Removed Preferences*

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Abstract

In formalizing a ‘veil of ignorance’ type procedure, this paper considers how an agent’s preferences over a set of alternatives change as he is placed at an increasing ‘distance’ from the consequences of his choices. A definition for such ‘removed preferences’ is presented and its properties studied. As an application, it is demonstrated that decreasingly impatient agents are ‘essentially’ exponential when distanced from the present, and that rank-dependent expected utility agents are ‘essentially’ expected utility when distanced from risk.

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1 Introduction

We consider how an agent's preferences over a set of alternatives change as he is increasingly *distanced* from the consequences of his choices. This paper formalizes and studies the notion of 'preferences from a distance', referring to it as a *removed preference.*

Motivation: A recent source of motivation for formally studying removed preference comes from the recent debate in welfare economics. Economics has traditionally adopted revealed preference as a guide for welfare policy. The hypothesized choice-welfare connection has had its early critics and there is renewed criticism due to the increased attention being paid to findings in psychology (such as mistakes, biases, anchoring, framing effects, impulse control problems) which has been accompanied with developments in descriptive theories of behavior. Researchers have been led to question whether revealed preference is an appropriate guide for welfare policy in a class of new models that study non-rational or constrained-rational behavior. In light of this, removed preferences are of potential interest from the point of view of non-standard welfare analysis.

Specifically, 'veil of ignorance' type reasoning (Rawls [23], Harsanyi [9]) is based on the idea that evaluations that have immediate consequences have less normative significance than those that have more distant (less immediate or certain) consequences for the agent. According to Rawls [23, pg 136], the veil's purpose is to "nullify the effect of specific contingencies which put men at odds and tempt them." Thus, an agent's preferences at time t about consumption at time $t + n$ may have some primacy over preferences at time $t+n$ (Ainslie [1], O'Donoghue and Rabin [20]). Alternatively, the preferences at time t over $t+n$ consumption may reflect the agent's normative assessment of his own choices at time t – an agent abusing drugs may normatively assess his current choice to be bad (such as if he finds himself overwhelmed by temptation), and reveal this by a preference for avoiding such choices in the future. However, despite the potential relevance of removed preference for welfare, this paper studies removed preferences without committing to any position on its welfare significance, leaving such evaluations to the reader.

A second source of motivation comes from the potential usefulness of removed preference in the axiomatization of decision models. Here the removed preference need have no welfare significance – it is just a mathematical object through which a representation may be identified. For instance, consider the following intertemporal choice model of preferences over (infinite horizon) consumption streams,

$$W(c_0, c_1, ...) = u(c_0) + \sum_{n=1}^{\infty}(\delta^n + \gamma^n)u(c_n), \qquad \delta > \gamma,$$

where discounting is non-exponential and exhibits negative time preference over part of the horizon when $\delta + \gamma > 1$. The latter may arise perhaps due to the effect of anticipation, which may drive an agent to postpone desirable consumption or expedite undesirable consumption (Loewenstein [14]). For any consumption vector $\mathbf{c}$ define $\mathbf{c}^{+t}$ as the vector that yields 0 in the first t periods and then pays according to the schedule $\mathbf{c}$ in subsequent periods. Then preferences over streams $\mathbf{c}$ from a distance of t are represented by $W(\mathbf{c}^{+t})$, which is ordinally equivalent to $\frac{W(\mathbf{c}^{+t})}{\delta^t}$,

$$\begin{aligned}\frac{W(\mathbf{c}^{+t})}{\delta^t} &= \frac{1}{\delta^t}\sum_{n=0}^{\infty}(\delta^{n+t} + \gamma^{n+t})u(c_n) \\ &= \sum_{n=0}^{\infty}\left[\delta^n + \gamma^n(\frac{\gamma}{\delta})^t\right]u(c_n).\end{aligned}$$

Since $\delta > \gamma$, we see that $\lim_{t\to\infty}\frac{W(\mathbf{c}^{+t})}{\delta^t} = \sum_{n=0}^{\infty}\delta^n u(c_n)$. That is, from an 'infinite distance', the agent ranks consumption streams according to the exponential discounting model. This observation then can serve as a starting point for providing foundations for such a model – it isolates an aspect of preference that must satisfy standard axioms and it identifies δ and u. For a more involved application of removed preferences, see the axiomatization of a model of temptation in Noor [19].

This paper: We define removed preferences in an abstract setting and formulate it in entirely choice-theoretic terms. Our primitive consists of a sequence of preferences $\{\succsim_n\}_{n=0}^{\infty}$ on a common domain $\mathcal{A}$, where n is some measure of distance. We define the removed preference as the set-theoretic limit inferior of the sequence – thus a is removed preferred to b if $a \succsim_n b$ for all large n. We study its properties and also those of related notions in an abstract setting. We then present two applications. Our first application is on time preference. Intertemporal choice experiments reveal that agents' discount functions are not exponential, but rather exhibit a property known as decreasing impatience. With distance defined as temporal distance and considering a general nonexponential discounting model, we show that decreasing impatience implies that agents' removed preferences 'essentially' feature exponential discounting, in a sense made precise.[1] Our second application is on risk preference. Experiments on choice under risk reveal particular violations of the vNM Independence axiom, such as the Allais paradox, which suggest nonlinear weighting of probabilities. Research also suggests that risk has visceral affects on decision makers, such as causing anxiety or dread. If distance from risk is to be understood as proximity to certainty, then distance can be achieved by scaling down the probabilities of non-zero outcomes. We find that in the rank-dependent utility model (of which prospect theory is a special case), the removed preference must 'essentially' be expected utility.

The paper is organized as follows. Sections 2 presents a definition of removed preferences. Section 3 studies the properties of removed preferences and its continuous extension under some basic restrictions. Section 4 describes an application that derives removed preferences from a model of intertemporal choice, and Section 5 does the same for a model of risk. All proofs are contained in appendices.

2 Removed Preferences

The idea of distancing is familiar to philosophers and psychologists, and is even part of common wisdom. For instance, when trying to demonstrate to a friend that his smoking is in fact against his own better judgment, we try to get him to view the act of smoking from a distance by asking him how he would feel about his children smoking. In philosophy, the 'original position' theories of justice (such as Rawls [23] and Harsanyi [9, 10]) derive an agent's notion of a 'just' social allocation by eliciting his views only after placing him behind a 'veil of ignorance' that distances him from his own personal identity, or his position in society. According to Rawls [23, pg 136], the veil's purpose is to "nullify the effect of specific contingencies which put men at odds and tempt them." According to Harsanyi [10, pg 316], an agent's choices

[1] Specifically, we show that a continuous extension of the removed preference, which we call the *closed removed preference*, must admit a representation with exponential discounting. We then show that the removed preference agrees with the closed removed preference on a dense set of pairs of alternatives.

reflect his 'ethical preferences' if they "indicate what social situation he would choose if he did not know what his personal position would be in the new situation chosen". Experiments in psychology have shown that a desire for immediate gratification is resisted more effectively when the object offering temptation is out of sight, and thus distant (Mischell and Ebbesen [18]). Psychologists such as Ainslie [1], Rachlin [21] and Rachlin and Green [22] have also argued that changes in behavior induced by temporal distancing, such as those studied in the literature on preference reversals and dynamic inconsistency (see Fredrick, Loewenstein and O'Donoghue [6] for a survey of the evidence), reveal that choices among temporally close alternatives were subject to a desire for immediate gratification. Ainslie [2] used the term 'long-run preference' to describe preferences from a (temporal) distance, in the absence of a desire for immediate gratification, and O' Donoghue and Rabin [20] formulated a welfare criterion for the quasi-hyperbolic discounting model in terms of preferences in a fictional, ex ante (thus distant) period.[2]

In this section we formalize the notion of 'preference from a distance'. The actual notion of distance used necessarily depends on the context. When dealing with temptation, for instance, temporal distance appears to be natural since introspection confirms that immediate temptations are harder to resist than temptations that lie in the future. One of the applications in this paper makes use of this. Another application, in a risk context, defines distance from risk in terms of proximity to certainty. Another possible notion relevant for temptation is that of 'availability': an item may be more tempting the more easily available it is, and so, the more distant an item is in terms of availability, the less tempting it may be. Finally, to the extent that a desire to look good in others' eyes may lead one to behave in a manner different from one's 'true' preference, the degree of anonymity of one's choices may serve as a notion of distance from the social context.

[2] Despite the intuitive appeal of the notion of distancing, one may argue for other welfare notions. For instance, in a multiple selves model, one may argue that some aggregation of the preferences of the multiple selves is an appropriate welfare criterion (Green and Hojman [7]), even if distancing isolates the preferences of a single self. We refrain from a philosophical argument, noting only that the notion of distance seems to have sufficient appeal on its own to deserve formal study.

2.1 Formalization

Let $\mathcal{A}$ be a set of alternatives. Let distance be some abstract notion – examples of distance were discussed in the previous section and are studied in our applications in the sequel. Let the "degree" n of distance be captured by the set of non-negative integers $\mathbb{N} \cup \{0\}$. Larger n represent greater degrees of distance. Suppose that the agent's choices at some fixed point in time are summarized by $\{\succsim_n\}_{n=0}^{\infty}$, a set of preference relations defined over $\mathcal{A}$. Each preference $\succsim_n$ ranks alternatives in $\mathcal{A}$ when these alternatives are distant by n degrees.[3] Then the agent's *removed preference* may be identified with a suitable limit point of the sequence $\{\succsim_n\}_{n=0}^{\infty}$.

To be formal, identify any binary relation B on $\mathcal{A}$ by identifying a binary relation B on $\mathcal{A}$ the graph $\Gamma(B) = \{(\mu, \eta) \in \mathcal{A} \times \mathcal{A} : \mu B \eta\}$. Then a binary relation is a subset of $\mathcal{A} \times \mathcal{A}$, and the space of all such binary relations is $2^{\mathcal{A} \times \mathcal{A}}$.[4] We claim that a natural definition for removed preference is the (set-theoretic) *limit inferior* of a sequence $\{\succsim_n\}_{n=0}^{\infty}$, given by $\succsim_\infty := \liminf \succsim_n := \cup_{n=0}^{\infty} \cap_{k=n}^{\infty} \succsim_k$. That is, $\mu \succsim_\infty \eta$ if and only if $\mu \succsim_n \eta$ for all but a finite number of n:

$$\mu \succsim_\infty \eta \iff \text{there exists } N \text{ s.t. } \mu \succsim_n \eta \text{ for all } n \geq N.$$

This requires, indeed, that $\mu \succsim_n \eta$ always holds with sufficient distance n. Thus the limit inferior serves as the natural candidate for the definition of removed preference.[5]

As we will see below, the removed preference is generally not continuous. This, along with the results we find in our applications, motivates us to also consider a continuous extension of the removed preference. Formally, suppose that $\mathcal{A}$ is a topological space and that $\mathcal{A} \times \mathcal{A}$ has the product topology. Identify any nontrivial continuous order on $\mathcal{A}$ with its graph, a nonempty

[3] Alternatively, one could take as primitive a single preference relation over the set $\mathcal{A} \times \mathbb{N} \cup \{0\}$ of distant rewards. However, this primitive embodies more information than we require.

