Temptation and Revealed Preference*

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Abstract

Gul and Pesendorfer [9] model the static behavior of an agent who ranks menus prior to the experience of temptation. This paper models the dynamic behavior of an agent whose ranking of menus itself is subject to temptation. The representation for the agent's dynamically inconsistent choice behavior views him as possessing a dynamically consistent view of what choices he "should" make (a normative preference) and being tempted by menus that contain tempting alternatives. Foundations for the model require a departure from Gul and Pesendorfer's idea that temptation creates a preference for commitment. Instead, it is hypothesized that distancing an agent from the consequences of his choices separates normative preference and temptation.

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1. Introduction

An agent may break his diet or abuse drugs while *simultaneously* telling himself that he really should not. Such instances suggest that choice is determined not by one, but two preference orderings: a *temptation preference* that captures the agent's desires, and a *normative preference* that captures his view of what choices he 'should' make. Choice behavior is the outcome of an aggregation of temptation preference and normative preference. The agent is said to experience temptation when his desires conflict with his normative preference. In order to write down a choice-theoretic model of temptation, a foundational question must be answered: what observable behavior identifies an agent who struggles with temptation and reveals his normative and temptation preferences?

Gul and Pesendorfer [9, 10] (henceforth GP) are the first to provide a choice-theoretic model of temptation. Their answer to the foundational question is based on the idea that *temptation creates a preference for commitment*: an agent who thinks he should choose a 'good' option g but anticipates being tempted by a 'bad' option b would avoid the latter. In particular he would strictly prefer $\{g\}$, the menu (choice problem) that commits him to g, rather than the menu $\{g,b\}$ that provides the flexibility of choosing b:

$$\{g\} \succ \{g,b\}.$$

This *preference for commitment* reveals the existence of temptation, and moreover reveals a normative preference for g and a temptation by b. Adopting an agent's preferences $\succsim$ over menus as their primitive, GP use such ideas to construct a model of temptation.

This paper studies an agent who may be tempted not just by alternatives in a menu, but also by menus themselves. Specifically, opportunities that lead to tempting consumption may themselves be tempting. For instance, the agent in the above example may be tempted by the menu $\{g,b\}$ because it offers b. Modelling such agents requires a substantial departure from GP's strategy for identifying temptation. Observe that if the temptation by the menu $\{g,b\}$ is strong enough, the agent would exhibit:

$$\{g\} \not\succ \{g, b\}.$$

That is, when the very act of choosing commitment requires self-control, it becomes possible that *temptation may induce the agent to refuse commitment*, contrary to GP's hypothesis. Indeed, applying their hypothesis here would lead the analyst to erroneously conclude that b does not tempt and may even be normatively superior to g. Evidently then, an alternative to GP is required in order to identify an agent's temptation and normative preference when menus tempt. One may consider extending GP's model by adopting a preference over menus of menus as the primitive. However, the logic of temptation by menus extends to this preference as well, and indeed, also to preferences over more complicated domains consisting of menus of menus.... up to all orders. The objective of this paper is to provide a choice-theoretic model that describes agents who are tempted by menus.

Distancing. Our answer to the foundational question is based on the idea that, at least in stationary environments, *normative preference is revealed when the agent is distanced from the consequences of his choices*. The idea is familiar to philosophers and psychologists, and is part of common wisdom. For instance, when trying to demonstrate to a friend that his smoking is against his better judgment, we try to get him to view the act of smoking from a distance by asking him how he would feel about his children smoking. The 'veil of ignorance' (Rawls [19]) in philosophy is a distancing tool. Psychologists have argued that reversals in choices induced by *temporal* distancing, such as the so-called 'preference reversals' and 'dynamic inconsistency' in the experimental literature on time-preference [7], reveal the existence of self-control problems.[1]

[1] Preference reversals and dynamic inconsistency reveal a loss in patience when rewards are brought closer to the present. For a large reward received at time $t + d$ and a smaller reward received at time t, subjects in experiments on preference reversals exhibit a preference for the small reward when $t = 0$ but reverse preferences when t is large. In experiments on dynamic inconsistency, subjects prefer the large reward when t is large, but switch preferences after t periods elapse.

We formalize the idea of temporal distancing in the following way. Suppose we observe how the agent chooses between delayed rewards. Derive a set of preference relations $\{\gtrapprox_t\}_{t=0}^{\infty}$ where each $\gtrapprox_t$ represents choices between consumption alternatives that are to be received t periods later. By the distancing hypothesis, as t grows, the influence of temptation on the agent's ranking $\gtrapprox_t$ of alternatives diminishes. That is, as t grows, the temptation component underlying $\gtrapprox_t$ becomes less significant, and so, each $\gtrapprox_t$ provides an increasingly better approximation of the agent's underlying normative preference. We identify normative preference with the (appropriately defined) limit:

$$\succsim^* \equiv \lim_{t\to\infty} \gtrapprox_t \, . \tag{1.1}$$

With the normative preference defined thus, temptation is naturally identified through *normatively inferior choices*. These ideas form the basis for building a choice-theoretic model of temptation.[2]

Our Model. The specific model we construct is a stationary infinite horizon dynamic model. In every period the agent faces a menu, from which he chooses immediate consumption and a menu for the next period. Choice is determined by a struggle between normative and temptation preferences (over consumption-menu pairs). Temptation preferences have a rich structure. The impact of temptation on choice is stronger for immediate consumption than future consumption. The agent is tempted by immediate consumption alternatives, may be tempted to over- or under-discount the future (relative to normative preference), and may be tempted by menus. A menu tempts to the extent that it offers an opportunity for future indulgence, but the model also permits normative considerations to affect the extent of temptation by a given menu. The discounting of menu-temptation may be nonexponential, in which case the temptation ranking of menus can reverse with delay.

[2] Note that the appeal of (1.1) relies on a stationary setup: if the agent, say, anticipates preference shocks in the future then those considerations will be reflected in $\succsim^*$, and therefore this ordering ceases to fully capture normative preference over current consumption.

Our agent is dynamically inconsistent. For instance, at time t the agent may plan to commit at $t + 2$ but, if given the opportunity at time $t + 1$, may deviate from this plan and postpone commitment till $t + 3$. Our agent is sophisticated in that he is fully aware of his future behavior. However, choice is not determined as the outcome of an intrapersonal game (Laibson [14]). Instead, choice in each period maximizes a recursively-defined (though not recursive) utility function.

Some of the components of our utility representation adopt the functional form of GP [9]. Nevertheless our model is different from existing infinite horizon versions of GP's model (GP [10], Krussel, Kurusçu and Smith [13] and Noor [15]) in two fundamental respects, described as contributions (b) and (c) below.

Summary of Contributions. This paper makes three main contributions:

(a) Foundations for temptation: While GP identify temptation by means of a preference for commitment, we introduce an alternative strategy that first derives a normative preference by looking at behavior from a distance, and then identifies temptation through gaps between normative preference and choice. This strategy permits us to study an agent whose behavior is contaminated by temptation in every period, and yet identify what does or does not constitute temptation. In particular, whereas GP's strategy identifies temptation experienced only in the next period(s), our's does so for the current period as well.

(b) Temptation by menus: We axiomatize a dynamic model of tempting menus. The model is an extension of GP [9] to an infinite horizon. Other extensions in the literature are by GP [10], Krussel et al [13] and Noor [15]. A key difference is that these models satisfy the so-called *Stationarity* axiom (see Section 5), which enables a relatively straightforward extension of GP to an infinite horizon, while in our model temptation by menus necessitates the violation of Stationarity. We exploit the distancing hypothesis to extend GP to an infinite horizon. The models in GP [10], Krussel at al [13] and Noor [15] have counterparts in our model, and as such, our model also unifies them.

(c) Foundations for sophistication: The literature emanating from GP [9] describes choices at one point in time – the ranking of menus in an ex ante period – and relies on an *interpretation* of the representation to describe subsequent dynamic choice behavior. In our model, we take dynamic choice behavior as our primitive. Thus, our model fully describes not only how the agent ranks menus over time but also what he chooses from them. While the literature assumes that the agent correctly anticipates future behavior, such sophistication produces a restriction on choice behavior in our model, and thus becomes a refutable hypothesis. We also prove a general result (Section 6.1) that shows how the 2-period model of GP [9] can be enriched so that sophistication can be given foundations in that model.

The paper proceeds as follows. Section 2 introduces our model. Sections 3 and 4 present axioms and representation theorems respectively. Section 5 relates this paper with the literature. Section 6 outlines the proof of our main representation theorem and Section 7 concludes. Proofs are relegated to the appendices and a supplementary appendix.

2. The Model

Given any compact metric space X, let $\Delta(X)$ denote the set of all probability measures on the Borel σ-algebra of X, endowed with the weak convergence topology ($\Delta(X)$ is compact and metrizable [1, Thm 14.11]); $\mathcal{K}(X)$ denotes the set of all nonempty compact subsets of X endowed with the Hausdorff topology ($\mathcal{K}(X)$ is a compact metric space [1, Thm 3.71(3)]). Generic elements of $\mathcal{K}(X)$ are x, y, z and those of $\Delta(X)$ are μ, η, ν. For $\alpha \in [0,1]$, $\alpha\mu + (1-\alpha)\eta \in \Delta(X)$ is the measure that assigns $\alpha\mu(A) + (1-\alpha)\eta(A)$ to each A in the Borel σ-algebra of X. Similarly, $\alpha x + (1-\alpha)y \equiv \{\alpha\mu + (1-\alpha)\eta : \mu \in x, \eta \in y\} \in \mathcal{K}(X)$ is a mixture of x and y. Denote these mixtures more simply by $\mu\alpha\nu$ and $x\alpha y$ respectively. Given a compact metric space C of consumption alternatives, GP [10] construct a space Z of infinite horizon menus. Each menu $z \in Z$ is a compact set of lotteries, where each lottery is a measure over current consumption and a continuation menu – Z is homeomorphic to $\mathcal{K}(\Delta(C \times Z))$, a compact metric space. See [10] for the formal definition of Z. Below we often write $\Delta(C \times Z)$ as Δ.

The primitive of our model is a closed-valued choice correspondence $\mathcal{C} : Z \rightsquigarrow \Delta$ where $\phi \neq \mathcal{C}(x) \subset x$ for all $x \in Z$. This is a time-invariant choice correspondence that describes the choices at any time $t = 1, 2....$ The time-line is given by:

$$\underset{(c,y)\in x}{\overset{t=1}{\bullet}} \text{———} \underset{(c',z)\in y}{\overset{t=2}{\bullet}} \text{———} \underset{(c'',z'')\in z}{\overset{t=3}{\bullet}} \text{—} \tag{2.1}$$

For any menu x faced in period 1, the agent chooses $(c, y) \in \mathcal{C}(x)$ say. He receives immediate consumption c, and a continuation menu y.[3] The continuation menu y is faced in period 2 and a choice is made from it. The process continues ad infinitum. All choice are interpreted as possibly subject to temptation.

For any pair of continuous linear functions $U, V : \Delta \to \mathbb{R}$, the GP representation is given by

$$W(x) := \max_{\mu \in x}\{U(\mu) + V(\mu) - \max_{\eta \in x} V(\eta)\}, \tag{2.2}$$

where the dependence of W on U, V is suppressed to ease notation. Our model takes the form of the following representation for $\mathcal{C}$.

Definition 2.1 (U-V Representation). *The choice correspondence $\mathcal{C}$ over Δ admits a U-V representation if there exist functions $U, V : \Delta \to \mathbb{R}$ such that $\mathcal{C}$ satisfies:*

$$\mathcal{C}(x) = \arg\max_{\mu \in x}\{U(\mu) + V(\mu)\}, \quad x \in Z. \tag{2.3}$$

and U, V satisfy the equations:

$$U(\mu) = \int_{C\times Z} (u(c) + \delta W(x))d\mu, \tag{2.4}$$

$$\text{and} \;\; V(\mu) = \int_{C\times Z} (v(c) + \left[\beta W(x) + \gamma \max_{\eta \in x} V(\eta)\right])d\mu, \tag{2.5}$$

for all $\mu \in \Delta$, where $W : Z \to \mathbb{R}$ is defined by (2.2), $u, v : C \longrightarrow \mathbb{R}$ are continuous functions and δ, γ, β are scalars satisfying $\delta \in (0, 1)$, $\gamma \in [0, \delta]$ and $\beta > \gamma - \delta$.

[3]More generally, if the alternative chosen from x is a nondegenerate lottery $\mu \in \Delta$, then the uncertainty plays out before the next period, yielding some (c, y). This leaves the agent with immediate consumption c and the menu y to face in period 2.

The representation is identified with the tuple $(u, v, \delta, \beta, \gamma)$. The model makes sense of the agent's (possibly temptation-ridden) choices $\mathcal{C}$ by asserting that the agent possesses normative and temptation preferences over Δ, reflected in U and V respectively. The representation (2.3) states that choice in any period is a compromise between the two: it maximizes the sum of U and V.

Normative (expected) utility U evaluates lotteries with a utility index $u(c) + \delta W(x)$ that comprises of utility u for immediate consumption, a discount factor δ and a utility W over continuation menus that has the familiar GP form (2.2). To remind the reader: the temptation opportunity cost $|V(\mu) - \max_{\eta \in x} V(\eta)|$ is interpreted as the self-control cost of choosing μ, and thus, W is a value function suggesting that the agent maximizes normative utility U net of self-control costs. An important observation is that the maximizer – the anticipated choice from x – maximizes $U + V$. *This is precisely what is described by (2.3).* Thus our agent is *sophisticated* in that her anticipated choices coincide with her actual choices.

Temptation (expected) utility V evaluates lotteries with a utility index $v(c) + \beta W(x) + \gamma \max_{\eta \in x} V(\eta)$ that evaluates immediate consumption by v and the continuation menu x according to the discounted utility:

$$\beta W(x) + \gamma \max_{\eta \in x} V(\eta). \tag{2.6}$$

There are two differences from how U evaluates continuation menus. First, the normative utility W of a continuation menu is discounted by β instead of δ.[4] Second, consideration is given to $\max_{\eta \in x} V(\eta)$, the *pure temptation value* of a menu. This is discounted by γ. The model requires $\gamma \leq \delta$, reflecting the intuitive idea that the temptation perspective is, in some sense, more myopic than the normative perspective.

The experience of temptation is suggested by a strict conflict between U and V. The experience of *temptation by menus*, or menu-temptation for short, is similarly suggested by a conflict in the ranking of continuation menus, such as (c, x) vs. (c, y). The nature of menu-temptation is determined in the model by

[4] The restriction $\beta > \gamma - \delta$ permits $\beta < 0$. Section 4 characterizes the special case that rules this out.

the parameters γ and β: When $\gamma = 0$ and $\beta \geq 0$, there is no menu-temptation:

$$U(c,x) \geq U(c,y) \Longrightarrow W(x) \geq W(y) \Longrightarrow V(c,x) \geq V(c,y).$$

When $\gamma > 0$ and $\beta = 0$, menu-temptation is determined completely by the tempting alternatives contained in it, whereas if $\beta > 0$ then the latter consideration is dampened by the normative value. Moreover, when $\beta > 0$, the relatively steeper discounting of the pure temptation value of a menu ($\gamma \leq \delta$) implies that a menu may cease to tempt if it is pushed into the future.

Some of these special cases have counterparts in the literature [10, 13, 15], and we give them related names:

Definition 2.2 (QSC, FT, DSC). *A U-V representation* $(u, v, \delta, \beta, \gamma)$ *is a Quasi-Hyperbolic Self-Control (QSC) representation if* $\gamma = 0$ *and* $\beta \geq 0$*, a Future Temptation (FT) representation if* $\gamma > 0$ *and* $\beta = 0$*, and a Dynamic Self-Control (DSC) agent if* $\beta = \gamma = 0$.

The description of and comparison with related literature is deferred to Section 5.

3. Foundations: Axioms

The following notation will aid exposition:

- Fix $\bar{c} \in C$ throughout. For any x, define $x^{+1} \equiv (\bar{c}, x)$ and inductively for $t > 1$, $x^{+t} = (\bar{c}, x^{+(t-1)})$. Then $x^{+t} \in \Delta$ is the alternative that yields menu x after $t > 0$ periods, and $\bar{c}$ in all periods between time 0 and t. We write $\{\mu\}^{+t}$ as μ^{+t} and identify μ^{+0} with μ. The reader should keep in mind that x^{+t} is not a menu, but a degenerate lottery that that yields a menu t periods later.
- Let $\gtrsim$ denote the revealed preference relation on Δ that is generated by choices from binary menus:

$$\mu \gtrsim \eta \Longleftrightarrow \mu \in \mathcal{C}(\{\mu, \eta\}). \tag{3.1}$$

The indifference relation $\approx$ and the strict preference relation $>$ are derived from $\gtrsim$ in the usual way. Consider the following axioms on $\mathcal{C}$. The quantifiers 'for all $\mu, \eta \in \Delta$, $x, y \in Z$, $c, c', c'' \in C$, and $\alpha \in [0,1]$' are suppressed.

