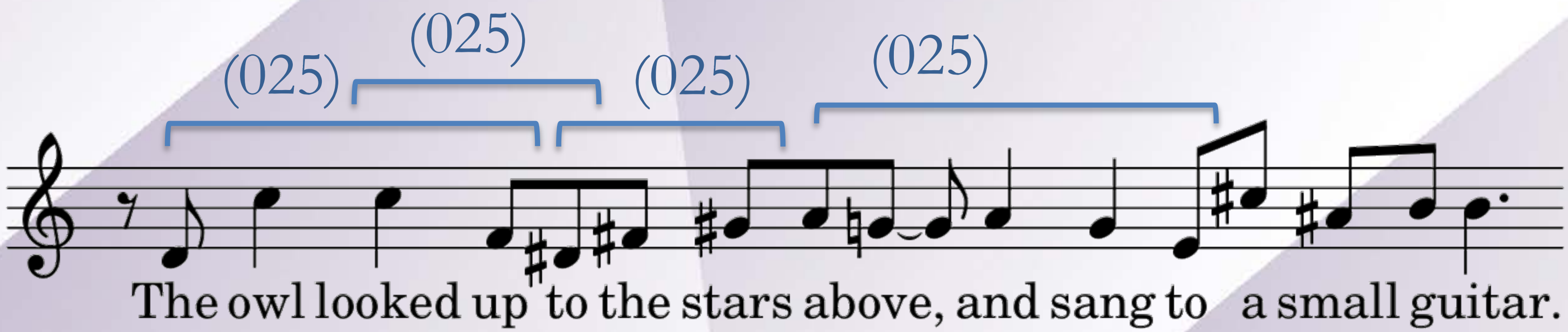
(025)s of P$_0$ in blue

Vertical (025)s in gray

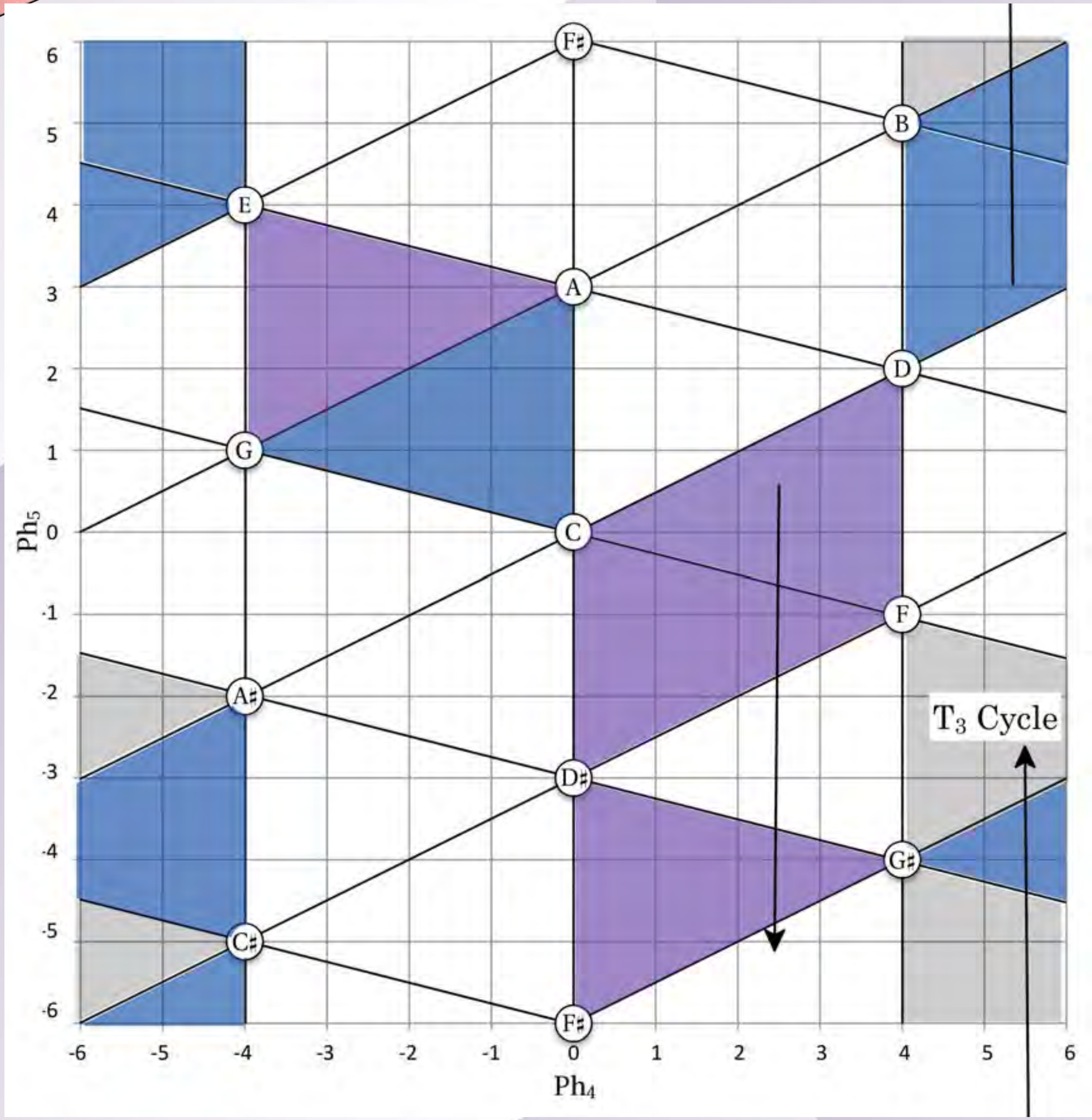
T$_3$ cycle (Octatonic)

Stravinsky: “Owl and the Pussycat”

I_0:



Stravinsky: "Owl and the Pussycat"

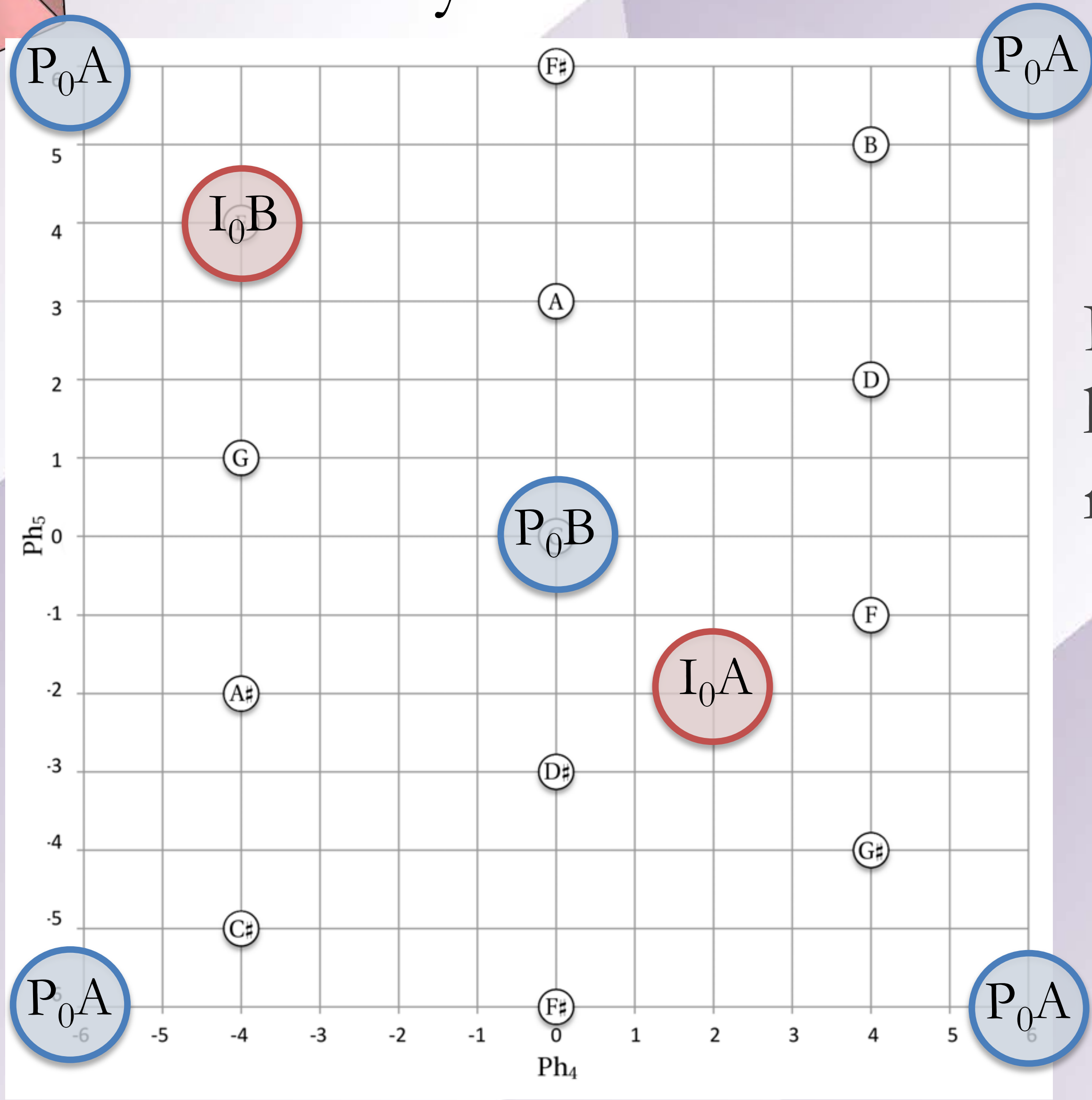


(025)s in P_0 in blue

Vertical (025)s in gray

(025)s in I_0 in purple

Stravinsky: “Owl and the Pussycat”

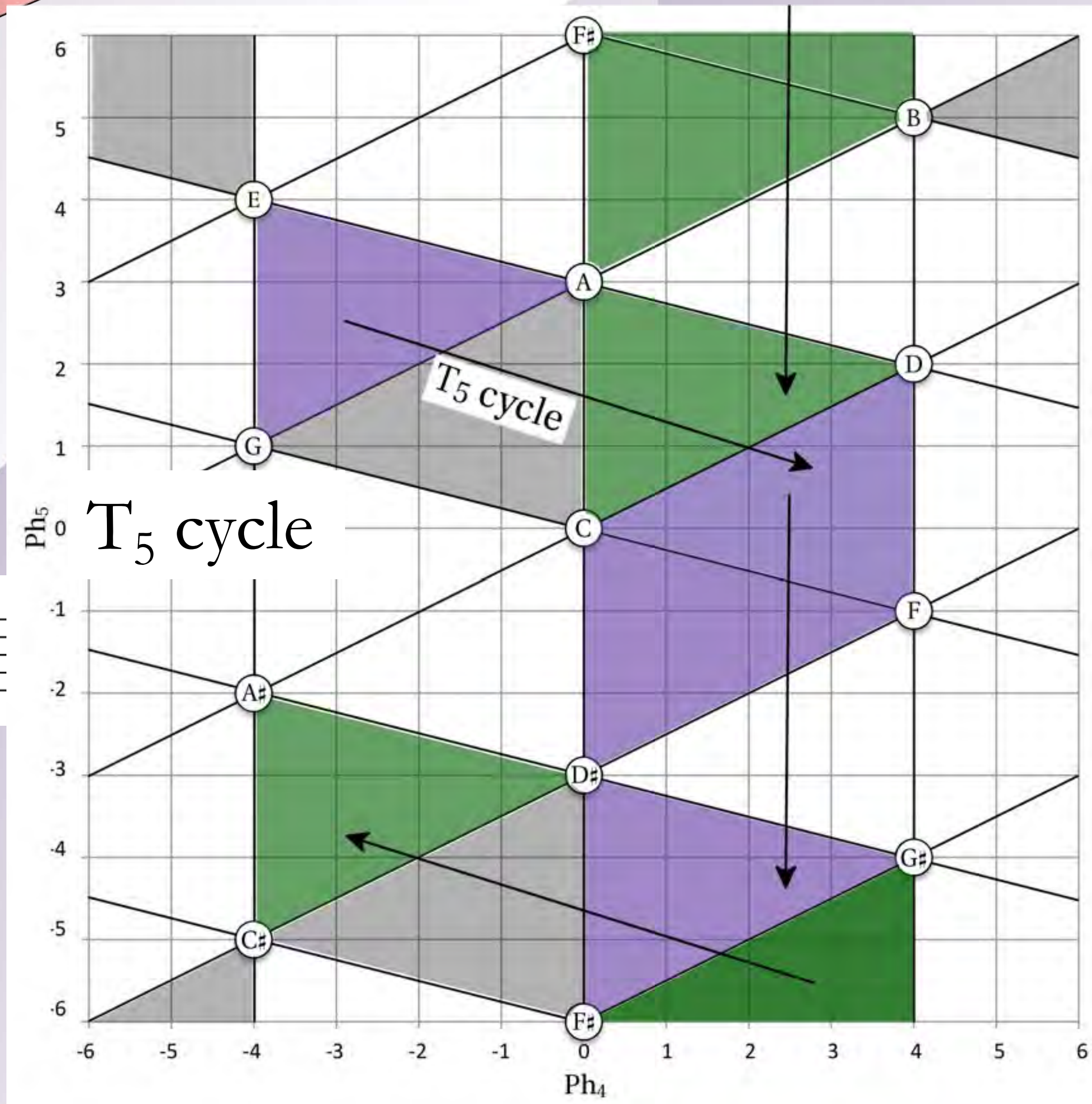


Positions of the hexachords for two row forms.

Stravinsky: "Owl and the Pussycat"

An isolated use of I_6 occurs at the pivot in the narrative

Stravinsky: "Owl and the Pussycat"

