

# NON-BAYESIAN LEARNING\*

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September 11, 2009

## Abstract

A series of experiments suggest that, compared to the Bayesian benchmark, people may either underreact or overreact to new information. We consider a setting where agents repeatedly process new data. Our main result shows a basic distinction between the long-run beliefs of agents who underreact to information and agents who overreact to information. Like Bayesian learners, non-Bayesian updaters who underreact to observations eventually forecast accurately. Hence, underreaction may be a transient phenomena. Non-Bayesian updaters who overreact to observations eventually forecast accurately with positive probability but may also, with positive probability, converge to incorrect forecasts. Hence, overreaction may have long-run consequences.

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## 1. INTRODUCTION

A central result in learning theory is that Bayesian forecasts eventually become accurate under suitable conditions, such as absolute continuity of the data generating process with respect to the agent’s beliefs (see Kalai and Lehrer [14]). Hence, multiple repetitions of Bayes’ rule may transform the historical record into a near perfect guide for the future. However, a series of experiments suggest that people may repeatedly process information using non-Bayesian heuristics (see Kahneman and Tversky [19] and surveys by Camerer [5] and Rabin [16]). These experiments contributed to a growing interest in the properties of non-Bayesian learning (see, for example, Gilboa and Jackson [13] and Gilboa, Postlewaite, and Schmeidler [11] and [12]).<sup>1</sup>

Departures from Bayesian updating can occur either because subjects tend to ignore the prior and overreact to the data (we refer to this bias as *overreaction*), or alternatively because subjects place excessive weight on prior beliefs and underreact to new observations (we refer to this bias as *underreaction*).

We investigate a non-Bayesian updater who faces a statistical inference problem and may either overreact or underreact to new data. Consider an agent who is trying to learn the true parameter in a set  $\Theta$ . Updating of beliefs in response to observations  $s_1, \dots, s_t$ , leads to posterior beliefs  $\{\mu_t\}$  where each  $\mu_t$  is a probability measure on  $\Theta$ . Bayesian updating leads to the process

$$\mu_{t+1} = BU(\mu_t; s_{t+1}),$$

where  $BU(\mu_t; s_{t+1})$  denotes the Bayesian update of  $\mu_t$  given the new observation  $s_{t+1}$ . A more general model is the process

$$\mu_{t+1} = (1 - \gamma_{t+1}) BU(\mu_t; s_{t+1}) + \gamma_{t+1} \mu_t, \quad (1.1)$$

where  $\gamma_{t+1} \leq 1$ . If  $\gamma_{t+1} = 0$  then the model reduces to standard Bayesian model. If  $\gamma_{t+1} > 0$  then the updating rule can be interpreted as attaching too much weight to the prior  $\mu_t$  and hence underreacting to observations. Conversely, if  $\gamma_{t+1} < 0$  then the updating rule can be interpreted as overreacting to observations.

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<sup>1</sup>The complexities of Bayesian procedures may make Bayesian updating rules excessively costly to implement in many practical applications. So, even agents who would prefer to use Bayes’ Rule often rely on simpler, non-Bayesian heuristics for updating beliefs (see, among others, Bala and Goyal [4]). Thus, there exists a normative motivation for analyzing non-Bayesian updating rules in addition to the positive motivation coming from laboratory and field experiments (see also Gilboa, Postlewaite, and Schmeidler [10] and Aragonés, Gilboa, Postlewaite, and Schmeidler [3] for normative motivations on non-Bayesian updating rules)

While there may be more than one way to provide a rationale for the non-Bayesian updating rules in (1.1), choice-theoretic foundations were provided in Epstein [7] and Epstein, Noor and Sandroni [8] in a axiomatic framework where an agent is self-aware of her biases and fully anticipates her updating behavior when formulating plans.<sup>2</sup> Foundations take the form of a representation theorem for suitably defined preferences such that both the prior and the way in which it is updated are subjective. In this paper, we describe the asymptotic properties of the process of beliefs defined by (1.1).