[4] In later sections we will deal with continuous binary relations (that is, those with a closed graph). When $\mathcal{A}$ is a topological space, the space of continuous binary relations on $\mathcal{A} \times \mathcal{A}$ (endowed with the product topology) is the set of closed subsets of $\mathcal{A} \times \mathcal{A}$.

[5] For perspective, note that the limit superior defined by $\cap_{n=0}^{\infty} \cup_{k=n}^{\infty} \succsim_k$ ranks μ higher than η if $\mu \succsim_n \eta$ for infinitely many n. This is not suitable for our purposes. The agent may, for instance, exhibit $\mu \succ_n \eta$ for all even n while $[\mu \not\succsim_n \eta$ and $\eta \not\succsim_n \mu]$ for all odd n. Then it cannot be said what the agent prefers from a sufficient distance. Yet the limit superior will strictly rank μ over η, an unwarranted conclusion for the removed preference.

closed subset of the set $\mathcal{A} \times \mathcal{A}$ endowed with the product topology. Identify the space of such binary relations on $\mathcal{A}$ with a subset of $\mathcal{P} = \mathcal{K}(\mathcal{A} \times \mathcal{A})$, the space of nonempty compact subsets of $\mathcal{A} \times \mathcal{A}$ endowed with the Hausdorff metric topology. The closure of the removed preference, referred to as the *closed removed preference*, is denoted:

$$\succsim^* := cl \succsim_\infty .$$

That is, $\mu \succsim^* \eta$ if and only if there exists sequences μ_i and η_i converging to μ and η respectively such that $\mu_i \succsim_\infty \eta_i$ for all i. The closed removed preference makes less conservative assertions about the agent's preferences from a distance. For instance, although $\mu \succ_n \eta$ for all n, suggesting a strict preference from a distance, it may well be that $\mu \sim^* \eta$. This would happen if in every neighborhood of μ and η there are μ_i and η_i such that $\mu_i \precsim_\infty \eta_i$.

3 Properties of $\succsim_\infty$ and $\succsim^*$

We explore properties of removed preference and closed removed preference under a simple 'single reversal' restriction on $\{\succsim_n\}_{n=0}^\infty$, which requires that distancing induce at most one complete reversal in preference.

3.1 Definitions

Consider a set of preference relations $\{\succsim_n\}_{n=0}^\infty$ over a compact metric space $\mathcal{A}$ where each preference $\succsim_n$ captures the ranking of alternatives in $\mathcal{A}$ when the alternatives are delayed by n periods. We will be interested in whether the usual assumptions on $\succsim_n$ carry over to the removed preference. Consider the following properties on $\{\succsim_n\}_{n=0}^\infty$.

Axiom 1 (Order) *$\succsim_n$ is complete and transitive for all n.*

Axiom 2 (Continuity) *The sets $\{\eta : \mu \succsim_n \eta\}$ and $\{\eta : \eta \succsim_n \mu\}$ are closed for all μ and n.*

Axiom 3 (Non-Triviality) *$\mu \succ_{n'} \eta$ and $\mu \prec_{n''} \eta$ for some μ,η and $n'' > n'$.*

We will also be interested in whether the vNM Independence condition holds from a distance. For this case (and only for this case), we suppose there is some compact metric space C and that $\mathcal{A} = \Delta(C)$ is the set of all probability measures on the Borel σ-algebra of C, endowed with the weak convergence topology; $\mathcal{A}$ is compact and metrizable [3]. For $\alpha \in [0,1]$ and $\mu, \eta \in \mathcal{A}$, the mixture $\alpha\mu + (1-\alpha)\eta \in \mathcal{A}$ is the measure that assigns $\alpha\mu(A) + (1-\alpha)\eta(A)$ to each A in the Borel σ-algebra of C. Now we can state:

Axiom 4 (Independence) *For all n,*

$$\mu \succ_n \eta \Longrightarrow \alpha\mu + (1-\alpha)\nu \succ_n \alpha\eta + (1-\alpha)\nu.$$

The next property imposes structure on $\{\succsim_n\}_{n=0}^{\infty}$ via the restriction that there can be no more than one reversal for any pair of rewards. The property ensures convergence of $\{\succsim_n\}_{n=0}^{\infty}$.

Axiom 5 (Reversal) *If $\mu \prec_n \eta$ (resp. $\mu \precsim_n \eta$) and $\mu \succsim_{n'} \eta$ (resp. $\mu \succ_{n'} \eta$) for some $n' > n$, then $\mu \succsim_{n''} \eta$ (resp. $\mu \succ_{n''} \eta$) for all $n'' > n'$.*

3.2 Convergence and Characterization

The first result in this section shows that under Order, Continuity and Reversal, the closed removed preference can in fact be computed as the limit of the sequence $\{\succsim_n\}_{n=0}^{\infty}$ with respect to the Hausdorff metric topology – this is the *closed limit*, and is denoted $\lim_{n\to\infty} \succsim_n$ (see Appendix A for details). As is demonstrated in subsequent sections, knowing that the two notions coincide proves useful in exploring properties of the closed removed preference. Though the closed limit is continuous, the two do not coincide in general.[6]

[6] To see this, let $\mathcal{A} = [0,1]$ (with the relative Euclidian topology) and suppose that each $\succsim_n$ is represented by the utility function u_n defined by

$$u_n(\mu) = \begin{cases} \mu & \text{if } \mu \neq 1 - \frac{1}{n} \\ 0 & \text{if } \mu = 1 - \frac{1}{n} \end{cases}, \quad \mu \in [0,1].$$

(Though u_n is discontinuous, $\succsim_n$ is continuous). Each $\succsim_n$ is strictly monotone (larger numbers are ranked higher) except that the point $(1-\frac{1}{n})$ is indifferent to 0. As n increases, this point tends to 1. The sequence $\{\succsim_n\}_{n=0}^{\infty}$ converges to a preference $\lim_{n\to\infty} \succsim_n$ that is not monotone: it ranks 1 and 0 as indifferent and every number in between is ranked strictly higher. In contrast, the closed removed preference $\succsim^*$ is strictly monotone: it is easy to see that the removed preference $\succsim_\infty$ is strictly monotone, and since it must therefore be continuous it must coincide with the closed removed preference.

Theorem 1 *Suppose that $\{\succsim_n\}$ satisfies Order, Continuity and Reversal. Then $\succsim_n$ converges and*

$$\succsim^* = \lim_{n\to\infty} \succsim_n .$$

Thus, under Order, Continuity and Reversal, closed limit is characterized by closed removed preference. Since $\succsim_\infty \subset \succsim^*$, the removed preference is distinct from the limit in general.

Next we characterize the difference between the removed preference and the closed removed preference under Order, Continuity and Reversal. A straightforward implication of Order and Reversal is the existence of a function $\tau : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ such that for each $(\mu, \eta) \in \mathcal{A} \times \mathcal{A}$, $\tau(\mu, \eta)$ is the number of periods that μ and η need to be delayed before a reversal is observed; if no reversal is observed, then $\tau(\mu, \eta) = 0$. For instance, if $\mu \succ_0 \eta$, $\mu \succsim_1 \eta$ and $\mu \prec_n \eta$ for all $n \geq 2$, then $\tau(\mu, \eta) = 2$. This *switching function* $\tau : \mathcal{A}\times\mathcal{A} \to \mathbb{R}$ is defined formally in Appendix B.

The set $\Omega \subset \mathcal{A} \times \mathcal{A}$ on which τ is upper semicontinuous is given by:

$$\Omega = \{(\mu, \eta) \in \mathcal{A}\times\mathcal{A} : \limsup_{i\to\infty} \tau(\mu_i, \eta_i) \leq \tau(\mu, \eta) \textit{ whenever } (\mu_i, \eta_i) \to (\mu, \eta)\}.$$

The next result identifies the connection between removed preferences and the close removed preference.

Theorem 2 *Suppose that $\{\succsim_n\}$ satisfies Order, Continuity and Reversal. Then, for any $\mu, \eta \in \mathcal{A}$, the following statements are equivalent:*
(a) $\mu \succ^* \eta$
(b) $\mu \succ_\infty \eta$ and $(\mu, \eta) \in \Omega$

By definition, $\mu \succ^* \eta$ implies $\mu \succ_\infty \eta$. However, the result demonstrates that the converse is not true in general. Intuitively, the reason is that preferences can reverse "at infinity", and that these reversals are not captured by $\succsim_\infty$. The behavior of τ in the neighborhood of a pair of reward associated with such a reversal suggests this terminology: for any μ, η such that $\mu \succ_\infty \eta$, we have $(\mu, \eta) \notin \Omega$ if and only if there exists a sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η) such that $\limsup_{i\to\infty} \tau(\mu_i, \eta_i) = \infty$ and $\eta_i \succsim_\infty \mu_i$ for all i. Appendix E shows this formally.

The above result shows us that any disagreement between removed preference and closed removed preference can only take the form $\mu \succ_\infty \eta$ and $\mu \sim^* \eta$, and that too only happens at points outside Ω. The next result shows that $\succsim_\infty$ and $\succsim^*$ agree on a dense set, that is, that the two are 'essentially' in agreement.

Theorem 3 *Suppose that* $\{\succsim_n\}$ *satisfies Order, Continuity and Reversal. Then* $\succsim_\infty$ *and* $\succsim^*$ *agree on a dense subset of* $\mathcal{A} \times \mathcal{A}$*, namely:*

$$S = \{(\mu, \eta) \in \mathcal{A} \times \mathcal{A} : \mu \succ^* \eta \text{ or } \mu \sim_\infty \eta\}.$$

We will establish counterparts of this particular result in our applications, and those will prove useful for the following reason: If $\succsim^*$ has a particular property, then there will be justification for claiming that $\succsim_\infty$ *essentially* satisfies that property. That is, we will be able to make claims about removed preference by studying properties of closed removed preference. In particular, the closed removed preference provides a means of understanding removed preference in a way that goes beyond a discussion of the specific restrictions satisfied by the latter. This is the main motivation for introducing closed removed preference.

3.3 Properties

Say that a binary relation $\succsim$ is non-trivial if $\mu \not\succsim \eta$ for some μ, η, and that $\succsim$ satisfies Independence if for all μ, η, ν, $\mu \succ \eta \Longrightarrow \alpha\mu + (1-\alpha)\nu \succ \alpha\eta + (1-\alpha)\nu$.

Theorem 4 *Suppose that* $\{\succsim_n\}$ *satisfies Order, Continuity and Reversal. Then the following statements hold:*

(a) $\succsim_\infty$ *and* $\succsim^*$ *are both complete*
(b) $\succsim^*$ *is continuous but* $\succsim_\infty$ *may not be.*
(c) $\succsim_\infty$*is transitive, and* $\succ^*$*is transitive but* $\sim^*$ *may not be.*
(d) If $\{\succsim_n\}$ *satisfies Non-triviality then* $\succsim^*$ *and* $\succsim_\infty$ *are non-trivial.*

The result demonstrates that familiar properties imposed on $\{\succsim_n\}$ may not always imply analogous properties for $\succsim^*$ and $\succsim_\infty$. The lack of continuity of $\succsim_\infty$ is due to the existence of "reversals at infinity" noted earlier. In particular, it is possible that $\mu \succ_\infty \eta$ and yet for there to exist a sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η) that satisfies $\eta_i \succsim_\infty \mu_i$. By contrast, $\succsim^*$ is continuous – it is a continuous extension of $\succsim_\infty$obtained by setting $\mu \sim^* \eta$ whenever there is a reversal at infinity for μ, η.