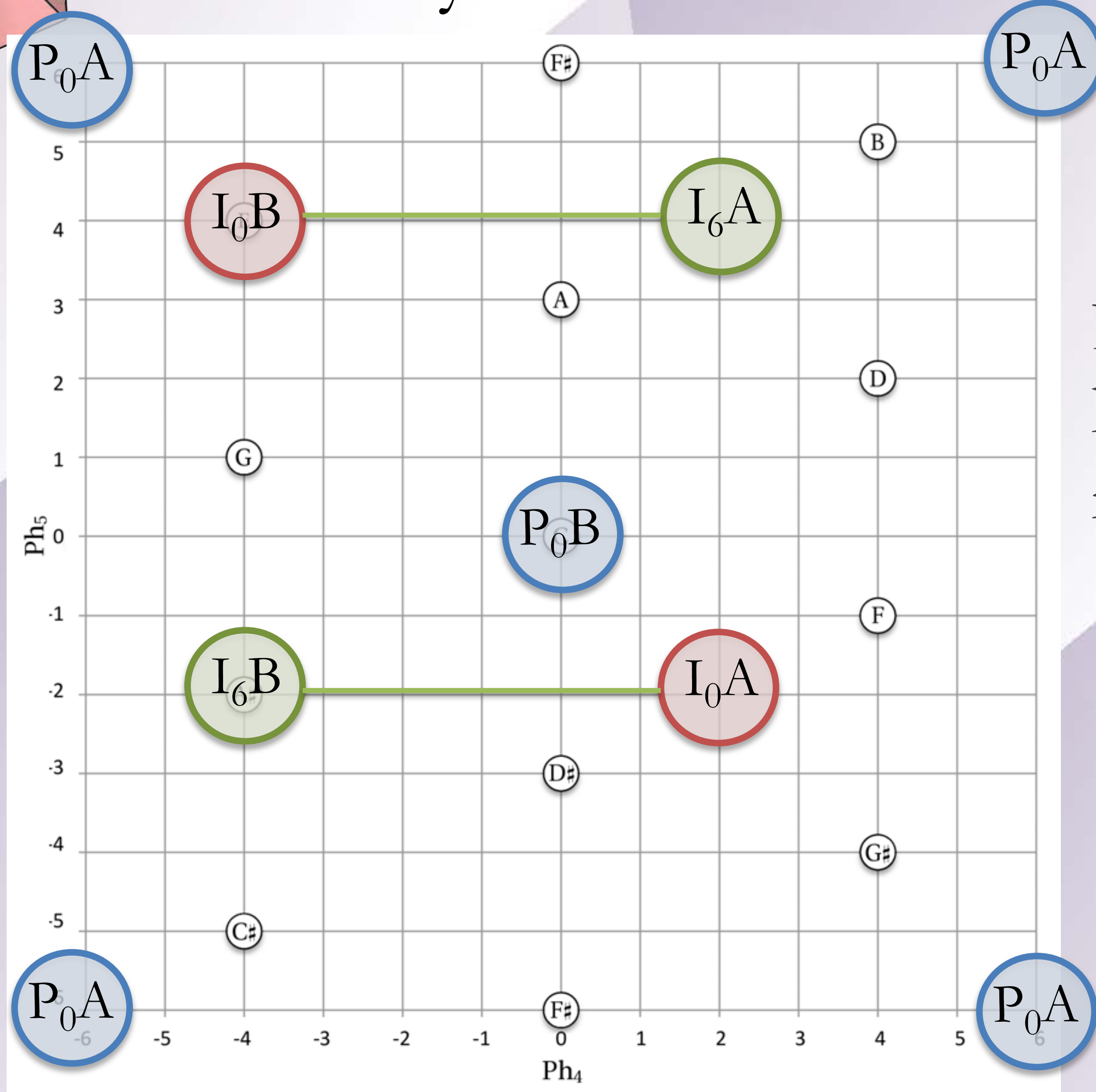


(025)s in I_0 in purple

(025)s in I_6 in green

Vertical (025)s in gray

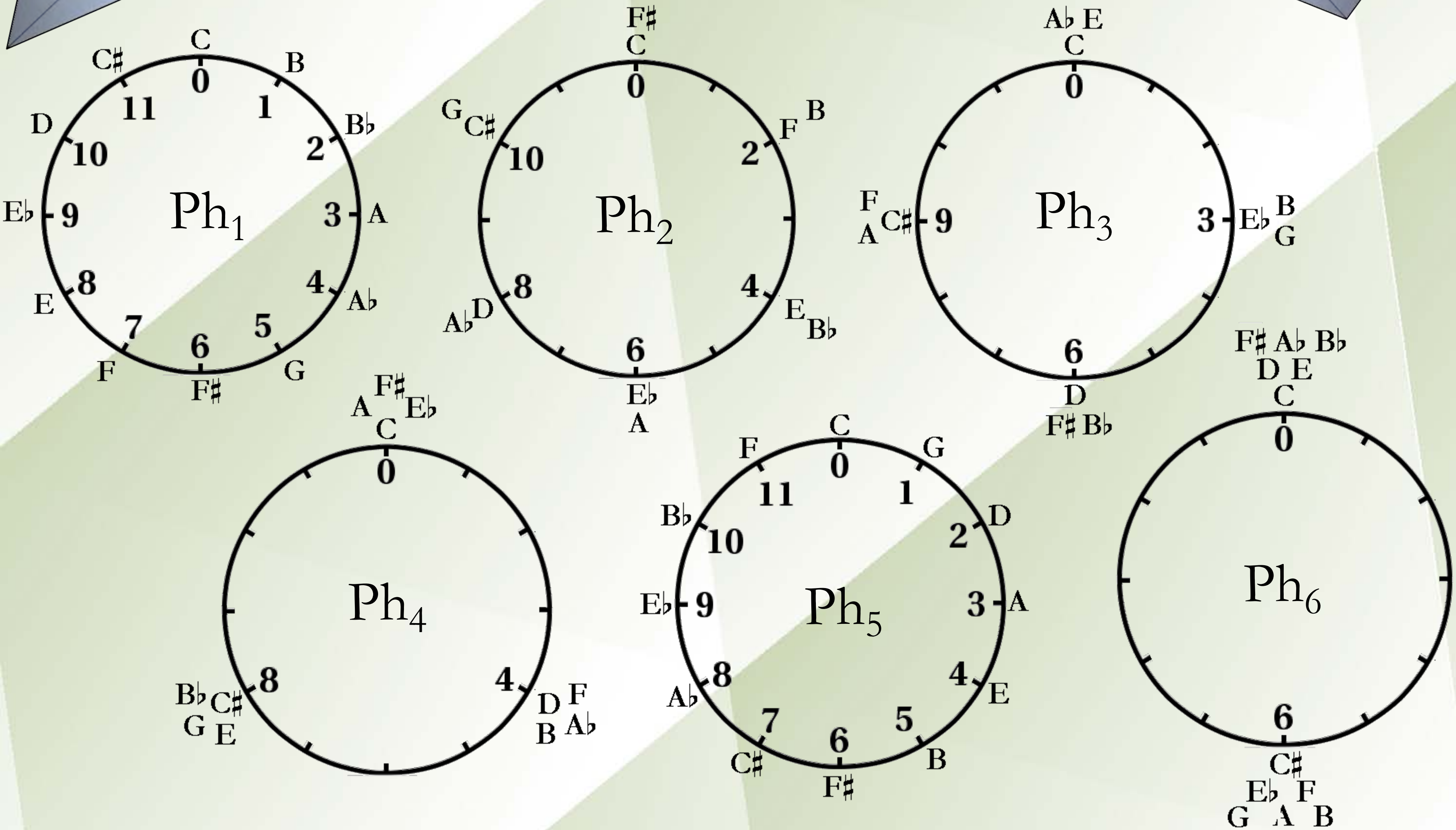
Stravinsky: "Owl and the Pussycat"



Positions of the hexachords for two row forms.

(2) Optimizing Spaces

One-Dimensional Phase Spaces



Optimizing Spaces

Available unique two-dimensional spaces (u = 12):

Ph_1-Ph_1	Ph_1-Ph_2	Ph_1-Ph_3	Ph_1-Ph_4	Ph_1-Ph_5	Ph_1-Ph_6
~~Ph_2-Ph_2~~	Ph_2-Ph_3	~~Ph_2-Ph_4~~	Ph_2-Ph_5	~~Ph_2-Ph_6~~	
~~Ph_3-Ph_3~~	Ph_3-Ph_4	Ph_3-Ph_5	~~Ph_3-Ph_6~~		
~~Ph_4-Ph_4~~	Ph_4-Ph_5	~~Ph_4-Ph_6~~			
Ph_5-Ph_5	Ph_5-Ph_6				
~~Ph_6-Ph_6~~					

Optimizing Spaces

Optimization criterion (for most compact regions):

Maximize the Fourier coefficients of given trichord

Optimizing Spaces

Trichord	Best space(s)	Other
(012)**	Ph_1-Ph_6	
(013)	Ph_1-Ph_4	Ph_1-Ph_5
(014)	Ph_1-Ph_3	Ph_3-Ph_4
(015)	Ph_2-Ph_3	
(016)	Ph_1-Ph_2, Ph_2-Ph_3, Ph_2-Ph_5	
(025)	Ph_4-Ph_5	Ph_1-Ph_5
(027)**	Ph_5-Ph_6	
(037)	Ph_3-Ph_5	Ph_3-Ph_4

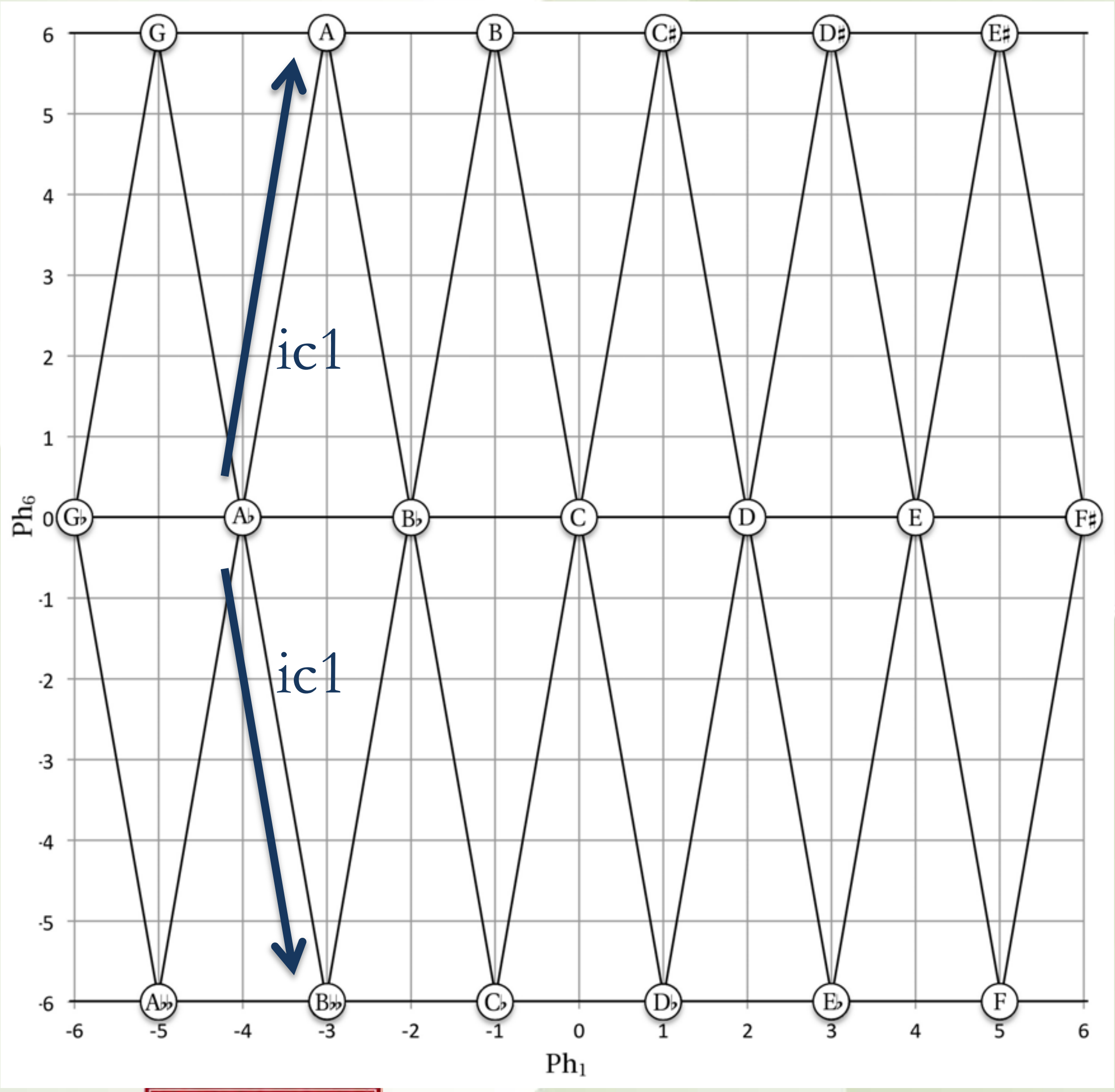
(5) Duplicated intervals and folding

Duplicated Intervals

Sets (012) and (027) have duplicated intervals (ic1 and ic5)

Nonetheless, toroidal (012) and (027) Tonnetze are possible because we can draw multiple distinct axes for the same interval.

(012) Tonnetz



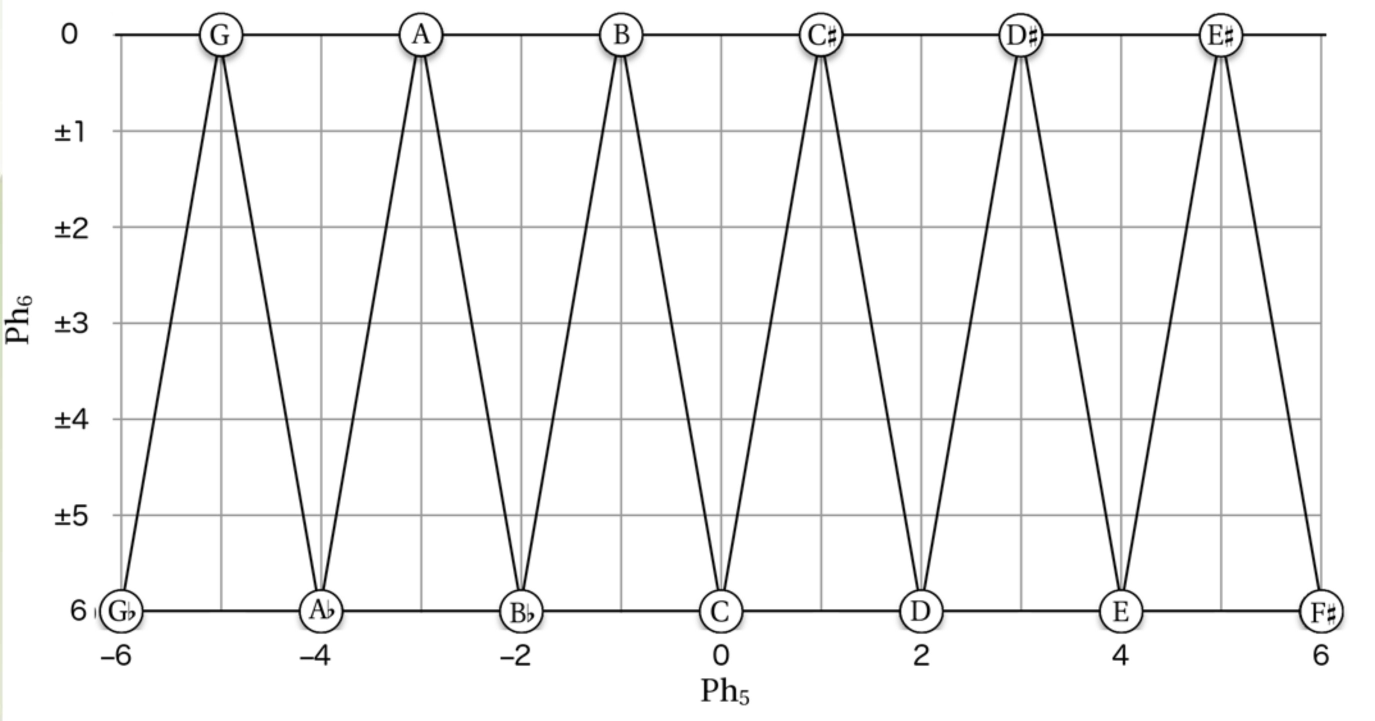
The two distinct axes may be used to represent distinct forms of the given interval.

For instance, here the ic1s are spelled as chromatic or diatonic semitones (enharmonic distinctions).

Folding and other topologies

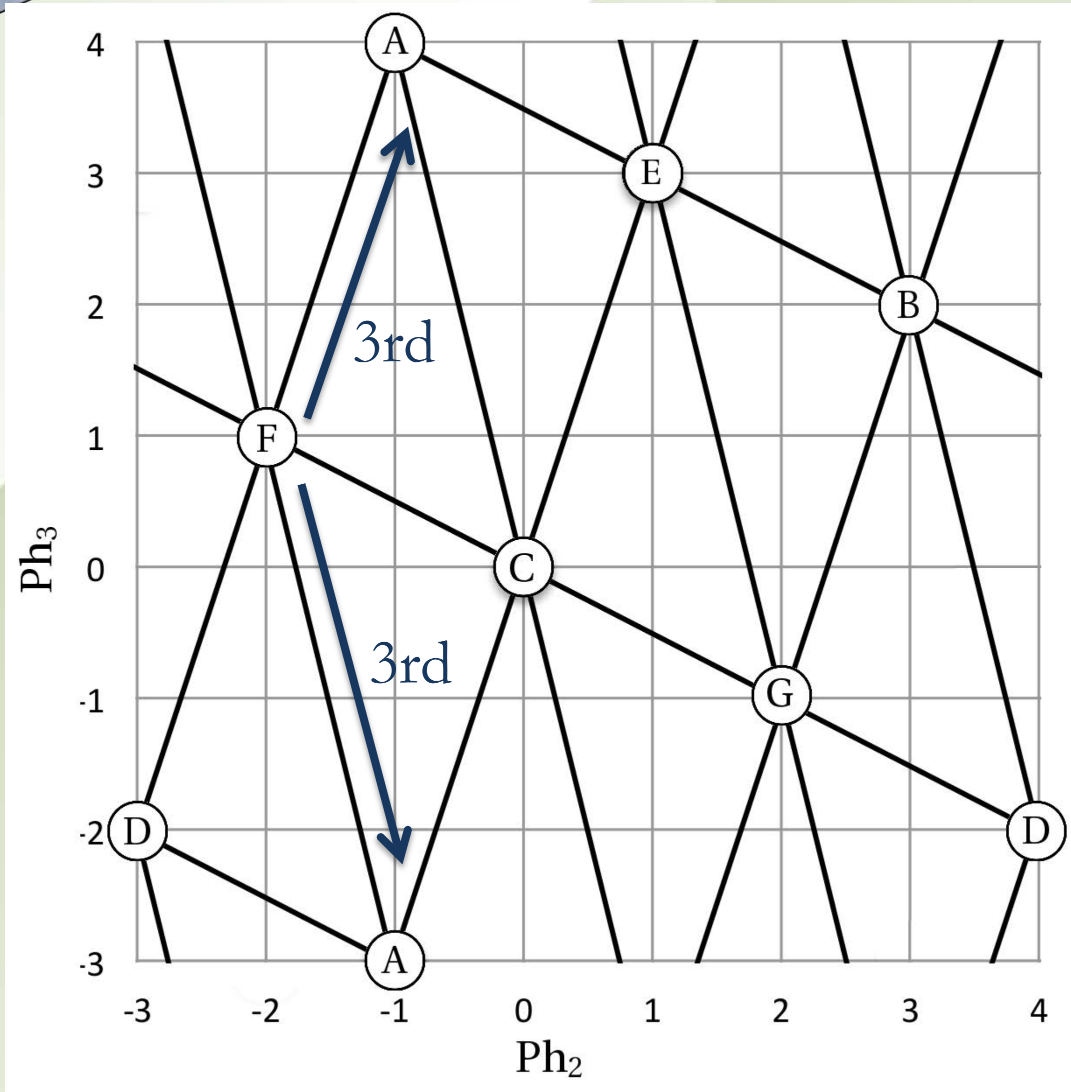
A Tonnetz with duplicated intervals can be folded to equate the duplicates.

The result is other kinds of topology derived from the basic toroidal one.



The folded (012) Tonnetz is a circular band with two boundaries.