3.1. Standard Axioms

The first set of axioms reflect that the agent has some features possessed by 'standard' agents: the agent behaves as if he maximizes a single, continuous, linear, additively separable utility function. Although such features might rule out certain forms of temptation, the representation theorems in the next section confirm that they are consistent with an interesting class of temptation models.

Axiom 1 (WARP). *If $\mu, \eta \in x \cap y$, $\mu \in \mathcal{C}(x)$ and $\eta \in \mathcal{C}(y)$, then $\mu \in \mathcal{C}(y)$.*

This is the familiar Weak Axiom of Revealed Preference. It is a minimal consistency requirement on choices. However, though WARP is a standard axiom in standard choice theory, it is not clear that it is appropriate for a theory of choice under temptation. While WARP is an expression of the agent using a menu-independent preference to guide his choices, intuition suggests that the degree of self-control an agent has may well depend on what is available in the menu.[5] The upshot is that the current model should be thought of one where self-control, or the relative weight between temptation and normative preference in the agent's choices, is *menu-independent.* This intuition assures us that temptation need not rule out WARP.

Axiom 2 (Continuity). *$\mathcal{C}(\cdot)$ is upper hemicontinuous.*

Upper hemicontinuity of $\mathcal{C}(\cdot)$ is implied by choices being determined by the maximization of a continuous preference. We impose upper hemicontinuity as an

[5] This is explored in Noor and Takeoka [17]. To illustrate, let s denote salad, b a burger and B a large burger. If b is not so tempting, the agent may apply self-control and choose s out of $\{s, b\}$. But when faced with $\{s, b, B\}$, the presence of a large burger B may whet his appetite for a burger, and in order to compromise between his craving for B and his normative preference for s, he may settle for b, thereby violating WARP.

axiom, with the intention of establishing that choices are determined in such a way. Formally, upper hemicontinuity is equivalent to the statement that if $\{x_n\}$ is a sequence of menus converging to x, and $\mu_n \in \mathcal{C}(x_n)$ for each n, then the sequence $\{\mu_n\}$ has a limit point in $\mathcal{C}(x)$.

Axiom 3 (Independence). $\mu > \eta \Longrightarrow \mu\alpha\nu > \eta\alpha\nu$.

This is the familiar Independence axiom. The next axiom is an explicit statement of the 'indifference to the timing of resolution of uncertainty' property that is implicitly assumed in GP [9].

Axiom 4 (Indifference to Timing). $x^{+t}\alpha y^{+t} \approx (x\alpha y)^{+t}$.

Under both rewards $x^{+t}\alpha y^{+t}$ and $(x\alpha y)^{+t}$, the agent faces x after t periods with probability α and y after t periods with probability $(1-\alpha)$. However, under $x^{+t}\alpha y^{+t}$, the uncertainty will be resolved today, whereas under $(x\alpha y)^{+t}$, the uncertainty will be resolved after t periods. That is, the two rewards differ only in the timing of resolution of uncertainty. Indifference between the rewards corresponds to indifference to the timing of resolution of uncertainty. The axiom rules out temptation that may be associated with the timing of resolution of uncertainty, such as anxiety.[6]

Axiom 5 (Separability). $(\frac{1}{2}(c,x)+\frac{1}{2}(c',x'))^{+t} \approx (\frac{1}{2}(c,x')+\frac{1}{2}(c',x))^{+t}$.

Separability states that when comparing two lotteries (delayed by $t \geq 0$ periods), the agent only cares about the marginal distributions on C and Z induced by the lotteries. That is, only marginals matter, and correlations between consumption and continuation menus do not affect the agent's choices. Separability is not consistent with addiction, where the value of a menu may well depend on what is consumed today.

[6] Indifference to Timing and Independence together imply the Set-Independence axiom of Dekel et al [4] and GP [9]: in our context it can be stated as $x^{+t} > y^{+t} \Longrightarrow (x\alpha z)^{+t} > (y\alpha z)^{+t}$. Noor and Takeoka [17] demonstrate that this axiom needs to be relaxed in order accommodate stories about anticipated choice from menus that violate WARP. This is also observed by Fudenberg and Levine [8], and Dekel et al [5] note that Set-Independence is also related with a stochastic version of WARP.

3.2. Main Axioms

There are four main axioms.

Axiom 6 (Set-Betweenness). $x^{+t} \gtrapprox y^{+t} \Longrightarrow x^{+t} \gtrapprox (x \cup y)^{+t} \gtrapprox y^{+t}$.

This adapts the Set-Betweenness axiom in GP [9] into our setting. GP [9] used the axiom to describe menu choice prior to the experience of temptation, and that interpretation is valid here when the ranking of x^{+t}, y^{+t} is not swayed by menu-temptation. But it is interesting that the axiom may be satisfied even when the agent is swayed by menu-temptation. To illustrate, suppose that $\{\mu\}^{+t} \gtrapprox \{\eta\}^{+t}$ is due to overwhelming temptation by the menu $\{\mu\}$. We assert that the agent would exhibit $\{\mu\}^{+t} \approx \{\mu, \eta\}^{+t} \gtrapprox \{\eta\}^{+t}$, consistent with Set-Betweenness.[7] The ranking $\{\mu, \eta\}^{+t} \gtrapprox \{\eta\}^{+t}$ would arise because the choice between $\{\mu, \eta\}^{+t}$ and $\{\eta\}^{+t}$ must also be overwhelmed by temptation – $\{\mu, \eta\}^{+t}$ is a tempting menu for the same reason that $\{\mu\}^{+t}$ is, and since $\{\mu\}^{+t}$ is chosen over $\{\eta\}^{+t}$, so is $\{\mu, \eta\}^{+t}$. The indifference between $\{\mu\}^{+t}$ and $\{\mu, \eta\}^{+t}$ reflects that the agent foresees being overwhelmed by μ in either menu.

Axiom 7 (Sophistication). *If* $\{\mu\}^{+t} > \{\eta\}^{+t}$ *then* $\{\mu, \eta\}^{+t} > \{\eta\}^{+t} \Longleftrightarrow \mu > \eta$.

As the name suggests, this axiom connects the agent's expectation of his future choices with his actual choices. Suppose that μ is preferred to η from a distance of t periods, $\{\mu\}^{+t} > \{\eta\}^{+t}$. Owing to this, if the anticipated choice from $\{\mu, \eta\}$ is μ, then he would exhibit $\{\mu, \eta\}^{+t} > \{\eta\}^{+t}$. The axiom states that the agent anticipates choosing μ from $\{\mu, \eta\}$ after t periods if and only if he actually does so (recall that $\gtrapprox$ reflects time-invariant choice, and so it describes also choice after t periods). That is, he is sophisticated in that he correctly anticipates future choices. It should be noted that this axiom is dynamic in that it relates choice across different times.

[7] The possibility that $\{\mu\}^{+t} > \{\mu, \eta\}^{+t}$ when $\{\mu\}$ is overwhelmingly tempting is ruled out by the Reversal axioms presented shortly – note that overwhelming temptation is associated with the existence of a reversal below.

From the perspective of one point in time, Sophistication also relates the ranking of different menus at different delays: if μ is ranked higher than η at delays t, t' (that is, $\{\mu\}^{+i} > \{\eta\}^{+i}$, $i = t, t'$) then $\{\mu, \eta\}^{+t} > \{\eta\}^{+t} \Longleftrightarrow \{\mu, \eta\}^{+t'} > \{\eta\}^{+t'}$. Two final axioms establish further such connections by describing how the agent's ranking between two alternatives might change, if at all, when the alternatives are pushed into the future.

Axiom 8 (Reversal). *If $\mu^{+t} < \eta^{+t}$ (resp. $\mu^{+t} \lessapprox \eta^{+t}$) and $\mu^{+t'} \gtrapprox \eta^{+t'}$ (resp. $\mu^{+t'} > \eta^{+t'}$) for some $t' > t$, then $\mu^{+t''} \gtrapprox \eta^{+t''}$ (resp. $\mu^{+t''} > \eta^{+t''}$) for all $t'' > t'$.*

The axiom states that pushing a pair of rewards into the future may lead the agent to reverse the way he ranks them, and if this happens, then the reversed ranking is maintained for all subsequent delays in the rewards. The axiom expresses the basic structure of 'preference reversals', a robust finding in the experimental psychology literature on time-preference; see [7] for a survey of the evidence. [8] An explanation given for preference reversals in the literature is that it is caused by a desire for immediate gratification. As in GP [10], we specifically view it as arising due to temptation: when two alternatives are pushed into the future, temptation is weaker and eventually *resistible*, and this induces a reversal. Observe that given the time-invariance of the primitive $\mathcal{C}$, Reversal implies that revealed preferences in our model are *dynamically inconsistent* in the sense that the agent may exhibit $(c, \{\mu\}) \gtrapprox (c, \{\eta\})$ at t but $\mu \not\gtrapprox \eta$ at $t + 1$.

As a simple consequence of Reversal we obtain a function $\tau : \Delta \times \Delta \to \mathbb{R}$ that defines the "switching point" of preference reversals, that is, $\tau(\mu, \eta)$ is the minimum number of periods that μ and η need to be delayed before a preference reversal is observed; if no reversal is observed, then $\tau(\mu, \eta) = 0$. For instance, if $\mu^{+t} \gtrapprox \eta^{+t}$ for all $t < T$ and $\mu^{+t} < \eta^{+t}$ for all $t \geq T$, then $\tau(\mu, \eta) = T$. See Appendix C for a precise definition of the function τ.

[8] Let s (resp. l) denote a consumption stream that gives a small (resp. large) reward immediately and $\bar{c}$ in all other periods. Typical preference reversals are expressed by the Reversal axiom when μ, η are of the form $\mu = s^{+0}$ and $\eta = l^{+d}$.

The final axiom makes specific statements about what menu rankings are not subject to reversals. Note that part (ii) presumes that A is a neighborhood of $(\{\mu\}^{+t}, \{\mu, \eta\}^{+t}) \in \Delta \times \Delta$ with respect to the product topology on $\Delta \times \Delta$.

Axiom 9 (Menu-Reversal). *(i) If* $\tau(\mu, \eta) = 0$ *then* $\tau(\{\mu\}^{+1}, \{\mu, \eta\}^{+1}) = 0$.
(ii) If $\{\mu\}^{+t} > \{\mu, \eta\}^{+t}$ *then* $\tau(A) = 0$ *for some neighborhood* A *of* $(\{\mu\}^{+t}, \{\mu, \eta\}^{+t})$.

Part (i) of the axiom states that if the ranking of μ and η does not reverse with delay, then neither must the ranking of $\{\mu\}^{+1}$ and $\{\mu, \eta\}^{+1}$. Intuitively, the lack of reversal indicates that the ranking of μ and η is either not subject to any temptation or it is subject to *resistible* temptation. In either case, the ranking of the menus (which are necessarily one period away) is either not subject to temptation or subject to resistible temptation. Therefore delaying the menus will not give rise to a reversal.[9]

Part (ii) of the axiom makes two statements. First, if $\{\mu\}^{+t} > \{\mu, \eta\}^{+t}$ then $\{\mu\}^{+t'} > \{\mu, \eta\}^{+t'}$ for all $t' \geq t$. That is, there is no reversal after a preference for commitment. Intuitively, the choice to commit is driven by the agent's normative considerations, and thus is not subject to a reversal. Second, the axiom says that there is no reversal also for any neighboring pair of alternatives. Since the ranking $\{\mu\}^{+t} > \{\mu, \eta\}^{+t}$ is driven by normative considerations, it is associated either with no temptation or with resistible temptation by $\{\mu, \eta\}^{+t}$. In either of these cases, continuity of underlying temptation and normative preferences implies there is either no temptation or resistible temptation in any neighboring pairs of rewards. Hence no reversals will be observed when these neighboring pairs of rewards are pushed into the future.

[9] The axiom may be weakened to hold for μ, η such that $\mu > \eta$. The case $\mu \approx \eta$ is implied by Set-Betweenness, and the case $\eta > \mu$ is implied by Sophistication.

4. Foundations: Representation Results

4.1. Main Theorem

Say that $\mathcal{C}$ is *nondegenerate* if (i) there exists μ, η, T such that $\mu < \eta$ and $\mu^{+T} > \eta^{+T}$, and moreover, (ii) there exists μ, η such that $\tau(A) = 0$ for a neighborhood A of (μ, η) and $\{\mu\}^{+1} \approx \{\mu, \eta\}^{+1} > \{\eta\}^{+1}$. Part (i) of this definition asserts the existence of a preference reversal, and part (ii) asserts the existence of μ, η that neither exhibit a preference reversal (and neither do neighboring rewards) nor give rise to a preference for commitment. This corresponds to the case in the model where U and V are nonconstant and affinely independent, and in particular where temptation is non-trivial.

The main result in this paper is the axiomatization of the U-V model (Def 2.1).

Theorem 4.1. *If a nondegenerate choice correspondence $\mathcal{C}$ satisfies Axioms 1-9 then it admits a U-V representation. Conversely, a choice correspondence $\mathcal{C}$ that admits a U-V representation also satisfies Axioms 1-9.*

The Theorem states that an agent's choices satisfy Axioms 1-9 if and only if it is as if they are the result of an aggregration of the functions U and V in Def 2.1. The order $\gtrapprox$ defined by (3.1) is represented by $U + V$. It is worth noting that Axioms 1-9 do not explicitly restrict the nature of temptation by menus – none of the axioms make the statement that, for instance, a menu is tempting only if it contains tempting items. Yet they produce the very special structure (2.6) on menu-temptation in the U-V representation that capture this property. The proof of the Theorem is discussed in detail in Section 6.

Next is a uniqueness result that assures us that all the U-V representations of $\mathcal{C}$ deliver the same normative and temptation preferences.

Theorem 4.2. *If a nondegenerate choice correspondence $\mathcal{C}$ admits two U-V representations $(u, v, \delta, \beta, \gamma)$ and $(u', v', \delta', \beta', \gamma')$ with respective normative and temptation utilities (U, V) and (U', V'), then there exist constants $a > 0$ and b_u, b_v such* *that* $U' = aU + b_u$ *and* $V' = aV + b_v$*. Moreover,* $\delta = \delta'$, $\beta = \beta'$, $\gamma = \gamma'$, $u' = au + (1-\delta)b_u$ *and* $v' = av + \beta b_v + (1-\gamma)b_v$*.*

For the question of when there exist functions (U, V) satisfying the equations in the U-V representation, the relevant proofs in [10, 15] can be adapted to show that the equations in Def 2.1 admit a *unique* continuous solution (U, V) when $\beta = 0$, that is, when the representation is either FT or DSC. More generally, however, a contraction mapping argument cannot be used because the U-V representation is not monotone in the appropriate sense.[10] Nevertheless, we can ensure that:

Theorem 4.3. *For any continuous functions* u, v *and scalars* δ, γ, β *as in Def 2.1, the equations (2.4)-(2.5) admit a continuous linear solution* (U, V)*.*

The proof of the theorem considers the space $\mathcal{F}\times\mathcal{F}$ of pairs of continuous linear functions on Δ endowed with the weak topology (induced by the norm dual of $\mathcal{F}\times\mathcal{F}$). We identify a compact subset of this space and establish that the mapping defined by the equations in Def 2.1 is continuous in the weak topology and a self-map on the compact subset. We then invoke the Brouwer-Schauder-Tychonoff fixed point theorem, which states that a continuous self-map on a compact convex subset of a locally convex linear topological space has a nonempty set of fixed points.

4.2. Special Cases

We characterize some subclasses that are of interest.

Theorem 4.4. *A nondegenerate choice correspondence* $\mathcal{C}$ *admits a* U-V *representation* $(u, v, \delta, \beta, \gamma)$ *with* $\beta \geq 0$ *if and only if it satisfies Axioms 1-9 and Weak Menu-Temptation Stationarity: If* $x^{+t} > y^{+t}$ *for all large* t*, then* $\{x^{+2}\}^{+t} > \{x^{+2}, y^{+2}\}^{+t}$ *for all large* $t \Longrightarrow \{x^{+1}\}^{+t} > \{x^{+1}, y^{+1}\}^{+t}$ *for all large* t*.*

[10] Krussel et al [13] were the first to note that such an issue arises for generalizations of GP [10]. The question of existence was left open.