The closed removed preference $\succsim^*$ turns out to have potentially intransitive indifference (see the appendix for an example). Specifically, it is possible that there is a reversal at infinity for both pairs μ, η and η, ν and yet for there to be no such reversal for the pair μ, ν, thereby permitting a situation where $\mu \sim^* \eta \sim^* \nu$ but $\mu \not\sim^* \nu$. In order to rule out such cases some structure is required on the behavior of the switching function τ across neighborhoods of various pairs of rewards – note that Order, Continuity and Reversal does very little in this regard. The applications we consider in subsequent sections have more structure: for instance, $\{\succsim_n\}$ is derived from a model which induces a uniformly convergent sequence of utility representations $\{u_n\}$. Indeed, $\succsim^*$ is transitive in each of the applications.

For the current section, with its focus on relatively weak structure, we consider an example of additional structure in the context of preferences $\{\succsim_n\}$ over a mixture space. Consider the property where for all μ, η, ν and $\alpha \in (0,1)$,

$$\mu \succ_\infty \eta \text{ and } (\mu, \eta) \in \Omega \Longrightarrow (\alpha\mu + (1-\alpha)\nu \ , \ \alpha\eta + (1-\alpha)\nu) \in \Omega \qquad (1)$$

This states that if there is no reversal at infinity for the pair of lotteries μ and η, then no reversal arises for pairs obtained by a common mixture with a third lottery (the condition is restricted to cases where $\mu \succ_\infty \eta$). In this sense, the behavior of τ around (μ, η) is similar to that around $(\alpha\mu + (1-\alpha)\nu, \alpha\eta + (1-\alpha)\nu)$. This would be implied if $\succsim^*$ satisfied Independence.

We can show:[7]

Theorem 5 *If $\{\succsim_n\}$ satisfies Order, Continuity, Reversal, Independence and (1) then $\succsim^*$ is transitive and satisfies Independence.*

4 Application to Time Preference

In this section we derive the removed preference $\succsim^*$ from a model based on the literature on decreasing impatience. The main finding of this literature in experimental psychology is that of *preference reversals* – see Fredrick, Loewenstein and O'Donoghue [6] for references. An example of a preference reversal is when an agent exhibits the following at some given point in time:

$$(\$100,\ now) \succ (\$120,\ 1\ month)$$
$$(\$100,\ 1\ year) \prec (\$120,\ 1\ year\ and\ 1\ month).$$

The first comparison reveals that the agent is impatient when rewards are near – he prefers a smaller earlier reward to a larger later one. The second

[7] For completeness, we also show the following in the appendix: If $\{\succsim_n\}$ satisfies Independence then $\succsim_\infty$ satisfies Independence. If, moreover, $\succsim^*$ is transitive, then $\succsim^*$ also satisfies Independence.

reveals that the agent's impatience decreases when the same rewards are pushed into the future by a common number of periods. The implied feature of time preference, that impatience diminishes as alternatives become more distant, is known as *decreasing impatience* and has inspired the literature on the desire for immediate gratification – this includes hyperbolic discounting (Laibson [13]) and temptation (Gul and Pesendorfer [8]). Given that a desire for immediate gratification is obviously stronger for more immediate rewards than more distant ones, we take temporal distancing as our notion of distancing. Below we outline a general model that exhibits decreasing impatience and then explore the implied properties of removed preference.

4.1 Assumptions

The primitive of the model captures how the agent ranks delayed consumption streams. Let C be some bounded interval in $\mathbb{R}_+$ including 0, and let the space of $T+1$-period horizon consumption streams be given by $\mathcal{A} = C^T$, which is endowed with the product topology. We impose $T < \infty$ (see footnote 10 for the reason). The primitive is $\{\succsim_n\}_{n=0}^{\infty}$, a set of preferences defined over $\mathcal{A}$ that captures the agent's ranking of delayed $T+1$-period consumption streams. More precisely, each $\succsim_n$ captures, from the perspective of some fixed period 0, the agent's ranking of streams of length $T+1$ that begin *after* n periods. We adopt three assumptions on $\{\succsim_n\}_{n=0}^{\infty}$:

Assumption 1 *There exist functions $u : C \to \mathbb{R}$ and $\phi : \mathbb{N} \cup \{0\} \to \mathbb{R}$ such that each $\succsim_n$ is represented by the utility function defined by:*

$$U_n(c_0, c_1, .., c_T) = \sum_{t=0}^{T} \phi(t+n) u(c_t), \tag{2}$$

for each $(c_0, c_1, .., c_T) \in C^{T+1}$.

That is, $\succsim_n$ is represented by a discounted utility model with discount function $\phi(\cdot)$. One can imagine that $\{\succsim_n\}_{n=0}^{\infty}$ was derived from a single preference over infinite horizon consumption streams with the discounted utility representation defined by ϕ and u (though this would require us to strengthen Assumption 3 below so that the infinite sum of discounted utilities is well-defined). Note how n appears in the function U_n: since the consumption stream $(c_0, c_1, .., c_T)$ begins after n periods, the t^{th} element of the stream (that is, c_t) is in fact $t+n$ periods away. Thus, its utility is discounted by $\phi(t+n)$.

Assumption 2 *The instantaneous utility function $u(\cdot)$ is strictly increasing and continuous and satisfies $u(0)=0$.*

Assumption 3 *The discount function $\phi(\cdot)$ is strictly positive and weakly decreasing in t, and $\frac{\phi(t+1)}{\phi(t)}$ is weakly increasing in t.*

The first part of the assumption states that a unit of consumption is worth weakly less the farther it is, and moreover, it is always worth something. The second part embodies (weakly) decreasing impatience since $\frac{\phi(t+1)}{\phi(t)}$ reflects how, from today's perspective, the agent weighs consumption in period $t+1$ relative to consumption in period t. The literature on preference reversals and decreasing impatience lends empirical support to this assumption. The class of discount functions that satisfy Assumption 3 includes both exponential discount functions δ^t, where $\delta \leq 1$, and generalized hyperbolic discount functions $\left(\frac{1}{1+\alpha t}\right)^{\frac{\lambda}{\alpha}}$, where $\alpha, \lambda > 0$ (Chung and Hernnstein [4], Loewenstein and Prelec [15]).

4.2 Results

Though the agent uses the discount function $\phi(\cdot)$ when evaluating consumption streams, he may, for instance, be 'too' impatient in his own opinion, perhaps due to a desire for earlier gratification as discussed in the literature on decreasing impatience. Thus the discount function that he uses in his decisions may be different from the discount function he would use when at a distance. We derive the representation for the agent's closed removed preference.

The results of Section 3 are not applicable here: although $\{\succsim_n\}_{n=0}^{\infty}$ satisfies Order and Continuity, it violates Reversal in general.[8]

[8] To see this, suppose $T=2$ and $u(c)=c$. For $\theta=0.49$, define the discount function $\phi(t)$ by $\phi(t)=\begin{cases}\theta^t\sqrt{t!} & \text{if } 0\leq t\leq 4\\ \phi(4) & \text{otherwise}\end{cases}$. Then ϕ satisfies Assumption 3. Take two deterministic streams (c_0,c_1,c_2) and $(c_0',c_1',c_2')=(c_0-0.445,c_1+1.5,c_2-1)$. Define $F(n):=U_n(c_0,c_1,c_2)-U_n(c_0',c_1',c_2')=\phi(n)0.445-\phi(n+1)1.5+\phi(n+2)$, and compute that $F(0)>0$, $F(1)<0$ and $F(2)>0$. It follows that the preference over (c_0,c_1,c_2) and (c_0',c_1',c_2') reverse more than once.

Theorem 6 *Under Assumptions 1-3, the closed removed preference* $\succsim^*$ *is represented by a utility function* $U^* : C^{T+1} \to \mathbb{R}$ *defined by:*

$$U^*(c_0, c_1, .., c_T) = \sum_{t=0}^{T} \gamma^t u(c_t), \quad \textit{for any } (c_0, c_1, .., c_T) \in C^{T+1},$$

where $\gamma \equiv \lim_{n\to\infty} \frac{\phi(1+n)}{\phi(n)}$ *is a well-defined limit. Moreover,* $\{\succsim_n\}_{n=0}^{\infty}$ *converges and* $\succsim^* = \lim_{t\to\infty} \succsim_t$.

Thus, under the assumptions, discounting by the closed removed preference is given by the *exponential* discount function:

$$\phi^*(t) = \gamma^t, \quad 0 \le t \le T.$$

Since the removed preference $\succsim_\infty$ agrees with $\succsim^*$ on a dense subset of $\mathcal{A} \times \mathcal{A}$,[9] we conclude that removed preference is *essentially* an exponential discounted utility model.

4.3 Examples

We study what commonly studied discount functions imply about ϕ^*. In what follows, we restrict the domain of $\phi(\cdot)$ to $\{0, 1, ..., T\}$ and implicitly assume the quantifiers 'for all $0 < \delta, \beta \le 1$', and 'for all $\alpha, \lambda > 0$' where appropriate.

- $\phi(t) = \delta^t$ implies $\phi^*(t) = \delta^t$.

That is, exponential discounting implies that there is no difference between an agent's actual and removed discount function. If a difference between actual and removed perspectives is attributed to temptation, the standard model thus precludes any temptation surrounding discounting.

- $\phi(t) = \begin{cases} 1 & \text{if } t = 0 \\ \beta\delta^t & \text{otherwise.} \end{cases}$ implies $\phi^*(t) = \delta^t$.

[9] The proof is as follows. By definition we have $[\mathbf{c} \sim_\infty \mathbf{c}' \Longrightarrow \mathbf{c} \sim^* \mathbf{c}']$ and $[\mathbf{c} \succ^* \mathbf{c}' \Longrightarrow \mathbf{c} \succ_\infty \mathbf{c}']$ for all $\mathbf{c}, \mathbf{c}' \in \mathcal{A}$, so $\succsim_\infty$ agrees with $\succsim^*$ on the subset $S = \{(\mathbf{c}, \mathbf{c}') \in \mathcal{A} \times \mathcal{A}$: $\mathbf{c} \succ^* \mathbf{c}'$ or $\mathbf{c} \sim_\infty \mathbf{c}'\}$. To see that S is dense, we note that U^* is nonconstant and strictly monotone. So whenever $\mathbf{c} \succsim^* \mathbf{c}'$ we can always find a sequence $(\mathbf{c}_i, \mathbf{c}'_i) \to (\mathbf{c}, \mathbf{c}')$ such that $\mathbf{c}_i \succ^* \mathbf{c}'_i$ for all i.

Thus, the quasi-hyperbolic discount function implies that the closed removed discount function sets $\beta = 1$. This is reminiscent of O' Donoghue and Rabin [20].

- $\phi(t) = \left(\frac{1}{1+\alpha t}\right)^{\frac{\lambda}{\alpha}}$ implies $\phi^*(t) = 1$.