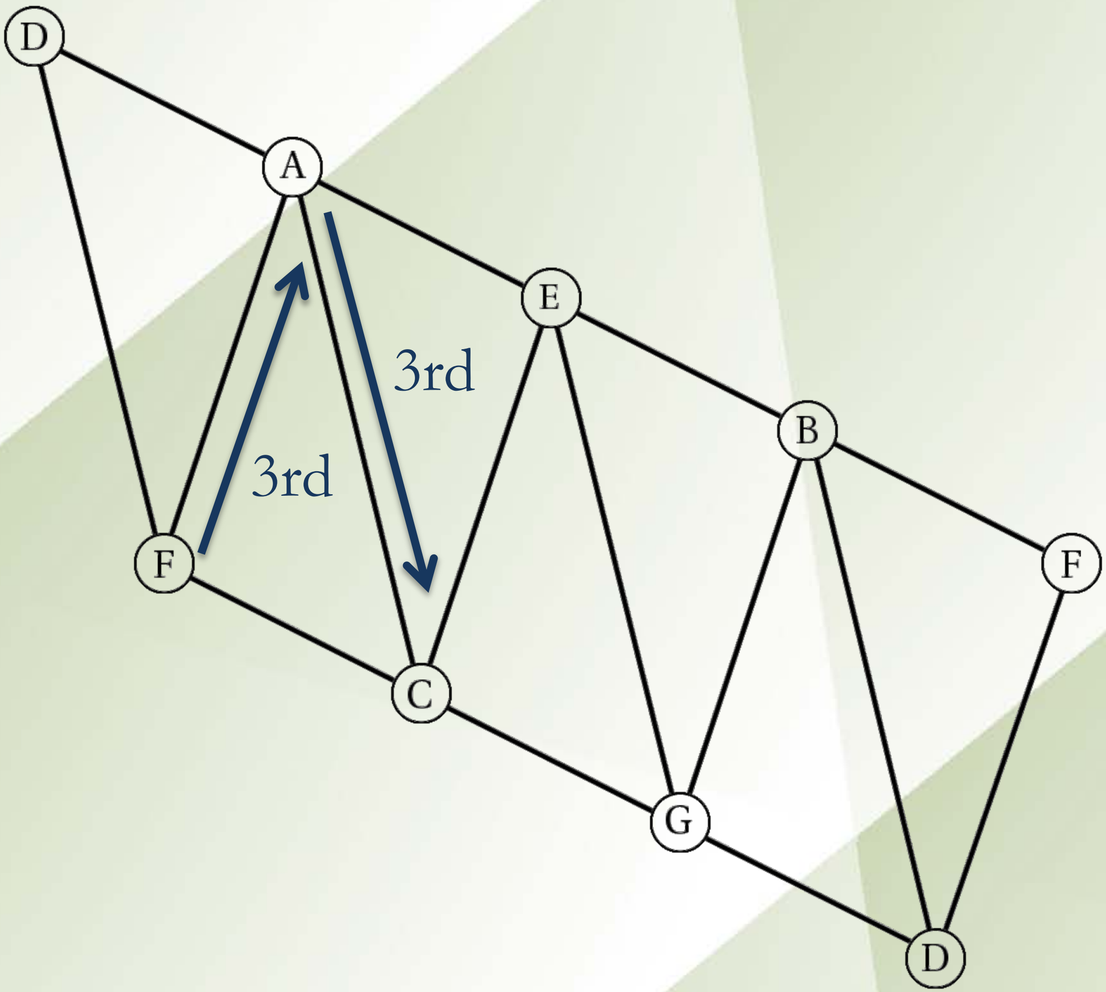
7-equal triad Tonnetz



In a scale of 7 equally spaced notes, triads have a duplicated interval (the 3rd).

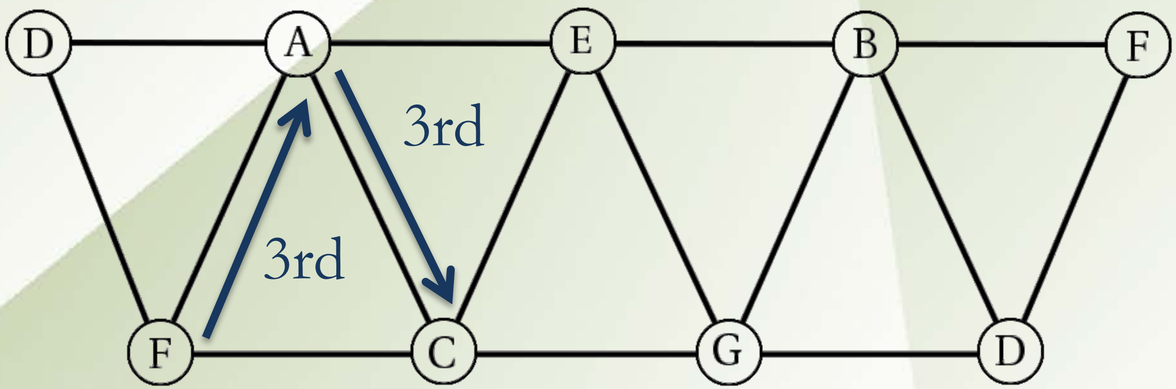
The Tonnetz therefore has two distinct intervallic axes for 3rds.

7-equal triad Tonnetz



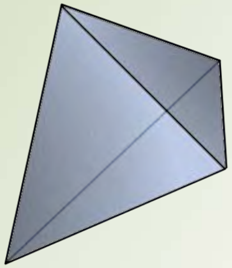
The space can be sheared and folded to equate the two kinds of thirds.

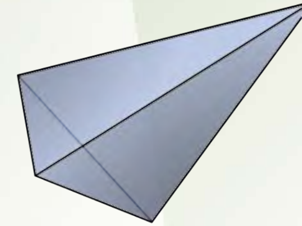
7-equal triad Tonnetz



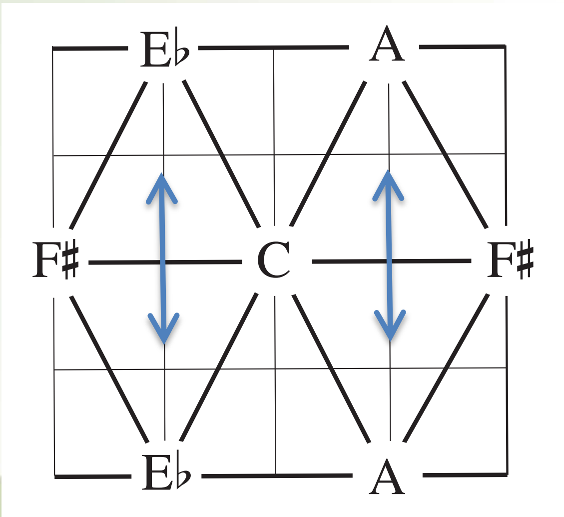
The space can be sheared and folded to equate the two kinds of thirds.

The result is Muzzulini (1995) and Mazzola's (2002) Möbius strip Tonnetz.



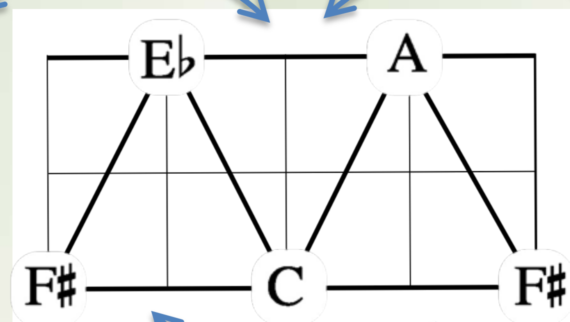


Spherical Tonnetz



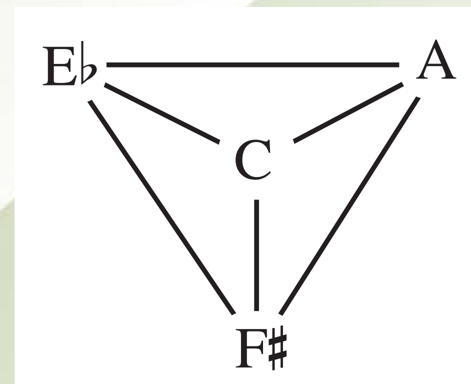
Diminished triad Tonnetz
has two kinds of
interval duplications

Fold to equate minor thirds . . .



Fold to equate tritones . . .

The result is a tetrahedral Tonnetz (a spherical topology)



This is a factor of Tymoczko’s (2012) seventh-chord Tonnetz

(4) Three-dimensional Tonnetze

Tetrachordal Tonnetz

Previous approaches (Gollin JMT 1998, Childs JMT 1998, Bigo CMJ 2015) take it as basic that a Tonnetz involves a single set class.

The result is networks of tetrahedra representing tetrachords, usually (0258)s, that intersect mostly in edges. Geometrically this means that there is empty space between the tetrahedra.

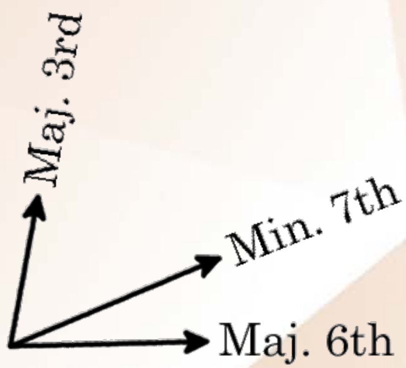
To make the Tonnetz a simplicial decomposition we instead completely fill the space with tetrahedra: this requires three set classes intersecting in shared trichords.

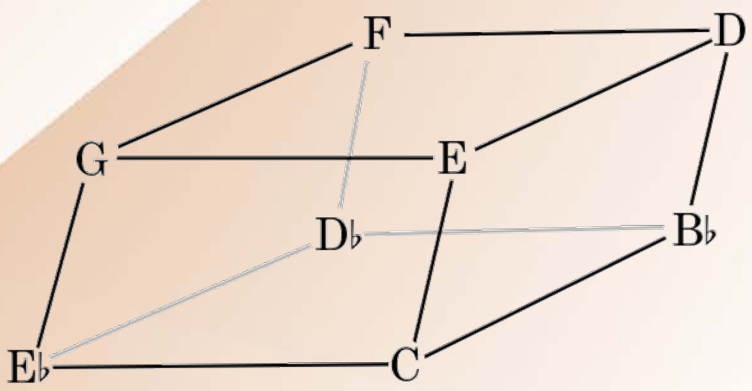
Tetrachordal Tonnetz: Construction

(1) Choose a three-dimensional phase space and any three non-parallel sets of intervallic axes

This partitions the space into parallelepipeds

Here are axes for ic2, ic3, and ic4 (in Z_{12})



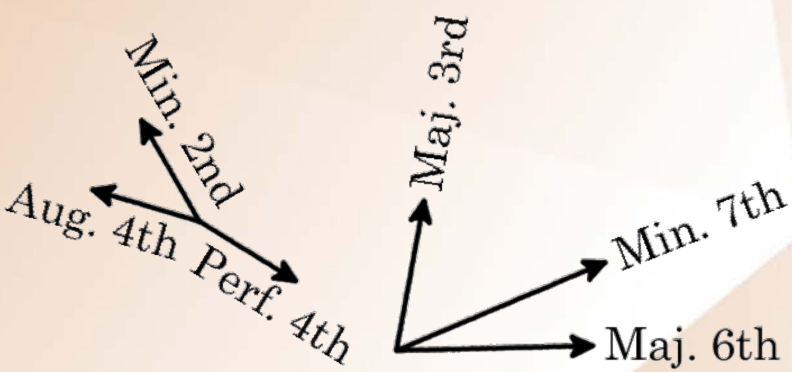


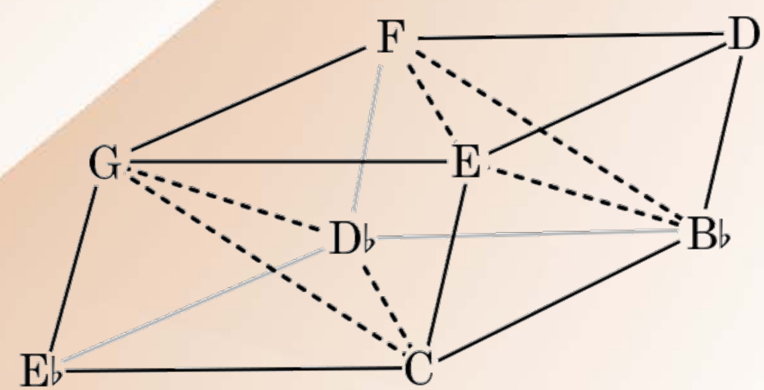
Tetrachordal Tonnetz: Construction

(2) Add three intervals to bisect the faces

The plane defined by these intervals creates tetrahedra for a single set class separated by octahedral regions. (Like Gollin's tetrachordal Tonnetz but with no shared faces!)

The new intervals in this example are ic1, ic6, and ic5. The set class is (0137) (an all-interval tetrachord).

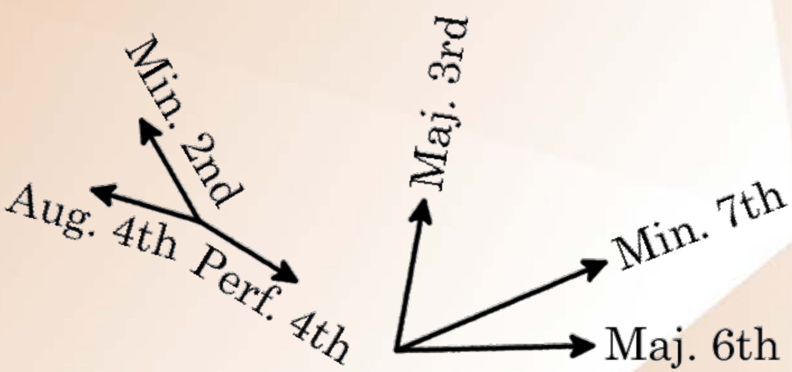


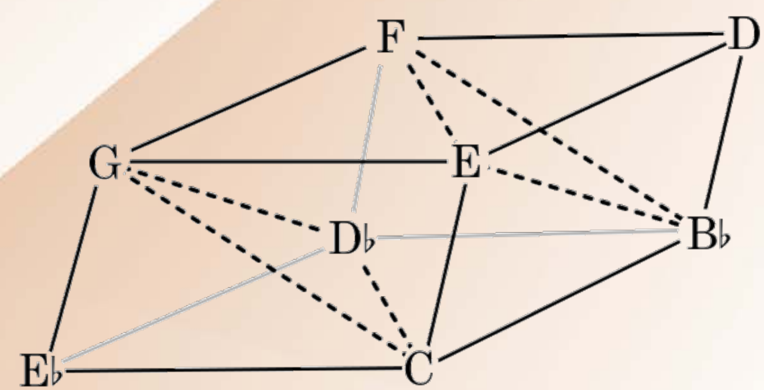


Tetrachordal Tonnetz: Construction

The space is now partitioned by four (sets of) planes, each of which is cut by the others into a two-dimensional Tonnetz based on one of the trichordal subsets of the given tetrachord.

In this example, there are planes with (037), (013), (026) and (016) Tonnetze.

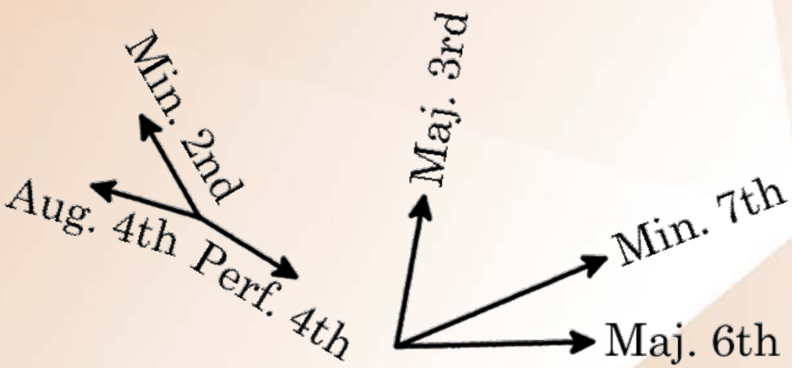


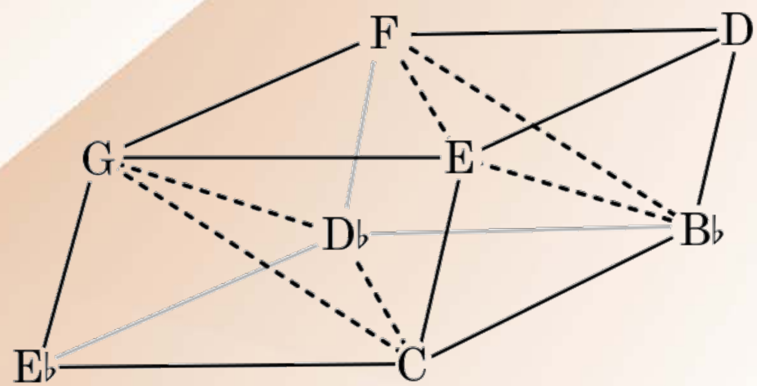


Tetrachordal Tonnetz: Construction

(3) The octahedral regions have three internal quadrilaterals. Bisecting any two of these with a single interval completes the partition into tetrachords. There are three possibilities.

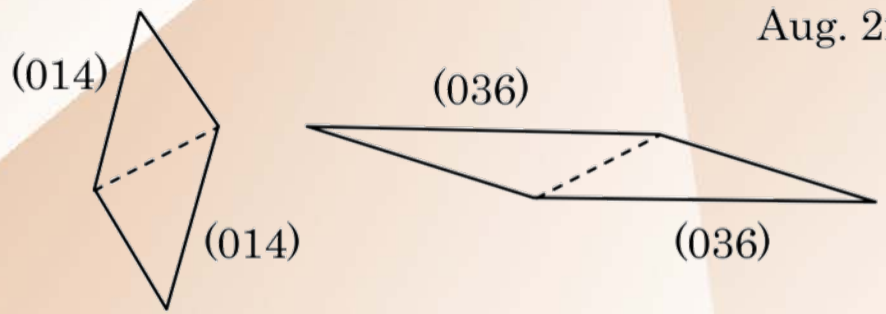
The internal quadrilaterals in this example are (FECD♭), (EB♭D♭G), and (FB♭CG).