The existence of a preference for commitment suggests the existence of temptation within a menu. Thus, the Weak Menu-Temptation Stationarity axiom states that, at any t, if y tempts the agent when it is available at $t+2$, then it tempts him also when it is available at $t+1$. That is, bringing a tempting menu closer to the present does not turn it into an untempting menu. This property is the behavioral meaning of the restriction $\beta \geq 0$ in the representation.

Observe that the axiom allows for the possibility that pushing a tempting menu into the future may not just make it easier to resist, but may make it altogether untempting. Consider the implication of a strong version of the axiom that rules this out, that is, that requires that a menu that tempts when t periods away also tempts when t' periods away, for any $t, t' > 0$.

Theorem 4.5. *A nondegenerate choice correspondence $\mathcal{C}$ admits either an FT representation or a QSC representation if and only if it satisfies Axioms 1-9 and Strong Menu-Temptation Stationarity: If $x^{+t} > y^{+t}$ for all large t, then $\{x^{+2}\}^{+t} > \{x^{+2}, y^{+2}\}^{+t}$ for all large $t \iff \{x^{+1}\}^{+t} > \{x^{+1}, y^{+1}\}^{+t}$ for all large t.*

This result characterizes the *union* of the QSC and FT classes of models, thereby identifying the behavior that is common to them. The last result determines precisely what is different between them.[11]

Theorem 4.6. *The following statements are equivalent for a nondegenerate choice correspondence $\mathcal{C}$ that satisfies Axioms 1-9:*

(i) $\mathcal{C}$ admits a QSC representation.

(ii) $\mathcal{C}$ satisfies Menus Do Not Tempt: For all $t > 0$, $\{x^{+1}, y^{+1}\}^{+t} \gtrsim \{x^{+1}\}^{+t}$.

(iii) $\tau \leq 1$.

[11] Note that in Thm 4.6, when $\mathcal{C}$ exhibits $\tau \leq 1$, Axiom 9 can be dropped from the hypothesis and Axioms 5-7 can be weakened by restricting their statements to hold only for $t = 1$. It is easily shown by invoking [10, Theorem 1] that if $\mathcal{C}$ admits a QSC representation, then it admits a DSC representation if and only if $\mathcal{C}$ satisfies the 'Temptation by Immediate Consumption' axiom in GP [10]. In this setting the axiom states that if η and υ both induce the same marginal distribution on C, and if $\{\mu\}^{+1} > \{\mu, \eta\}^{+1} > \{\eta\}^{+1}$ and $\{\mu\}^{+1} > \{\mu, \upsilon\}^{+1} > \{\upsilon\}$, then $\{\mu, \eta\}^{+1} \approx \{\mu, \upsilon\}^{+1}$.

Thus, the key behaviors associated with QSC agents are that they exhibit no preference for delayed commitment (as captured by the Menus Do Not Tempt axiom), and that the switching point of preference reversals is at most one period away. The latter is intuitive for QSC agents: if μ is overwhelmingly tempting and $\mu > \eta$, and if only immediate consumption tempts, then a single period delay in both rewards removes temptation and leads to a reversal, $\mu^{+1} < \eta^{+1}$.

The result tells us that in the presence of Axioms 1-9 and Strong Menu-Temptation Stationarity, observing one instance of a preference for delayed commitment, or one instance of a preference reversal with a switching point at two or more periods delay is equivalent to the existence of an FT representation. Note that this axiomatization of the FT model differs significantly from the axiomatization in [15], even after accounting for the different primitives: most notably, unlike any of the axioms we impose here, the key axiom in [15] (called "Temptation Stationarity") explicitly identifies the source of menu-temptation as being temptation within a menu. Our Strong Menu-Temptation Stationarity corresponds to a substantial weakening of that key axiom: it imposes stationarity of menu-temptation but without identifying the source of menu-temptation.

4.3. Foundations for Normative Preference

According to our interpretation of the representation, the U-V agent behaves as if he struggles with two preferences, represented by the normative utility U and temptation utility V. Theorem 4.2 assures us that these functions represent unique preferences, and thus there is a unique normative and temptation preference associated with the model. In this subsection we identify the behavioral underpinnings of the normative preference.

Derive a sequence of preference relations $\{\gtrsim_t\}_{t=0}^{\infty}$ over Δ, where for each $t \geq 0$ and $\mu, \eta \in \Delta$,

$$\mu \gtrsim_t \eta \Longleftrightarrow \mu^{+t} \in \mathcal{C}(\{\mu^{+t}, \eta^{+t}\}).$$

Thus, the preference $\gtrsim_t$ ranks μ and η when both rewards are to be received t

periods later. Define the preference $\succsim^*$ over Δ by[12]

$$\succsim^* \equiv \lim_{t\to\infty} \gtrsim_t \, . \tag{4.1}$$

Refer to $\succsim^*$ as *the normative preference derived from $\mathcal{C}$.* It captures the agent's ranking of alternatives as the alternatives are distanced from him. The next Theorem tells us that $\succsim^*$ is the preference represented by the normative utility U.

Theorem 4.7. *If $\mathcal{C}$ admits an U-V representation with normative utility U and if $\succsim^*$ is the normative preference derived from $\mathcal{C}$, then U represents $\succsim^*$.*

Thus, $\succsim^*$ constitutes the empirical foundations for the ranking underlying U. The justification for referring to $\succsim^*$ (resp. U) as normative preference (resp. normative utility) lies in the intuitive idea that the influence of temptation on choice is reduced when the agent is separated from the consequences of his choices, and consequently such choices are guided by the agent's view of what he *should* do.

Recall that for the U-V agent, choice maximizes $U + V$. In a sense, V fills the gap between choice $\mathcal{C}$ and normative preference $\succsim^*$. In our model, choice is determined by the normative perspective and visceral influences, and thus the gap between $\mathcal{C}$ and $\succsim^*$ is naturally attributed to temptation. This is the justification for referring to V as temptation utility.

5. Perspective and Related Literature

Our model is an extension of GP [9] to an infinite horizon. In this section we compare it to existing infinite horizon extensions, namely GP [10], Krussel, Kurusçu

[12] To be formal, say that a binary relation B on Δ is nonempty if $\mu B \eta$ for some $\mu, \eta \in \Delta$. Following Hildenbrand [11], identify any nonempty continuous binary relation on Δ with its graph, a nonempty compact subset of $\Delta\times\Delta$. Thus, the space of nonempty continuous preferences on Δ can be identified with $\mathcal{P} = \mathcal{K}(\Delta \times \Delta)$, the space of nonempty compact subsets of $\Delta \times \Delta$ endowed with the Hausdorff metric topology. See Appendix B for details. Hence $\{\gtrsim_t\}_{t=0}^{\infty}$ is a sequence in $\mathcal{P}$ and its limit in the Hausdorff metric topology is $\succsim^* \equiv \lim_{t\to\infty} \gtrsim_t$.

and Smith [13] and Noor [15]. These models are special cases of the following class of models:

Definition 5.1 (W Representation). *A preference $\succsim$ over Z admits a W representation if it admits a GP representation*

$$W(x) = \max_{\mu \in x}\{U(\mu) + V(\mu) - \max_{\eta \in x} V(\eta)\}$$

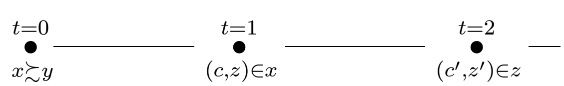
such that the functions $U, V : \Delta \to \mathbb{R}$ take the form

$$U(\mu) = \int_{C \times Z} (u(c) + \delta W(x))\, d\mu(c, x) \quad \text{and} \quad V(\mu) = \int_{C \times Z} \left(v(c) + \widehat{V}(x)\right) d\mu(c, x),$$

where $\delta \in (0, 1)$, $u, v : C \to \mathbb{R}$ are continuous and $\widehat{V} : Z \to \mathbb{R}$ is continuous and linear.

Unlike the U-V model where the primitive is a revealed preference over Δ, the primitive of a W model is a preference $\succsim$ over the set Z of menus. The functional forms for U and V are similar in the two models, except that in the W model the temptation utility $\widehat{V}$ from continuation menus lacks structure. The Dynamic Self-Control (DSC) model of GP [10] takes $\widehat{V} = 0$, the (non-axiomatic) generalization of DSC by Krussel, Kurusçu and Smith [13] takes $\widehat{V} = \beta W$ for $\beta \geq 0$, and the Future Temptation model of Noor [15] takes $\widehat{V} = \gamma \max_{\eta \in x} V(\eta)$ for $0 < \gamma < \delta$. These functional forms for $\widehat{V}$ were discussed at the end of Section 2.

A benchmark comparison between the U-V and W models can be provided in terms of the following time-line:



The W model describes choice in a period 0, where the preference $\succsim$ is used to guide choice of a menu x. The representation implicitly tells a story about subsequent choice: the selected menu x is faced in period 1 and a choice $(c, z) \in x$ is made by maximizing $U + V$; immediate consumption c and a continuation menu z is obtained, and a choice $(c', z') \in z$ is made in period 2 by maximizing

$U+V$, so on and so forth. While these period $t>0$ choices are derived from the *interpretation* of the representation W, the U-V model can be understood as one explicitly of period $t>0$ choices: it describes an agent that maximizes $U+V$ in any menu at any $t>0$. In a sense, our model describes the revealed preference implications for period $t>0$ choice of some W-model. Indeed, observe that a W appears as a component in the U-V representation. This can be interpreted as a representation of preferences in a *hypothetical* period 0 where the agent ranks menus prior to the experience of temptation.

Behavioral Comparison 1: A peculiar feature of the W-model is a generic asymmetry between the agent's ex ante and ex post ranking of menus. Observe that period $t>0$ choice from $\{(c,x),(c,y)\}$ maximizes $U+V$, and in particular,

$$(U+V)(c,x) \geq (U+V)(c,y) \Longleftrightarrow W(x)+\frac{1}{\delta}\widehat{V}(x) \geq W(y)+\frac{1}{\delta}\widehat{V}(y).$$

That is, menus are ranked in period $t>0$ by $W+\frac{1}{\delta}\widehat{V}$. On the other hand, in period 0, the preference $\succsim$ ranks menus by W. Evidently, this asymmetry arises due to the menu-temptation $\widehat{V}$ experienced in periods $t>0$, which is seemingly absent in period 0. That is, it appears that $\succsim$ is not subject to the same menu-temptation that it identifies for subsequent periods. Indeed, the primitive $\succsim$ of the W-model appears to describe behavior in a period 0 that is *special* relative to all subsequent periods $t>0$.[13]

Our motivation for not pursuing a W model of menu-temptation is our view that this 'special period 0' feature is a problem from the perspective of foundations. If we take the common interpretation in the literature that period 0 behavior reflects how the agent ranks menus *if* he were not tempted by menus,[14] then it is not obvious how to answer questions such as: How is such a ranking identified? Is it even observable? More generally, the 'special period 0' feature is

[13] The asymmetry does not exist when $\widehat{V}$ is constant or a positive affine transformation of W, that is, $\widehat{V}=\beta W$ for $\beta \geq 0$. Observe that this corresponds to GP [10] and Krussel, Kurusçu and Smith [13]. Thus, period 0 is not special *only* when menus do *not* tempt.

[14] The literature typically regards period 0 as reflecting the agent's preferences in a 'cold' state.

problematic because it is not clear how to justify any particular interpretation of it. Yet the interpretation of the representation depends heavily on how the special period 0 is interpreted. If period 0 reflects behavior in the absence of temptation, then the representation suggests what the agent's normative and temptation preferences and anticipated choices are, but if menu-temptation contaminates period 0 behavior, then it is less clear what the components of the representation reflect.

The special period 0 is a consequence of relying on a preference for commitment for identifying temptation although the agent being studied is subject to menu-temptation (recall the discussion in the Introduction). By relying on an alternative method of identifying temptation we avoid the asymmetry between behavior in period 0 and all subsequent periods. Indeed, our agent's behavior is time-invariant.

Behavioral Comparison 2: Another distinguishing property of the ranking $\succsim$ of menus in the W model is the *Stationarity axiom*: for all $x, y \in Z$,

$$x \succsim y \Longleftrightarrow \{(c, x)\} \succsim \{(c, y)\}. \tag{5.1}$$

This adapts the standard stationarity condition (Koopmans [12]) to a preference-over-menus setting. This is a key property of the W model, and thus also of [10, 13, 15].[15] In contrast, the U-V model violates Stationarity: given the distancing hypothesis, menu-temptation may cause the ranking of tomorrow's menus to differ from the ranking of more distant menus. Moreover, there may never exist a minimum distance after which the rankings always agree for all pairs of menus. In the absence of Stationarity, the job of relating the ranking of menus at different delays is done by the Reversal and Menu-Reversal axioms, and indirectly also by the Sophistication axiom.

Stationarity enables a relatively straightforward extension of GP [9] to an infinite horizon. Roughly, if W_1 is a linear function representing $\succsim$ that can be

[15] To connect back with our observation about the 'special period 0', it should be noted that while Stationarity is satisfied by period 0 behavior, it may be violated in all subsequent periods $t > 0$. Intuitively, how the menu-temptation component $\widehat{V}$ changes as a menu is delayed determines whether Stationarity holds in periods $t > 0$, but this component is altogether absent in period 0.

written as $W_1(\{(c,x)\}) = u(c) + W_2(x)$, where W_2 is also linear, then Stationarity implies that W_1 and W_2 are ordinally equivalent, and thus by linearity, they are in fact cardinally equivalent. Ignoring constants, we can thus write $W_1(\{(c,x)\}) = u(c) + \delta W_1(x)$ for some $\delta > 0$, a key step in establishing an infinite horizon extension of GP's representation. Since our model violates Stationarity, we need to find an alternative means of establishing our representation. Our idea is to make use of the distancing hypothesis – see the next section.

Comparison of Primitives: Our primitive consists of a choice correspondence over menus that satisfies WARP. Thus, our primitive is effectively a complete and transitive revealed preference relation $\gtrapprox$ over $\Delta = \Delta(C \times Z)$. Observe that Z can be identified with a subdomain of Δ, and thus the restriction $\gtrapprox|_Z$ is a preference over menus Z. The induced representation for $\gtrapprox|_Z$ is

$$x \longmapsto W(x) + \frac{\gamma}{\delta + \beta} \max_{\eta \in x} V(\eta). \tag{5.2}$$

where the components have the form in Def 2.1. The representation suggests that the agent's ranking of menus is a compromise between the normative evaluation (reflected in $W(x)$) and menu-temptation evaluation (reflected in $\max_{\eta \in x} V(\eta)$).

Instead of taking dynamic choice as a primitive, we could have axiomatized a single preference over menus that admits this representation (5.2).[16] The resulting model would lie squarely within the literature on menu choice (Dekel, Lipman and Rusticini [5], GP [9]), to which the W model also belongs. While this observation reveals that dynamic choice is strictly speaking not necessary to write down or axiomatize a model of tempting menus, we argue that such a primitive *should* be adopted:

1. As is, (5.2) represents static choice, and therefore any dynamic (or ex post) behavior derived from the model is a mere *interpretation* of the representation, based on an *assumption* that the agent is sophisticated and his accurately anticipated ex post choices affect his ranking of menus in a particular way. While

[16] We thank an anonymous referee for pointing this out.

dynamic choice and sophistication are notions of central interest, they lack empirical foundations when the primitive consists of a preference over menus – this problem is common to the literature on menu choice, including Dekel et al [5] and GP [9]. In our model, dynamic choice behavior is the primitive data and not derived from the representation, and sophistication is a refutable hypothesis rather than an assumption. Thus, our use of dynamic choice as a primitive permits us to fill the gaps that remain when a preference $\succsim$ over menus is taken as the primitive. See Section 6.1 below for a general result on this.

2. More pertinent to menu-temptation is the observation that without dynamic choice, our use of the distancing hypothesis would not be defensible. The agent exhibits reversals but for all we know these may be driven by anticipated preference shocks. Due to possibly time-variant choices, the distancing hypothesis would lose its appeal: When behavior is time-varying, we have to allow that the agent's normative evaluations may be time-varying, and as a result it is harder to justify identifying normative preference over current alternatives from the ranking of distant alternatives.[17] Being able to do so is important because the distancing hypothesis is the backbone of our model: it is needed to identify temptation, and in particular, since our agent may experience temptation even if he does not exhibit a preference for commitment, the hypothesis is required to identify menu-temptation.

3. Finally, dynamic choice is a very useful technical tool. Since our primitive $\mathcal{C}$ simultaneously describes choice of (continuation) menus and choice from any menu, restrictions on $\mathcal{C}$ have implications for both. As a result, our model is characterized with relatively simple restrictions despite the very rich structure on the representation. If we were to axiomatize (5.2), restrictions could only be imposed on period 0 preference $\succsim$ over menus. Indeed, the restrictions would need to be strong enough in order to get the rich structure on (implied) choice from

[17] The issue is not resolved if we suppose that the primitive for (5.2) is a dynamic but time-invariant preference over menus. This is because ex post choice involves not just menus but rather (c, x) pairs, and even if the ranking of menus (obtained by fixing c and varying x) is time-invariant, ex post choice of (c, x) pairs can still be time-varying.

menus.