This follows from[10]

$$\gamma \equiv \lim_{n\to\infty} \frac{\phi(1+n)}{\phi(n)} = \lim_{n\to\infty} \left(\frac{1+\alpha n}{1+\alpha n+\alpha}\right)^{\frac{\lambda}{\alpha}} = \left(\lim_{n\to\infty} \frac{1+\alpha n}{1+\alpha n+\alpha}\right)^{\frac{\lambda}{\alpha}} = 1.$$

The result states that the hyperbolic discount function $\left(\frac{1}{1+\alpha t}\right)^{\frac{\lambda}{\alpha}}$, a special case of which has been used to fit experimental data (Mazur [17]), implies that the agent's removed perspective weights the present and the future equally. This holds regardless of the values of the parameters α, λ (the agent's idiosyncrasies).

5 Application to Risk Preference

A well-known finding in the experimental literature on risk preference is the *certainty effect*, also known as the Allais paradox (Kahneman and Tversky [12] for references. An example of a certainty effect is when an agent exhibits:

$$(\$3000,\ \text{probability } 1) \succ (\$4000,\ \text{probability } 0.8)$$
$$(\$3000,\ \text{probability } 0.25) \prec (\$4000,\ \text{probability } 0.2).$$

The second pair of lotteries is obtained by scaling down probabilities by 0.25 in the first pair. This behavior is incompatible with expected utility theory since it violates Independence, specifically the fact that for any $\alpha \in (0,1)$, lotteries p, q and the degenerate lottery 0 that yields \$0,

$$p \succsim q \Longleftrightarrow \alpha p + (1-\alpha)0 \succsim \alpha q + (1-\alpha)0.$$

Various nonexpected utility theories have been built to accommodate the certainty effect. These suggest that the certainty effect is an intrinsic aspect of preferences. An alternative suggestion is that the certainty effect is an

[10] The fact that $\phi^*(\cdot)$ could be 1 for an interesting class of discount functions led us to restrict attention to finite T.

emotional response arising from a dread for risk - see Loewenstein et al [16] for a review of evidence suggesting that risk invokes anxiety and dread in decision-makers. To the extent that an analyst may be interested in an agent's dispassionate evaluation of risk, he may seek to study the agent's risk preferences from a 'distance'. Increasing distance from risk - that is, reducing dread - is naturally understood as increasing the proximity to certainty. Thus a candidate definition of distance in this context is the uniform scaling down of probabilities of nonzero outcomes, that is, by mixing lotteries with 0. Intuitively, by mixing a lottery p with 0, the likelihood of receiving or losing a reward reduces and in turn so does the 'size of the risk' and its associated dread.

5.1 Assumptions

Let Δ be the set of finite support lotteries over some arbitrary space X. A generic lottery is p. For any $x \in X$, let x also denote the lottery yielding x with probability 1. Let $\succsim$ be a binary relation on Δ.

The first assumption asserts the existence of (what we will interpret as) 0 consumption and also asserts non-triviality of the preference.

Assumption 1 *There exists some element $0 \in \Delta$ so that $x \succsim 0$ for any $x \in \Delta$. Moreover, there exists $x \in \Delta$ such that $x \succ 0$.*

For any given $p \in \Delta$, label the non-zero outcomes $x_1, ..., x_{k_p}$ so that $x_{k_p} \succsim ... \succsim x_1$ (the dependence on p is suppressed to ease notation here and in the appendix). We assume that the preference admits a rank-dependent utility representation, which generalizes expected utility theory by permitting nonlinear weighting of probabilities while preserving monotonicity with respect to first order stochastic dominance. (Cumulative) prospect theory is a special case.

Assumption 2 *There exists a function $u : X \to \mathbb{R}$ with $u(0) = 0$ and a strictly increasing, twice continuously differentiable function $v : [0, 1] \to [0, 1]$ with $v(0) = 0$ and $v(1) = 1$ such that $\succsim$ is represented by the Rank-Dependent Utility:*

$$V(p) = \sum_{j=1}^{k_p} [v(\sum_{i=1}^{j} p(x_i)) - v(\sum_{i=1}^{j-1} p(x_i))]u(x_j),$$

for each $p \in \Delta$.

Experimental studies on probability weighting support the following assumption.

Assumption 3 *The function* $v : [0,1] \to [0,1]$ *is strictly convex on a neighborhood of* 0.

We now define our sequence of preferences indexed by distance, where distance is obtained by mixing a lottery with 0. For each n, define $\succsim_n$ over Δ by

$$p \succsim_n q \iff \frac{1}{n}p + \frac{n-1}{n}0 \succsim \frac{1}{n}q + \frac{n-1}{n}0.$$

As before, $\succsim^*$ is the closed removed preference generated by $\{\succsim_n\}$. The main result of this section is that, under the assumptions, $\succsim^*$ is expected utility with respect to the utility index u. That is, from a distance, the agent weights probabilities linearly.

Theorem 7 *Under Assumptions 1-3,* $\succsim^*$ *is represented by* $U^*(p) = \sum p(x_i)u(x_i)$.

Since removed preference $\succsim_\infty$ agrees with $\succsim^*$ on a dense subset of $\mathcal{A}\times\mathcal{A}$,[11] we therefore conclude that removed preference is *essentially* expected utility.

6 Conclusion

This paper formalizes the notion of removed preference. When studied in the context of well-known non-standard utility models, it is shown to 'essentially' possess the properties of standard models – we show that it is essentially expected utility in a non-expected utility model and essentially exponential discounting in a non-exponential model. We show that closed removed preference is a useful notion for understanding removed preference.

[11] The proof is as in footnote 9 except that mixture linearity of U^* is exploited here in place of strict monotonicity.

A Appendix: Topology on $\mathcal{P}$

When is $\mathcal{A}$ is compact and metrizable, $\mathcal{A} \times \mathcal{A}$ is compact and metrizable under the product topology. Let d be a metric that generates the topology on $\mathcal{A} \times \mathcal{A}$. Denote the space of nonempty compact subsets of $\mathcal{A} \times \mathcal{A}$ by $\mathcal{P}$. For any $A, B \in \mathcal{P}$, let $d(a, B) = \inf_{b \in B} d(a, b)$ and $d(b, A) = \inf_{a \in A} d(b, a)$. The Hausdorff metric h_d induced by d is defined by

$$h_d(A, B) = \max\{\sup d(a, B), \sup d(b, A)\},$$

for all $A, B \in \mathcal{P}$. An ε-ball centered at A is defined by

$$B(A, \varepsilon) = \{B : h_d(A, B) < \varepsilon\}.$$

The Hausdorff metric topology on $\mathcal{P}$ is the topology for which the collection of balls $\{B(A, \varepsilon)\}_{A \in \mathcal{P}, \varepsilon \in (0, \infty)}$ is a base.

View the elements of $\mathcal{P}$ as binary relations on $\mathcal{A}$ by identifying a binary relation B on $\mathcal{A}$ with $\Gamma(B)$, the graph of B:

$$\Gamma(B) = \{(\mu, \eta) \in \mathcal{A} \times \mathcal{A} : \mu B \eta\}.$$

If B is a weak order (complete and transitive binary relation) then $\Gamma(B)$ is nonempty. If B is also continuous then $\Gamma(B)$ is closed, and hence compact.[12] Thus, the set of continuous weak orders on $\mathcal{A}$ is a subset of $\mathcal{P}$.

By [3, Thm 3.71(3)], compactness of $\mathcal{A} \times \mathcal{A}$ implies that $\mathcal{P}$ is compact. Also, under compactness of $\mathcal{A} \times \mathcal{A}$, $\Gamma(B)$ is the Hausdorff metric limit of a sequence $\{\Gamma(B_n)\} \subset \mathcal{P}$ if and only if $\Gamma(B)$ is the 'closed limit' of $\{\Gamma(B_n)\}$ [3, Thm 3.79]. To define the closed limit of a sequence $\{\Gamma(B_n)\}$, first define the topological limit superior $Ls\Gamma(B_n)$ and topological limit inferior $Li\Gamma(B_n)$ of the sequence $\{\Gamma(B_n)\}$:

$$Ls\Gamma(B_n) = \{a \in \mathcal{A} \times \mathcal{A} : \text{for every neighborhood } V \text{ of } a,$$
$$V \cap \Gamma(B_n) \neq \phi \text{ for infinitely many } n\}$$
$$Li\Gamma(B_n) = \{a \in \mathcal{A} \times \mathcal{A} : \text{for every neighborhood } V \text{ of } a,$$
$$V \cap \Gamma(B_n) \neq \phi \text{ for all but a finite number of } n\}.$$

The sequence $\{\Gamma(B_n)\}$ converges to a closed limit $\Gamma(B)$ if $\Gamma(B) = Ls\Gamma(B_n) = Li\Gamma(B_n)$.

[12] To show that $\Gamma(B)$ is closed if B is a continuous weak order, use [5, Lemma 5.1 and Exercise 3.16]. Note that the space of lotteries $\mathcal{A}$ is connected, and moreover, it is separable since it is compact metric.

B Appendix: Definition of τ

A straightforward implication of Order and Reversal is the existence of a function $\tau : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ such that for each $(\mu, \eta) \in \mathcal{A} \times \mathcal{A}$, $\tau(\mu, \eta)$ is the number of periods that μ and η need to be delayed before a preference reversal is observed; if no reversal is observed, then $\tau(\mu, \eta) = 0$. For instance, if $\mu \succ_0 \eta$, $\mu \succsim_1 \eta$ and $\mu \prec_n \eta$ for all $n \geq 2$, then $\tau(\mu, \eta) = 2$. More precisely, for any $\{\succsim_n\}_{n=0}^{\infty}$ that satisfies Order (in fact, completeness of each $\succsim_n$ is all we need), define the function $\tau : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ in the following way. First consider any $(\mu, \eta) \in \mathcal{A} \times \mathcal{A}$ such that $\mu \succsim_0 \eta$. If $\mu \sim_n \eta$ for all n or $\mu \succ_n \eta$ for all n, then define $\tau(\mu, \eta) = 0$, and if there exists T such that $\mu \prec_T \eta$, then define

$$\tau(\mu, \eta) = \min\{n : \mu \prec_n \eta\}.$$

If $\mu \succ_0 \eta$, and there exists n such that $\mu \sim_n \eta$ and there is no n' such that $\mu \prec_{n'} \eta$, then define

$$\tau(\mu, \eta) = \min\{n : \mu \sim_n \eta\}.$$

Finally, let $\tau(\mu, \eta) = \tau(\eta, \mu)$ for all μ, η.

Lemma 1 *Suppose $\{\succsim_n\}_{n=0}^{\infty}$ satisfies Order and Reversal and take any μ, η such that $\mu \succsim_0 \eta$. If $\tau(\mu, \eta) = 0$ then $\mu \sim_n \eta$ for all n or $\mu \succ_n \eta$ for all n. If $\tau(\mu, \eta) > 0$ then only one of the following holds:*
(a) $\mu \succ_n \eta$ for $n < \tau(\mu, \eta)$ and $\mu \prec_n \eta$ for all $n \geq \tau(\mu, \eta)$;
(b) $\mu \succ_n \eta$ for $n < \tau(\mu, \eta)$ and $\mu \sim_n \eta$ for all $n \geq \tau(\mu, \eta)$;
(c) There is $0 \leq N < \tau(\mu, \eta)$ such that $\mu \succ_n \eta$ for all $n < N$, $\mu \sim_n \eta$ for all $N \leq n < \tau(\mu, \eta)$, and $\mu \prec_n \eta$ for all $n \geq \tau(\mu, \eta)$.