We show that, like Bayesian updating, multiple repetitions of non-Bayesian updating rules that underreact to observations eventually lead to accurate forecasts (e.g., forecasts close to the actual data generating process).<sup>3</sup> Thus, non-Bayesian updaters who underreact to the data eventually forecast accurately. The case in which the agent overreacts to the data is quite different. Multiple repetitions of non-Bayesian updating rules that overreact to the observations eventually lead to accurate forecast with positive probability. In some cases, however, with strictly positive probability, non-Bayesian updaters become certain that a false parameter is true and thus converge to incorrect forecasts. Hence, overreaction may not be a transient phenomena. It may have long-run implications.

Our results suggest a fundamental difference between underreacting and overreacting to new data. Bayesian and underreacting agents eventually forecast as if they have uncovered the data generating process. However, there is a broader range of possible long-run forecasts for agents who overreact to new observations. These agents may eventually forecast accurately, but they may also permanently forecast incorrectly. Unlike Bayesian and underreacting agents, the ultimate fate of overreacting agents is not entirely pre-determined by the data generating process itself. It also depends on the historical record.

The paper proceeds as follows: In section 2, we present the main concepts and the basic results on overreaction and underreaction. In addition, in section 2, we also consider a bias akin to the hot hand fallacy and show that, like the underreaction bias, agents who update beliefs consistently with this bias also eventually uncover the data generating process. Section 3 concludes. Proofs are in the appendix.

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<sup>2</sup>Hence, an agent may revise their beliefs in a non-Bayesian way if she is aware of her biases. See also Ali [2] for a model of learning about self-control.

<sup>3</sup>This result requires an important qualifier. It is also needed that the weights  $\gamma_{t+1}$  for prior and Bayesian posterior depend only on the observations realized until period  $t$  and not upon the observation  $s_{t+1}$  realized at period  $t + 1$ .

## 2. Basic Model and Results

Time is discrete and varies over  $t = 0, 1, 2, \dots$ . Uncertainty is represented by a (finite) period state space  $S$ . One element  $s_t$  of  $S$  is realized at each period  $t$ . Thus, the complete uncertainty is represented by the full state space  $\Omega = \prod_{t=1}^{\infty} S_t$ , where  $S_t = S$  for all  $t > 0$ . Let  $\Delta(S)$  be the set of probability measures on  $S$ . A stochastic process  $(X_t)$  on  $\Omega$  is *adapted* if  $X_t$  is measurable with respect to the  $\sigma$ -algebra  $\mathcal{S}_t$  that is generated by all sets of the form  $\{s_1\} \times \dots \times \{s_t\} \times \prod_{t+1}^{\infty} S_t$ . Unless otherwise noted, stochastic processes  $(X_t)$  on  $\prod_{t=1}^{\infty} S_t$  are adapted.

Let  $\Theta$  denote a countable set of possible parameters. The prior belief over  $\Theta$  is  $\mu_0 \in \Delta(\Theta)$ , where  $\Delta\Theta$  is the set of probability measure over  $\Theta$ . The  $\sigma$ -algebra associated with  $\Theta$  is suppressed.

Conditional on parameter  $\theta$ , at each time period  $t \geq 0$ , an observation  $s_t \in S$  is independently generated according to the likelihood function  $\ell(s_t | \theta)$ . Let  $\theta^* \in \Theta$  be the actual parameter determining the data generating process. We define a probability triple  $(\Omega, \mathcal{S}, \mathbb{P}^*)$ , where  $\mathcal{S}$  is the smallest  $\sigma$ -field containing all  $\mathcal{S}_t$  for  $t > 0$  and  $\mathbb{P}^*$  is the probability measure induced over sample paths in  $\Omega$  by parameter  $\theta^*$ . That is,  $\mathbb{P}^* = \otimes_{t=1}^{\infty} \ell(\cdot | \theta^*)$ . We use  $\mathbb{E}^*[\cdot]$  to denote the expectation operator associated with  $\mathbb{P}^*$ .