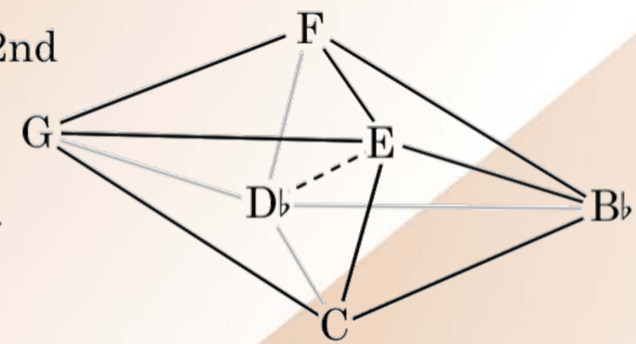


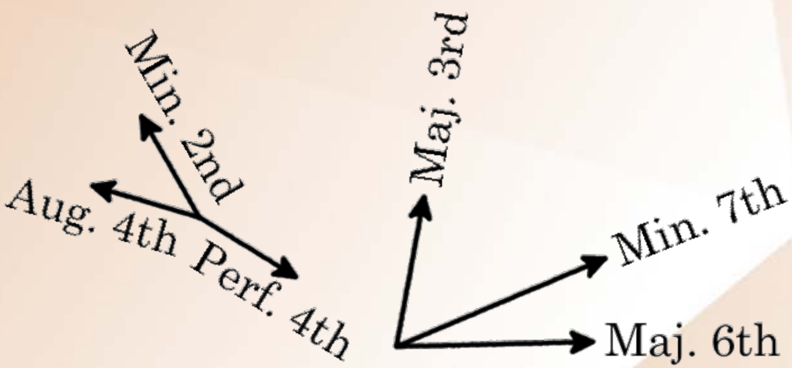


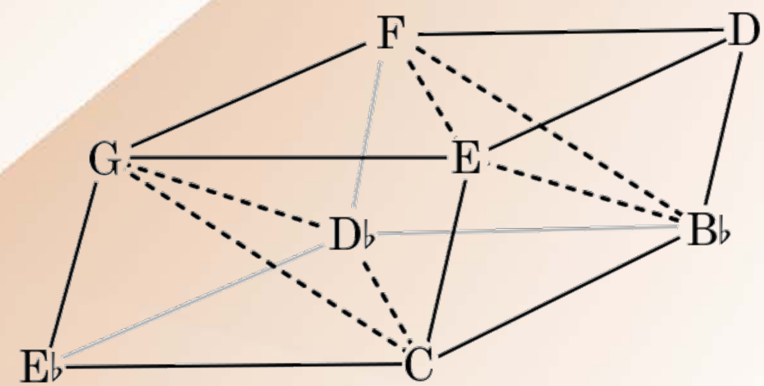
Tetrachordal Tonnetz: Construction

The first possibility adds an aug-2nd axis (distinct from the min-3rd axis!) creating (014) and (036) planes.



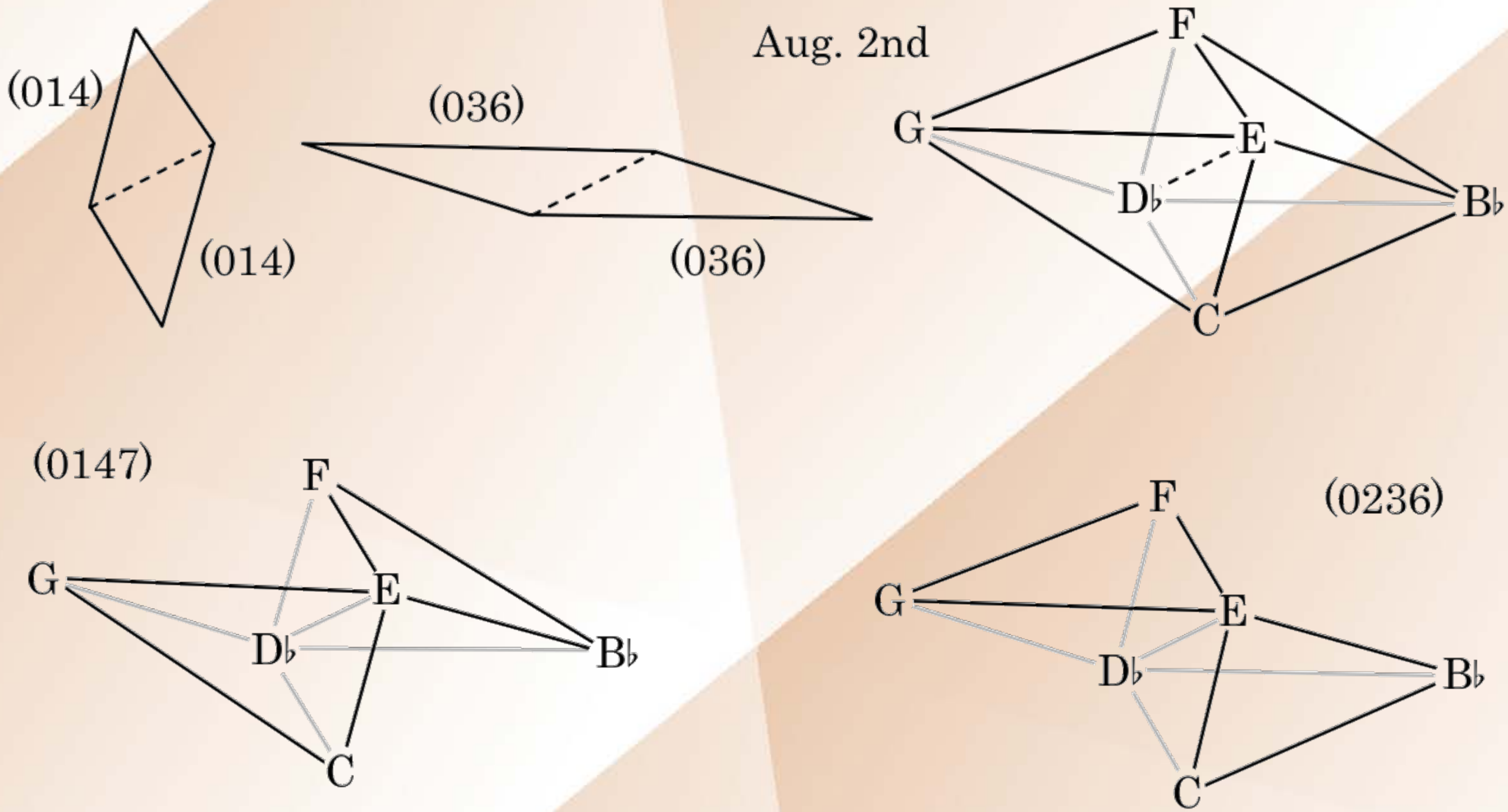






Tetrachordal Tonnetz: Construction

These add new tetrahedral (0147) and (0236) regions



Properties and Classes of Three-Dimensional Tonnetze

- Three tetrachord types
- Six planes with trichordal Tonnetze:
  Each trichord is shared by two tetrachords
- Seven intervallic axes
  - –Four are shared by all tetrachords and three trichordal planes
  - –Three are shared by two tetrachords and two trichordal planes

In Z_{12} there are only six interval classes, so at least one must be duplicated.

Tonnetze can be classified according to where (in which group of intervallic axes) duplications occur.

Classes of 3-D Tonnetz in $\mathbf{Z}_{12}$

Class	Properties	Tetrachords	Duplications	Omits	Space
A	One sym. trichord	(0125) (0126) (0146)	ic1		$Ph_{1,2,6}$
		(0237) (0157) (0137)	ic5		$Ph_{2,5,6}$
		(0136) (0236) (0146)	ic3		$Ph_{1,2,4}$
		(0147) (0236) (0137)	ic3		$Ph_{2,3,4}$ / $Ph_{3,4,6}$
		(0147) (0258) (0146)	ic3		$Ph_{2,3,4}$ / $Ph_{3,4,6}$
		(0136) (0258) (0137)	ic3		$Ph_{2,4,5}$
B	Two sym. trichords	(0124) (0125) (0135)	ic1, ic2	ic6	$Ph_{1,3,6}$
		(0135) (0237) (0247)	ic5, ic2	ic6	$Ph_{3,5,6}$
C	Two sym. and one dup. trichord	(0126) (0127) (0157)	ic1, ic5, (016)	ic3	$Ph_{1,2,6}$ or $Ph_{2,5,6}$

Classes of 3-D Tonnetz in $\mathbf{Z}_{12}$

Class	Properties	Tetrachords	Duplications	Omits	Space
D	Dup. tetrachord	(0134), (0236) × 2	ic1, ic3, (013), (014)	ic5	$Ph_{1,4,6}$
	+ one sym. trichord	(0145), (0125) × 2	ic1, ic4, (014), (015)	ic6	$Ph_{1,3,6}$
		(0156), (0126) × 2	ic1, ic5, (015), (016)	ic3	$Ph_{1,2,6}$
		(0156), (0157) × 2	ic1, ic5, (015), (016)	ic3	$Ph_{2,5,6}$
		(0235), (0135) × 2	ic2, ic3, (013), (025)	ic6	$Ph_{1,3,6}$ or $Ph_{3,5,6}$
		(0235), (0136) × 2	ic2, ic3, (013), (025)	ic4	$Ph_{2,3,4}$
		(0347), (0147) × 2	ic3, ic4, (014), (037)	ic2	$Ph_{2,3,4}$
		(0158), (0237) × 2	ic4, ic5, (015), (037)	ic6	$Ph_{3,5,6}$
		(0358), (0258) × 2	ic3, ic5, (025), (037)	ic1	$Ph_{4,5,6}$

Classes of 3-D Tonnetz in $\mathbf{Z}_{12}$

Class	Properties	Tetrachords	Duplications	Omits	Space
E	Dup. tetrachord	(0145), (0148) × 2	ic4 (× 3), ic1, (014), (015)	ic2, ic6	$Ph_{1,2,6}$
	& Aug. triad	(0347), (0148) × 2	ic4 (× 3), ic3, (014), (037)	ic2, ic6	$Ph_{1,4,6}$ / $Ph_{4,5,6}$
		(0158), (0148) × 2	ic4 (× 3), ic5, (015), (037)	ic2, ic6	$Ph_{2,5,6}$
F	Dup. tetrachord &	(0134), (0124) × 2	ic1, ic3, ic2, (013), (014)	ic5, ic6	$Ph_{1,3,6}$
	two sym. trichords	(0358), (0247) × 2	ic5, ic3, ic2, (025), (037)	ic1, ic6	$Ph_{3,5,6}$
G	Dup. tetrachord &	(0123), (0124) × 2	ic1 (× 3), ic2, (012), (013)	ic5, ic6	$Ph_{1,5,6}$
	sym. tetrachord	(0257), (0247) × 2	ic5 (× 3), ic2, (025), (027)	ic1, ic6	$Ph_{1,5,6}$

JI scalar Tonnetz

Class D: (0235), (0135) × 2

Optimal space: Ph_2-Ph_3-Ph_6

Duplicated ic2s and ic3s may be interpreted as just vs. Pythagorean intervals (10/9 vs. 9/8 and 6/5 vs. 32/27)

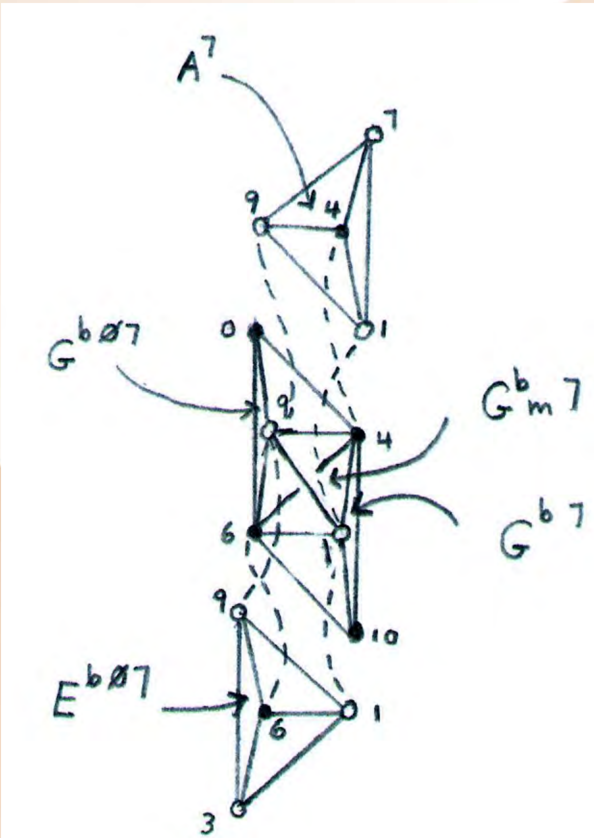
j(0135): $\underbrace{s + p2}_{j3} + j2$ or $j2 + \underbrace{p2 + s}_{j3}$

p(0135): $\underbrace{s + j2}_{p3} + p2$ or $p2 + \underbrace{j2 + s}_{p3}$

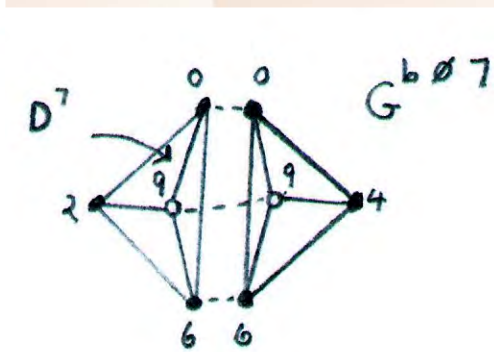
(0235): j2 + s + p2 or p2 + s + j2

p3 j3 j3 p3

Douthett's Tetrahedral Tonnetz



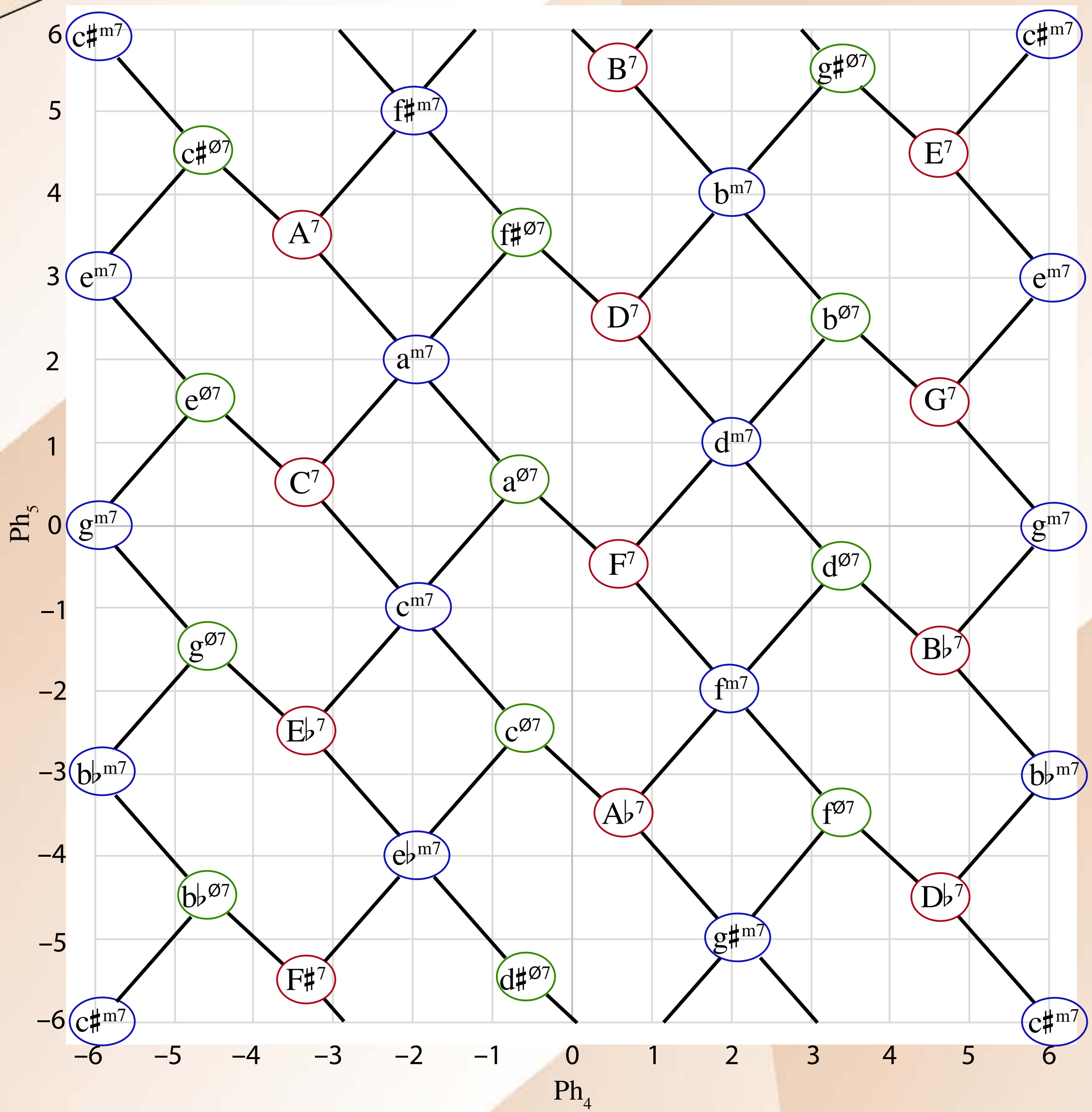
This can be derived from class D Tonnetz on (0358), (0258) × 2 by equating duplicated ic3 and ic5 axes. The result is cyclic in two dimensions with the (026)-Tonnetz planes forming a boundary in the third dimension.



From unpublished ms., 1997

Douthett's Tonnetz still has duplicated tritones. Folding these creates Tymoczko's four-note voice-leading Tonnetz.