Proof. The case with $\tau(\mu, \eta) = 0$ follows from the definition of $\tau(\cdot)$. For the second part, first consider the case where $\mu \succ_0 \eta$ and suppose $\tau(\mu, \eta) > 0$, so that there is n such that $\mu \precsim_n \eta$. Let $N^* > 0$ be the first integer for which preferences reverse; thus, $\mu \succ_n \eta$ for $n < N^*$ and $\mu \precsim_{N^*} \eta$. By Reversal, we must have $\mu \precsim_n \eta$ for all $n \geq N^*$. If $\mu \prec_n \eta$ for all $n \geq N^*$, then given the definition of $\tau(\cdot)$, we have $\tau(\mu, \eta) = N^*$ and we are in case (a) in the statement of the Lemma. Similarly, if $\mu \sim_n \eta$ for all $n \geq N^*$ then we are in case (b).

If we are in neither case, then given that we must have $\mu \precsim_n \eta$ for all $n \geq N^*$, there exist $n', n'' \geq N^*$ such that $\mu \sim_{n'} \eta$ and $\mu \prec_{n''} \eta$. By Reversal, $\mu \succ_0 \eta$ and $\mu \prec_{n''} \eta$ implies $\mu \prec_n \eta$ for all $n \geq n''$, and so $n' < n''$. For the same reason, it must also be that $\mu \sim_{N^*} \eta$. Let N^{**} be the first integer larger than N^* for which indifference turns into strict preference, that is $\mu \sim_n \eta$ for $N^* \leq n < N^{**}$ and $\mu \prec_{N^{**}} \eta$. Reversal ensures $\mu \prec_n \eta$ for all $n \geq N^{**}$. Given the definition of $\tau(\cdot)$, we have $\tau(\mu, \eta) = N^{**}$ and we are in case (c).

The above established the result for μ, η such that $\mu \succ_0 \eta$ and $\tau(\mu, \eta) > 0$. Now consider μ, η such that $\mu \sim_0 \eta$ and $\tau(\mu, \eta) > 0$. Let N^* be the first integer for which preferences reverse, that is, $\mu \sim_n \eta$ for $n < N^*$ and wlog, $\mu \prec_{N^*} \eta$. By Reversal, $\mu \prec_n \eta$ for all $n \geq N^*$. By definition of $\tau(\cdot)$, $\tau(\mu, \eta) = T^*$ and we are in case (c) in the Lemma. This completes the proof. ■

C Appendix: Proof of Theorem 1

Assume that $\{\succsim_n\}_{n=0}^{\infty}$ satisfies Order, Continuity and Reversal, and define the switching function $\tau(\cdot)$ as in Appendix B. Since each $\succsim_n$ is a continuous weak order, $\{\Gamma(\succsim_n)\}$ is a sequence in $\mathcal{P}$. We show that the closed limit of $\succsim_n$ as n goes to infinity is the closed removed preference $\succsim^*$ over $\mathcal{A}$:

$$\mu \succsim^* \eta \Longleftrightarrow \exists \text{ sequence } \{(\mu_i, \eta_i)\} \text{ converging to } (\mu, \eta) \text{ s.t. } \forall n,\ \mu_i \succsim_\infty \eta_i. \tag{3}$$

Note that

$$\mu \succsim_\infty \eta \Longleftrightarrow \mu \succsim_{\tau(\mu,\eta)} \eta,$$

Note also that $\Gamma(\succsim^*) = \overline{\Gamma(\succsim_\infty)}$, where $\overline{\Gamma(\succsim_\infty)}$ denotes the closure of the graph of $\succsim_\infty$, and finally, observe that the definition of $\succsim^*$ directly implies:

$$\mu \succsim_\infty \eta \Longrightarrow \mu \succsim^* \eta \tag{4}$$

We establish simultaneously the existence of a closed limit and the fact that it is characterized by $\succsim^*$.

Step 1: $Ls\Gamma(\succsim_n) \subset \Gamma(\succsim^*)$.

Suppose $(\mu^*, \eta^*) \notin \Gamma(\succsim^*)$. Closedness of $\Gamma(\succsim^*)$ implies that there exists a neighborhood U_1 of (μ^*, η^*) such that $U_1 \cap \Gamma(\succsim^*) = \phi$. By the contrapositive of (4),

$$U_1 \cap \Gamma(\succsim_\infty) = \phi. \tag{5}$$

In particular, $(\mu^*, \eta^*) \notin \Gamma(\succsim_{\tau^*})$ where $\tau^* \equiv \tau(\mu^*, \eta^*)$. However, $\Gamma(\succsim_{\tau^*})$ is closed and so there exists a neighborhood U_2 of (μ^*, η^*) such that

$$U_2 \cap \Gamma(\succsim_{\tau^*}) = \phi. \tag{6}$$

Let $V = U_1 \cap U_2$ and observe that V is a neighborhood of (μ^*, η^*).

Given (5), the post-reversal preference for any $(\mu, \eta) \in V$ ranks η strictly higher than μ, that is,

$$\mu \prec_{\tau(\mu,\eta)} \eta \text{ for all } (\mu, \eta) \in V. \tag{7}$$

But by (6) it is also true that $\mu \prec_{\tau^*} \eta$ for all $(\mu, \eta) \in V$. It follows from Reversal that

$$\tau(\mu, \eta) \leq \tau^* \text{ for all } (\mu, \eta) \in V,$$

and hence (7) implies $\mu \prec_n \eta$ for all $(\mu, \eta) \in V$ and $n \geq \tau^*$. That is, $V \cap \Gamma(\succsim_n) = \phi$ for all $n \geq \tau^*$. Conclude that $(\mu^*, \eta^*) \notin Ls\Gamma(\succsim_t)$ since V is a neighborhood of (μ^*, η^*) that does not intersect with infinitely many $\succsim_t$.

Step 2: $\Gamma(\succsim^*) \subset Li\Gamma(\succsim_n)$.

Since $\Gamma(\succsim_\infty) \subset Li\Gamma(\succsim_n)$ and $Li\Gamma(\succsim_n)$ is closed [3, Lemma 3.67], it follows that $\Gamma(\succsim^*) = \overline{\Gamma(\succsim_\infty)} \subset Li\Gamma(\succsim_n)$, as desired.

By Steps 1 and 2, $Ls\Gamma(\succsim_n) \subset \Gamma(\succsim^*) \subset Li\Gamma(\succsim_n)$. Hence,

$$Li\Gamma(\succsim_n) = Ls\Gamma(\succsim_n) = \Gamma(\succsim^*).$$

This completes the proof.

D Appendix: Proof of Theorem 2

The set Ω of points in $\mathcal{A} \times \mathcal{A}$ on which τ is upper semicontinuous is defined by

$$\Omega = \{(\mu, \eta) \in \mathcal{A} \times \mathcal{A} : (\mu_i, \eta_i) \to (\mu, \eta) \Longrightarrow \limsup_{i \to \infty} \tau(\mu_i, \eta_i) \leq \tau(\mu, \eta)\}.$$

For later, note that since $\tau(\mu, \eta) = \tau(\eta, \mu)$ for all μ, η by definition of $\tau(\cdot)$, it follows that $(\mu, \eta) \in \Omega$ implies $(\eta, \mu) \in \Omega$ as well.

We want to show: $\mu \succ^* \eta \Longleftrightarrow [\mu \succ_\infty \eta \text{ and } (\mu, \eta) \in \Omega]$

Proof. $\Longleftarrow$: Take μ and η such that $\mu \succ_\infty \eta$ and $(\mu, \eta) \in \Omega$. Given Lemma 1, $\mu \succ_\infty \eta$ implies $\mu \succ_{\tau(\mu,\eta)+1} \eta$. Since $\succsim_{\tau(\mu,\eta)+1}$ is continuous, for every sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η) there exists I such that

$$\mu_i \succ_{\tau(\mu,\eta)+1} \eta_i, \text{ for all } i \geq I.$$

By hypothesis, $\limsup_{i\to\infty} \tau(\mu_i, \eta_i) \leq \tau(\mu, \eta)$. Therefore, there exists I' such that

$$\tau(\mu, \eta) + 1 > \tau(\mu_i, \eta_i), \text{ for all } i \geq I'.$$

It follows that for all $i \geq I'$, the ranking of (μ_i, η_i) by $\succsim_{\tau(\mu_i,\eta_i)}$ must agree with that by $\succsim_{\tau(\mu,\eta)+1}$, and so,

$$\mu_i \succ_{\tau(\mu_i,\eta_i)} \eta_i, \quad \text{for all } i \geq \max\{I, I'\}.$$

But $\mu_i \succ_{\tau(\mu_i,\eta_i)} \eta_i$ is equivalent to $\mu_i \succ_\infty \eta_i$. This establishes that for any sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η), there exists J such that $\mu_i \succ_\infty \eta_i$ for all $i \geq J$. In particular, there is no sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η) such that $\eta_i \succsim_\infty \mu_i$ for all i. Thus $\eta \not\succsim^* \mu$, as desired.

$\Longrightarrow$: Take μ, η such that $\mu \succ^* \eta$. Then (4) yields

$$\mu \succ_\infty \eta \tag{8}$$

thus establishing the first assertion in the implication. To establish the second assertion, take any sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η). Since $\mu \succ^* \eta$ and since $\succsim^*$ is continuous (Theorem 1), there exists I such that

$$\mu_i \succ^* \eta_i, \quad \text{for all } i \geq I.$$

By (4),

$$\mu_i \succ_\infty \eta_i, \quad \text{for all } i \geq I. \tag{9}$$

Without loss of generality, let $I = 1$. Suppose by way of contradiction that

$$\limsup_{i\to\infty} \tau(\mu_i, \eta_i) > \tau(\mu, \eta).$$

Then, there exists a subsequence $\{(\mu_{i_m}, \eta_{i_m})\} \subset \{(\mu_i, \eta_i)\}$ where for all m,

$$\tau(\mu_{i_m}, \eta_{i_m}) > \tau(\mu, \eta) \geq 0. \tag{10}$$

By construction, $\mu_{i_m} \succ_\infty \eta_{i_m}$ for all m, that is, $\mu_{i_m} \succ_{\tau(\mu_{i_m},\eta_{i_m})} \eta_{i_m}$ for all m. Thus, by Lemma 1 and (10),

$$\eta_{i_m} \succsim_{\tau(\mu,\eta)} \mu_{i_m}, \quad \text{for all } m.$$

However, since $\succsim_{\tau(\mu,\eta)}$ is continuous and $(\mu_{i_m}, \eta_{i_m}) \to (\mu, \eta)$, we have $\eta \succsim_{\tau(\mu,\eta)} \mu$. This is equivalent to $\eta \succsim_\infty \mu$, which contradicts (8). ■

E Appendix: Reversal at Infinity and Proof of Thm 3

E.1 Reversal at Infinity

We show that $\mu \succ_\infty \eta$ and $(\mu, \eta) \notin \Omega$ implies that there exists a sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η) such that $\limsup_{i\to\infty} \tau(\mu_i, \eta_i) = \infty$ and $\eta_i \succsim_\infty \mu_i$ for all i.