6. Proof Outline for Theorem 4.1

Before presenting the proof of Theorem 4.1 we first describe a lemma. While this simplifies the exposition of one step in the proof, the lemma is of independent interest as it shows how foundations can be provided for the interpretation of the GP [9] representation.

6.1. A Result: Foundations for Sophistication

Consider the *two period* model of GP [9]. As noted earlier, the interpretation of GP's representation (2.2) assumes that the agent is sophisticated in the sense of correctly anticipating ex-post choices. However, there is nothing in the primitives that can justify this 'sophistication assumption'.[18] A lemma used in the proof of Theorem 4.1 provides the missing foundations for this assumption in GP's model. We state it below as a theorem.

Extend GP's model to include ex-post choice. Let C be some compact metric space, and $Z_2 = \mathcal{K}(\Delta(C))$ the set of nonempty compact menus. Take a preference $\succsim$ over Z_2 and a closed-valued choice correspondence $\mathcal{C} : Z_2 \rightsquigarrow \Delta(C)$ where, for all $x \in Z$, $\mathcal{C}(x) \neq \phi$ and $\mathcal{C}(x) \subset x$. The preference $\succsim$ captures period 0 preference over menus, and the choice correspondence $\mathcal{C}$ captures period 1 choice from any menu. Say that $\succsim$ *is nontrivial* if there exist μ, η such that $\{\mu, \eta\} \succ \{\eta\}$. The following is a representation theorem for our 'extended GP model'.

Theorem 6.1. *Consider a non-trivial preference $\succsim$ and a choice correspondence $\mathcal{C}$ over Z_2 such that $\succsim$ admits a GP representation $(\overline{U}, \overline{V})$ and $\mathcal{C}$ is rationalized by a vNM preference. Then $\mathcal{C}$ admits the representation*

$$\mathcal{C}(x) = \arg\max_{\mu \in x} \{\overline{U}(\mu) + \overline{V}(\mu)\}, \quad x \in Z_2,$$

[18] Indeed, as recently demonstrated by Dekel and Lipman [3], GP's axioms for a preference over menus also characterize another model with very different implications for ex post choice.

if and only if, for any μ, η *such that* $\{\mu\} \succ \{\eta\}$,

$$\{\mu, \eta\} \succ \{\eta\} \Longleftrightarrow \mathcal{C}(\{\mu, \eta\}) = \{\mu\}.$$

Thus, the exhaustive testable implications of GP's model for choice in both periods 0 and 1 are given by GP's axioms for $\succsim$, rationalizability by a vNM preference for $\mathcal{C}$ and the noted joint restriction (the counterpart of the Sophistication axiom) on $\succsim$ and $\mathcal{C}$. Indeed, these behaviors imply that the agent behaves as in the interpretation of GP's representation: period 1 choice maximizes $\overline{U} + \overline{V}$. We thus obtain as a theorem what GP propose as an interpretation.

The result can be used directly in conjunction with the axiomatizations of GP [10], Krussel et al [13] and Noor [15] to provide dynamic extensions of these models. However, while this poses no issue for the extension of models without menu-temptation (namely GP [10] and Krussel et al [13]), extensions of models with menu-temptation will still possess the problematic 'special period 0' property noted in the previous section. Given the question of what period 0 behavior reflects and whether it corresponds to any observed behavior, the question of empirical foundations for such models would therefore remain.

6.2. Proof Outline for Main Theorem

The proof of Theorem 4.1 has three broad steps that show:

(i) there is a preference $\succsim$ over menus Z with a W-representation (as defined in Section 5) that can be derived from $\mathcal{C}$,

(ii) temptation utility over menus $\widehat{V}$ in this representation has the desired form (2.6),

(iii) the ex-post choice suggested by the W-representation is exactly the original choice correspondence $\mathcal{C}$.

In the proof, steps (ii) and (iii) are completed simultaneously in the final lemma.

The sequence of preferences $\{\gtrsim_t\}_{t=0}^{\infty}$ defined in Section 4.3 is derived from $\mathcal{C}$ and it produces a normative preference $\succsim^*$ over Δ via (4.1). Existence of $\succsim^*$ is ensured by WARP, Continuity and Reversal alone. Roughly speaking, the single-reversal property underlying Reversal implies increased agreement between $\gtrsim_t$ and $\gtrsim_{t+1}$ as t grows, and gives rise to convergence in the limit. The limit preference $\succsim^*$ is complete, transitive and continuous. A connection with $\mathcal{C}$ exploited below is given by:

$$\mu \succ^* \eta \implies \mu >_t \eta \text{ for all large } t. \tag{6.1}$$

The candidate preference $\succsim$ over menus Z is defined as the induced normative preference over menus: for all $x, y \in Z$,

$$x \succsim y \Longleftrightarrow x^{+1} \succsim^* y^{+1}.$$

Axioms 1-4 and Set-Betweenness imply that $\succsim$ satisfies the GP [9] axioms and thus admits a GP representation (2.2) with some U, V. This together with Axiom 5 implies additive separability of U, V. Reversal implies that $\succsim$ satisfies the Stationarity axiom (5.1); roughly speaking, Reversal implies that there are "no reversals in the limit". Under these conditions, Lemma D.5 shows that $\succsim$ is a W model (Def 5.1).[19]

At this point, we also obtain a key 'partial rationalizability' result: Lemma D.6 shows that there is $\xi \geq 1$ s.t. for all x,

$$\mathcal{C}(x) = \arg\max_{\mu \in x}\{\xi U(\mu) + V(\mu)\}.$$

The argument makes use of (6.1). This implies that if $\{\mu, \eta\} \succ \{\eta\}$ (in which case $\{\mu\} \succ \{\eta\}$ also holds), then $\{\mu, \eta\}^{+t} > \{\eta\}^{+t}$ and $\{\mu\}^{+t} > \{\eta\}^{+t}$ for all large t. By Sophistication $\mu > \eta$ follows. That is, whenever the limit W agent normatively

[19] The proof in an earlier version of this paper was relatively simpler and proceeded from this point as follows: additional axioms imposed further desired structure on W, and one axiom connected $\mathcal{C}$ with $\succsim$ in a way that Thm 6.1 could be invoked directly to complete the proof. The axiom connecting $\mathcal{C}$ with $\succsim$ was unattractive since $\succsim$ is a *limit* ranking. The current proof exploits the observation discussed next in the text: specifically, an intimate connection between $\mathcal{C}$ and $\succsim$ already exists due to (6.1) and Sophistication. This allows us to axiomatize the representation with fewer and simpler axioms. See also the discussion after Thm 4.1.

prefers μ over η and anticipates choosing μ over η then our agent chooses μ over η. After exploiting Continuity, this generates the statement:

$$U(\mu) \geq U(\eta) \text{ and } U(\mu) + V(\mu) \geq U(\eta) + V(\eta) \Longrightarrow \mu \gtrapprox \eta.$$

Using Harsanyi's aggregation theorem we obtain the result. Full rationalizability ($\xi = 1$) is obtained only at the end of the proof. The partial rationalizability result is a crucial step because it enables *the axioms on $\mathcal{C}$ to directly translate into restrictions on U and V*. The remainder of the proof relies heavily on this.

The first step (Lemma D.7) toward obtaining the functional form for $\widehat{V}$ exploits the fact that the $\gtrapprox$-ranking of alternatives of the form x^{+1} has two different cardinally equivalent representations, specifically (Rep. A) and (Rep. B) below.

To derive Rep. A, begin by noting that Set-Betweenness implies that the ranking admits a GP representation:

$$x \mapsto \max_{\mu \in x}\{\widetilde{U}(\mu) + \widetilde{V}(\mu)\} - \max_{\eta \in x} \widetilde{V}(\eta).$$

By Theorem 6.1 presented in the previous subsection, Sophistication implies that $\widetilde{U} + \widetilde{V}$ represents $\gtrapprox$ and thus by the partial rationalizability result, it is cardinally equivalent to $\xi U + V$. Moreover, by Menu-Reversal (ii), whenever the agent exhibits a preference for commitment $\{\mu\}^{+1} > \{\mu, \eta\}^{+1}$ then so does the limit preference $\{\mu\} \succ \{\mu, \eta\}$. This fact is used to show that $\widetilde{V} = \alpha U + \alpha' V$. Thus, the $\gtrapprox$-ranking of period 1 menus is represented by

$$x \mapsto \max_{\mu \in x}\{\xi U(\mu) + V(\mu)\} - \max_{\eta \in x}\{\alpha U(\eta) + \alpha' V(\eta)\}. \qquad \text{(Rep. A)}$$

The derivation of Rep. B is based on the fact that the partial rationalizability result implies that this ranking is also represented by $x \mapsto \xi U(x^{+1}) + V(x^{+1})$. In fact, given the functional form of U, V in the W representation, $\xi U(x^{+1}) + V(x^{+1})$ is ordinally equivalent to $\xi \delta W(x) + \widehat{V}(x)$. Therefore, (Rep. A) is ordinally equivalent to

$$x \mapsto \xi \delta W(x) + \widehat{V}(x). \qquad \text{(Rep. B)}$$

But linearity of Rep. A and Rep. B implies cardinal equivalence, and so, by writing down the affine transformation and rearranging terms we find that $\widehat{V}$ must take the following form with some $\lambda > 0$ (ignoring constants):

$$\begin{aligned}\widehat{V}(x) &= \lambda \max_{\mu\in x}\{\xi U(\mu) + V(\mu)\} - \xi\delta \max_{\mu\in x}\{U(\mu) + V(\mu)\} \\ &\quad +\xi\delta \max_{\eta\in x} V(\eta) - \max_{\mu\in x}\{\alpha U(\mu) + \alpha' V(\mu)\}, \qquad x \in Z.\end{aligned}$$

This yields a functional form for $\widehat{V}$.

The remainder of the proof shows that by the partial rationalizability result, Reversal and Set-Betweenness, it must be that $\widehat{V}$ reduces to the desired form (for some appropriately defined β, γ). Menu-Reversal (i) is used in the process to rule out one additional form for $\widehat{V}$ that is consistent with the other axioms but nonintuitive. Reversal helps to establish $0 \leq \gamma \leq \delta$. Finally, $\xi \neq 1$ gives rise to a violation of Set-Betweenness, and therefore we must have $\xi = 1$. At this point, full rationalizability obtains, and the proof is complete.

6.3. Necessity and Summarizing Comments

Reversal: This ensures the existence of a limit of a sequence of preferences. To see that Reversal is implied, note that in the model, for all μ, η and t,

$$\mu^{+t} \gtrapprox \eta^{+t} \Longleftrightarrow U(\mu) + D_t V(\mu) \geq U(\eta) + D_t V(\eta) \tag{6.2}$$

where $D_t = \frac{\left(\frac{\gamma}{\delta}\right)^t}{1+\frac{\beta}{\delta}\left[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i\right]}$, adopting the convention that $\sum_{i=0}^{-1}\left(\frac{\gamma}{\delta}\right)^i = 0$. This can be established with a proof by induction. The model requires $0 \leq \gamma \leq \delta$ and $\beta > \gamma - \delta$, and these imply that $D_t \geq 0$, $D_0 = 1$ and $D_t \searrow 0$. Thus, whenever U and V strictly disagree on the ranking of μ, η, delaying the two alternatives leads to at most one reversal in the ranking. When there is no strict disagreement then there is no reversal.

Set-Betweenness: As one would expect, this axiom is responsible for the GP form for the normative preference over menus. However, this feature could be achieved by requiring merely that Set-Betweenness hold for sufficiently distant menus. Imposing Set-Betweenness also for the ranking of more immediate menus places a lot of structure on choices, and is largely responsible for a functional form for $\widehat{V}$ that is not more general.

The necessity of Set-Betweenness is established by using (6.2) to see that for any x, y and $t > 0$,

$$x^{+t} \gtrapprox y^{+t} \Longleftrightarrow W(x) + \frac{D_t}{\delta}\widehat{V}(y) \geq W(y) + \frac{D_t}{\delta}\widehat{V}(\eta). \tag{6.3}$$

Inserting the functional form for W and $\widehat{V}$, and defining $a_t := (1 + \frac{D_t \beta}{\delta}) > 0$ and $b_t := 1 + \frac{D_t(\beta-\gamma)}{\delta} > 0$, we obtain:

$$x^{+t} \gtrapprox y^{+t} \Longleftrightarrow \max_{\mu \in x}\{U_t + V_t - \max_{\eta \in x} V_t\} \geq \max_{\mu \in y}\{U_t + V_t - \max_{\eta \in y} V_t\},$$

where $U_t(\mu) = U(\mu) + (1 - \frac{b_t}{a_t})V(\mu)$ and $V_t(\mu) = \frac{b_t}{a_t}V(\mu)$. That is, the ranking of period t menus admits a GP [9] representation, and hence Set-Betweenness is implied.

Sophistication: This is our only dynamic axiom. It plays a part in establishing the partial rationalizability result (Lemma D.6). Theorem 6.1 (which makes use of Sophistication) is used to get a connection between $\gtrapprox$ and the $\gtrapprox$-ranking of menus, and this eventually plays a part in getting the functional form for $\widehat{V}$. The proof for necessity uses the observation derived above for Set-Betweenness and invokes Theorem 6.1.

Menu-Reversal: If Menu-Reversal(i) is dropped then Theorem 4.1 generalizes to permit $\widehat{V}$ to also take an additional possible form. This additional form is nonintuitive, as a violation of the very intuitive Menu-Reversal(i) axiom would suggest. Menu-Reversal(ii) ensures that an immediate preference for commitment $\{\mu\}^{+1} > \{\mu, \eta\}^{+1}$ implies a preference for commitment in the limit $\{\mu\} \succ \{\mu, \eta\}$. This ensures that the functional form of $\widehat{V}$ includes only functions that are linear combinations of U and V. (Set-Betweenness then does the rest of the job).

One may ask why Menu-Reversal(ii) requires a statement about the neighorhood of a pair of menus. The answer is that in general, for any μ, η, a strict preference $\mu^{+t} > \eta^{+t}$ for all t does not imply strict preference in the limit. The nature of the distant preference between *neighboring* pairs of rewards plays a role in how μ, η are ranked in the limit (Lemma C.3(b)). Thus, a statement about neighborhood pairs of rewards is required in order to directly make a statement about the limit preferences. The necessity of Menu-Reversal is verified in the Supplementary Appendix.

7. Concluding Remarks

We conclude with a comment on welfare. The standard revealed preference criterion suggests that welfare policy should be guided by $\gtrapprox$. The concept of normative preference lends itself as an alternative welfare criterion. The agent's view of what he should or should not choose is his *own* definition of his welfare. Thus, his normative preference is a subjective welfare criterion. If an analyst believes that this is the appropriate criterion for welfare policy, and if he takes the position that distancing is an appropriate tool (that serves as a veil of ignorance of sorts [19]) for determining normative preference, then the ranking $\succsim^*$ defined in (4.1) would guide welfare analysis in our model while the revealed preference criterion would be viewed as contaminated with temptation.

A. Appendix: Proof of Theorem 6.1

Prove sufficiency of the axioms. By GP, $\succsim$ is represented by (2.2) for some continuous linear functions $U, V : \Delta \to \mathbb{R}$. The first lemma collects some simple facts about the representation [15], and the second establishes the result.

Lemma A.1. *For all* x, y,

(a) $x \succ x \cup y \Longleftrightarrow \max_y V > \max_x V$ *and* $W(x) > W(y)$.

(b) $x \cup y \succ y \Longleftrightarrow \max_x U + V > \max_y U + V$ *and* $W(x) > W(y)$.

(c) $x \succ x \cup y \succ y \Longleftrightarrow \max_x U + V > \max_y U + V$ *and* $\max_y V > \max_x V$.

Lemma A.2. $\mu \gtrapprox \eta \Longleftrightarrow U(\mu) + V(\mu) \geq U(\eta) + V(\eta)$.

Proof. By hypothesis there exists ρ, ν such that $\{\rho, \nu\} \succ \{\nu\}$. By Set-Betweenness, $\{\rho\} \succ \{\nu\}$ and by Lemma A.1(b) and Sophistication, $\rho > \nu$ and $U(\rho) + V(\rho) > U(\nu) + V(\nu)$. These observations will be used to prove the result.