Douthett's Tetrahedral *Tonnetz*



Projection of Douthett's Tonnetz onto the Ph_4-Ph_5 boundary

Example: Aug-2nd *Tonnetz*

Class A: (0137), (0147), (0236)

Optimal space: Ph_2-Ph_3-Ph_4 or Ph_3-Ph_4-Ph_6

Duplicated ic3s may be interpreted as min 3rd/aug 2nd

(0137): Diatonic subset, maj. triad + #4th (IV + $\hat{7}$) or
min. triad + $\flat$2nd ($V^{7(no5th)}$ + $\hat{3}$)

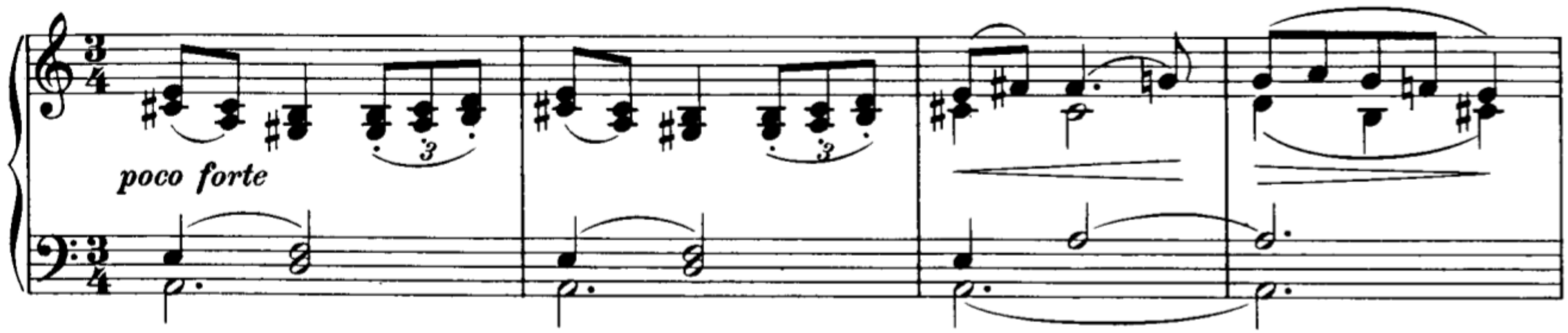
(0147): Non-diatonic, maj. triad + $\flat$2nd (V + $\flat\hat{6}$) or
min. triad + #4th (iv + #$\hat{7}$)

(0236): Non-diatonic, harmonic minor scale segment:
$\flat\hat{6}$-$\hat{7}$-$\hat{1}$-$\hat{2}$ or $\hat{4}$-$\hat{5}$-$\flat\hat{6}$-$\hat{7}$

(2) Example of analysis with a three-dimensional Tonnetz

Brahms, Sarabande in A minor, WoO 5/1

Brahms Sarabande WoO 5: A part



Triadic: A maj.————————————————————————(G maj.—A maj.)
Dim. 7th: [2]——————————————————————[1]———————————————

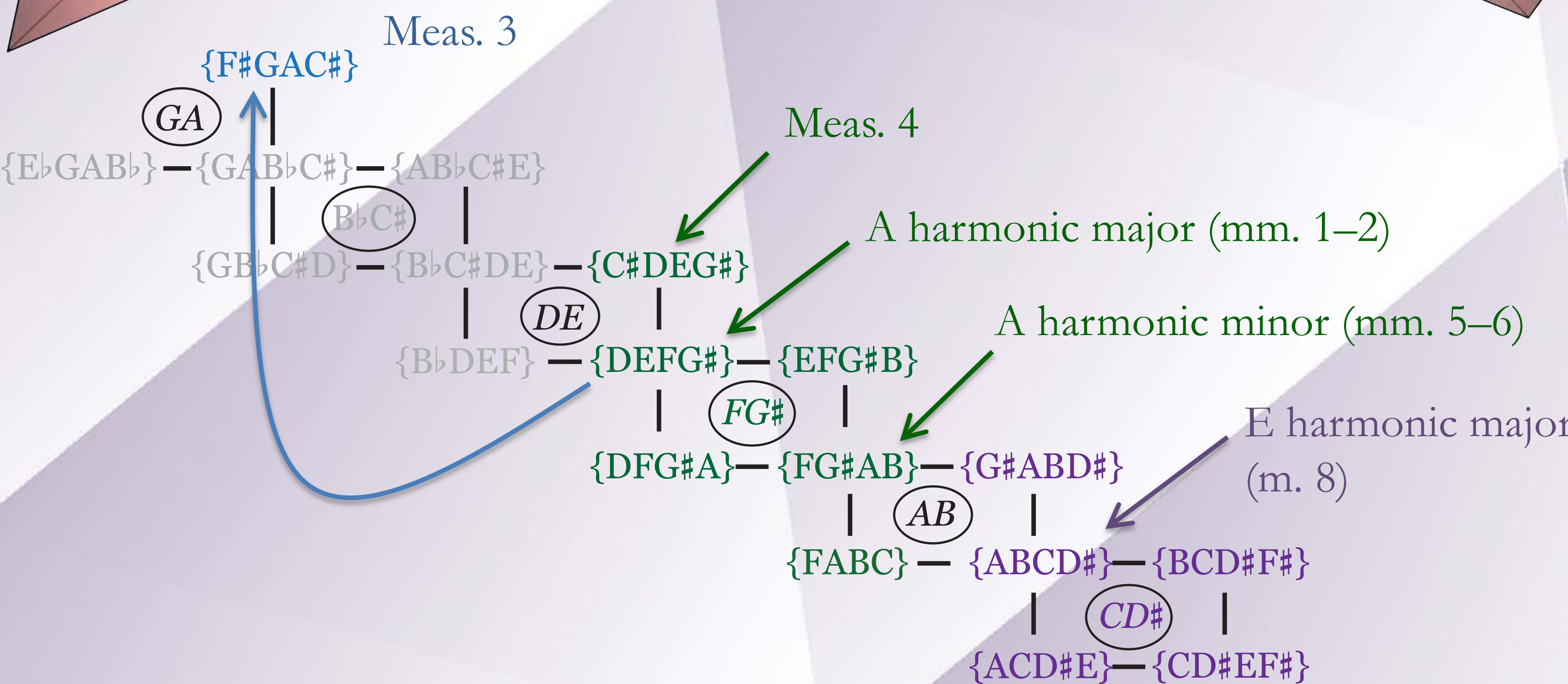
Triadic: A min.——————————————————————————D maj.——————E maj.
Dim. 7th: [2]————————————————————————[1]————————[0]——

Example: Brahms Sarabande

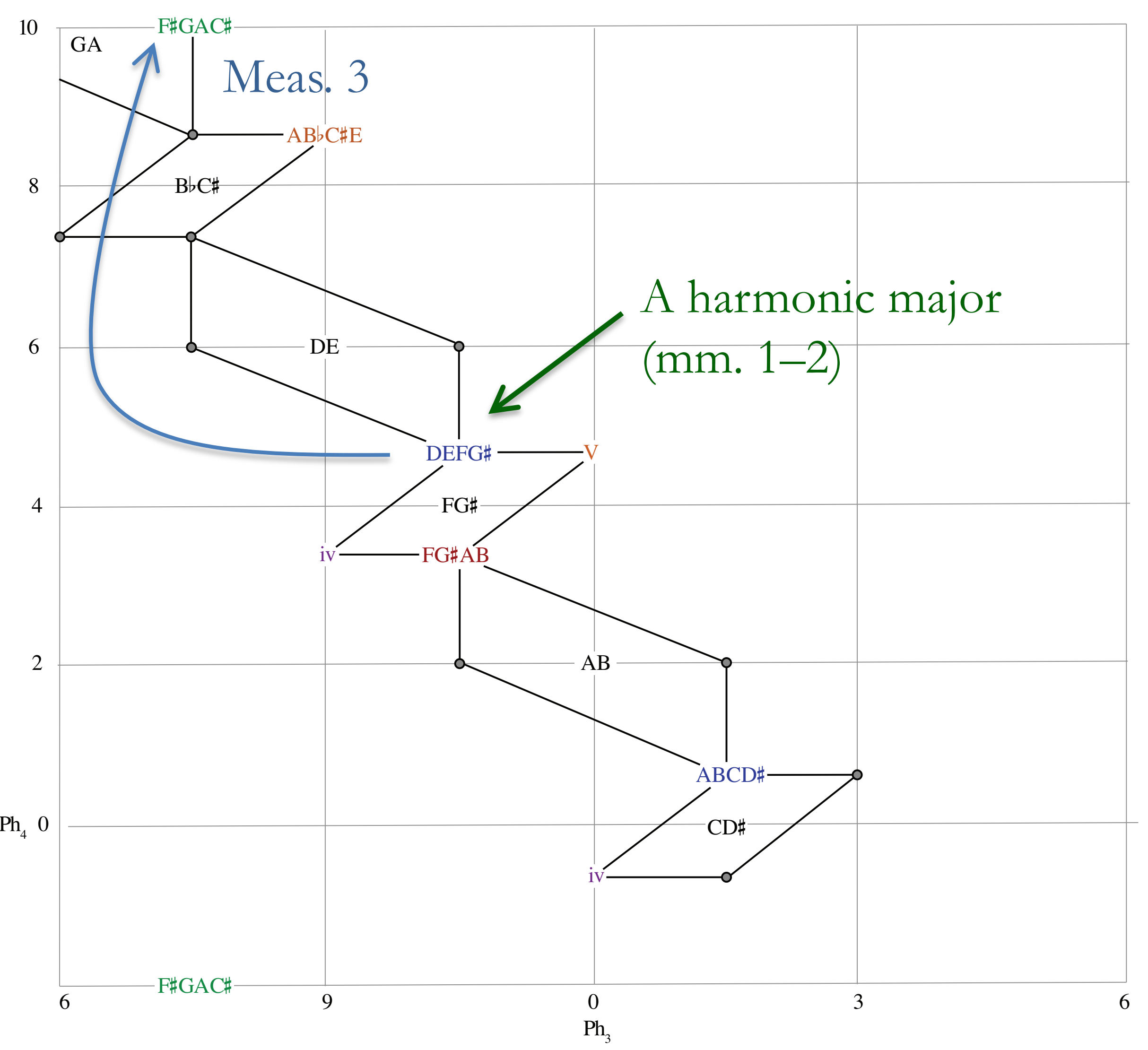
Positions of triads and scales in Ph_3/Ph_4 space.

Note the Ph_4 separation of triads and scales.

Example: Brahms Sarabande



Example: Brahms Sarabande



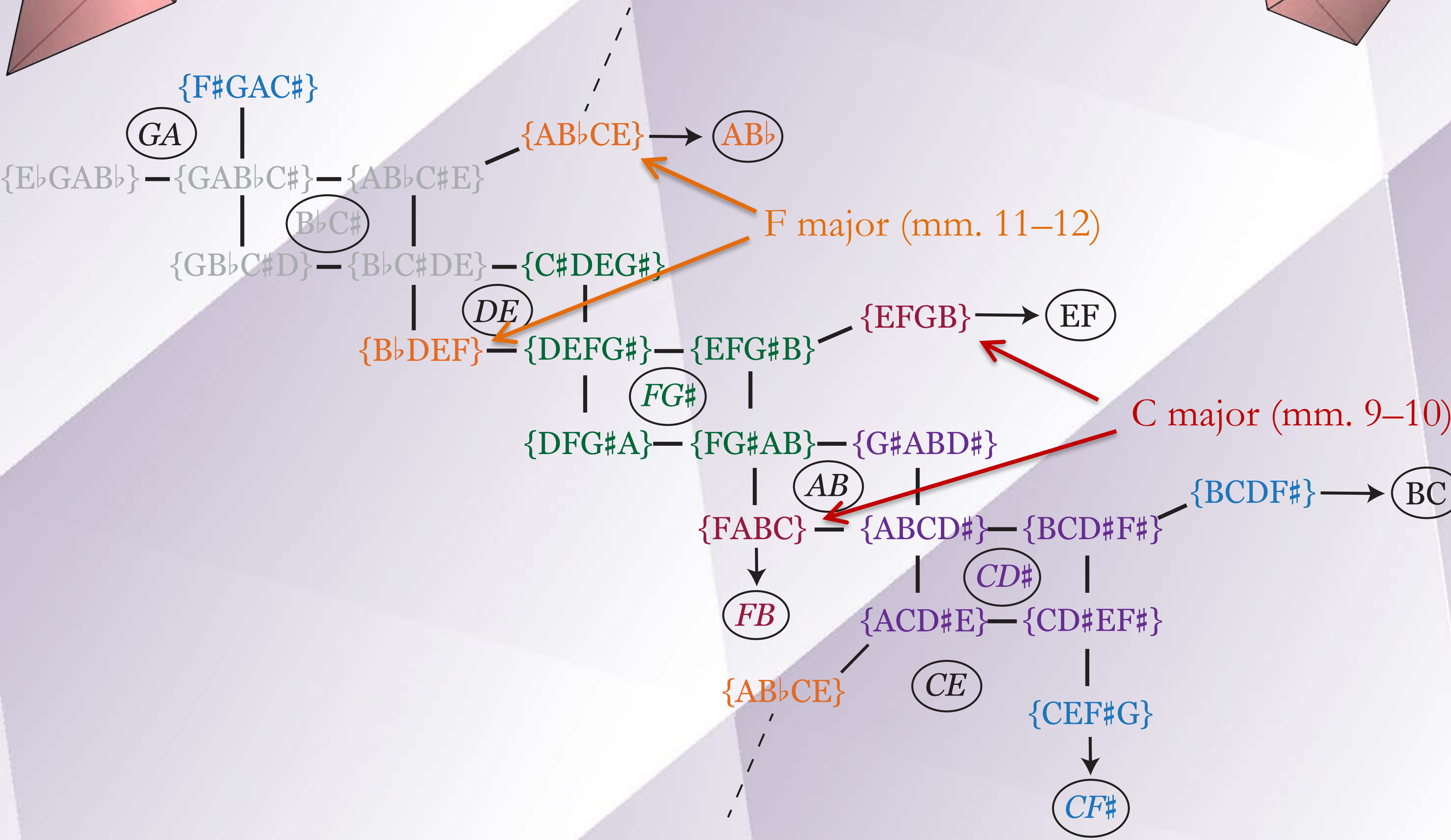
Network for A part projected into Ph$_3$/Ph$_4$ space

Brahms Sarabande WoO 5: B part

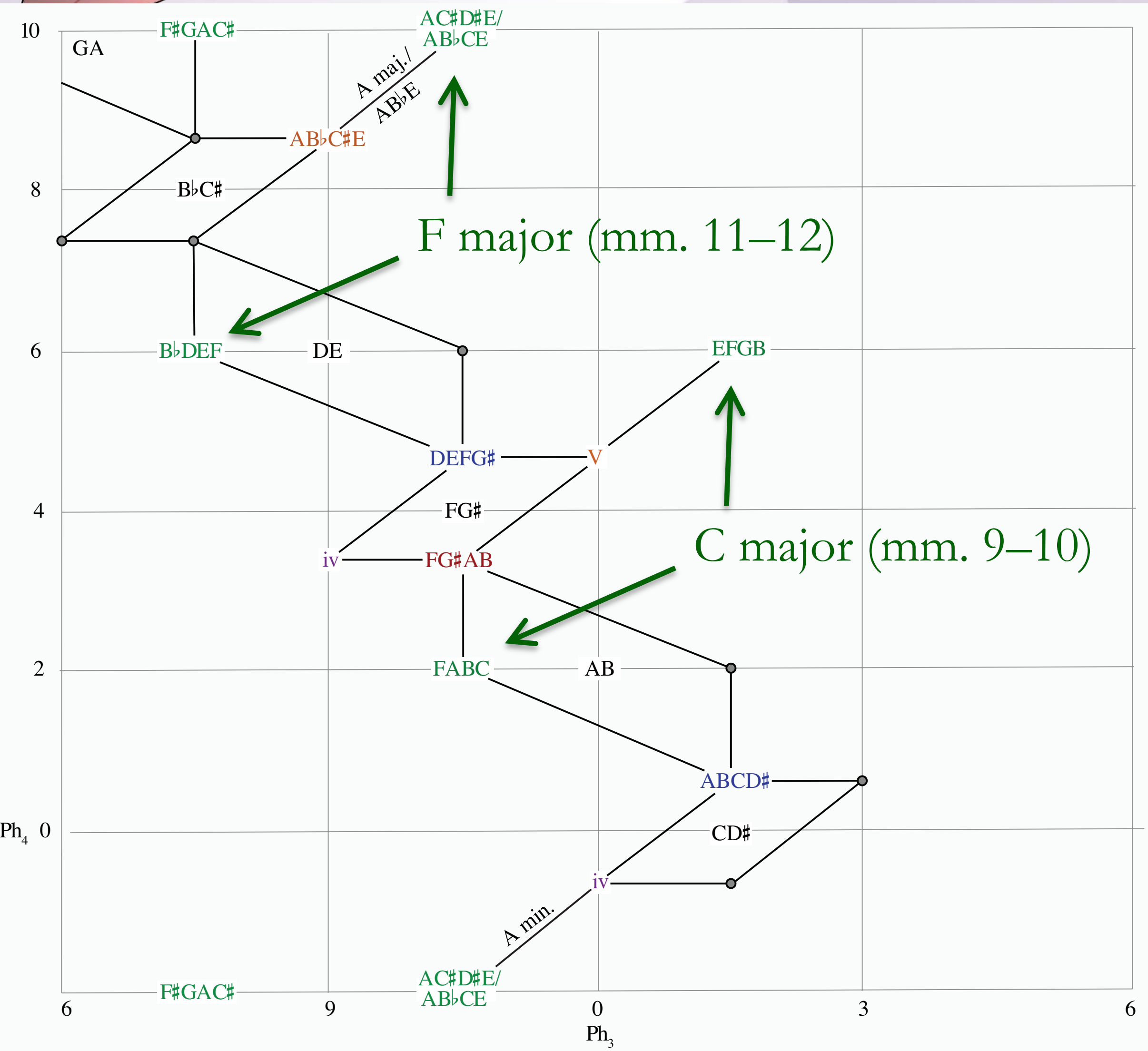
Triadic: A min./C maj.————————————————————————F maj.——————————
Dim. 7th: [2]—————————————————————————————[1]——————————————

