Decreasing Impatience and the Magnitude Effect Jointly Contradict Exponential Discounting*

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Abstract

The experimental literature on time preference finds that the manner in which subjects discount money (as opposed to utility) exhibits properties known as Decreasing Impatience and the Magnitude Effect. While these findings are often referred to as anomalies for the Exponential Discounting model, several authors have demonstrated that each of these qualitative findings can be explained by the curvature of utility and thus are not anomalies. We prove that, under basic regularity conditions, the two findings jointly imply the existence of Preference Reversals, and thus jointly contradict the Exponential Discounting model.

Keywords: Time Preference, Preference Reversals, Decreasing Impatience, Magnitude Effect, DU anomalies.

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1 Introduction

The manner in which an agent trades-off money across time is captured in his preferences $\succsim$ over *dated rewards*; a dated reward is a pair (m, t) that specifies a reward \$$m$ to be received after a delay of t periods. The standard theory of intertemporal choice – the Exponential Discounted Utility (EDU) model – posits that the agent's preferences $\succsim$ admit the representation:

$$U(m, t) = \delta^t u(m), \tag{1}$$

where $u(m)$ is a strictly increasing utility function that evaluates the reward and δ^t is an exponential discount function with $\delta \in (0, 1)$ that evaluates the delay.

There is a sizable experimental literature that explores the descriptive validity of the EDU model (see [3, 5] for a review). Two well-known robust experimental findings are *Decreasing Impatience* (greater patience in more distant trade-offs between rewards) and the *Magnitude Effect* (greater patience toward larger rewards). These findings are often referred to as anomalies for the EDU model, but in fact neither has been shown to qualitatively contradict the EDU model. While the EDU model is a theory of the discounting of the *utility* from money, the findings concern only how subjects directly discount *money*, and indeed, several authors have shown that each of these behaviors can arise purely due to the *curvature of the utility u for money* [4, 5, 6].[1] See Section 2.2 below for a demonstration.

This note shows that although Decreasing Impatience and the Magnitude Effect are individually consistent with the EDU model, *they jointly contradict it*. This is a general result, proved in an abstract ordinal setting under only very basic regularity conditions. In particular, we make no use of utility functions (let alone assume properties like differentiability) nor make assumptions about risk preferences, etc. Our results also show that Decreasing Impatience and the Magnitude Effect bear a relationship with a third experimental finding, known as *Preference Reversals*.

[1] The literature that views the findings as anomalies argues that the curvature of utility required to fit the data implies unreasonable degrees of risk aversion. The weakness in this argument is that it presumes that subjects respect Expected Utility theory (or at least, that the curvature of utility characterizes risk attitudes). We take the position that a given theory of intertemporal choice should be judged by its implications for intertemporal choice behavior only, and in particular, it should be judged independently of any theory of choice in another domain.

2 Main Result

Suppose that time is discrete and given by $\mathcal{T} = \{0, 1, 2, ...\}$. Let the nondegenerate real interval $\mathcal{M} = [0, M]$ represent the set of possible rewards. The set of *dated rewards* is $X = \mathcal{M} \times \mathcal{T}$.[2] The primitive is a preference relation $\succsim$ over X. The majority of experiments elicit such preference data [3, 5], and in particular, the experimental findings that motivate us are explicitly properties of such preference data.

2.1 Definitions

The EDU model is formally defined by:

Definition 1 (EDU Representation) *The preference $\succsim$ admits an EDU representation if there exists a strictly increasing continuous utility function $u : \mathcal{M} \to \mathbb{R}_+$ with $u(0) = 0$ and a discount factor $\delta \in (0, 1)$ such that $\succsim$ is represented by the function $U(m, t) = \delta^t u(m)$ defined for all $(m, t) \in X$.*

We restrict attention to preferences that satisfy the following regularity conditions.