$\Longrightarrow$: Suppose that $\mu \gtrapprox \eta$. If $\{\eta\} \succ \{\mu\}$, then by Sophistication, $\{\eta, \mu\} \not\succ \{\mu\}$ and it follows by Lemma A.1(b) that $U(\mu)+V(\mu) \geq U(\eta)+V(\eta)$. If, on the other hand, $\{\mu\} \succsim \{\eta\}$, then by Set-Independence and Independence,

$$\{\mu\alpha\rho\} \succ \{\eta\alpha\nu\} \text{ and } \mu\alpha\rho > \eta\alpha\nu,$$

for all $\alpha \in (0, 1)$. By Sophistication and Lemma A.1(b), $U(\mu\alpha\rho) + V(\mu\alpha\rho) > U(\eta\alpha\nu) + V(\eta\alpha\nu)$ for all $\alpha \in (0, 1)$. By continuity of $U + V$, it follows that $U(\mu) + V(\mu) \geq U(\eta) + V(\eta)$, as desired.

$\Longleftarrow$: Suppose that $U(\mu) + V(\mu) \geq U(\eta) + V(\eta)$. If $\{\eta\} \succ \{\mu\}$, then by Lemma A.1(b), $\{\eta, \mu\} \not\succ \{\mu\}$ and by Sophistication, $\mu \gtrapprox \eta$. If, on the other hand, $\{\mu\} \succsim \{\eta\}$, then for all $\alpha \in (0, 1)$,

$$\{\mu\alpha\rho\} \succ \{\eta\alpha\nu\} \text{ and } U_t(\mu\alpha\rho) + V_t(\mu\alpha\rho) > U_t(\eta\alpha\nu) + V_t(\eta\alpha\nu).$$

By Lemma A.1(b) and Sophistication, $\mu\alpha\rho > \eta\alpha\nu$ for all $\alpha \in (0, 1)$. Thus, continuity of $\gtrapprox$ implies $\mu \gtrapprox \eta$. ■

B. Appendix: Topology on $\mathcal{P}$

Since is Δ is compact and metrizable, $\Delta \times \Delta$ is compact and metrizable under the product topology. Let d be a metric that generates the topology on $\Delta \times \Delta$. Denote the space of nonempty compact subsets of $\Delta \times \Delta$ by $\mathcal{P}$. For any $A, B \in \mathcal{P}$, let $d(a, B) = \inf_{b \in B} d(a, b)$ and $d(b, A) = \inf_{a \in A} d(b, a)$. The Hausdorff metric h_d induced by d is defined by $h_d(A, B) = \max\{\sup d(a, B), \sup d(b, A)\}$, for all $A, B \in \mathcal{P}$. An ε-ball centered at A is defined by $B(A, \varepsilon) = \{B : h_d(A, B) < \varepsilon\}$.The Hausdorff metric topology on $\mathcal{P}$ is the topology for which the collection of balls $\{B(A, \varepsilon)\}_{A \in \mathcal{P}, \varepsilon \in (0,\infty)}$ is a base.

View the set $\mathcal{P}$ as the space of nonempty and continuous binary relations on Δ by identifying any such binary relation B on Δ with $\Gamma(B)$, the graph of B:

$$\Gamma(B) = \{(\mu, \eta) \in \Delta \times \Delta : \mu B \eta\}.$$

If B is a weak order (complete and transitive binary relation) then $\Gamma(B)$ is non-empty. Given that Δ is a connected separable space, if B is also continuous then $\Gamma(B)$ is closed, and hence compact. Thus, the set of continuous weak orders on Δ is a subset of $\mathcal{P}$.

By [1, Thm 3.71(3)], compactness of $\Delta \times \Delta$ implies that $\mathcal{P}$ is compact. Also, under compactness of $\Delta \times \Delta$, $\Gamma(B)$ is the Hausdorff metric limit of a sequence $\{\Gamma(B_n)\} \subset \mathcal{P}$ if and only if $\Gamma(B)$ is the 'closed limit' of $\{\Gamma(B_n)\}$ [1, Thm 3.79]. To define the closed limit of a sequence $\{\Gamma(B_n)\}$, first define the topological limit superior $Ls\Gamma(B_n) := \{a \in \Delta \times \Delta :$ for every neighborhood V of a, $V \cap \Gamma(B_n) \neq \phi$ for infinitely many $n\}$ and topological limit inferior $Li\Gamma(B_n) := \{a \in \Delta \times \Delta :$ for every neighborhood V of a, $V \cap \Gamma(B_n) \neq \phi$ for all but a finite number of $n\}$. The sequence $\{\Gamma(B_n)\}$ converges to a closed limit $\Gamma(B)$ if $\Gamma(B) = Ls\Gamma(B_n) = Li\Gamma(B_n)$.

C. Appendix: Normative Preference

This appendix collects some results from [16]. Take as given a set of preference relations $\{\gtrapprox_t\}_{t=0}^{\infty}$ on some set Δ of lotteries, defined over some compact metric space, that is endowed with the weak convergence topology. For each μ, η, the preference $\gtrapprox_t$ captures how the agent ranks the rewards μ, η when they are to be received t periods later. Normative preference $\succsim^*$ over Δ is defined by $\succsim^* = \lim_{t\to\infty} \gtrapprox_t$, as in Section 4.3.

Consider the following axioms on $\{\gtrapprox_t\}$.

Axiom A1 (Order*) $\gtrapprox_t$ *is complete and transitive, for all* t.
Axiom A2 (Continuity*) $\{\eta : \mu \gtrapprox_t \eta\}$ *and* $\{\eta : \eta \gtrapprox_t \mu\}$ *are closed for all* t.
Axiom A3 (Reversal*) *If* $\mu <_t \eta$ *(resp.* $\mu \lessapprox_t \eta$*) and* $\mu \gtrapprox_{t'} \eta$ *(resp.* $\mu >_{t'} \eta$*) for some* $t' > t$*, then* $\mu \gtrapprox_{t''} \eta$ *(resp.* $\mu >_{t''} \eta$*) for all* $t'' > t'$*.*
Axiom A4 (Independence*) $\mu >_t \eta \Longrightarrow \mu\alpha\nu >_t \eta\alpha\nu$, for all t.

Define the function $\tau : \Delta \times \Delta \to \mathbb{R}$ which captures the time at which a reversal takes place for each (μ, η) in the following way: First take any $(\mu, \eta) \in \Delta \times \Delta$ such that $\mu \gtrapprox_0 \eta$. If $\mu \approx_t \eta$ for all t or $\mu >_t \eta$ for all t, then define $\tau(\mu, \eta) = 0$. If there exists T such that $\mu <_T \eta$, then define $\tau(\mu,\eta) = \min\{t : \mu <_t \eta\}$. If $\mu >_0 \eta$ and there exists T such that $\mu \approx_t \eta$ for all $t \geq T$, then define $\tau(\mu,\eta) = \min\{t : \mu \approx_t \eta\}$. Finally, to cover all the remaining cases, let $\tau(\mu,\eta) = \tau(\eta,\mu)$ for all μ,η. The following results are proved in [16] (except for Lemma C.3(d)) and will be used later.

Lemma C.1. *Suppose $\{\gtrapprox_t\}_{t=0}^{\infty}$ satisfies A1 and A3 and take any μ,η such that $\mu \gtrapprox_0 \eta$. If $\tau(\mu,\eta) = 0$ then $\mu \approx_t \eta$ for all t or $\mu >_t \eta$ for all t. If $\tau(\mu,\eta) > 0$ then only one of the following holds:*
(a) $\mu >_t \eta$ for $t < \tau(\mu,\eta)$ and $\mu <_t \eta$ for all $t \geq \tau(\mu,\eta)$;

(b) $\mu >_t \eta$ for $t < \tau(\mu,\eta)$ and $\mu \approx_t \eta$ for all $t \geq \tau(\mu,\eta)$;

(c) There is $0 \leq T < \tau(\mu,\eta)$ such that $\mu >_t \eta$ for all $t < T$, $\mu \approx_t \eta$ for all $T \leq t < \tau(\mu,\eta)$, and $\mu <_t \eta$ for all $t \geq \tau(\mu,\eta)$.

Lemma C.2. *If $\{\gtrapprox_t\}$ satisfies A1-3, then $\succsim^*$ is well-defined, complete, transitive and continuous. If $\{\gtrapprox_t\}$ also satisfies A4, then $\succsim^*$ also satisfies vNM independence.*

The last lemma provides two characterizations of normative preference $\succsim^*$ and collects some observations. Part (c) of the lemma is implied by (a). Let Ω be the subset of $\Delta \times \Delta$ on which τ is upper semicontinuous, that is,

$$\Omega = \{(\mu,\eta) \in \Delta \times \Delta : (\mu_n,\eta_n) \to (\mu,\eta) \Longrightarrow \limsup_{n\to\infty} \tau(\mu_n,\eta_n) \leq \tau(\mu,\eta)\}. \quad \text{(C.1)}$$

Lemma C.3. *(a) $\mu \succ^* \eta \Longleftrightarrow [\mu >_{\tau(\mu,\eta)} \eta$ and $(\mu,\eta) \in \Omega]$*

(b) $\mu \succsim^* \eta \Longleftrightarrow \exists$ a sequence $\{(\mu_n,\eta_n)\}$ that converges to (μ,η) and $\mu_n \gtrapprox_{\tau(\mu_n,\eta_n)} \eta_n$ for all n.

(c) $\mu \gtrapprox_{\tau(\mu,\eta)} \eta \Longrightarrow \mu \succsim^* \eta$.

(d) $\mu <_0 \eta$ and $\mu >_t \eta$ for some $t \Longrightarrow \mu \succ^* \eta$.

Proof. We prove part (d). Letting $T = \tau(\mu,\eta)$ we have $\mu >_T \eta$. Take any sequence $\{(\mu_n,\eta_n)\}$ that converges to (μ,η). By Continuity*, for sufficiently large n we have $\mu_n <_0 \eta_n$ and $\mu_n >_T \eta_n$. It follows that $\tau(\mu_n,\eta_n) \leq T = \tau(\mu,\eta)$ and thus $(\mu,\eta) \in \Omega$. Invoke part (a) to get the result. ■

D. Appendix: Proof of Theorem 4.1

Necessity is readily or already verified for most of the axioms. The necessity of Menu-Reversal is established in the Supplementary Appendix. The proof of sufficiency is divided into subsections. We start with a simple lemma. Define the choice correspondence $\mathcal{C}^*(\cdot, \gtrapprox)$ by $\mathcal{C}^*(x, \gtrapprox) \equiv \{\mu \in x : \mu \gtrapprox \eta \text{ for all } \eta \in x\}$. Say that a preference $\gtrapprox$ over Δ *generates* $\mathcal{C}(\cdot)$ if $\mathcal{C}(x) = \mathcal{C}^*(x, \gtrapprox)$ for all x.

Lemma D.1. *$\gtrapprox$ is the unique preference relation that generates $\mathcal{C}(\cdot)$. Furthermore $\gtrapprox$ satisfies the vNM axioms, and there exists μ, η s.t. $\{\mu\}^{+t} > \{\mu, \eta\}^{+t}$ for all large $t > 0$.*

Proof. The first two assertions are standard. The last follows from the fact that by nondegeneracy of $\mathcal{C}$ there exists μ, η such that $\mu < \eta$ and $\mu^{+t} > \eta^{+t}$ for large t, and by Sophistication, Set-Betweenness and transitivity it is implied that $\mu^{+t} > \{\mu, \eta\}^{+t} \approx \eta^{+t}$ for all large t. ■

D.1. Properties of Normative Preference $\succsim^*$

Define $\gtrapprox_t$ over Δ for each $t \geq 0$ and $\mu, \eta \in \Delta$ by

$$\mu \gtrapprox_t \eta \Longleftrightarrow \mu^{+t} \gtrapprox \eta^{+t}.$$

Since $\mathcal{C}(\cdot)$ satisfies WARP, Continuity and Reversal, $\{\gtrapprox_t\}$ satisfies the conditions in Lemma C.2. Thus, there is a well-defined normative preference $\succsim^* \equiv \lim_{t\to\infty} \gtrapprox_t$ over Δ and a well-defined function $\tau : \Delta \times \Delta \to \mathbb{R}$ as in Lemma C.1.

Lemma D.2. *$\succsim^*$ satisfies (i) the vNM axioms; (ii) a separability property: for all c, c', x, x', $[(c, x)\frac{1}{2}(c', x') \sim^* (c, x')\frac{1}{2}(c', x)]$, and (iii) an indifference to timing property: for all μ, η and α, $[\mu^{+1}\alpha\eta^{+1} \sim^* (\mu\alpha\eta)^{+1}]$.*

Proof. Lemma C.2 establishes that $\succsim^*$ is complete, transitive and continuous. Independence and Indifference to Timing imply that each $\gtrapprox_t$ satisfies vNM independence. Thus by Lemma C.2 the limit $\succsim^*$ satisfies vNM independence. By the Separability axiom, $(\frac{1}{2}(c,x)+\frac{1}{2}(c',x'))^{+t} \approx (\frac{1}{2}(c,x')+\frac{1}{2}(c',x))^{+t}$ for all t and hence by Lemma C.3(c), $(c,x)\frac{1}{2}(c',x') \sim^* (c,x')\frac{1}{2}(c',x)$. The indifference to timing property follows similarly from the Strong Indifference to Timing property of $\gtrapprox_t$ proved in the Supplementary Appendix. ■

Lemma D.3. *$\succsim^*$ satisfies a stationarity property: for any c,μ,η,*

$$\mu \succsim^* \eta \Longleftrightarrow (c,\mu) \succsim^* (c,\eta).$$

Proof. We prove this in a series of steps.

Step 1: For any c,c',μ,η, $[(c,\mu) \succsim^* (c,\eta) \Longleftrightarrow (c',\mu) \succsim^* (c',\eta)]$.

Suppose by way of contradiction that $(c,\mu) \succsim^* (c,\eta)$ and $(c',\mu) \prec^* (c',\eta)$. Since $\succsim^*$ satisfies the vNM axioms, $(c,\mu)\frac{1}{2}(c',\eta) \succ^* (c,\eta)\frac{1}{2}(c',\mu)$. But this contradicts the separability property in Lemma D.2.

Step 2: For any c,c',μ,η, $[\mu \succsim^* \eta \Longrightarrow (c,\mu) \succsim^* (c,\eta)]$.

If $\mu \succsim^* \eta$, then by Lemma C.3(b), there exists a sequence $\{(\mu_n,\eta_n)\}$ such that $(\mu_n,\eta_n) \to (\mu,\eta)$ and $\mu_n \gtrapprox_{\tau(\mu_n,\eta_n)} \eta_n$ for all n. It follows by definitions[20] that $\{(\mu_n^{+1},\eta_n^{+1})\}$ is a sequence such that $(\mu_n^{+1},\eta_n^{+1}) \to (\mu^{+1},\eta^{+1})$ and $\mu_n^{+1} \gtrapprox_{\tau(\mu_n^{+1},\eta_n^{+1})} \eta_n^{+1}$ for all n. But then by Lemma C.3(b), $\mu^{+1} \succsim^* \eta^{+1}$. Apply step 1.

Step 3: The result.

By step 1, it suffices to show $\mu \succsim^* \eta \Longleftrightarrow (\bar{c},\mu) \succsim^* (\bar{c},\eta)$. Define a binary relation $\succsim^{**}$ over Δ by $\mu \succsim^{**} \eta \Longleftrightarrow (\bar{c},\mu) \succsim^* (\bar{c},\eta)$. We need to show $\mu \succsim^* \mu' \Longleftrightarrow \mu \succsim^{**} \mu'$. This follows from the facts that, first, $\mu \succsim^* \eta \Longrightarrow \mu \succsim^{**} \eta$ (step 1), second, $\succsim^{**}$ satisfies the vNM axioms (by the first and last assertion in Lemma D.2), and third, that $\succsim^{**}$ is non-trivial, that is, there exist $\rho,\nu \in \Delta$ such that $\rho \succ^{**} \nu$. To see non-triviality, recall that by Lemma D.1 there is x,y such that $x^{+t} > (x \cup y)^{+t}$ for $t>1$. Menu-Reversal(ii) implies $\tau(x^{+t},(x\cup y)^{+t})=0$ and τ is continuous at $(x^{+t},(x\cup y)^{+t})$, and so $(x^{+t},(x\cup y)^{+t}) \in \Omega$. By Lemma

[20]Specifically, it follows from the fact that $\mu \gtrapprox_{\tau(\mu,\eta)} \eta \Longleftrightarrow \mu^{+1} \gtrapprox_{\tau(\mu^{+1},\eta^{+1})} \eta^{+1}$. This holds by definition of τ if $\tau(\mu,\eta)=0$ (in which case $\tau(\mu^{+1},\eta^{+1})=0$ as well), and when $\tau(\mu,\eta)>0$, then note that $\mu^{+\tau(\mu,\eta)}=(\bar{c},\mu)^{+(\tau(\mu,\eta)-1)}$ and $\eta^{+\tau(\mu,\eta)}=(\bar{c},\eta)^{+(\tau(\mu,\eta)-1)}$, in which case the assertion follows from the easily proved fact that $\tau(\mu^{+1},\eta^{+1})=\tau(\mu,\eta)-1$ whenever $\tau(\mu,\eta)>0$.