Triadic: A min.————————A maj.————————————————(D maj.)—G maj.——————A maj.
Dim. 7th: [2]———————————————[1]—————————————————[0]——————————[2]———

Example: Brahms Sarabande

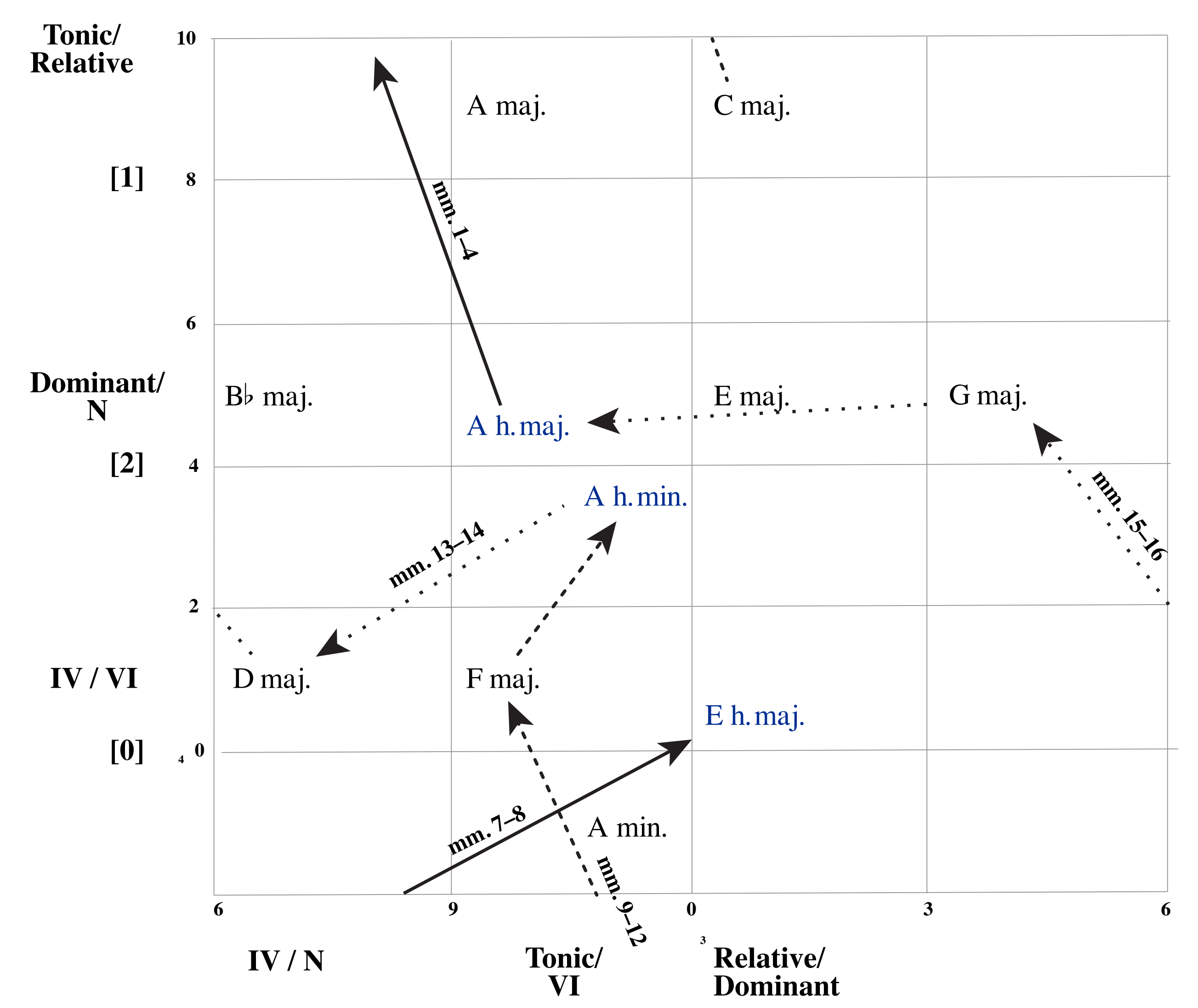


Example: Brahms Sarabande



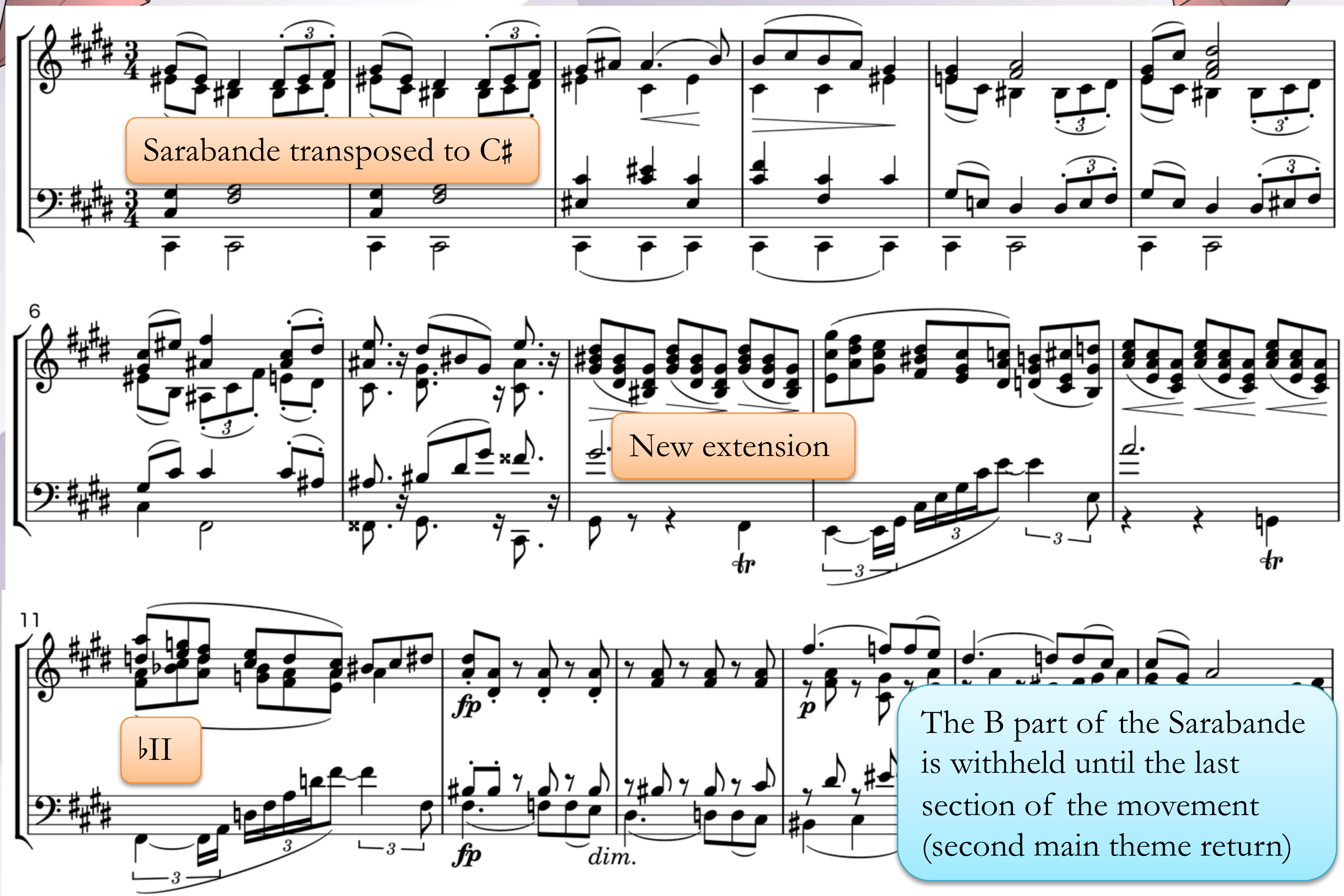
Network for the complete Sarabande projected into Ph_3/Ph_4 space

Example: Brahms Sarabande



Triads and scales in Ph_3/Ph_4 space

Op. 88 String Quintet, Grave ed Appassionato (ii)

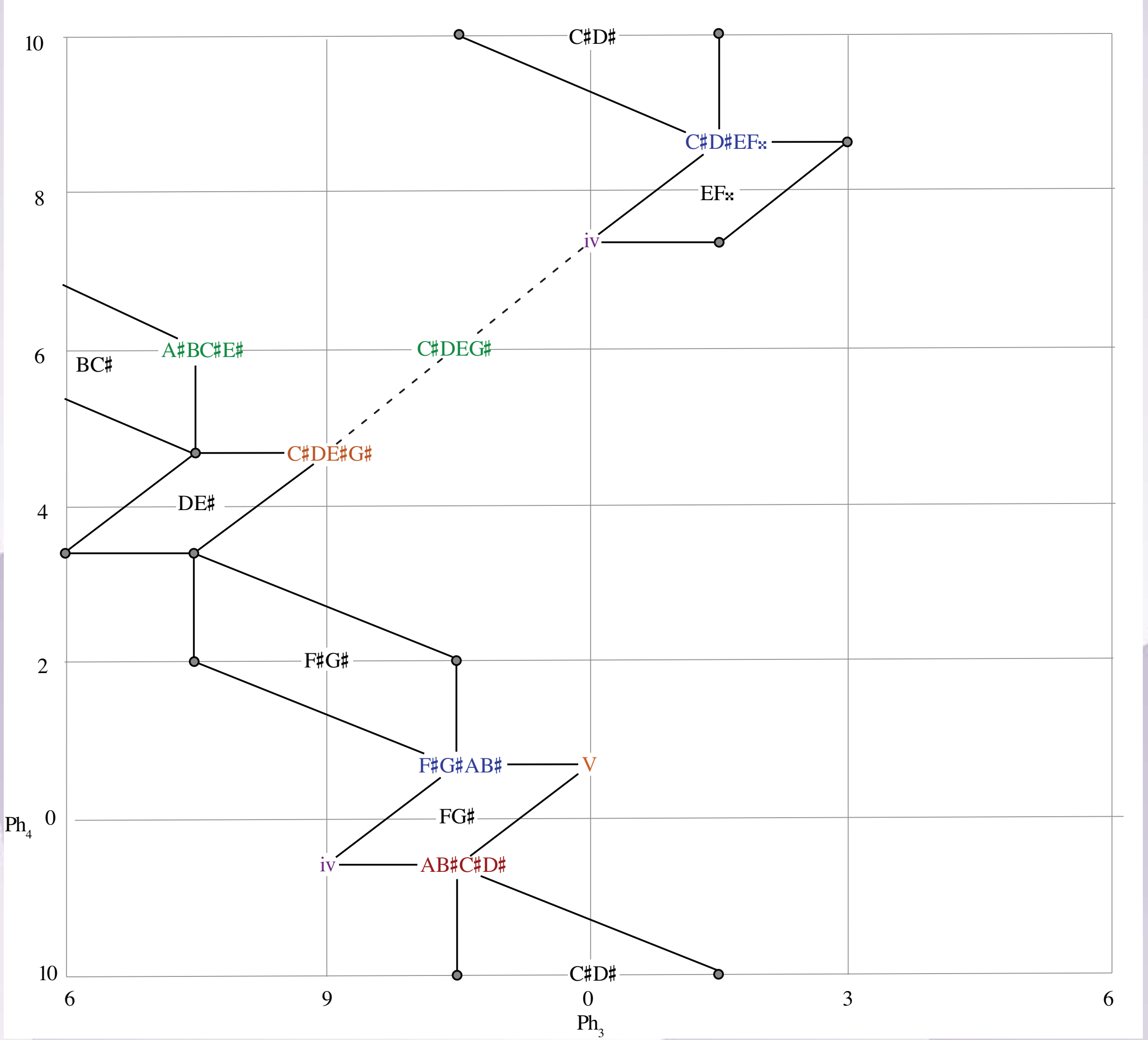


Op. 88 String Quintet, Grave ed Appassionato (ii)

The first return of the theme extends the first four measures with the subdominant minor (which fills in the Tonnetz gaps)

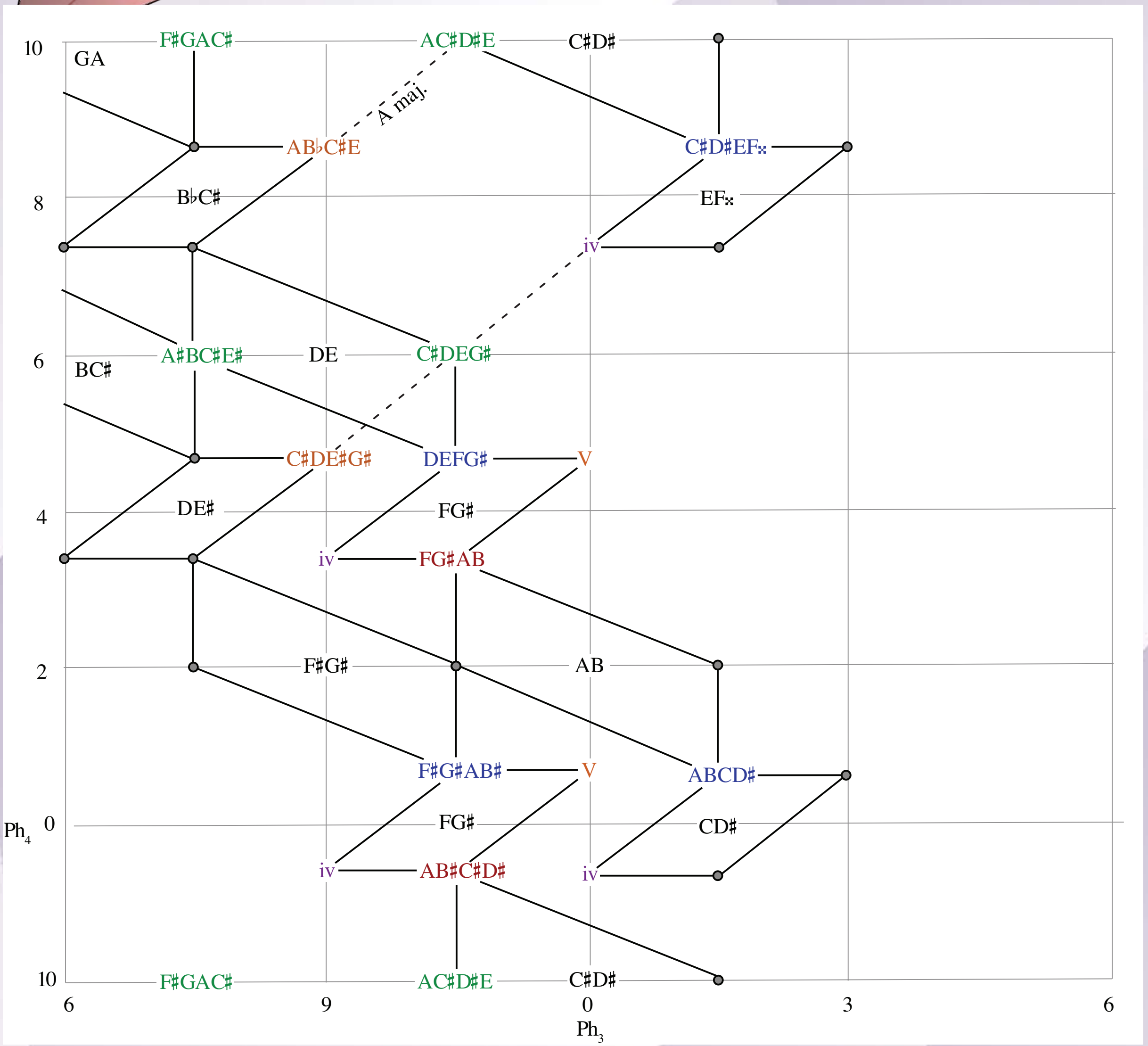
The ending further underscores the rhetorical importance of ♭II

Op. 88 String Quintet, Grave ed Appassionato (ii)



In the Op. 88 Quintet the A part of the Sarabande theme is transposed to C♯

Op. 88 String Quintet, Grave ed Appassionato (ii)



At the third return, the theme begins transposed to A (the key of the Sarabande)

Summary

- DFT phase spaces emerge naturally from the idea of a generalized geometric Tonnetz.
- The DFT can be used to enrich analytical applications of generalized Tonnetze and to find optimum spaces.
- Toroidal spaces can duplicate intervals—i.e., multiple axes representing the same interval.
- Non-toroidal Tonnetz can be derived by folding toroidal ones.
- Generalizing the Tonnetz as triangulation (simplicial decomposition) of a toroidal space to three dimensions gives a tetrahedral Tonnetz on three tetrachord types, formed by six planes and seven intervallic axes

Generalized Trichordal and Tetrachordal Tonnetze: Geometry and Analytical Application