Definition 2 (Regularity) *A preference $\succsim$ over X is regular if it satisfies:*
1. Order: $\succsim$ is complete and transitive.
2. Continuity: For each (m, t), the sets $\{(m', t') : (m', t') \succsim (m, t)\}$ and $\{(m', t') : (m, t) \succsim (m', t')\}$ are closed.
3. Impatience: For all t and $m > 0$, $(m, t) \succ (m, t + 1)$ and $(0, t) \sim (0, t + 1)$.
4. Monotonicity: For any t, if $m > m'$ then $(m, t) \succ (m', t)$.

Order and Continuity are standard. Impatience states that 'earlier is better' and that the timing of a \$0 reward is a matter of indifference for the agent. Monotonicity states that 'more is better'.

Decreasing Impatience and the Magnitude Effect are properties of how subjects discount money directly. Formally, let $\psi(m, t)$ denote the *present equivalent* of a reward (m, t), which is defined by:

$$(\psi(m, t), 0) \sim (m, t). \tag{2}$$

[2] $\mathcal{M}$ is endowed with the Euclidean subspace topology, $\mathcal{T}$ with the discrete topology and X with the product topology.

Then the *money-discount function* $\phi : \mathcal{M}\backslash\{0\} \times \mathcal{T} \to \mathbb{R}_+$ is defined for each $m > 0$ and t by

$$\phi(m,t) := \frac{\psi(m,t)}{m}. \tag{3}$$

For instance, if the agent finds \$80 now just as good as \$100 at time t, then the \$100 reward loses 20% of its money value (as opposed to utility) due to the t period delay, and we say that $(100,t)$ is discounted by a (t-period) money-discount factor of $\phi(100,t) = \frac{80}{100} = 0.8$.

Definition 3 (Decreasing Impatience) $\succsim$ *exhibits Decreasing Impatience for Money if* $\frac{\phi(m,t+\tau)}{\phi(m,t)}$ *is weakly increasing and nonconstant in* t *for any* $m, \tau > 0$.

The exponential discounting function $D(t) = \delta^t$ has the property that $\frac{D(t+1)}{D(t)}$ is constant for all t. This may be referred to as "Constant Impatience", as the manner in which the agent trades-off rewards at time t and $t+1$ is independent of t. Decreasing Impatience is the property that the agent becomes more patient in his trade-offs between t and $t+1$ as t increases. That is, there is greater patience in more distant trade-offs. The hyperbolic discount function $D(t) = \frac{1}{1+t}$ exhibits Decreasing Impatience.

Definition 4 (Magnitude Effect) $\succsim$ *exhibits the Magnitude Effect for Money if* $\phi(m,t)$ *is weakly increasing in* m *for any* t.

The exponential and hyperbolic discount functions exhibit "magnitude-independent discounting", as the degree of patience is independent of the size of the rewards being considered. The Magnitude Effect is the property that larger rewards are treated with greater patience. The reader should observe, however, that we define the Magnitude Effect in a way so as to include magnitude-independence (that is, constant $\phi(\cdot,t)$) as a special case.

2.2 Result

Decreasing Impatience and the Magnitude Effect are individually consistent with the EDU model [4, 5, 6]. To see this, observe that if the agent respects the EDU model then the definition (2) of the present equivalent $\psi(m,t)$

implies $u(\psi(m,t)) = \delta^t u(m)$, and thus $\psi(m,t) = u^{-1}(\delta^t u(m))$. In particular, the money-discount function, defined by (3), is given by:

$$\phi(m,t) = \frac{u^{-1}(\delta^t u(m))}{m},$$

for all $m, t > 0$. Clearly then, the money-discount function ϕ is *generically* nonexponential and magnitude-dependent although the underlying utility-discount function δ^t is exponential and magnitude-independent. It is easily verified that Decreasing Impatience is exhibited, for instance, when we take log utility $u(m) = \ln m$ and restrict attention to rewards $m > 1$. Loewenstein and Prelec [4] identify a sufficient condition on u (called 'subproportionality') for it to give rise to the Magnitude Effect.