By definition, $(\mu, \eta) \notin \Omega$ implies that there exists a sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η) and $\limsup_{i\to\infty} \tau(\mu_i, \eta_i) > \tau(\mu, \eta)$. Without loss of generality, $\tau(\mu_i, \eta_i) > \tau(\mu, \eta)$ for all i. Suppose by way of contradiction that $\limsup_{i\to\infty} \tau(\mu_i, \eta_i) = T < \infty$. Thus, there exists I such that for all $i \geq I$, $T + 1 > \tau(\mu_i, \eta_i)$. Also, for large enough i, $\mu_i \succ_{\tau(\mu,\eta)} \eta_i$, and since

$$T + 1 > \tau(\mu_i, \eta_i) > \tau(\mu, \eta),$$

it follows that for all large enough i, $\eta_i \succsim_{T+1} \mu_i$; this is because $\tau(\mu_i, \eta_i) > \tau(\mu, \eta)$ implies $\eta_i \succsim_{\tau(\mu_i,\eta_i)} \mu_i$, and $T + 1 > \tau(\mu_i, \eta_i)$ implies $\eta_i \succsim_{T+1} \mu_i$. By continuity of $\succsim_{T+1}$, $\eta \succsim_{T+1} \mu$. But since $T+1 > \tau(\mu, \eta)$, this contradicts the hypothesis that $\mu \succ_{\tau(\mu,\eta)} \eta$. Therefore, $\limsup_{i\to\infty} \tau(\mu_i, \eta_i) = \infty$.

E.2 Proof of Thm 3

By definition we have $[\mu \sim_\infty \eta \Longrightarrow \mu \sim^* \eta]$ and $[\mu \succ^* \eta \Longrightarrow \mu \succ_\infty \eta]$ for all $\mu, \eta \in \mathcal{A}$, so $\succsim_\infty$ agrees with $\succsim^*$ on the subset S defined in the statement of the theorem. To see that S is dense, begin by noting that disagreements can only take the form of $\mu \succ_\infty \eta$ and $\mu \sim^* \eta$. In this case there is a reversal at infinity, that is, there exists a sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η) such that $\limsup_{i\to\infty} \tau(\mu_i, \eta_i) = \infty$ and $\eta_i \succsim_\infty \mu_i$ for all i.

If $\eta_{i_m} \sim_\infty \mu_{i_m}$ for some subsequence then we are done. If not, then wlog we can suppose that $\eta_i \succ_\infty \mu_i$ for all i. Since $\mu \succ_\infty \eta$, there is an earliest N such that $\mu \succ_N \eta$. Continuity implies that $\mu_i \succ_N \eta_i$ for large i. We claim that $(\mu_i, \eta_i) \in \Omega$ for all large i. Indeed, for any large i, we have $\mu_i \succ_N \eta_i$ and $\eta_i \succ_\infty \mu_i$, and letting $\tau := \tau(\mu_i, \eta_i)$ we have $\eta_i \succ_\tau \mu_i$. Now take any sequence $(\mu_i^m, \eta_i^m) \to (\mu_i, \eta_i)$. By Continuity, $\eta_i^m \succ_\tau \mu_i^m$ and $\mu_i^m \succ_N \eta_i^m$ for large m, and Reversal implies that $\tau(\mu_i^m, \eta_i^m) \leq \tau(\mu_i, \eta_i)$ for large m. Consequently $\limsup_{m\to\infty} \tau(\mu_i^m, \eta_i^m) \leq \tau(\mu_i, \eta_i)$ and in turn, $(\mu_i, \eta_i) \in \Omega$, as desired.

To conclude the proof, we note that by Theorem 2, $(\mu_i, \eta_i) \in \Omega$ and $\eta_i \succ_\infty \mu_i$ for all large i implies that we have a sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η) such that $\mu_i \succ^* \eta_i$ for all large i. Therefore S is dense in $\mathcal{A} \times \mathcal{A}$.

F Appendix: Proof of Theorems 4 and 5

That $\succsim_\infty$ is not necessarily continuous follows from Theorem 2. In particular, if $\mu \succ_\infty \eta$ and $(\mu, \eta) \notin \Omega$, then there exists a sequence $(\mu_i, \eta_i) \to (\mu, \eta)$ s.t. $\mu_i \precsim_\infty \eta_i$ for all i. The remaining implications for $\succsim_\infty$ are straightforward to establish. Below we establish properties of $\succsim^*$ only.

F.1 Proof of Thm 4(a)-(c)

To establish completeness, take any μ, η and suppose $\eta \not\succsim^* \mu$. By (4), $\eta \not\succsim_\infty \mu$, and by completeness of $\succsim_\infty$, $\mu \succsim_\infty \eta$. Then, again by (4), $\mu \succsim^* \eta$.

To establish continuity, we show that $\{\eta : \eta \succsim^* \mu\}$ is closed; the other case holds by an analogous argument. Take a sequence $\{\nu_i\}$ such that $\nu_i \succsim^* \mu$ for all i and $\nu_i \to \nu$. Also, for each j let $V_j \subset \mathcal{A} \times \mathcal{A}$ be a ball of radius 2^{-j} that contains (ν, μ). Because $\nu_i \to \nu$, for every j there exists i such that $(\nu_i, \mu) \in V_j$. Furthermore, $\nu_i \succsim^* \mu$ and the definition of $\succsim^*$ imply the existence a sequence $\{(\nu'_m, \mu'_m)\}$ such that $(\nu'_m, \mu'_m) \to (\nu_i, \mu)$ and $\nu'_m \succsim_\infty \mu'_m$ for all m. Since V_j is also a neighborhood of (ν_i, μ), for each j there exists m_j such that $(\nu'_{m_j}, \mu'_{m_j}) \in V_j$. By construction, $\nu'_{m_j} \succsim_\infty \mu'_{m_j}$ for each j and furthermore, $(\nu'_{m_j}, \mu'_{m_j}) \to (\nu, \mu)$ as $j \to \infty$. Thus $\nu \succsim^* \mu$, as desired.

A counterexample for transitivity is as follows.

Example 1 *Let $\mathcal{A} = [0, 2]$ (with the relative Euclidian topology) and suppose that each $\succsim_n$ is represented by the utility function u_n defined by 1*

$$u_n(\mu) = \begin{cases} n\mu & \text{if } \mu \leq \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \leq \mu \leq 1 \\ 1 + n(\mu - 1) & \text{if } 1 \leq \mu \leq 1 + \frac{1}{n} \\ 2 & \text{if } 1 + \frac{1}{n} \leq \mu \leq 2 \end{cases}, \quad \mu \in [0, 2].$$

Qualitatively, the utility function takes the following shape: u_n increases from 0 to 1 as μ goes from 0 to $\frac{1}{n}$ and then stays constant at 1 until $\mu = 1$. Then there is another increase from 1 to 2 as μ goes from 1 to $1 + \frac{1}{n}$ and stays constant at 2 until $\mu = 2$. As n increases, the upward sloping portions are *restricted to a smaller subdomain, and the pointwise limit is the step function*

$$u_\infty(\mu) = \begin{cases} 0 & \text{if } \mu = 0 \\ 1 & \text{if } 0 \leq \mu \leq 1 \\ 2 & \text{if } 1 < \mu \leq 2 \end{cases}, \quad \mu \in [0, 2].$$

In fact this utility function represents the removed preference $\succsim_\infty$. We show that $\succsim^*$ is intransitive. Observe that $2 \succ_\infty 1 \succ_\infty 0$. However, there is a sequence $(\mu_i, \eta_i) = (1 + \frac{1}{i-1}, 2) \to (1, 2)$ with $\mu_i \sim_\infty \eta_i$ for large i, and similarly there is also a sequence $(\mu_i, \eta_i) \to (0, 1)$ with $\mu_i \sim_\infty \eta_i$ for large i. Thus $2 \sim^* 1 \sim^* 0$. Yet, for any $(\mu_i, \eta_i) \to (0, 2)$, we have $\eta_i \succ_\infty \mu_i$ and thus $2 \succ^* 0$, an intransitivity.

Therefore indifference may be intransitive. We observe next that strict preference is transitive. For later use, we also make an additional observation.

Take any μ, η, ν.

Claim 1 *$\mu \succ^* \eta \succ^* \nu \Longrightarrow \mu \succ^* \nu$.*

Proof. Then $\mu \succ_\infty \eta \succ_\infty \nu$ and by the (obvious) transitivity of $\succsim_\infty$ we have $\mu \succ_\infty \nu$ and thus $\mu \succsim^* \nu$. Suppose by way of contradiction that $\mu \sim^* \nu$. Then there exists a sequence $(\mu_i, \nu_i) \to (\mu, \nu)$ such that $\nu_i \succsim_\infty \mu_i$ for each i. Since $\mu \succ^* \eta$, by definition it must be that for all large i we have $\mu_i \succ_\infty \eta$. By transitivity of $\succsim_\infty$, $\nu_i \succsim_\infty \eta$ for all large i. But this contradicts $\eta \succ^* \nu$. ■

Claim 2 *$[\mu \succ^* \eta \sim^* \nu \Longrightarrow \mu \succsim^* \nu]$ and $[\mu \sim^* \eta \succ^* \nu \Longrightarrow \mu \succsim^* \nu]$.*

Proof. Consider $\mu \succ^* \eta \sim^* \nu$ (the argument for the other case is analogous). Suppose by way of contradiction that $\nu \succ^* \mu$. Then $\nu \succ^* \mu \succ^* \eta$ and so by the previous lemma, $\nu \succ^* \eta$, a contradiction. ■

F.2 Proof of Thm 4(d)

We show that $\mu \succ_{n'} \eta$ and $\mu \prec_{n''} \eta$ for $n'' > n'$ implies $\mu \prec_\infty \eta$ and $(\mu, \eta) \in \Omega$. It then follows from Theorem 2 that $\mu \prec^* \eta$, as desired. Given $\mu \succ_{n'} \eta$ and $\mu \prec_{n''} \eta$ for $n'' > n'$, Lemma 1 implies $\mu \prec_{\tau(\mu,\eta)} \eta$, which in turn implies the first assertion $\mu \prec_\infty \eta$. To show that $(\mu, \eta) \in \Omega$, first observe that by Reversal, $\mu \succ_{n'} \eta$ and $\mu \prec_{n''} \eta$ for $n'' > n'$ implies $\mu \succ_0 \eta$. Now take any sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η). By Continuity, there exists I such that for all $i \geq I$, $\mu_i \succ_0 \eta_i$ and $\mu_i \prec_{\tau(\mu,\eta)} \eta_i$. It follows by Reversal that for all $i \geq I$, $\tau(\mu_i, \eta_i) \leq \tau(\mu, \eta)$, and so, $\limsup_{i\to\infty} \tau(\mu_i, \eta_i) \leq \tau(\mu, \eta)$. That is, $(\mu, \eta) \in \Omega$, as was to be shown.

F.3 Proof of Thm 5

Lemma 2 *If $\{\succsim_n\}$ satisfies Order, Continuity and Independence, then*
(i) $\mu \succ^* \eta \Longrightarrow \mu\alpha\nu \succsim^* \eta\alpha\nu$.
(ii) $\mu \sim^* \eta \Longrightarrow \mu\alpha\nu \sim^* \eta\alpha\nu$.
(iii) $\mu\alpha\nu \succ^* \eta\alpha\nu \Longrightarrow \mu \succ^* \eta$.