C.3(a), $x^{+t} \succ^* (x \cup y)^{+t}$. It follows that $x^{+(t-1)} \succ^{**} (x \cup y)^{+(t-1)}$, that is, $\succsim^{**}$ is non-trivial. ■

D.2. Properties of Normative Menu-Preference $\succsim$

Consider the preference $\succsim$ over Z defined by

$$x \succsim y \Longleftrightarrow (c, x) \succsim^* (c, y), \tag{D.1}$$

for some $c \in C$. By step 1 in the proof of Lemma D.3, the preference $\succsim$ is invariant to the choice of c. This subsection shows that $\succsim$ has a W representation (as defined in Section 5) and highlights a connection with $\mathcal{C}$.

Lemma D.4. *(i)* $\{\mu\}^{+t} > \{\mu, \eta\}^{+t} \Longrightarrow \{\mu\} \succ \{\mu, \eta\}$.
(ii) There exists μ, μ', η, η' *s.t.* $\{\mu\} \succ \{\mu, \eta\}$ *and* $\{\mu'\} \sim \{\mu', \eta'\} \succ \{\eta'\}$.

Proof. (i) If $\{\mu\}^{+t} > \{\mu, \eta\}^{+t}$ then Menu-Reversal(ii) implies $\tau(\{\mu\}^{+t}, \{\mu, \eta\}^{+t}) = 0$ and τ is continuous at $(\{\mu\}^{+t}, \{\mu, \eta\}^{+t})$, and so $(\{\mu\}^{+t}, \{\mu, \eta\}^{+t}) \in \Omega$. Hence, by Lemma C.3(a), $\{\mu\}^{+t} \succ^* \{\mu, \eta\}^{+t}$. Repeated application of Lemma D.3 yields $\{\mu\}^{+1} \succ^* \{\mu, \eta\}^{+1}$, and the result follows.

(ii) By Lemma D.1 there is μ', η' such that $\{\mu'\}^{+t} > \{\mu', \eta'\}^{+t}$ for $t \geq 1$. The above result implies $\{\mu'\} \succ \{\mu', \eta'\}$, thus establishing the first part of the statement. To show the second part, recall that by nondegeneracy of $\mathcal{C}$ there is μ, η such that $\tau(A) = 0$ for a neighborhood A of (μ, η) and $\{\mu\}^{+1} \approx \{\mu, \eta\}^{+1} > \{\eta\}^{+1}$. By Sophistication, $\mu > \eta$ and since $\tau(A) = 0$ for a neighborhood A of (μ, η), Lemmas C.3(a) and D.3 imply $\{\mu\} \succ \{\eta\}$. By Menu-Reversal(i), $\tau(\mu, \eta) = 0$ and $\{\mu\}^{+1} \approx \{\mu, \eta\}^{+1}$ implies $\{\mu\}^{+t} \approx \{\mu, \eta\}^{+t}$ for all t. Lemma C.3(c) implies $\{\mu\}^{+1} \sim^* \{\mu, \eta\}^{+1}$ and thus $\{\mu\} \sim \{\mu, \eta\}$. However, since $\{\mu\} \succ \{\eta\}$, transitivity implies $\{\mu\} \sim \{\mu, \eta\} \succ \{\eta\}$ as desired. ■

Lemma D.5. $\succsim$ *admits the representation*

$$\begin{aligned} W(x) &= \max_{\mu \in x} U(\mu) + V(\mu) - \max_{\eta \in x} V(\eta), \text{ where for } \mu \in \Delta, \\ U(\mu) &= \int_{C \times Z} (u(c) + \delta W(y))\, d\mu \text{ and } V(\mu) = \int_{C \times Z} \left(v(c) + \widehat{V}(y)\right) d\mu, \end{aligned}$$

and where $\delta \in (0,1)$, the functions u, v are continuous, $W, \widehat{V}$ are continuous and linear, and U and V are nonconstant and affinely independent.

Proof. The result is obtained by confirming that $\succsim$ satisfies Axioms 1-6 in [15]. The analogs of the three vNM axioms for $\succsim$ follow from Lemmas D.2 and D.3, and the Stationarity property $[z \succsim z' \Longleftrightarrow \{(c,z)\} \succsim \{(c,z')\}]$ follows from Lemmas D.3 and step 1 in the proof of Lemma D.3. The Supplementary Appendix confirms that (i) the Set-Betweenness property $[x \succsim y \Longrightarrow x \succsim x \cup y \succsim y]$, (ii) a Strong Indifference to Timing condition and (iii) a nondegeneracy property (there exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$) are also satisfied. Therefore, we can argue as in the proof of [10, Thm 1] to establish the existence of the desired representation (alternatively, see the proof of [15, Thm 3.1]). The noted nondegeneracy property implies that U and V are nonconstant and affinely independent (apply Lemma A.1). ■

The next lemma establishes an important connection between U, V and $\mathcal{C}$.

Lemma D.6. *There is $\xi \geq 1$ s.t. $\mathcal{C}(x) = \arg\max_{\mu \in x}\{\xi U(\mu) + V(\mu)\}$ for all x.*

Proof. We know that $\mathcal{C}$ is rationalized by $\gtrapprox$ and that $\gtrapprox$ admits a nonconstant continuous linear representation – denote this by $w : \Delta \to \mathbb{R}$. If w is a positive affine transformation of $U+V$ then the result holds trivially with $\xi = 1$. Consider next the case where w is not a positive affine transformation of $U+V$. To ease notation write $UV(\cdot)$ instead of $U(\cdot) + V(\cdot)$.

Since UV is nonconstant and linear, there exist μ', η' s.t. $UV(\mu') > UV(\eta')$ and $w(\eta') \geq w(\mu')$.[21] Show that it must be that $U(\eta') \geq U(\mu')$. If $U(\mu') > U(\eta')$ holds by way of contradiction, then we have that

[21] Since UV is not a positive affine transformation of w, there is μ, η s.t. either $UV(\mu) > UV(\eta)$ and $w(\mu) \leq w(\eta)$, or $UV(\mu) \geq UV(\eta)$ and $w(\mu) < w(\eta)$. If the second case holds with $UV(\mu) = UV(\eta)$ then the nonconstancy and linearity of UV implies the existence of μ', η' that takes us into the first case (with strict inequalities). An aside: This kind of reasoning will be applied below to show the existence of μ, η on whose ranking two nonconstant linear functions strictly disagree.

$U(\mu') > U(\eta')$ and $UV(\mu') > UV(\eta')$
$\Longrightarrow \{\mu', \eta'\} \succ \{\eta'\}$
$\Longrightarrow \{\mu', \eta'\}^{+t} > \{\eta'\}^{+t}$ for large t by Lemma C.3(c)
$\Longrightarrow \mu' > \eta'$ by Sophistication, contradicting $w(\eta') \geq w(\mu')$.
Therefore, for every μ, η,

$$-UV(\eta) > -UV(\mu) \text{ and } w(\eta) \geq w(\mu) \Longrightarrow U(\eta) \geq U(\mu).$$

If $-UV(\eta) = -UV(\mu)$ and $w(\eta) \geq w(\mu)$, then linearity and continuity of the functions and the fact that there exists μ', η' s.t. $UV(\mu') > UV(\eta')$ and $w(\eta') \geq w(\mu')$ can be used to show $U(\eta) \geq U(\mu)$. Therefore the weak pareto condition holds:

$$-UV(\eta) \geq -UV(\mu) \text{ and } w(\eta) \geq w(\mu) \Longrightarrow U(\eta) \geq U(\mu).$$

By Harsanyi's aggregation theorem [2], there is $\alpha, \beta \geq 0$ s.t. $U = -\alpha UV + \beta w$. Since U and V are affinely independent, $\beta > 0$. Moreover, by nondegeneracy we know there is μ, η s.t. $\mu < \eta$ and $\mu^{+t} > \eta^{+t}$ for all large t. Thus $U(\mu) > U(\eta)$ and $w(\mu) < w(\eta)$, which implies $\alpha > 0$. Taking an affine transformation of w if necessary, we can write $w = \xi U + V$ for some $\xi > 1$. ■

D.3. Structure on $\widehat{V}$ and Rationalizability

Lemma D.7. *Under Axioms 1-9, there exist $\lambda > 0$ and β, γ such that $\xi(\lambda - \delta) - \beta \geq 0$ and $\lambda - \gamma \geq 0$ and for all $x \in Z$,*

$$\begin{aligned}\widehat{V}(x) &= \lambda \max_{\mu \in x}\{\xi U(\mu) + V(\mu)\} - \xi\delta \max_{\mu \in x}\{U(\mu) + V(\mu)\} \\ &\quad + \xi\delta \max_{\eta \in x} V(\eta) - \max_{\mu \in x}\{(\xi(\lambda - \delta) - \beta)U(\mu) + (\lambda - \gamma)V(\mu)\} - \theta.\end{aligned}$$

Proof. Consider the revealed preference over next-period menus, that is, the ranking $\gtrapprox^1$ over Z defined by $x \gtrapprox^1 y \iff x^{+1} \gtrapprox y^{+1}$. Since the ranking is complete, transitive, continuous and satisfies Independence and Set-Betweenness, it must admit a GP representation [9]:

$$\widetilde{W}(x) = \max_{\mu \in x}\{\widetilde{U}(\mu) + \widetilde{V}(\mu)\} - \max_{\eta \in x} \widetilde{V}(\eta).$$

By Sophistication, Lemma D.6 and Theorem 6.1, $\widehat{U}+\widehat{V}$ is cardinally equivalent to $\xi U+V$. We claim also that $\widetilde{V}=\alpha U+\alpha' V$ for $\alpha,\alpha'\geq 0$. Use Harsanyi's theorem to establish this. First suppose that $U(\mu)>U(\eta)$ and $V(\mu)\geq V(\eta)$. Then $\xi U(\mu)+V(\mu)>\xi U(\eta)+V(\eta)$ and also $\widetilde{U}(\mu)+\widetilde{V}(\mu)>\widetilde{U}(\eta)+\widetilde{V}(\eta)$. Suppose by way of contradiction that $\widetilde{V}(\eta)>\widetilde{V}(\mu)$. Then the representation $\widetilde{W}$ implies $\{\mu\}>^1\{\mu,\eta\}$. By Lemma D.4, this implies $\{\mu\}\succ\{\mu,\eta\}$, which in turn implies $V(\mu)\not\geq V(\eta)$, a contradiction. This establishes that

$$U(\mu)>U(\eta) \text{ and } V(\mu)\geq V(\eta)\Longrightarrow \widetilde{V}(\mu)\geq\widetilde{V}(\eta).$$

This holds also when $U(\mu)=U(\eta)$, as can be established by a limit argument that exploits linearity and the fact that, by Lemma D.4(ii), there exists μ',η' satisfying $\{\mu'\}\sim\{\mu',\eta'\}\succ\{\eta'\}$. Thus the weak pareto property holds, and Harsanyi's theorem implies $\widetilde{V}=\alpha U+\alpha' V+k$ for some $\alpha,\alpha'\geq 0$ and k. The preceding shows that $\gtrsim^1$ admits the representation:

$$\widetilde{W}(x)=\max_x\{\xi U+V\}-\max_x\{\alpha U+\alpha' V\}+k$$

Given Lemma D.6, we also know that the function $\xi\delta W+\widehat{V}$ is a representation for $\gtrsim^1$. Thus $\xi\delta W+\widehat{V}$ and $\widetilde{W}$ are cardinally equivalent, and so, redefining α,α' if necessary, there is $\lambda>0$ and θ such that for all x,

$$\xi\delta W(x)+\widehat{V}(x)=\lambda\max_{\mu\in x}\{\xi U(\mu)+V(\mu)\}-\max_{\mu\in x}\{\alpha U(\mu)+\alpha' V(\mu)\}-\theta.$$

Rearranging yields

$$\widehat{V}(x)=\lambda\max_{\mu\in x}\{\xi U(\mu)+V(\mu)\}-\xi\delta\max_{\mu\in x}\{U(\mu)+V(\mu)\}+\xi\delta\max_{\eta\in x}V(\eta)-\max_{\mu\in x}\{\alpha U(\mu)+\alpha' V(\mu)\}-\theta.$$

Define β and γ such that $\xi(\lambda-\delta)-\beta=\alpha\geq 0$ and $\lambda-\gamma=\alpha'\geq 0$, and we get the desired representation. ■

Below we adopt the convention that $\sum_{i=0}^{-1}=0$.

Lemma D.8. *For all x,y,t,*

$$x^{+t}\gtrsim y^{+t}\Longleftrightarrow W(x)+B_{t-1}\widehat{V}(x)\geq W(y)+B_{t-1}\widehat{V}(y), \tag{D.2}$$

where $B_t := \frac{\gamma^t}{\xi\delta^{t+1}+\beta\delta^t[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]} = \frac{1}{\xi\delta}\frac{\left(\frac{\gamma}{\delta}\right)^t}{1+\frac{\beta}{\xi\delta}[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]}$. *Moreover, if* $\widehat{V}$ *is nonconstant and affinely independent of* W*, then*

(i) $B_t \geq 0$ *for all* t.

(ii) B_t *is nonconstant and nonincreasing. In particular,* $B_t < \frac{1}{\xi\delta}$ *for all large* t.

Proof. In the previous lemma, the constants β and γ were chosen so that the temptation utility of a delayed menu $\widehat{V}(x^{+1})$ is cardinally equivalent to $\beta\delta W(x)+\gamma\widehat{V}(x)$. An induction argument establishes that for all $t \geq 1$, $\widehat{V}(x^{+t})$ is cardinally equivalent to:[22]

$$\beta\delta^t[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]W(x) + \gamma^t\widehat{V}(x).$$

This can be used to get a representation (D.2) for the ranking of future menus by noting that for any $t \geq 0$, $x^{+t+1} \gtrsim y^{+t+1} \iff \xi U(x^{+t+1})+V(x^{+t+1}) \geq \xi U(y^{+t+1})+V(y^{+t+1})$

$\iff \xi\delta^{t+1}W(x) + \widehat{V}(x^{+t}) \geq \xi\delta^{t+1}W(y) + \widehat{V}(y^{+t})$

$\iff W(x) + B_t\widehat{V}(x) \geq W(y) + B_t\widehat{V}(y)$ where B_t is as in the statement of the lemma.

Proof of (i): Since $\widehat{V}$ is nonconstant and affinely independent of W, there exist x^*, y^* s.t. $W(x^*) = W(y^*)$ and $\widehat{V}(x^*) < \widehat{V}(y^*)$.[23] By (D.2), $x^{*+1} < y^{*+1}$. Lemma C.3(d) rules out $x^{*+t} > y^{*+t}$ for any t. Thus $x^{*+t} \lesssim y^{*+t}$ for all t. It follows from (D.2) that $0 = \frac{W(x^*)-W(y^*)}{\widehat{V}(y^*)-\widehat{V}(x^*)} \leq B_t$ for all t.

Proof of (ii): Since $\widehat{V}$ is nonconstant and affinely independent of W, there exist x^*, y^*, x', y' s.t. $W(x^*) = W(y^*)$, $\widehat{V}(x^*) < \widehat{V}(y^*)$, $\widehat{V}(x') = \widehat{V}(y')$ and $W(x') >$

[22]Proof: Lemma D.7 implies that $\widehat{V}(x^{+1})$ is cardinally equivalent to $\beta\delta W(x) + \gamma\widehat{V}(x)$. Assuming the induction hypothesis that $\widehat{V}(x^{+t})$ is cardinally equivalent to $\beta\delta^t[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]W(x) + \gamma^t\widehat{V}(x)$, we see that $\widehat{V}(x^{+t+1})$ is equivalent to $\beta\delta W(x^{+t}) + \gamma\widehat{V}(x^{+t})$ which itself is equivalent to

$\beta\delta^{t+1}W + \gamma\beta\delta^t[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]W(x) + \gamma^{t+1}\widehat{V}(\mu)$

$= \beta\delta^{t+1}[1 + \frac{\gamma}{\delta}\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]W(x) + \gamma^{t+1}\widehat{V}(\mu)$

$= \beta\delta^{t+1}[\sum_{i=0}^{t}\left(\frac{\gamma}{\delta}\right)^i]W(x) + \gamma^{t+1}\widehat{V}(\mu)$, completing the induction argument.