In order to determine what contradicts the EDU model, one must first behaviorally identify the class of preferences $\succsim$ that admit an EDU representation. Such an analysis is conducted by Fishburn and Rubinstein [2, Thm 2]:

Theorem (Fishburn and Rubinstein (1982)) *A regular preference $\succsim$ admits an EDU representation if and only if it satisfies Stationarity: for all s, l, τ and for all t,*

$$(s,0) \succsim (l,\tau) \Longleftrightarrow (s,t) \succsim (l,t+\tau).$$

Stationarity states that the preference between $(s,0)$ and (l,τ) does not reverse if any common delay t is applied to both rewards. Our main result states that Decreasing Impatience and the Magnitude Effect are sufficient conditions on the money-discount function that ensure the existence of reversals in preference.

Theorem 1 *If a regular preference $\succsim$ exhibits Decreasing Impatience and the Magnitude Effect, then $\succsim$ violates Stationarity. In particular, for every $l, \tau > 0$ there exists $0 < s < l$ and T such that*

$$(s,0) \succsim (l,\tau) \text{ and } (s,t) \prec (l,t+\tau) \text{ for all } t \geq T.$$

Thus, the Magnitude Effect and Decreasing Impatience jointly contradict the EDU model. The Theorem is a corollary of a more general result proved in the appendix. The trick in the proof is the observation that, under regularity, $\succsim$ can be represented by the function

$$V(m,t) = \phi(m,t) \cdot m.$$

That is, if we *pretend* that the agent possesses linear utility, then we are assured that he behaves *as if* he discounts money according to the money-discount function. This enables us to directly connect properties of the money-discount function ϕ with behavior $\succsim$.

3 Preference Reversals

Subjects in experiments are often observed to exhibit the following kind of reversals in preference [3]:

$$(100,\ \text{now}) \succ (120,\ 3\ \text{months}) \text{ and } (100,\ 1\ \text{year}) \prec (120,\ 1\ \text{year}\ 3\ \text{months}).$$

That is, the agent prefers a smaller earlier reward when it is available immediately but reverses preferences in favor of the larger later reward when both rewards are delayed by a common number of periods. This is known as a *Preference Reversal* or a *Common Difference Effect*. A notable feature of Theorem 1 is that the violation of Stationarity it ensures is of the nature of Preference Reversals: for various pairs of rewards, the larger later reward is preferred for any sufficient delay.

While the Theorem ensures such reversals only for some set of pairs of rewards, it also allows preferences to switch more than once before settling on a strict preference for the larger later reward. Preference Reversals are more typically understood to involve no more than one reversal – this is due to the presumption that they arise from utility-discount functions with features such as those captured by hyperbolic discounting [3, 4]. Below, we identify sufficient conditions for the no-more-than-one-reversal property to hold in every violation of Stationarity.

Consider the following restriction:

Definition 5 (Strong Magnitude Effect) $\succsim$ *exhibits a Strong Magnitude Effect for Money if* $\frac{\phi(m,t+1)}{\phi(m,t)}$ *is weakly increasing in* m *for any* t.

While the Magnitude Effect is based on how the agent feels about rewards of different magnitudes at any given time t, the Strong Magnitude Effect is based on how he feels about rewards at time $t+1$ relative to t. It requires greater relative patience (across any two given consecutive periods) with respect to larger rewards. A simple proof by induction establishes that, under regularity (specifically, under the condition that $\phi(m,0)=1$), the Strong Magnitude Effect is indeed stronger than the Magnitude Effect.[3]

The following result strengthens the hypothesis of Theorem 1 by replacing the Magnitude Effect with the Strong Magnitude Effect. This yields the stronger conclusion that $\succsim$ exhibits a *global* no-more-than-one-reversal property.

Theorem 2 *If a regular preference $\succsim$ satisfies Decreasing Impatience and a Strong Magnitude Effect, then $\succsim$ violates Stationarity, and moreover, for every $s, l, \tau > 0$ and T,*

$$\begin{aligned} (s,T) \ &\prec\ (l,T+\tau) \Longrightarrow (s,t) \prec (l,t+\tau) \text{ for all } t \geq T, \\ \text{and } (s,T) \ &\sim\ (l,T+\tau) \Longrightarrow (s,t) \precsim (l,t+\tau) \text{ for all } t \geq T. \end{aligned}$$

Thus, under the hypothesis, if the agent exhibits $(s,0) \succsim (l,\tau)$ and $(s,T) \prec (l,T+\tau)$, then there are no further reversals. Moreover, if $(s,0) \prec (l,\tau)$ then there is no reversal.