Proof. Proof of (i): Given the definition of $\succsim^*$ and the fact that $\succsim_\infty$ satisfies Independence, $\mu \succ^* \eta \Longrightarrow \mu \succ_\infty \eta \Longrightarrow \mu\alpha\nu \succ_\infty \eta\alpha\nu \Longrightarrow \mu\alpha\nu \succsim^* \eta\alpha\nu$.

Proof of (ii): If $\mu \sim_\infty \eta$ then the claim follows from Independence and the definition of $\succsim^*$. If $\mu \not\sim_\infty \eta$ then $\mu \sim^* \eta$ implies that $(\mu, \eta) \notin \Omega$. That is, there exists a sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η) such that $\eta_i \succsim_\infty \mu_i$. But then $\{(\mu_i\alpha\nu, \eta_i\alpha\nu)\}$ is a sequence that converges to $(\mu\alpha\nu, \eta\alpha\nu)$ such that $\eta_i\alpha\nu \succsim_\infty \mu_i\alpha\nu$. Thus $(\mu\alpha\nu, \eta\alpha\nu) \notin \Omega$.

Proof of (iii): Suppose $\mu\alpha\nu \succ^* \eta\alpha\nu$ and $\mu \not\succ^* \eta$. Then by (i) and (ii), $\mu\alpha\nu \precsim^* \eta\alpha\nu$, a contradiction. ■

Lemma 3 *If $\{\succsim_n\}$ satisfies Order, Continuity, Independence and (1), then:*
(a) $\succsim^*$ satisfies Independence.
(b) $\succsim^*$ is transitive.

Proof. Proof of (a). Given the previous Lemma, all that needs to be shown is that $\mu \succ^* \eta \Longrightarrow \mu\alpha\nu \succ^* \eta\alpha\nu$. But this follows easily from (1).

Proof of (b). Given claims 1 and 2, we just need to establish transitivity of indifference, that is, $[\mu \sim^* \eta \sim^* \nu \Longrightarrow \mu \sim^* \nu]$. Suppose by way of contradiction that $\mu \not\sim^* \nu$. Wlog, suppose $\mu \succ^* \nu$. Consider each of the following subcases (observe that $\mu \succ^* \nu$ implies $\mu \succ_\infty \nu$, and that transitivity of $\succsim_\infty$ implies that these subcases are exhaustive).

(i) $\mu \succ_\infty \eta \precsim_\infty \nu$.

Observe that $\mu \succ_\infty \eta$ and $\mu \sim^* \eta$ implies that there exists a sequence $\{(\mu_i, \eta_i)\}$ that converges to (μ, η) such that $\eta_i \succsim_\infty \mu_i$. Since $\nu \succsim_\infty \eta$, Independence implies $\eta_i \frac{1}{2} \nu \succsim_\infty \mu_i \frac{1}{2} \eta$ and thus $\eta_i \frac{1}{2} \nu \succsim^* \mu_i \frac{1}{2} \eta$ for all i. By continuity $\eta\frac{1}{2}\nu \succsim^* \mu\frac{1}{2}\eta$, which by an application (a) implies $\nu \succsim^* \mu$, a contradiction.

(ii) $\eta \succ_\infty \nu$

Since $\eta \sim^* \nu$ and $\eta \succ_\infty \nu$, there is a reversal at infinity, and thus there exists the usual sequence $\nu_i \succsim_\infty \eta_i$. If $\eta \succsim_\infty \mu$ then by Independence, $\eta\frac{1}{2}\nu_i \succsim_\infty \mu\frac{1}{2}\eta_i$ and thus $\eta\frac{1}{2}\nu_i \succsim^* \mu\frac{1}{2}\eta_i$ for all i, but then by continuity, $\eta\frac{1}{2}\nu \succsim^* \mu\frac{1}{2}\eta$, and part (a) implies $\nu \succsim^* \mu$, a contradiction. If on the other hand $\mu \succ_\infty \eta$, then $\mu \sim^* \eta$ implies that there exists a sequence $\eta_i' \succsim_\infty \mu_i$. By Independence, $\eta_i'\frac{1}{2}\nu_i \succsim_\infty \mu_i\frac{1}{2}\eta_i$ and thus $\eta_i'\frac{1}{2}\nu_i \succsim^* \mu_i\frac{1}{2}\eta_i$ for all i, but then by continuity, $\eta\frac{1}{2}\nu \succsim^* \mu\frac{1}{2}\eta$, and (a) implies $\nu \succsim^* \mu$, a contradiction. ■

F.4 Proof of Claims in footnote 7

The fact that $\succsim_\infty$ satisfies Independence is straightforward to establish. We show below that if $\succsim^*$ is transitive then it must satisfy Independence as well. This is proved in 5 steps, and will make use of Theorem 2.

Step 1: $\mu \succ_n \eta \Longleftrightarrow \mu\alpha\nu \succ_n \eta\alpha\nu$, for all n.

Axioms 1, 2 and 3 together imply this stronger version of Independence.

Step 2: $\tau(\mu,\eta) = \tau(\mu\alpha\nu, \eta\alpha\nu)$ and $\mu \succ_{\tau(\mu,\eta)} \eta \Longleftrightarrow \mu\alpha\nu \succ_{\tau(\mu\alpha\nu,\eta\alpha\nu)} \eta\alpha\nu$.

This follows from Step 1.

Step 3: $(\mu,\eta) \notin \Omega \Longrightarrow (\mu\alpha\nu, \eta\alpha\nu) \notin \Omega$.

If $\{(\mu_i,\eta_i)\}$ is a sequence that converges to (μ,η) and

$$\limsup_{i\to\infty} \tau(\mu_i,\eta_i) > \tau(\mu,\eta),$$

then $\{(\mu_i\alpha\nu, \eta_i\alpha\nu)\}$ is a sequence that converges to $(\mu\alpha\nu, \eta\alpha\nu)$ and, by the first assertion in Step 2,

$$\limsup_{i\to\infty} \tau(\mu_i\alpha\nu, \eta_i\alpha\nu) > \tau(\mu\alpha\nu, \eta\alpha\nu).$$

Thus, $(\mu\alpha\nu, \eta\alpha\nu) \notin \Omega$.

Step 4: $\mu \sim^* \eta \Longrightarrow \mu\alpha\nu \sim^* \eta\alpha\nu$.

Suppose $\mu \sim^* \eta$. Then

$\mu \sim^* \eta$

$\Longrightarrow \mu \sim_{\tau(\mu,\eta)} \eta$ or $(\mu,\eta) \notin \Omega$ by Theorem 2

$\Longrightarrow \mu\alpha\nu \sim_{\tau(\mu\alpha\nu,\eta\alpha\nu)} \eta\alpha\nu$ or $(\mu\alpha\nu, \eta\alpha\nu) \notin \Omega$ by Steps 2 and 3

$\Longrightarrow \mu\alpha\nu \sim^* \eta\alpha\nu$, as desired.

Step 5: $\mu \succ^* \eta \Longrightarrow \mu\alpha\nu \succ^* \eta\alpha\nu$.

By the main theorem in Herstein and Milnor [11], under completeness, transitivity and continuity of $\succsim^*$, Step 4 implies the result.

G Appendix: Proof of Theorem 6

Note that each $\succsim_n$ is also represented by the utility function

$$\widehat{U}_n(c_0, c_1, .., c_T) = \sum_{t=0}^{T} \frac{\phi(t+n)}{\phi(n)} u(c_t),$$

which is obtained by dividing $U_n(\cdot)$ by the strictly positive constant, $\phi(n)$. Define the function $U^* : C^{T+1} \to \mathbb{R}$ by:

$$U^*(c_0, c_1, .., c_T) = \sum_{t=0}^{T} \phi^*(t) u(c_t),$$
$$\text{where } \phi^*(t) \equiv \lim_{n\to\infty} \frac{\phi(t+n)}{\phi(n)}.$$

The first lemma shows this function is well-defined.

Lemma 4 *For each t, $\lim_{n\to\infty} \frac{\phi(t+n)}{\phi(n)}$ exists. Moreover, $\widehat{U}_n$ converges uniformly to U^*.*

Proof. Step 1: $\frac{\phi(t+n)}{\phi(n)}$ is bounded above by 1.

By assumption, $\phi(\cdot)$ is decreasing. Therefore, for all n and t, $\phi(t+n) \leq \phi(n)$.

Step 2: $\frac{\phi(t+n)}{\phi(n)}$ is weakly increasing in n.

Note that $\frac{\phi(t+n)}{\phi(n)} = \frac{\phi(t+n)}{\phi(t+n-1)} \cdot \frac{\phi(t+n-1)}{\phi(t+n-2)} \cdots \frac{\phi(n+2)}{\phi(n+1)} \cdot \frac{\phi(n+1)}{\phi(n)}$, and $\frac{\phi(t+n+1)}{\phi(n+1)} = \frac{\phi(t+n+1)}{\phi(t+n)} \frac{\phi(t+n)}{\phi(t+n-1)} \cdot \frac{\phi(t+n-1)}{\phi(t+n-2)} \cdots \frac{\phi(n+2)}{\phi(n+1)}$, and therefore,

$$\frac{\frac{\phi(t+n+1)}{\phi(n+1)}}{\frac{\phi(t+n)}{\phi(n)}} = \frac{\frac{\phi(t+n+1)}{\phi(t+n)}}{\frac{\phi(n+1)}{\phi(n)}}.$$

But by assumption, $\frac{\phi(r+2)}{\phi(r+1)} \geq \frac{\phi(r+1)}{\phi(r)}$ for any integer r. Thus, $\frac{\frac{\phi(t+n+1)}{\phi(t+n)}}{\frac{\phi(n+1)}{\phi(n)}} \geq 1$ and in particular, $\frac{\frac{\phi(t+n+1)}{\phi(n+1)}}{\frac{\phi(t+n)}{\phi(n)}} \geq 1$ for all t and n.

Step 3: The sequence $\{\widehat{U}_n\}$ converges uniformly to U^*.

By Steps 1 and 2, for each t, the sequence $\{\frac{\phi(t+n)}{\phi(n)}\}_{n=0}^{\infty}$ is a bounded, increasing sequence. Thus, $\phi^*(t) \equiv \lim_{n\to\infty} \frac{\phi(t+n)}{\phi(n)}$ is well-defined. Observe further that $\{\widehat{U}_n\}$ is a sequence of continuous real functions defined on a compact space C^{T+1}, and that it is monotone increasing and converges pointwise to U^*. By Dini's Theorem [3, Theorem 2.62], the convergence is uniform. ■

Lemma 5 $\phi^*(t) = \gamma^t > 0$ *for each* $0 \le t \le T$.