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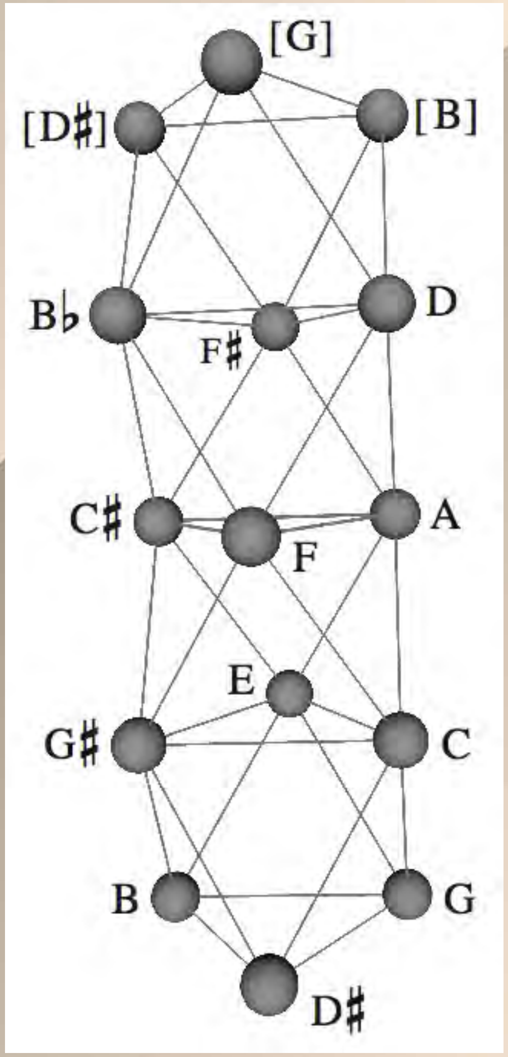
SMT 2017, Arlington, VA

Example: Tymoczko's triadic *Tonnetz*

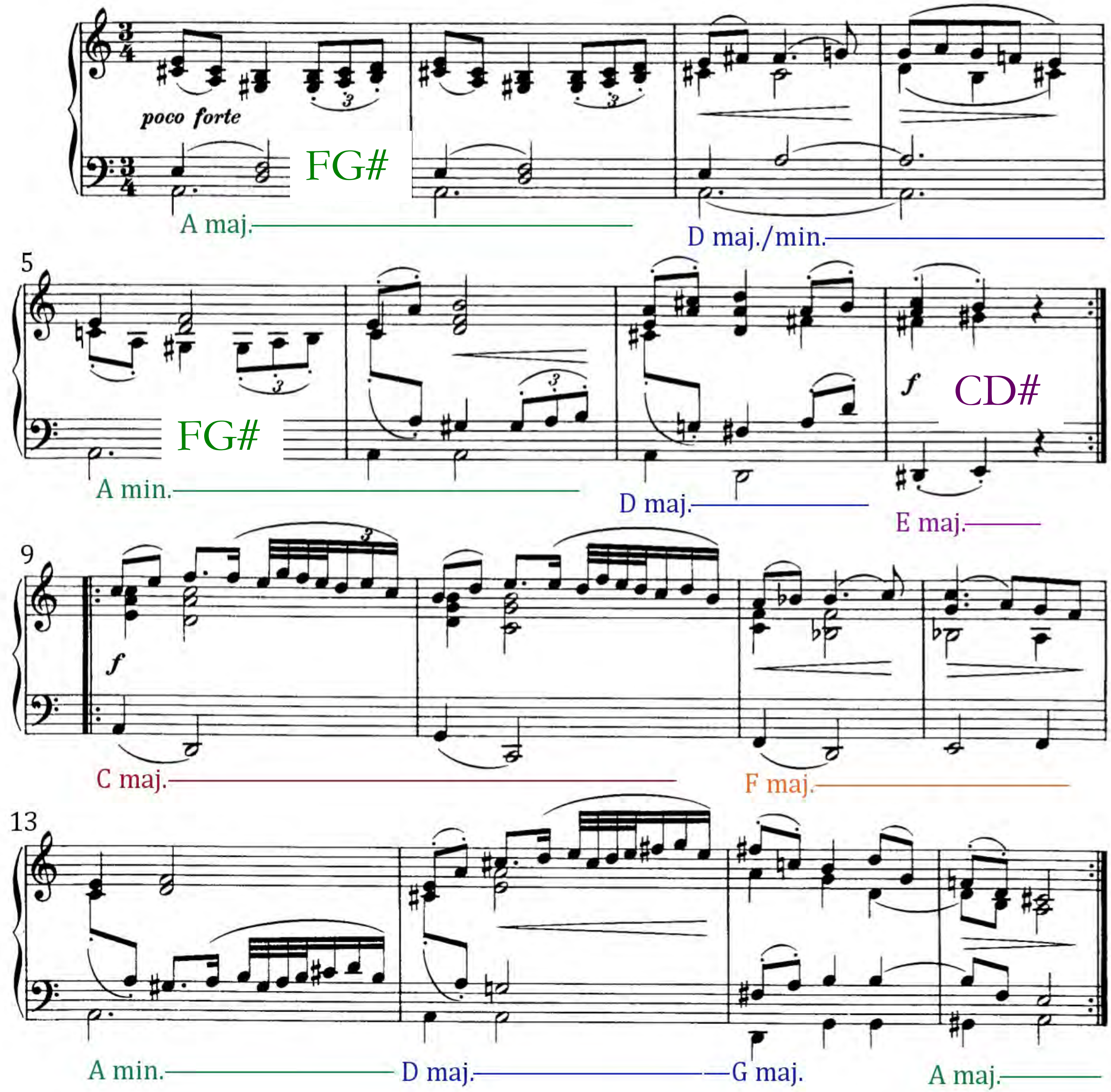


- As a two-dimensional space (surface of the figure) it is toroidal. It satisfies transposability through translations.

- Embedded in a three-dimensional space it satisfies transposability through rotations and screw rotations.

- For Tymoczko, the embedding three-dimensional space has an important role in the voice-leading intent of the figure, but to the transposability of vertices and the lattice it is superfluous.

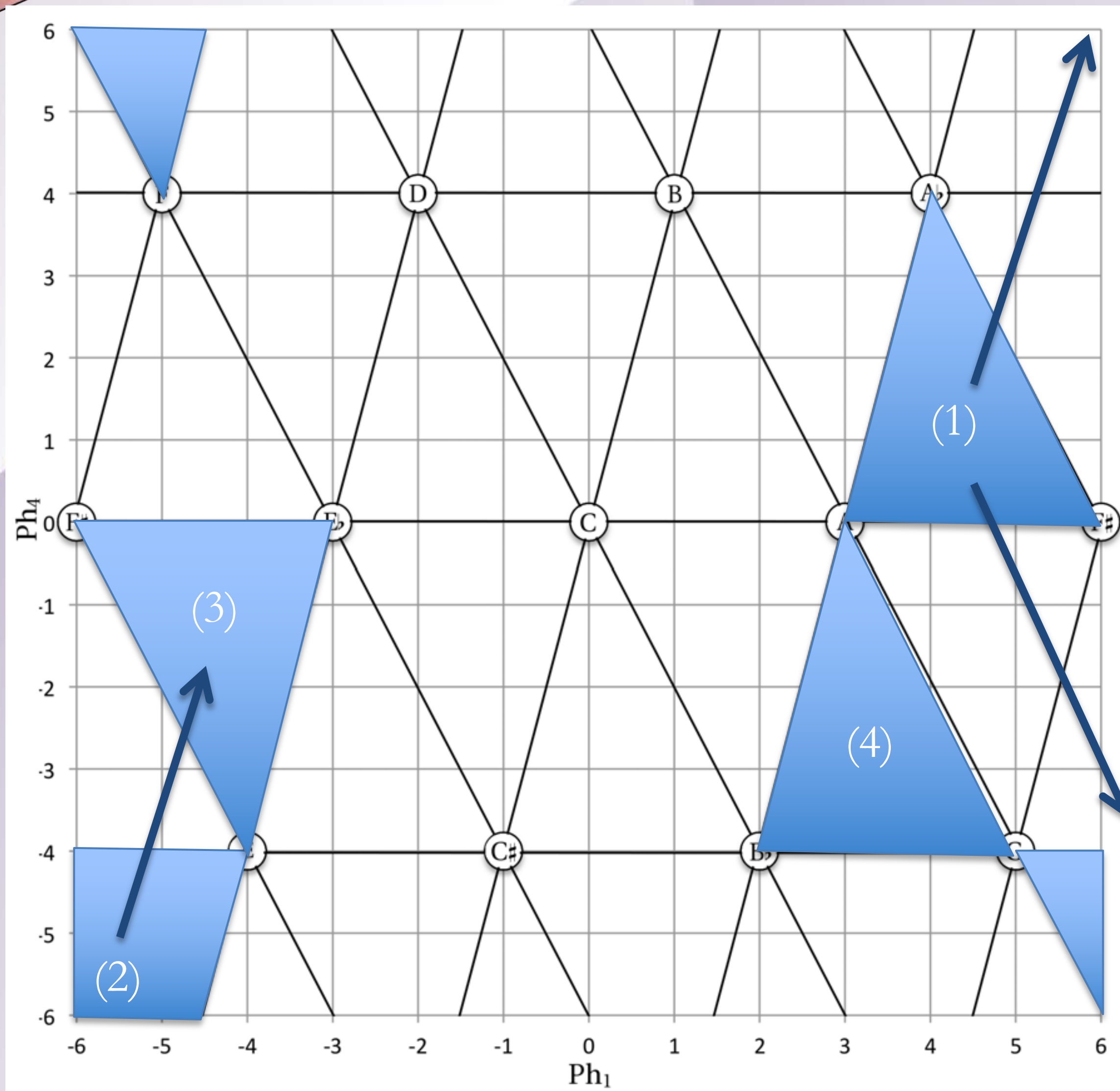
Brahms, Sarabande 1, WoO5



Shostakovich: String Quartet 12

Beginning of second movement, (013)s:

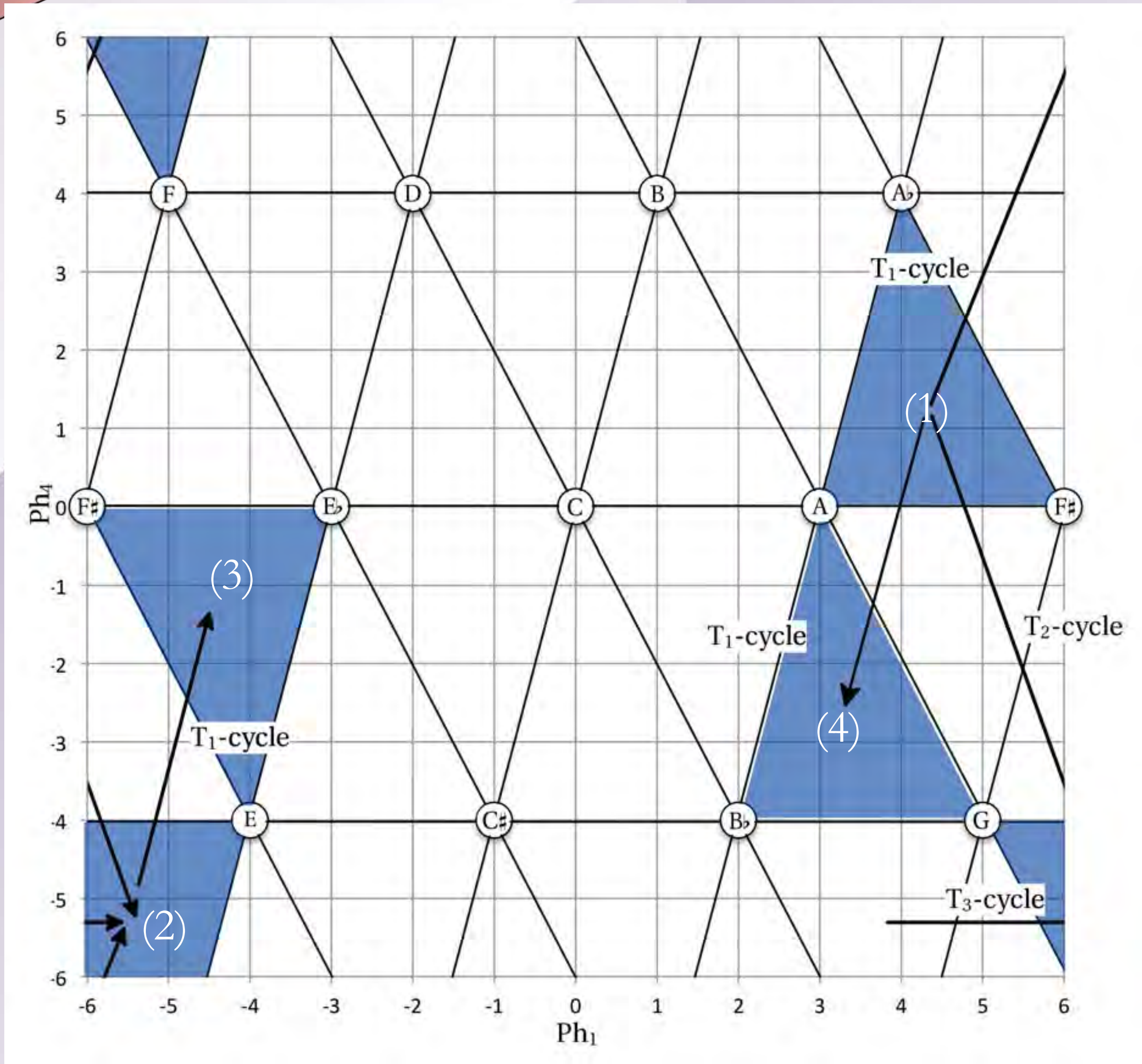
Shostakovich: String Quartet 12



The melody initially focuses on disjunct and T_1 relationships between (013)s

We can identify cycles (T_1 or T_2) for the disjunct relationship

Shostakovich: String Quartet 12



The melody initially focuses on disjunct and T_1 relationships between (013)s

We can identify cycles (T_1 or T_2) for the disjunct relationship

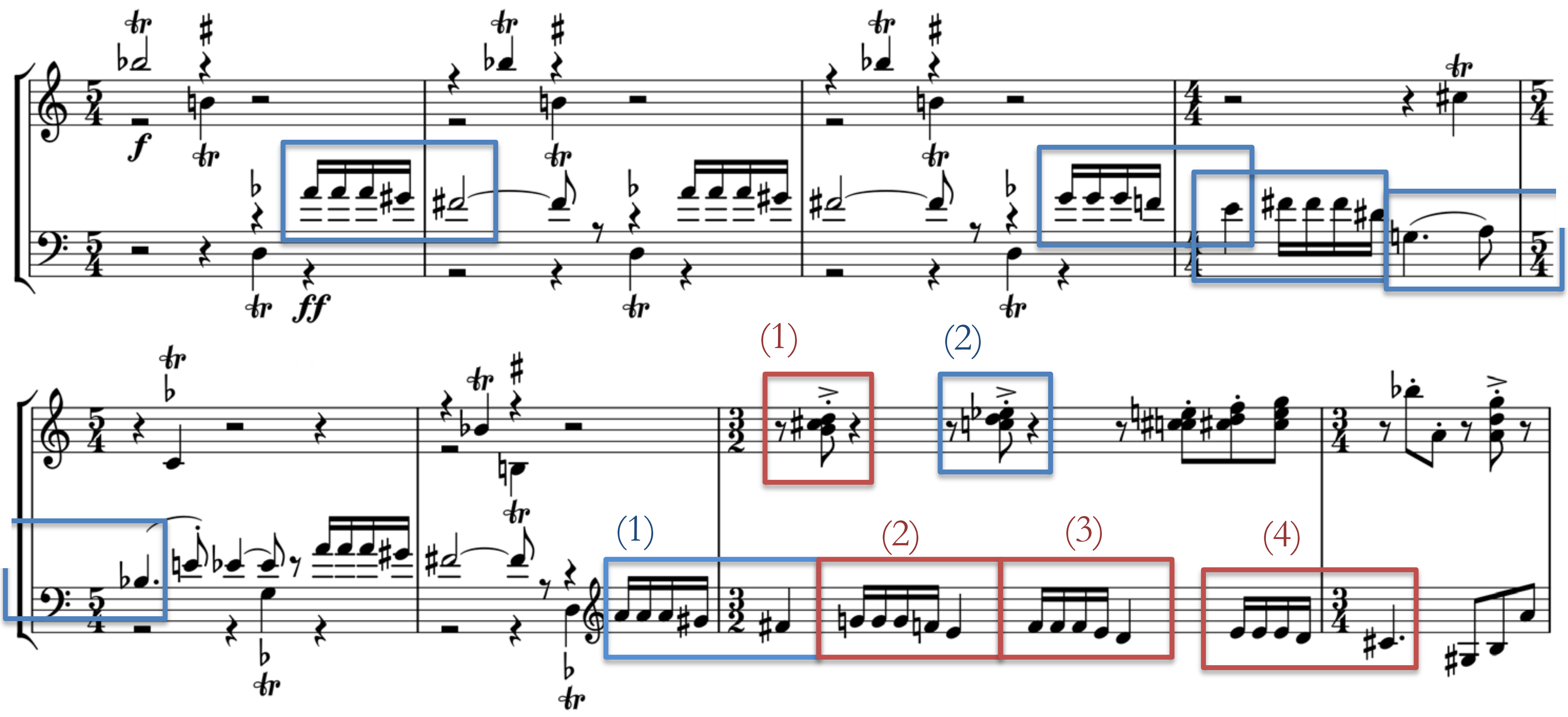
Shostakovich: String Quartet 12

T_2 cycle:

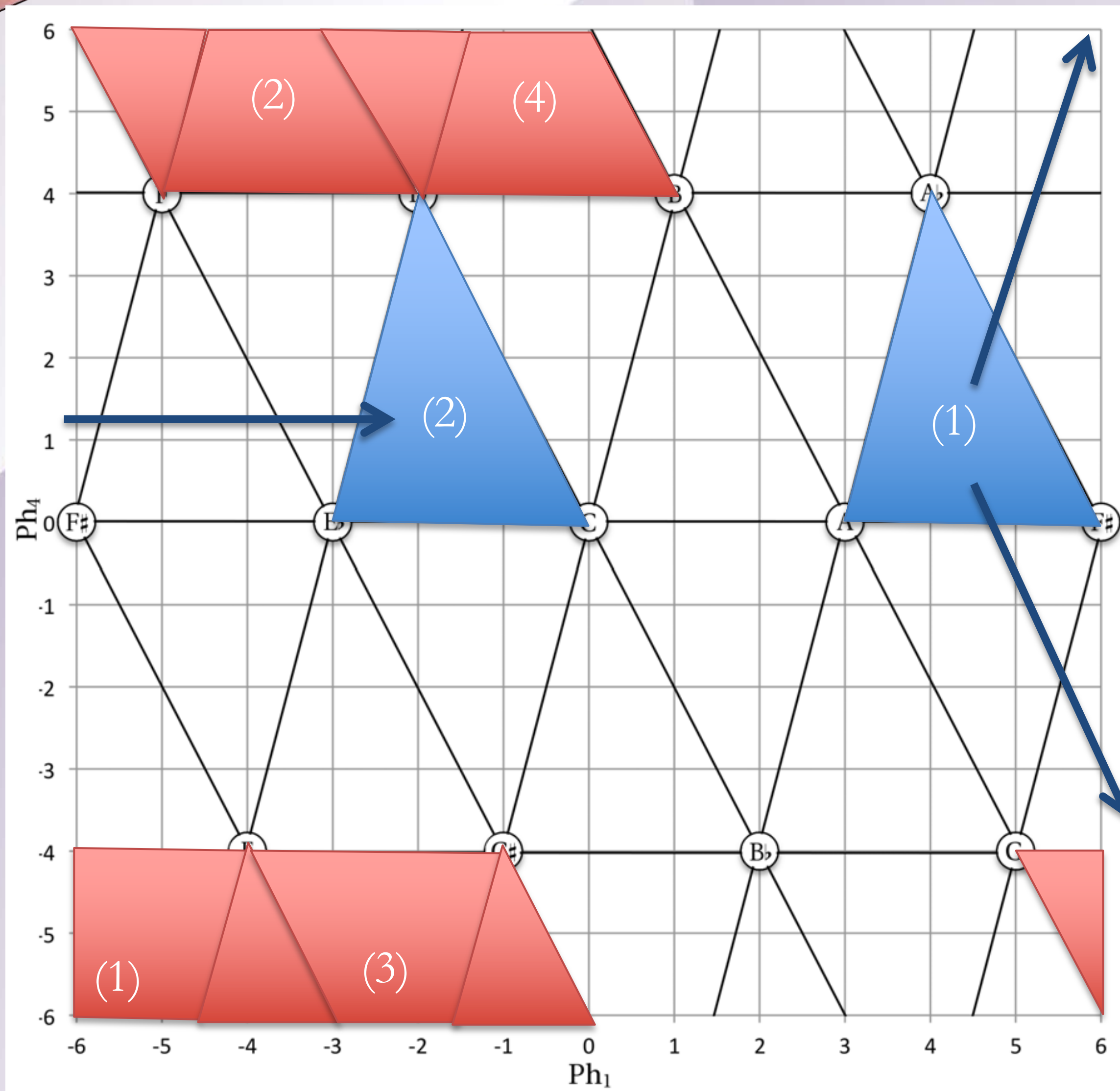
T_1 cycle:

Shostakovich: String Quartet 12

Beginning of second movement, (013)s:



Shostakovich: String Quartet 12



The melody initially focuses on disjunct and T_1 relationships between (013)s

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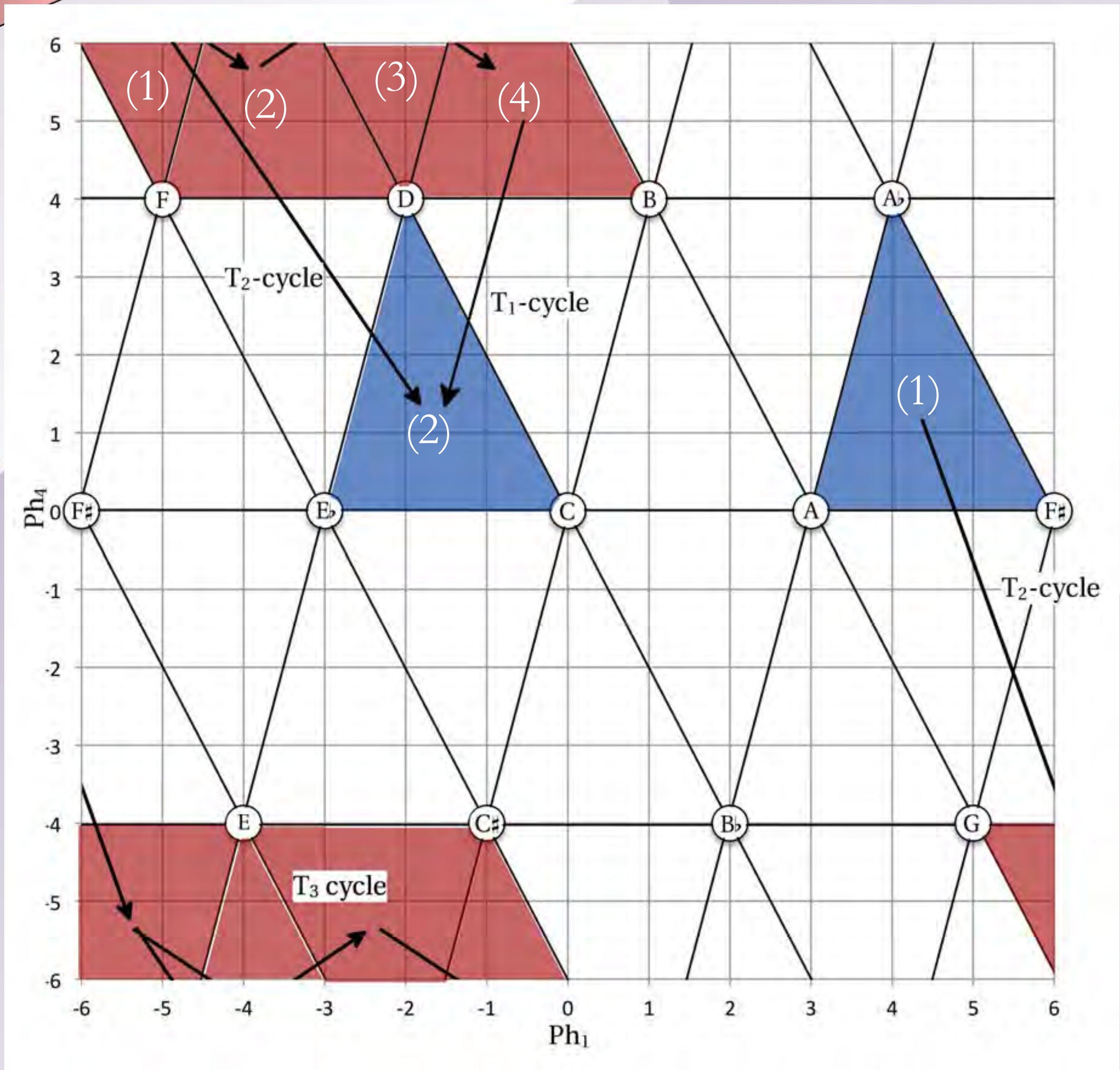
Shostakovich: String Quartet 12



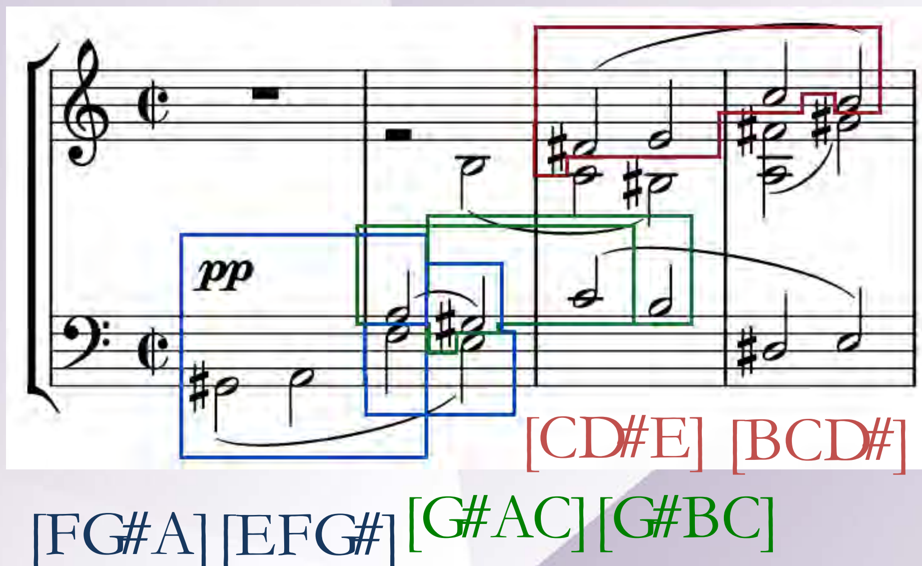
Meas. 7 extends the previous relationships and also expresses part of a conjunct T_3 (octatonic) cycle.

(4) Example: Optimal and non-optimal spaces

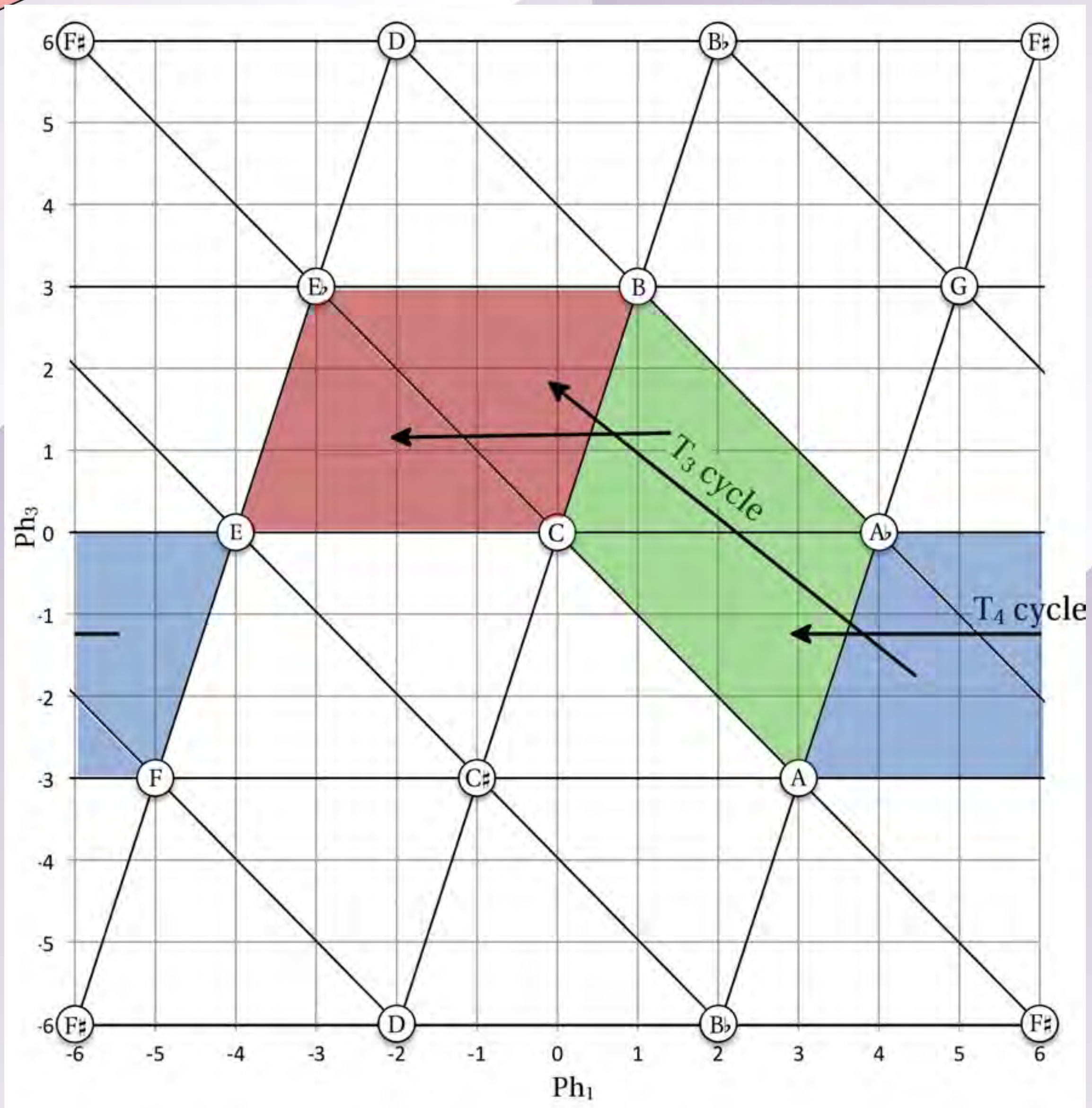
(014) *Tonnetz*, Beethoven String Quartet Op. 132

Beethoven, Op. 132 String Quartet

Network of (014)s in the opening of the first movement

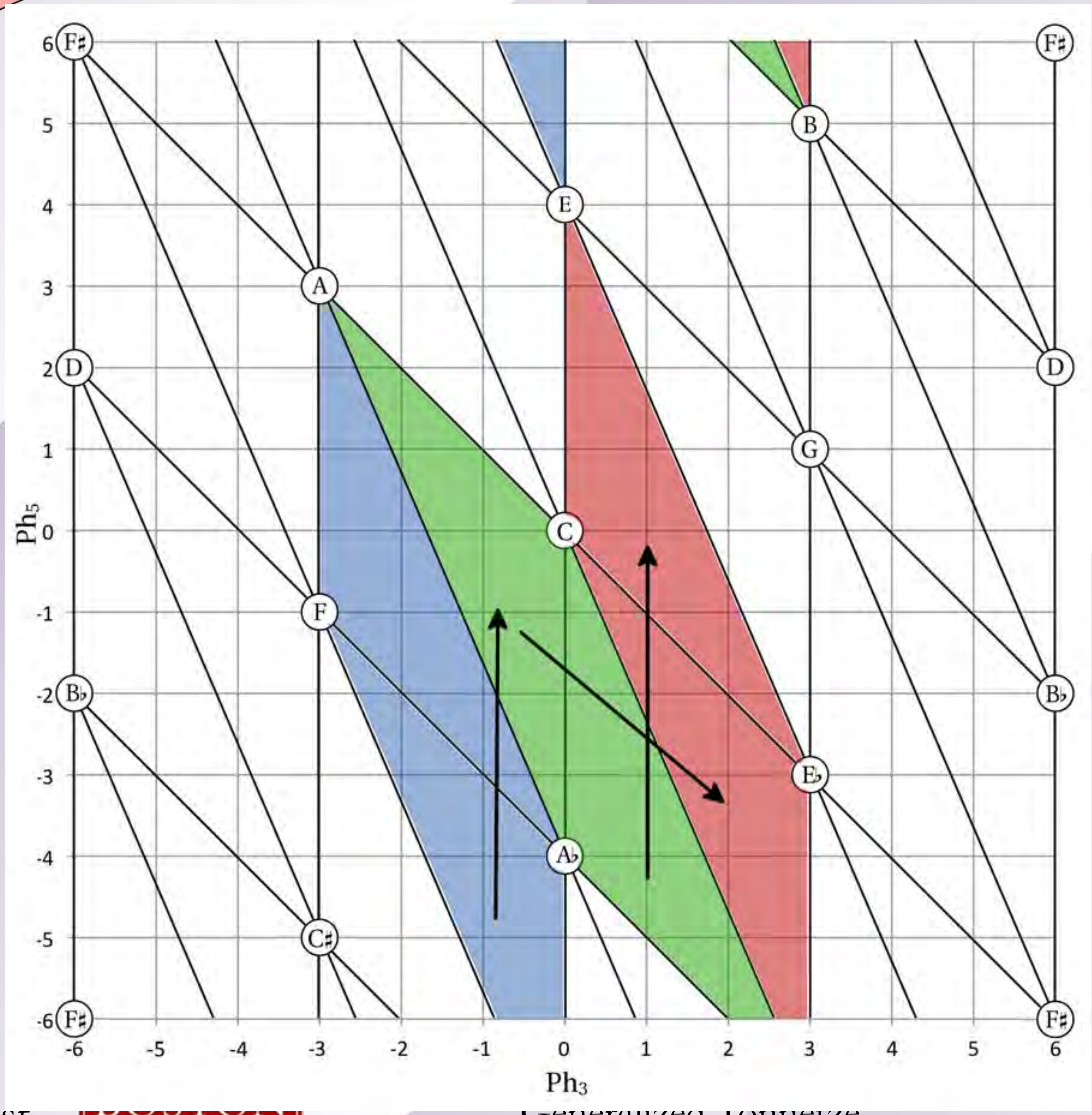


Beethoven, Op. 132 String Quartet



Chain of (014)s in optimal chromatic-triadic (Ph_1-Ph_3) space

Beethoven, Op. 132 String Quartet



Chain of (014)s in non-optimal triadic-diatonic (Ph_3-Ph_5) space