4 Conclusion

Our results contribute toward a better understanding of the available evidence on intertemporal choice by establishing a connection between three distinct findings in experiments, and by establishing that two of the findings jointly acquire the status of anomalies for the EDU model.[4] Our results also have value in guiding future experimental work. Recent experimental research recognizes the importance of accounting for the curvature of utility

[3] To see this, begin by observing that according to the Strong Magnitude Effect, $l > s$ implies $\frac{\phi(l,t+1)}{\phi(l,t)} \geq \frac{\phi(s,t+1)}{\phi(s,t)}$ for all t. Since $\phi(m,0)=1$, it follows immediately that $\phi(\cdot,1)$ is weakly increasing. Assuming the induction hypothesis that $\phi(\cdot,n)$ is weakly increasing for some $n>0$, the Strong Magnitude Effect yields that $l > s \Longrightarrow \frac{\phi(l,n+1)}{\phi(l,n)} \geq \frac{\phi(s,n+1)}{\phi(s,n)} \Longrightarrow \phi(l,n+1) \geq \phi(s,n+1)\frac{\phi(l,n)}{\phi(s,n)} \Longrightarrow \phi(l,n+1) \geq \phi(s,n+1)$ since $\frac{\phi(l,n)}{\phi(s,n)} \geq 1$ by the induction hypothesis. Indeed, $\phi(\cdot,n+1)$ is weakly increasing, and this proves that $\phi(\cdot,t)$ is weakly increasing for all t. That is, the Magnitude Effect holds.

[4] The practical application of this result is subject to the same considerations as that of the result that Stationarity characterizes the EDU model: In experimental settings, factors that are absent in the abstract EDU model but affect behavior (such as uncertainty about future endowments) need to be accounted for before such results can be invoked. See [6], and also [5] for a recent review of the critique of experiments.

when studying time preferences.[5] A natural question is whether this may lead to the disappearance of findings such as Decreasing Impatience and the Magnitude Effect. Indeed, Andersen et al [1] remark that "many 'discounting anomalies' have been suggested in the literature, and it is unclear a priori how the proper accounting for concave utility functions affects these anomalies". Our results help formulate priors by providing reason to believe that even after proper accounting for curvature of utility, all the 'discounting anomalies' may not disappear.

A Appendix: Proof of Theorem 1

The Theorem is a corollary of a more general result we prove here. Consider the following property.

Definition 6 (Eventually Lower Impatience) *A preference $\succsim$ over X exhibits Eventually Lower Impatience for Money for some $m, \tau > 0$ if $\phi(m,\tau) < \lim_{t\to\infty} \inf \frac{\phi(m,t+\tau)}{\phi(m,t)}$.*

Note that if we assume regularity then $\phi(m,0) = 1$, and Eventually Lower Impatience can be rewritten to state that $\frac{\phi(m,\tau)}{\phi(m,0)} < \lim_{t\to\infty} \inf \frac{\phi(m,t+\tau)}{\phi(m,t)}$. Thus, under regularity, the property is a substantial weakening of Decreasing Impatience. We prove the following Theorem.

Theorem 3 *If a regular preference $\succsim$ exhibits Eventually Lower Impatience for some $l, \tau > 0$ and the Magnitude Effect, then there exists $0 < s < l$ and T such that*

$$(s, 0) \succsim (l, \tau) \text{ and } (s, t) \prec (l, t+\tau) \text{ for all } t \geq T.$$

Proof. We prove the result in a series of steps.
Step 1: *Show that for each (m,t) there exists $\psi(m,t)$ such that $(\psi(m,t), 0) \sim (m,t)$.*

Transitivity, Monotonicity and Impatience ensure that $(m,0) \succsim (m,t) \succsim (0,0)$. Continuity and Monotonicity then imply the existence of a unique $\psi(m,t) \in \mathcal{M}$ s.t. $(\psi(m,t),0) \sim (m,t)$.