Proof. For any k, $\{\frac{\phi(1+n+k)}{\phi(n+k)}\}_{n=0}^{\infty}$ is a subsequence of the increasing sequence $\{\frac{\phi(1+n)}{\phi(n)}\}_{n=0}^{\infty}$ that we showed to be convergent in Step 3 of Lemma 4. Hence, for all k,

$$\lim_{n\to\infty} \frac{\phi(1+n)}{\phi(n)} = \lim_{k\to\infty} \frac{\phi(1+n+k)}{\phi(n+k)}.$$

Define $\gamma \equiv \lim_{n\to\infty} \frac{\phi(1+n)}{\phi(n)} > 0$ and recall that we defined $\phi^*(t) \equiv \lim_{n\to\infty} \frac{\phi(t+n)}{\phi(n)}$ above. For any t,

$$\frac{\phi(t+n)}{\phi(n)} = \Pi_{r=0}^{t-1} \frac{\phi(t+n-r)}{\phi(t+n-r-1)}.$$

Therefore,

$\phi^*(t) = \lim_{n\to\infty} \frac{\phi(t+n)}{\phi(n)} = \lim_{n\to\infty} \Pi_{r=0}^{t-1} \frac{\phi(t+n-r)}{\phi(t+n-r-1)} = \Pi_{r=0}^{t-1} \lim_{n\to\infty} \frac{\phi(t+n-r)}{\phi(t+n-r-1)}$ $= \Pi_{r=0}^{t-1} \lim_{n\to\infty} \frac{\phi(1+n)}{\phi(n)} = \Pi_{r=0}^{t-1}\gamma = \gamma^t$, as desired. ■

Lemma 6 *The closed limit* $\lim_{n\to\infty} \Gamma(\succsim_n)$ *exists and is represented by* U^*

Proof. Since u is nonconstant and C is bounded, there is a best and worst stream $\rho, \nu \in C^{T+1}$ and $U^*(\rho) > U^*(\nu)$. Therefore, for every $\mu, \eta \in C^{T+1}$,

$$U^*(\mu) \ge U^*(\eta) \Longrightarrow U^*(\mu\alpha\rho) > U^*(\eta\alpha\nu), \text{ for all } \alpha \in (0,1)\,, \qquad (11)$$

where the stream $\mu\alpha\rho$ is the pointwise mixture of the streams μ and ρ, that is, $(\alpha\mu_1 + (1-\alpha)\rho_1,\, \alpha\mu_2 + (1-\alpha)\rho_2, ...)$. This observation will be used below. Let $\succsim'$ be the preference relation represented by U^*. As in Appendix A, identify any binary relation B on $\mathcal{A}$ with its graph $\Gamma(B) \subset \mathcal{A} \times \mathcal{A}$. We show that $\Gamma(\succsim') = \lim_{n\to\infty} \Gamma(\succsim_n)$.

First establish $Ls\Gamma(\succsim_n) \subset \Gamma(\succsim')$. If $(\mu, \eta) \in Ls\Gamma(\succsim_n)$ then there is a subsequence $\{\Gamma(\succsim_{n(m)})\}$ and a sequence $\{(\mu_m, \eta_m)\}$ that converges to (μ, η) such that $(\mu_m, \eta_m) \in \Gamma(\succsim_{n(m)})$ for each m. Therefore, for each m,

$$U_{n(m)}(\mu_m) \geq U_{n(m)}(\eta_m).$$

Since $U_{n(m)}$ converges to U^* uniformly, it follows that $U^*(\mu) \geq U^*(\eta)$. Hence $(\mu, \eta) \in \Gamma(\succsim')$, as desired.

Next establish $\Gamma(\succsim') \subset Li\Gamma(\succsim_n)$. Let $(\mu, \eta) \in \Gamma(\succsim')$ and take any neighborhood O of (μ, η). By (11), there exists $\alpha \in (0, 1]$ s.t $(\mu\alpha\rho, \eta\alpha\nu) \in O$ and $U^*(\mu\alpha\rho) > U^*(\eta\alpha\nu)$. By Step 3 of Lemma 4, there exists $T < \infty$ such that $U_n(\mu\alpha\rho) > U_n(\eta\alpha\nu)$ for all $n \geq T$, that is, $(\mu\alpha\rho, \eta\alpha\nu) \in \Gamma(\succsim_n)$ for all $n \geq T$. Hence $V \cap \Gamma(\succsim_n) \neq \phi$ for all but a finite number of n, that is, $(\mu, \eta) \in Li\Gamma(\succsim_n)$. ■

Lemma 7 *For any* $(c_0, c_1, .., c_T)$ *and* $(c'_0, c'_1, .., c'_T)$,

$$U^*(c_0, c_1, .., c_T) > U^*(c'_0, c'_1, .., c'_T) \Longrightarrow (c_0, c_1, .., c_T) \succ_\infty (c'_0, c'_1, .., c'_T).$$

Proof. This follows from the fact that $\widehat{U}_n$ converges to U^* pointwise, and that $\widehat{U}_n$ represents $\succsim_n$. ■

Lemma 8 $\succsim^*$ *is represented by* U^*.

Proof. Denote consumption streams by c, c'. Given observation (11), $U^*(\mu) \geq U^*(\eta)$ implies that there is a sequence $(\mu_n, \eta_n) \to (\mu, \eta)$ s.t. $U^*(\mu_n) > U^*(\eta_n)$ and thus by the previous lemma, $\mu_n \succ_\infty \eta_n$. Therefore $\mu \succsim^* \eta$. Next suppose $U^*(\mu) > U^*(\eta)$. Since U^* is continuous, for any sequence $(\mu_n, \eta_n) \to (\mu, \eta)$ it must be that $U^*(\mu_n) > U^*(\eta_n)$ for all large n, and thus $\mu_n \succ_\infty \eta_n$. It follows that $\mu \succ^* \eta$. ■

H Appendix: Proof of Theorem 7

Recall that $\succsim$ is represented by

$$V(p) = v(1 - \sum_{j=1}^{k_p} p(x_i))u(0) + \sum_{j=1}^{k_p} [v(\sum_{i=1}^{j} p(x_i)) - v(\sum_{i=1}^{j-1} p(x_i))]u(x_j), \qquad p \in \Delta,$$

and $u(\cdot)$ is normalized so that $u(0) = 0$. Observe that we can write

$$\begin{aligned}
V(\frac{1}{n}p + \frac{n-1}{n}0) &= v(1 - \frac{1}{n}[1 - p(0)])u(0) + \sum_{j=1}^{k_p}[v(\frac{1}{n}\sum_{i=1}^{j} p(x_i)) - v(\frac{1}{n}\sum_{i=1}^{j-1} p(x_i))]u(x_j). \\
&= \sum_{j=1}^{k_p}[v(\frac{1}{n}\sum_{i=1}^{j} p(x_i)) - v(\frac{1}{n}\sum_{i=1}^{j-1} p(x_i))]u(x_j) \\
&= \frac{v'(0)}{n}\sum_{j=1}^{k_p}\frac{n}{v'(0)}[v(\frac{1}{n}\sum_{i=1}^{j} p(x_i)) - v(\frac{1}{n}\sum_{i=1}^{j-1} p(x_i))]u(x_j) \\
&= \frac{v'(0)}{n}\sum_{j=1}^{k_p}\pi_n^j u(x_j) \\
&\equiv V_n(p)
\end{aligned}$$

where $\pi_n^j \equiv \frac{n}{v'(0)}[v(\frac{1}{n}\sum_{i=1}^{j} p(x_i)) - v(\frac{1}{n}\sum_{i=1}^{j-1} p(x_i))]$. Therefore, $\succsim_n$ is represented by $\hat{V}_n$. Define $\hat{V}_n$ by $\hat{V}_n(p) = \sum_{i=1}^{j}\pi_n^j u(x_j)$. Note that $V_n(p) \geq V_n(q) \iff \hat{V}_n(p) \geq \hat{V}_n(q)$.

We show that $\hat{V}_n \to \hat{V}$ pointwise and that $\hat{V}$ is expected utility. Take a first order Taylor's expansion of $v(\cdot)$ around $\frac{1}{n}\sum_{i=1}^{j-1} p(x_i)$ to get

$$v(x) = v(\frac{1}{n}\sum_{i=1}^{j-1} p(x_i)) + v'(\frac{1}{n}\sum_{i=1}^{j-1} p(x_i))[x - \frac{1}{n}\sum_{i=1}^{j-1} p(x_i)] + v''(x')\frac{[x - \frac{1}{n}\sum_{i=j+1}^{k_p} p(x_i)]^2}{2}$$

for some $x' \in [\frac{1}{n}\sum_{i=1}^{j-1} p(x_i), \frac{1}{n}\sum_{i=1}^{j} p(x_i)]$. For any $n > 0$, let $\bar{v}''_n$ be the maximum of $v''(\cdot)$ on $[0, \frac{1}{n}]$. This maximum is well defined and attained because $v''(\cdot)$ is continuous by hypothesis and $[0, \frac{1}{n}]$ is compact. Further, $0 \leq v''(x') \leq \bar{v}''_n$ for any x' in $[\frac{1}{n}\sum_{i=1}^{j-1} p(x_i), \frac{1}{n}\sum_{i=1}^{j} p(x_i)]$. By taking $x = \frac{1}{n}\sum_{i=1}^{j} p(x_i)$,

$$\pi_n^j \leq p(x_j)\frac{v'(\frac{1}{n})}{v'(0)} + \frac{\bar{v}''_n p(x_j)^2}{2v'(0)n} \to p(x_j),$$

since $v'(\frac{1}{n}\sum_{i=j+1}^{k_p} p(x_i)) \leq v'(\frac{1}{n})$. Moreover,

$$\pi_n^j \geq p(x_j)\frac{v'(0)}{v'(0)} = p(x_j),$$

since $v'(\frac{1}{n}\sum_{i=1}^{j-1} p(x_i)) \geq v'(0)$. Both these inequalities follow from $v''(\cdot) \geq 0$. Therefore, $\pi_n^j \to p(x_j)$ for all p and all $j \leq k_p$. Defining $U^*(p) = \sum_{i=1}^{k_p} u(x_i)p(x_i)$, the above shows that $\hat{V}_n(p) \to U^*(p)$ for any p. Note that $U^*(\cdot)$ is expected utility.

Finally, we show that the closed removed preference $\succsim^*$ is represented by U^*. Let $\widehat{\succsim}$ be the preference represented by U^*. We first observe that $U^*(p) > U^*(q)$ implies $p \succ_\infty q$. Suppose $U^*(p) > U^*(q)$. Since $\hat{V}_n(p) \to U^*(p)$ and $\hat{V}_n(q) \to U^*(q)$, $\exists N$ so that $\hat{V}_n(p) > \hat{V}_n(q)$ all $n > N$, and hence $p \succ_\infty q$, as desired. In particular, $p \succsim^* q$ imples $p \succsim_\infty q$ implies $U^*(p) \geq U^*(q)$. To conclude that $\succsim^*$ is represented by U^*, we show that $p \succ^* q$ implies $U^*(p) > U^*(q)$. So suppose $p \succ^* q$ and suppose by way of contradiction that $U^*(q) \geq U^*(p)$. By Assumption 1 (which implies that u and thus U^* is nonconstant) and the fact that U^* is expected utility, there exists a sequence $(p_n, q_n) \to (p, q)$ such that $U^*(q_i) > U^*(p_i)$ for all i (these can be constructed by mixing q with the U^*-best lottery and p with the U^*-worst lottery). By the observation we just made, it follows that $q_i \succ_\infty p_i$ for all i but, by definition of $\succsim^*$ as the closure of $\succsim_\infty$, this contradicts the hypothesis that $p \succ^* q$. This completes the proof.

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