[23]Note that if $[W(x) = W(y) \implies \widehat{V}(x) = \widehat{V}(y)]$ for all x, y, then W and $\widehat{V}$ are affinely dependent.

$W(y')$ (see footnote 23). Note that

$$\frac{W(x^*\alpha x') - W(y^*\alpha y')}{\widehat{V}(y^*\alpha y') - \widehat{V}(x^*\alpha x')} = \frac{(1-\alpha)}{\alpha}\frac{W(x') - W(y')}{\widehat{V}(y^*) - \widehat{V}(x^*)} \geq 0.$$

Now suppose by way of contradiction that B_t is not nonincreasing: $B_T > B_{T'}$ for some $T > T' \geq 0$. There is some $\alpha \in (0,1)$ for which $B_T > \frac{W(x^*\alpha x')-W(y^*\alpha y')}{\widehat{V}(y^*\alpha y')-\widehat{V}(x^*\alpha x')} > B_{T'}$ and thus there is a reversal: $(x^*\alpha x')^{+T'+1} > (y^*\alpha y')^{+T'+1}$ and $(x^*\alpha x')^{+T+1} < (y^*\alpha y')^{+T+1}$. However, $\alpha \in (0,1)$ implies $W(x^*\alpha x') > W(y^*\alpha y')$, while by Lemma C.3(d), the reversal implies $W(x^*\alpha x') < W(y^*\alpha y')$, a contradiction. Thus B_t is nonincreasing.

To complete the step, we need to show that B_t is nonconstant. Suppose that it is constant, $B_t = B_0 = \frac{1}{\xi\delta} := B > 0$ for all t. Since $W, \widehat{V}$ are nonconstant and affinely independent, $B > 0$ implies that $W + B\widehat{V}$ and W are nonconstant and ordinally distinct. Thus there is μ, η where $W(\mu) + B\widehat{V}(\mu) < W(\eta) + B\widehat{V}(\eta)$ and $W(\mu) > W(\eta)$ (argue as in footnote 21, for instance). But then by (D.2) we have $\mu^{+t} < \eta^{+t}$ for all large t but $W(\mu) > W(\eta)$, violating Lemma C.3(c). Conclude that B_t is nonconstant, and therefore that $B_t < B_0 = \frac{1}{\xi\delta}$ for all large t. ■

Lemma D.9. *If $\widehat{V}$ is nonconstant and affinely independent of W, then $0 \leq \gamma \leq \delta$.*

Proof. Step 1: $\gamma \geq 0$.

Suppose by contradiction that $\gamma < 0$, and consider three cases. We use the identities:

$$\sum_{i=0}^{t}\left(\frac{\gamma}{\delta}\right)^i = \begin{cases} (1+\frac{\gamma}{\delta})(\sum_{i=0}^{\frac{t-1}{2}}\left(\frac{\gamma}{\delta}\right)^{2i}) & \text{for odd } t > 0 \\ (1+\frac{\gamma}{\delta})(\sum_{i=0}^{\frac{t-2}{2}}\left(\frac{\gamma}{\delta}\right)^{2i}) + \left(\frac{\gamma}{\delta}\right)^t & \text{for even } t > 0. \end{cases}$$

Case i: $\frac{|\gamma|}{\delta} = 1$.

The last statement of Lemma D.8(ii) requires

$$\left(\frac{\gamma}{\delta}\right)^t < 1 + \frac{\beta}{\xi\delta}[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i] \text{ for all large } t. \tag{D.3}$$

This cannot be satisfied for large even t if $\frac{|\gamma|}{\delta} = 1$ and $\beta \leq 0$. If $\beta > 0$ then $\sum_{i=0}^{t}\left(\frac{\gamma}{\delta}\right)^i = 0$ for all odd t and $\sum_{i=0}^{t}\left(\frac{\gamma}{\delta}\right)^i = 1$ for all even t. Thus $B_t = \frac{1}{\xi\delta}\frac{\left(\frac{\gamma}{\delta}\right)^t}{1+\frac{\beta}{\xi\delta}[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]}$ fluctuates between $\frac{1}{\xi\delta}$ and $\frac{1}{\xi\delta}\frac{1}{1+\frac{\beta}{\xi\delta}}$, violating Lemma D.8(ii).

Case ii: $\frac{|\gamma|}{\delta} \neq 1$ and $\beta \geq 0$

If $\frac{|\gamma|}{\delta} > 1$ then the term $\frac{\beta}{\xi\delta}[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]$ is negative for all even t, and thus, for all large even t, while the left hand side of (D.3) is strictly larger than 1, the right hand side is strictly less than 1, a contradiction.

If $\frac{|\gamma|}{\delta} < 1$ then $[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]$ is positive and increasing *for all* t. Consequently, since $\gamma < 0$, B_t is negative for some odd t, contradicting Lemma D.8(i).

Case iii: $\frac{|\gamma|}{\delta} \neq 1$ and $\beta < 0$

If $\frac{|\gamma|}{\delta} < 1$ then $[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]$ is positive for all t, increasing for all odd t, and increasing for all even t. Moreover, the limit is the same regardless of whether we consider the subsequence corresponding to all odd t or that corresponding to all even t. If the limit of $1 + \frac{\beta}{\xi\delta}[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]$ is non-negative (resp. negative), then $1 + \frac{\beta}{\xi\delta}[\sum_{i=0}^{t-1}\left(\frac{\gamma}{\delta}\right)^i]$ is positive for some odd t (resp. negative for some even t) and B_t is negative, contradicting Lemma D.8(i).

Suppose $\frac{|\gamma|}{\delta} > 1$. Then

$$B_t = \frac{1}{\xi\delta}\frac{\left(\frac{\gamma}{\delta}\right)^t}{1+\frac{\beta}{\xi\delta}\frac{1-\left(\frac{\gamma}{\delta}\right)^{t-1}}{1-\frac{\gamma}{\delta}}} = \frac{1}{\xi\delta}\frac{1}{\frac{1}{\left(\frac{\gamma}{\delta}\right)^t}+\frac{1}{\left(\frac{\gamma}{\delta}\right)^t}\frac{\beta}{\xi\delta}\frac{1}{1-\frac{\gamma}{\delta}}-\frac{\beta}{\xi\delta}\frac{\left(\frac{\gamma}{\delta}\right)^{-1}}{1-\frac{\gamma}{\delta}}} \to \frac{1}{\xi\delta}\frac{1}{-\frac{\beta}{\xi\delta}\frac{\left(\frac{\gamma}{\delta}\right)^{-1}}{1-\frac{\gamma}{\delta}}} = \frac{\gamma}{\delta\beta}(\frac{\gamma}{\delta}-1)$$

Denote the limit by $B > 0$. Since W,$\widehat{V}$ are nonconstant and affinely independent, B implies that $W + B\widehat{V}$ and W are nonconstant and ordinally distinct. Thus there is μ, η where $W(\mu) + B\widehat{V}(\mu) < W(\eta) + B\widehat{V}(\eta)$ and $W(\mu) > W(\eta)$. But then by (D.2) we have $\mu^{+t} < \eta^{+t}$ for all large t but $W(\mu) > W(\eta)$, violating Lemma C.3(c).

Step 2: $\frac{\gamma}{\delta} \leq 1$.

If $\beta \leq 0$, then $\frac{\gamma}{\delta} \geq 1$ contradicts (D.3). If $\beta > 0$, then suppose by way of contradiction that $\frac{\gamma}{\delta} > 1$. As at the end of case iii above, show that $B_t \to \frac{\gamma}{\delta\beta}(\frac{\gamma}{\delta} - 1) := B > 0$ and argue similarly to establish a contradiction. ■

Lemma D.10. *Under Axioms 1-9, for all $x \in Z$,*

$$\widehat{V}(x) = \beta W(x) + \gamma \max_{\eta \in x} V(\eta) - \theta.$$

Moreover, $\xi = 1$ and thus $\mathcal{C}(x) = \arg\max_{\mu \in x}\{U(\mu) + V(\mu)\}$ for all x.

Proof. If $\widehat{V}$ is constant or an affine transformation of W, the result holds with $\gamma = 0$. Henceforth suppose that $\widehat{V}$ is nonconstant and not an affine transformation of W. To ease notation let $\alpha := \xi(\lambda - \delta) - \beta \geq 0$ and $\alpha' := (\lambda - \gamma) \geq 0$. Also denote the $\gtrsim$-ranking of t-period menus by the binary relation $\gtrsim^t$ over Z, that is, $x \gtrsim^t y \Longleftrightarrow x^{+t} \gtrsim y^{+t}$ for $t > 0$. As in Lemma D.8, for any $t > 0$, the preference $\gtrsim^t$ is represented by the function:

$x \mapsto W(x) + B_{t-1}\widehat{V}(x) =$

$$\begin{aligned}&(1 - B_{t-1}\xi\delta) \max_{\mu \in x}\{U(\mu) + V(\mu)\} - (1 - B_{t-1}\xi\delta) \max_{\eta \in x} V(\eta)\\ &+B_{t-1}\lambda \max_{\mu \in x}\{\xi U(\mu) + V(\mu)\} - B_{t-1} \max_{\mu \in x}\{\alpha U(\mu) + \alpha' V(\mu)\} - \theta\end{aligned}$$

Consider the following cases:

Case A: ($\gamma = 0$). In this case $B_t = 0$ for all $t > 0$, and so $\gtrsim^{t+1}$ is represented by W for all $t > 0$ (note that $B_0 = \frac{1}{\xi\delta}$). By Sophistication and Theorem 6.1, $\xi = 1$. If $\alpha = 0$ then $\beta = (\lambda - \delta)$ and from the structure in Lemma D.7 we get

$$\widehat{V}(x) = (\lambda - \delta) \max_{x}\{U + V\} + (\delta - \lambda) \max_{\eta \in x} V(\eta) - \theta = \beta W - \theta.$$

We show next that $\alpha > 0$ is impossible. Note that $\gtrsim^1$ and $\gtrsim^2$ are respectively represented by

$$x \mapsto \max_{x}\{U + V\} - \max_{x}\{\frac{\alpha}{\lambda}U + V\} \quad \text{and} \quad x \mapsto \max_{x}\{U + V\} - \max_{x} V.$$

Since $\frac{\alpha}{\lambda} > 0$ and U, V are affinely independent, there exists μ, η s.t. $U(\mu) + V(\mu) > U(\eta) + V(\eta)$, $\frac{\alpha}{\lambda}U(\mu) + V(\mu) > \frac{\alpha}{\lambda}U(\eta) + V(\eta)$ and $V(\mu) < V(\eta)$.[24] It

[24] Argue as in the proof of Lemma D.8(ii) to show that there exists $\mu', \eta', \mu'', \eta''$ s.t. $U(\mu') = U(\eta'), V(\mu') < V(\eta'), U(\mu'') > U(\eta'')$, and $V(\mu'') = V(\eta'')$, and that for any θ, $\frac{U(\mu'\theta\mu'')-U(\eta'\theta\eta'')}{V(\eta'\theta\eta'')-V(\mu'\theta\mu'')} = \frac{(1-\theta)}{\theta}\frac{U(\mu'')-U(\eta'')}{V(\eta')-V(\mu')} := f(\theta)$. Then choosing θ s.t. $f(\theta) > \max\{1, \frac{\lambda}{\alpha}\}$ gives rise to the desired $\mu = \mu'\theta\mu''$ and $\eta = \eta'\theta\eta''$.

follows from the representations that $\{\mu\}^{+1} \approx \{\mu, \eta\}^{+1}$ and $\{\mu\}^{+2} > \{\mu, \eta\}^{+2}$, and hence $\tau(\{\mu\}^{+1}, \{\mu, \eta\}^{+1}) \neq 0$. We show that $\tau(\mu, \eta) = 0$, thereby establishing a contradiction to Menu-Reversal(i): Note that $U(\mu) + V(\mu) > U(\eta) + V(\eta)$ and $\xi \geq 1$ implies by Lemma D.6 that $\mu > \eta$. Moreover, since $U(\mu) > U(\eta)$, Lemma C.3(c) then implies $\tau(\mu, \eta) = 0$, as desired.

Case B: ($\gamma > 0$). In this case $B_t > 0$ for all t, and from Lemma D.8(ii) we know that $1 - B_t\xi\delta > 0$ for all large t. Consider two possibilities:

Case B(i): $\alpha U + \alpha' V$ is a positive affine transformation of $\xi U + V$.

First suppose $\xi = 1$. Then $(\lambda - \delta) - \beta = (\lambda - \gamma)$ and therefore, $\beta = -\delta + \gamma$. Moreover,

$\widehat{V}(x) = \lambda \max_x\{U+V\} - \delta \max_x\{U+V\} + \delta \max_x V - (\lambda-\gamma) \max_x\{U+V\} - \theta$
$= \beta W(x) + \gamma \max_x V - \theta$, as desired.

Next we show that $\xi > 1$ is impossible. First note that

$$B_t\lambda \max_{\mu\in x}\{\xi U(\mu) + V(\mu)\} - B_t \max_{\mu\in x}\{\alpha U(\mu) + \alpha' V(\mu)\} \neq 0.$$

For if equality holds, then $\alpha U + \alpha' V = \lambda\xi U + \lambda V$ and the fact that U and V are affinely independent then imply $\alpha' = \lambda$, and thus $(\lambda - \gamma) = \lambda$, which is not possible given the hypothesis $\gamma > 0$. Thus, it must be that for some $k \neq 0$,

$$B_t\lambda \max_{\mu\in x}\{\xi U(\mu) + V(\mu)\} - B_t \max_{\mu\in x}\{\alpha U(\mu) + \alpha' V(\mu)\} = k \max_{\mu\in x}\{\xi U(\mu) + V(\mu)\}.$$

Note that $\xi U + V$ is neither an affine transformation of $U + V$ or V, as U and V are affinely independent. We show that Set-Betweenness must be violated.

Since the set of simple lotteries (lotteries with finite support) is a dense subset of Δ, we can find a finite set of simple lotteries on which $U + V$, V and $\xi U + V$ are nonconstant and mutually affinely independent (for each pair of functions take two lotteries which the functions rank equivalently and two which they rank differently). Denote the (finite) union of the (finite) supports of these simple lotteries by $A \subset C \times Z$. Viewing A as just some finite set, one can then restrict attention to the subdomain of menus consisting of nonempty compact subsets of $\Delta(A)$ and apply the argument in [5, Lemma 1] to establish a violation of Set-Betweenness.

Case B(ii): $\alpha U + \alpha' V$ is not a positive affine transformation of $\xi U + V$.

First we show that $\xi > 1$ is impossible. The argument is similar to that in the previous case. Depending on whether $\alpha U + \alpha' V$ is a positive affine transformation of V or $U + V$ or neither, we have either two distinct positive states or two distinct negative states. In either case, since the distinct states are mutually affinely independent, the argument in the previous case yields a contradiction to Set-Betweenness.

Next consider the case $\xi = 1$. If $\alpha U + \alpha' V$ is a positive affine transformation of V then the fact that U and V are affinely independent implies $\alpha = 0$ and thus $(\lambda - \delta) - \beta = 0$ and therefore $\beta = \lambda - \delta$. Moreover,

$\widehat{V}(x) = \lambda \max_x\{U + V\} - \delta \max_x\{U + V\} + \delta \max_x V - \max_x(\lambda - \gamma)V - \theta$

$= \beta W(x) + \gamma \max_x V - \theta$, as desired.

On the other hand if $\alpha U + \alpha' V$ is not a positive affine transformation of V then we have a case with one positive state and two distinct negative states. Arguing as at the end of case B(i) yields a contradiction to Set-Betweenness. ■

Lemma D.11. *$\beta > \gamma - \delta$.*

Proof. From the representation, we see that the induced representation for the ranking $\succsim^1$ of next-period menus is

$\delta W(x) + \widehat{V}(x) = (\delta + \beta)W(x) + \gamma \max_x V$

$= (\delta + \beta) \max_x\{U + V\} - (\delta + \beta - \gamma) \max_x V.$

Set-Betweenness, nondegeneracy and $\gamma \geq 0$ implies $(\delta+\beta) > 0$ and $(\delta+\beta-\gamma) > 0$. ■

We have shown $0 \leq \gamma \leq \delta$ and $\beta > \gamma - \delta$; $\succsim$ is represented by $U + V$; for all c, x,

$$V(c, x) = v(c) + \widehat{V}(x) = v(c) + [\beta W(x) + \gamma \max_{\eta \in x} V(\eta)] - \theta.$$

We can take $\theta = 0$ wlog since temptation utility appears in both a negative and positive term in the function W. This completes the proof.