[5] Anderson et al [1] show that by accounting for the curvature of utility, one obtains lower and more reasonable estimates of discount rates than those yielded under the assumption of linearity of utility.

Step 2: *Show that $\succsim$ admits a utility representation U defined by $U(m,t) = \phi(m,t) \cdot m$. Moreover, $\phi(m,t)$ is strictly decreasing in its second argument and satisfies $\phi(m,0) = 1$ for any $m > 0$.*

By Order, Monotonicity, the existence of present equivalents and by definition of ϕ,

$(m,t) \succsim (m',t') \Longleftrightarrow (\psi(m,t),0) \succsim (\psi(m',t'),0) \Longleftrightarrow \psi(m,t) \geq \psi(m',t')$

$\Longleftrightarrow \phi(m,t)m \geq \phi(m',t')m'$, as desired. The other claims follow from the fact that by Impatience and Monotonicity, $\psi(m,t)$ is strictly decreasing in t and satisfies $\psi(m,0) = m$.

Step 3: *The result*

By Eventually Lower Impatience, there is $l, \tau > 0$ such that $\phi(l,\tau) < \lim_{t\to\infty} \inf \frac{\phi(l,t+\tau)}{\phi(l,t)}$. By Impatience, $\phi(l,\tau) < 1$ and thus there exists $0 < s < l$ s.t. $\phi(l,\tau) < \frac{s}{l} < \lim_{t\to\infty} \inf \frac{\phi(l,t+\tau)}{\phi(l,t)}$. Moreover, by definition of $\liminf$, there exists T s.t. $\frac{s}{l} < \frac{\phi(l,t+\tau)}{\phi(l,t)}$ for all $t \geq T$. By the Magnitude Effect, $\frac{\phi(l,t)}{\phi(s,t)} \geq 1$. It follows that for all $t \geq T$,

$$\phi(l,\tau) < \frac{s}{l} < \frac{\phi(l,t+\tau)}{\phi(l,t)} \frac{\phi(l,t)}{\phi(s,t)} = \frac{\phi(l,t+\tau)}{\phi(s,t)}.$$

In particular, $\phi(l,\tau)l < s = \phi(s,0)s$ and $\phi(s,t)s < \phi(l,t+\tau)l$ for all $t \geq T$. The assertion follows by Step 2. ■

B Appendix: Proof of Theorem 2

Under regularity, the Magnitude Effect is implied, and so Theorem 1 yields that Stationarity is violated. We show that for any s, l and τ there can be no more than one reversal. As in step 2 of the proof of Theorem 3, any regular preference $\succsim$ admits a utility representation U defined by $U(m,t) = \phi(m,t) \cdot m$. By the representation,

$$(s,t) \succsim (l,t+\tau) \Longleftrightarrow \frac{\phi(l,t+\tau)}{\phi(s,t)} \leq \frac{s}{l}.$$

The Theorem is proved once it is established that $\frac{\phi(l,t+\tau)}{\phi(s,t)}$ is weakly increasing in t. Take any $s < l$ and any t. By the Strong Magnitude Effect and Decreasing Impatience, we have that for any τ,

$\frac{\phi(s,t+1)}{\phi(s,t)} \leq \frac{\phi(l,t+1)}{\phi(l,t)} \leq \frac{\phi(l,t+\tau+1)}{\phi(l,t+\tau)} \Longrightarrow \frac{\phi(s,t+1)}{\phi(s,t)} \leq \frac{\phi(l,t+\tau+1)}{\phi(l,t+\tau)}$

$\Longrightarrow \frac{\phi(l,t+\tau)}{\phi(s,t)} \leq \frac{\phi(l,[t+1]+\tau)}{\phi(s,[t+1])}$. Conclude that $\frac{\phi(l,t+\tau)}{\phi(s,t)}$ is increasing in t. ■

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