We now define the measures  $\mu_t$  on the parameter set  $\Theta$  by induction. The prior  $\mu_0$  on  $\Theta$  is defined. Suppose that  $\mu_t$  has been constructed and define  $\mu_{t+1}$  by

$$\mu_{t+1} = (1 - \gamma_{t+1}) BU(\mu_t; s_{t+1}) + \gamma_{t+1} \mu_t, \quad (2.1)$$

where  $BU(\mu_t; s_{t+1})(\cdot)$  is the Bayesian update of  $\mu_t$  given the new observation  $s_{t+1}$  at period  $t+1$  and  $\gamma_{t+1}$  is  $\mathcal{S}_{t+1}$ -measurable process such that  $\gamma_t \leq 1$ . As mentioned in the introduction,  $\gamma_{t+1}$  are weights given to the Bayesian update of  $\mu_t$  and the prior belief  $\mu_t$  at period  $t+1$ . So, if  $\gamma_{t+1}$  is positive then the posterior belief  $\mu_{t+1}$  is a mixture of the Bayesian update  $BU(\mu_t; s_{t+1})$ , which incorporates the Bayesian response to the new observation  $s_{t+1}$ , and the prior  $\mu_t$ , which does not respond to the new observation  $s_{t+1}$  at all. In a natural sense, therefore, an agent with positive weight  $\gamma_{t+1} \geq 0$  *underreacts* to data. Similarly, if the weight  $\gamma_{t+1}$  is negative then the Bayesian update  $BU(\mu_t; s_{t+1})$  is a mixture of the posterior belief  $\mu_{t+1}$  and the prior  $\mu_t$ , which suggests *overreaction* to new data. Clearly, if  $\gamma_{t+1} = 0$  then the model reduces to the Bayesian updating rule. We refer to equation 2.1 as the *law of motion* for beliefs about parameters. Finally, define

$$m_t(\cdot) = \int_{\Theta} \ell(\cdot | \theta) d\mu_t.$$

as the belief at period  $t$  over observations at period  $t + 1$  given measure  $\mu_t$  over the parameters in  $\Theta$ .

## 2.1. Learning with underreaction and overreaction

We now turn to the question of what is learned in the long run. Learning may either signify learning the true parameter or learning to forecast future outcomes.<sup>4</sup> The latter kind of learning is more relevant to choice behavior and thus is our focus.

**Definition 2.1.** *Forecasts are eventually accurate on a path  $s^\infty \in \Omega$  if, along that path,*

$$m_t(\cdot) \longrightarrow \ell(\cdot | \theta^*) \quad \text{as } t \longrightarrow \infty.$$

That is, if forecasts are eventually accurate then, in the long-run, agents' beliefs converge to the data generating process.

**Theorem 2.2.** *Assume (2.1). Let  $\mu_0(\theta^*) > 0$ .*

(a) *Suppose that  $\gamma_{t+1} \geq 0$  (i.e., underreaction) and that  $\gamma_{t+1}$  is  $\mathcal{S}_t$ -measurable. Then, forecasts are eventually accurate  $\mathbb{P}^*$  - a.s.*

(b) *Suppose that  $\gamma_t \leq 1 - \epsilon$  for some  $\epsilon > 0$  (i.e., underreaction is allowed and overreaction is also allowed in different periods) and that  $\gamma_{t+1}$  is  $\mathcal{S}_t$ -measurable. Then forecasts are eventually accurate with  $\mathbb{P}^*$ -strictly positive probability.*

(c) *There exists a model  $(S, \Theta, \ell, \mu_0)$  and weights  $\gamma_t = \gamma < 0$  (hence overreaction) and a false parameter  $\theta \neq \theta^*$  such that*

$$m_t(\cdot) \longrightarrow \ell(\cdot | \theta) \quad \text{as } t \longrightarrow \infty,$$

*with  $\mathbb{P}^*$ -strictly positive probability. In these cases, the forecast are eventually based on a wrong parameter.*

(d) *Assume that the weights  $\gamma_{t+1} \geq 0$  are still positive (i.e., underreaction), but that they may depend upon observations at period  $t + 1$  (i.e.,  $\gamma_{t+1}$  is  $\mathcal{S}_{t+1}$ -measurable). Then, there exist a model  $(S, \Theta, \ell, \mu_0)$  and a false parameter  $\theta \neq \theta^*$  such that*

$$m_t(\cdot) \longrightarrow \ell(\cdot | \theta) \quad \text{as } t \longrightarrow \infty,$$

*with  $\mathbb{P}^*$ -strictly positive probability.*

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<sup>4</sup>See Lehrer and Smorodinsky [15] for the distinction between these two kinds of learning.