E. Appendix: Proof of Thm 4.3

Consider the set $\mathcal{F}$ of continuous linear functions on Δ. The sup norm $||\cdot||$ makes $\mathcal{F}$ a Banach space. Endow $\mathcal{F}\times\mathcal{F}$ with the norm defined by $||\ (F,G)\ ||=||\ F\ ||$ $+\ ||\ G\ ||$, for all $(F,G)\in\mathcal{F}\times\mathcal{F}$. Then $\mathcal{F}\times\mathcal{F}$ is a Banach Space. For $a=u,v$, let $\overline{c_a}$ and $\underline{c_a}$ denote the a-best and a-worst consumption in C, and define:

$$\overline{U}=\sum_{t=0}^{\infty}\delta^t u(\overline{c_a}),\qquad \overline{V}=\sum_{t=0}^{\infty}\gamma^t v(\overline{c_v})+\beta\sum_{t=0}^{\infty}D(t)u(\overline{c_u})$$

where $D(t+1)=\delta^t\sum_{i=0}^{t}(\frac{\gamma}{\delta})^i$, adopting the convention that $\sum_{i=0}^{0}=1$ and $\sum_{i=0}^{-1}=0$.

Let $\mathcal{F}_U=\{U\in\mathcal{F}\colon\ \underline{U}\leq U\leq\overline{U}\}$ and similarly $\mathcal{F}_V=\{V\in\mathcal{F}\colon\ \underline{V}\leq V\leq\overline{V}\}$. Define

$$X=\mathcal{F}_U\times\mathcal{F}_V.$$

The first lemma establishes compactness of X when $\mathcal{F}\times\mathcal{F}$ has the *weak topology* (induced by the norm dual of $\mathcal{F}\times\mathcal{F}$):

For any subset V of a normed vector space, denote by V^* the norm dual of V, the set of all norm-continuous and linear functionals on V. The weak topology on V is the weakest topology for which all the functionals in V^* are continuous. A net v_α in V converges weakly (ie, wrt the weak topology) to v if and only if $f(v_\alpha)\to f(v)$ for all $f\in V^*$.

Lemma E.1. *X is a nonempty, bounded, convex, compact subset of $\mathcal{F}\times\mathcal{F}$ in the weak topology.*

Proof. Step 1: Show that the weak topology of a product is the product of weak topologies.

We need to show that the weak topology on $\mathcal{F}\times\mathcal{F}$ is identical to the product topology on $\mathcal{F}\times\mathcal{F}$ with the weak topology on $\mathcal{F}$. The key observation is that any norm-continuous linear functional $L\in(\mathcal{F}\times\mathcal{F})^*$ can be written as a sum of norm-continuous linear functionals $l,l'\in\mathcal{F}^*$, that is, $L(f,g)=l(f)+l'(g)$. This follows from linearity and the fact that, for any fixed f',g', $(f,g)+(f',g')=(f,g')+(f',g)$.

Take a net (f_α, g_α) in $\mathcal{F} \times \mathcal{F}$. If $(f_\alpha, g_\alpha) \to (f, g)$ in the weak topology then $L(f_\alpha, g_\alpha) \to L(f, g)$ for every norm-continuous linear function $L \in (\mathcal{F} \times \mathcal{F})^*$. By taking all L that are constant either in the first or second argument, this implies that $f_\alpha \to f$ and $g_\alpha \to g$ in the weak topology on $\mathcal{F}$. Thus, $(f_\alpha, g_\alpha) \to (f, g)$ in the product topology. To prove the converse, suppose $(f_\alpha, g_\alpha) \to (f, g)$ in the product topology. Then $l(f_\alpha) \to l(f)$ and $l'(g_\alpha) \to l'(g)$ for all $l, l' \in \mathcal{F}^*$. By our observation above, it follows that $L(f_\alpha, g_\alpha) \to L(f, g)$ for all $L \in (\mathcal{F} \times \mathcal{F})^*$, and thus $(f_\alpha, g_\alpha) \to (f, g)$ weakly.

Step 2: Prove the lemma.

Nonemptiness and convexity are evident. (Norm) boundedness follows from the Banach-Steinhaus theorem. To establish weak compactness, by step 1 it suffices to show that $\mathcal{F}_U$ and $\mathcal{F}_V$ are weakly compact subsets of $\mathcal{F}$. We prove this for $\mathcal{F}_U$ below, and an anologous argument implies it for $\mathcal{F}_V$.

Convexity and boundedness, and the fact that Δ is compact metric and that a convex subset of a locally convex linear topological space is weakly closed if and only if it is closed [6, Thm V.3.13, pg 422] together imply the following statement: $\mathcal{F}_U$ is compact in the weak topology on $\mathcal{F}$ if $\mathcal{F}_U$ is a compact set of continuous linear functions when $\mathcal{F}$ has the topology of pointwise convergence [6, Thm IV.6.14, pg 269].[25]

To this end, begin by isometrically embedding Δ in the linear space $ca(C)$ of finite Borel signed measures on C, normed by the total variation norm.[26] Let $\mathcal{F}^e$ denote the set of all linear and continuous functions on $ca(C)$, and endow it with the topology of pointwise convergence. Define $\mathcal{F}_U^e = \{U \in \mathcal{F}^e : \underline{U} \leq U \leq \overline{U}\}$. Given [1, Corollary 6.23, pg 248], the pointwise limit of a sequence f_n in $\mathcal{F}_U^e$ lies in $\mathcal{F}_U^e$. Thus $\mathcal{F}_U^e$ is closed in topology of pointwise convergence on $\mathcal{F}^e$. But

[25] [6, Thm IV.6.14, pg 269] states that for a set F of continuous functions on a compact hausdorff space, the weak closure of F is weakly compact iff F is norm-bounded and its closure in the pointwise convergence topology is a compact set of continous functions in this topology.

[26] Since C is compact, $ca(C)$ is isometrically isomorphic to the topological dual $B(C)^*$ of the space $B(C)$ of continuous functions on C (normed by sup-norm). Use this duality and endow $ca(C)$ with weak*-topology $\sigma(B(C)^*, B(C))$. This topology induces the topology of weak convergence on Δ.

$\mathcal{F}_U^e \subset [\underline{U}, \overline{U}]^{ca(C)}$, that is, $\mathcal{F}_U^e$ is a closed subset of a compact set, and thus itself compact.

Finally, show that $\mathcal{F}_U$ is compact when $\mathcal{F}$ has the topology of pointwise convergence. Define the function $\Theta : \mathcal{F}^e \rightarrow \mathcal{F}$ by $\Theta(f) = f|_\Delta$, the restriction of f to Δ. It is obvious that Θ is continuous. Therefore compactness of $\mathcal{F}_U^e$ in $\mathcal{F}^e$ implies compactness of $\mathcal{F}_U$ in $\mathcal{F}$. This completes the proof. ■

Define the function $\Gamma : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ by

$$\Gamma(U,V)(\mu) = \begin{pmatrix} \int_{C\times Z}(u(c) + \delta\,[\max_{\nu\in x}\{U+V\} - \max_{\upsilon\in x} V])d\mu(c,x), \\ \int_{C\times Z}(v(c) + [\beta\max_{\nu\in x}\{U+V\} + (\gamma-\beta)\max_{\upsilon\in x} V(\eta)])d\mu(c,x) \end{pmatrix}.$$

Lemma E.2. $\Gamma|_X$ *is a self-map on* X.

Proof. Take any $(U,V) \in X$. Write $\Gamma(U,V) = (\Gamma U, \Gamma V)$. The Maximum theorem implies that $(\Gamma U, \Gamma V)$ is a pair of continuous functions. Linearity is evident. The fact that $\underline{U} \leq \Gamma(U) \leq \overline{U}$ is readily determined, given the GP functional form used. To see that $\underline{V} \leq \Gamma(V) \leq \overline{V}$, observe that $\beta\max_{\nu\in x}\{U + V\} + (\gamma-\beta)\max_{\upsilon\in x} V(\eta) = \beta W(x) + \gamma\max_{\upsilon\in x} V(\eta)$, and thus

$$\begin{aligned}
&\Gamma(V) \leq v(\overline{c_v}) + \beta\overline{U} + \gamma\overline{V} \\
&= v(\overline{c_v}) + \beta\textstyle\sum_{t=0}^{\infty}\delta^t u(\overline{c_a}) + \gamma\left[\sum_{t=0}^{\infty}\gamma^t v(\overline{c_v}) + \beta\sum_{t=0}^{\infty}[\delta^{t-1}\sum_{i=0}^{t-1}(\frac{\gamma}{\delta})^i]u(\overline{c_u})\right] \\
&= v(\overline{c_v}) + \textstyle\sum_{t=1}^{\infty}\gamma^t v(\overline{c_v}) + \beta u(\overline{c_a})\left[\sum_{t=0}^{\infty}\delta^t + \gamma\sum_{t=0}^{\infty}[\delta^{t-1}\sum_{i=0}^{t-1}(\frac{\gamma}{\delta})^i]\right] \\
&= \textstyle\sum_{t=0}^{\infty}\gamma^t v(\overline{c_v}) + u(\overline{c_a})\beta\sum_{t=0}^{\infty}[\delta^t + \gamma\delta^{t-1}\sum_{i=0}^{t-1}(\frac{\gamma}{\delta})^i] \\
&= \textstyle\sum_{t=0}^{\infty}\gamma^t v(\overline{c_v}) + u(\overline{c_a})\beta\sum_{t=0}^{\infty}[\delta^t + \delta^t\sum_{i=1}^{t}(\frac{\gamma}{\delta})^i] \\
&= \textstyle\sum_{t=0}^{\infty}\gamma^t v(\overline{c_v}) + u(\overline{c_a})\beta\sum_{t=0}^{\infty}[\delta^t\sum_{i=0}^{t}(\frac{\gamma}{\delta})^i] \\
&= \textstyle\sum_{t=0}^{\infty}\gamma^t v(\overline{c_v}) + u(\overline{c_u})\beta\sum_{t=0}^{\infty}D(t+1) =^* \overline{V}
\end{aligned}$$

where $=^*$ follows from the fact that $\sum_{t=0}^{\infty}D(t+1) = \sum_{t=0}^{\infty}D(t)$ since $D(0) = \delta^{-1}\sum_{i=0}^{-1}(\frac{\gamma}{\delta})^i = 0$. An analogous argument yields $\underline{V} \leq \Gamma(V)$. Thus $\Gamma(U,V) \in X$. ■

Lemma E.3. $\Gamma|_X$ *is continuous with respect to the weak topology.*

Proof. By [1, Thm 6.21, pg 247], norm-to-norm continuity of a function between two normed spaces is equivalent to weak-to-weak continuity. Below we establish that Γ is sup-norm continuous.[27] It then follows that Γ, and in turn $\Gamma|_X$, is weakly continuous.

Begin with a preliminary observation. Take any $f \in \mathcal{F}$ and consider the problem $\sup_{\mu\in\Delta} \left|\int(\max_{\upsilon\in x} f)d\mu(c,x)\right|$. Note that the objective function is insensitive to any c yielded by μ. Since $x \mapsto \max_{\upsilon\in x} f$ is continuous, the sup is achieved as a max for some μ. Because of the expected utility form, μ is degenerate on some (c,x) wlog. Denoting the maximizer of f in x^* by η_x, we see therefore that

$$\sup_{\mu\in\Delta}\left|\int(\max_{\upsilon\in x} f)d\mu(c,x)\right| = \sup_{(c,x)\in\Delta}\left|\max_{\upsilon\in x} f\right| = \sup_{(c,x)\in\Delta}|f(\eta_x)| = \sup_{\eta\in\Delta}|f(\eta)| =\| f \| .$$

Thus the sup is in fact the norm of f.

To prove the lemma, suppose $\| U_n - U \| \to 0$ and $\| V_n - V \| \to 0$. Take $U = V = 0$ wlog. Write $\Gamma(U_n, V_n)$ as $(\Gamma U_n, \Gamma V_n)$. Observe that

$\| \Gamma(U_n, V_n) - \Gamma(0,0) \|$

$=\| \Gamma U_n - \Gamma 0 \| + \| \Gamma V_n - \Gamma 0 \|$

$=\| \int_{C\times Z} \delta\,[\max_x\{U_n + V_n\} - \max_x V_n])d\mu(c,x) \|$

$+ \| \int_{C\times Z} [\beta \max_{\nu\in x}\{U+V\} + (\gamma-\beta)\max_{\upsilon\in x} V(\eta)])d\mu(c,x) \|$. Using the triangle inequality and the observation above, we see that $\| \Gamma(U_n, V_n) - \Gamma(0,0) \| \to 0$. Thus Γ is sup-norm continuous. ■

To complete the proof, we invoke the Brouwer-Schauder-Tychonoff fixed point theorem [6, Thm V.10.5, pg 456], which states that a continuous self-map on a compact convex subset of a locally convex linear topological space has a nonempty set of fixed points.

F. Proof of Theorems 4.2, 4.4-4.6

See the Supplementary Appendix.

[27] By [1, Thm 6.30, pg 252], the weak and norm topologies on a finite dimensional space coincide. Since the range of Γ is $\mathbb{R}$, we need to be concerned only with the topology on its domain.

G. Appendix: Proof of Thm 4.7

Let $\gtrapprox$ be the preference relation that is represented by $\varphi : \Delta \to \mathbb{R}$ defined by $\varphi(\mu) = U(\mu) + V(\mu)$ for all $\mu \in \Delta$. Given Theorem 4.2 we can assume $U, V \geq 0$. For each $t > 0$, define $\gtrapprox_t$ on Δ by $\mu \gtrapprox_t \eta \iff \mu^{+t} \gtrapprox \eta^{+t}$. We saw in (6.2) that $\gtrapprox_t$ is represented by the function $\varphi_t : \Delta \to \mathbb{R}$ defined by $\varphi_t(\mu) = U(\mu) + D_t V(\mu)$ for all $\mu \in \Delta$, where $D_t \searrow 0$.

Lemma G.1. *The sequence $\{\varphi_t\}$ uniformly converges to U.*

Proof. The sequence $\{\varphi_t\}$ is a sequence of continuous real functions defined on a compact space Δ. Since $D_t \searrow 0$, the sequence is monotone (decreasing) and φ_t converges pointwise to the continuous function U. Therefore, by Dini's Theorem [1, Theorem 2.62], the convergence is uniform. ■

Since U is nonconstant, there exists $\rho, \nu \in \Delta$ s.t. $U(\rho) > U(\nu)$. By linearity of U,

$$U(\mu) \geq U(\eta) \Longrightarrow U(\mu\alpha\rho) > U(\eta\alpha\nu), \text{ for all } \alpha \in (0,1)\,. \tag{G.1}$$

This observation will be used in the next lemma. Let $\succsim_U$ be the preference relation represented by U. As in Appendix B, identify any binary relation B on Δ with its graph $\Gamma(B) \subset \Delta \times \Delta$.

Lemma G.2. $\succsim_U = \succsim^*$.

Proof. As $\lim_{t\to\infty} \Gamma(\gtrapprox_t) \equiv \Gamma(\succsim^*)$, it suffices to show that $\Gamma(\succsim_U) = \lim_{t\to\infty} \Gamma(\gtrapprox_t)$. First establish $Ls\Gamma(\gtrapprox_t) \subset \Gamma(\succsim_U)$. If $(\mu, \eta) \in Ls\Gamma(\gtrapprox_t)$ then there is a subsequence $\{\Gamma(\gtrapprox_{t(n)})\}$ and a sequence $\{(\mu_n, \eta_n)\}$ that converges to (μ, η) such that $(\mu_n, \eta_n) \in \Gamma(\gtrapprox_{t(n)})$ for each n. Therefore, for each n, $\varphi_{t(n)}(\mu_n) \geq \varphi_{t(n)}(\eta_n)$. Since $\varphi_{t(n)}$ converges to U uniformly, it follows that $U(\mu) \geq U(\eta)$. Hence $(\mu, \eta) \in \Gamma(\succsim_U)$, as desired.

Next establish $\Gamma(\succsim_U) \subset Li\Gamma(\gtrapprox_t)$. Let $(\mu, \eta) \in \Gamma(\succsim_U)$ and take any neighborhood V of (μ, η). By (G.1), there exists $\alpha \in (0, 1]$ s.t. $(\mu\alpha\rho, \eta\alpha\nu) \in V$ and $U(\mu\alpha\rho) > U(\eta\alpha\nu)$. By Lemma G.1, there exists $T < \infty$ such that $\varphi_t(\mu\alpha\rho) >$

$\varphi_t(\eta\alpha\nu)$ for all $t \geq T$, that is, $(\mu\alpha\rho, \eta\alpha\nu) \in \Gamma(\gtrsim_t)$ for all $t \geq T$. Hence, $V \cap \Gamma(\gtrsim_t) \neq \phi$ for all but a finite number of t, that is, $(\mu, \eta) \in Li\Gamma(\gtrsim_t)$. This completes the proof. ■

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