Assume that before any data are observed the prior belief puts positive weight on the true parameter. That is, assume that  $\mu_0(\theta^*) > 0$ . Then, the basic result in Bayesian updating holds : multiple repetition of Bayes' Rule leads to accurate forecasts. This result is central in the Bayesian literature because it shows that the mere repetition of Bayes' Rule eventually transforms the historical record into a near perfect guide for the future. Part (a) of the Theorem 1 generalizes the Bayesian result to underreaction. Multiple repetitions of non-Bayesian updating rules in (2.1) that underreact to the new observations (and the measurability assumption on the weights), eventually produce good forecasting. So, in the case of underreaction, agent's forecasts converge to rational expectations although the available information is not processed by the Bayesian laws of probability.

Part (b) shows that, with positive probability, non-Bayesian forecasts are eventually accurate. This applies to both underreaction and overreaction. Perhaps surprisingly, the results hold even if the forecaster sometimes overreacts and sometimes underreacts to new information.

Parts (c) and (d) are based on examples. The example in part (c) shows that convergence to wrong forecasts may occur for overreactors. The weight  $\gamma_t$  is constant, but negative, corresponding to a forecaster that sufficiently overreacts to new information. In the example, the forecasts converge, but not necessarily to the data generating process. The forecasts may be eventually accurate, but they may also be eventually incorrect (i.e., correspond to a wrong parameter). Hence, whether overreacting updating rules eventually converge to the data generating process may not be pre-determined from the outset. It depends upon the realized historical record.

In example in part (d), the weight  $\gamma_{t+1}$  is positive corresponding to underreaction, but it depends on the current signal and, therefore,  $\gamma_{t+1}$  is  $\mathcal{S}_{t+1}$ -measurable. As in the case of overreaction, forecasts may eventually converge to an incorrect limit. Moreover, we also show that wrong long-run forecasts are at least as likely to occur as are accurate forecasts. This example shows that even non-Bayesian updaters who underreact to the new observations may eventually forecast based on a wrong parameter if the weights depend upon the new observation. Hence, the learning result of part (a) by underreaction depends not only on the fact that beliefs of underreacting agents are linear combinations of a prior and a Bayesian posterior, but also on the assumptions that the weight of this linear combination (which may depend on the past history and so, may change over time) does not depend upon the most recent observation.

## 2.2. Intuition behind Theorem 2.2

Let  $\mu_t(\theta^*)$  be the probability that  $\mu_t$  assigns to the true parameter  $\theta^*$ . The expected value (according to the data generating process) of the Bayesian update of  $\mu_t(\theta^*)$  (given new information) is greater than  $\mu_t(\theta^*)$  itself. This submartingale property ensures that, in the Bayesian case,  $\mu_t(\theta^*)$  must converge to some value and cannot remain in endless random fluctuations. The submartingale property follows because under the Bayesian paradigm future changes in beliefs that can be predicted are incorporated in current beliefs. It is immediate from the linear structure in (2.1) that this basic submartingale property still holds in our model as long as the weight between prior and Bayesian posterior depends upon the history only up to period  $t$ . Hence, with this assumption,  $\mu_t(\theta^*)$  must also converge and, as in the Bayesian case, cannot remain in endless random fluctuations.<sup>5</sup> This convergence result holds even if overreaction and underreaction occur in different periods. In the case of underreaction,  $\mu_t(\theta^*)$  tends to grow and so, forecasts are eventually accurate. In the case of sufficiently strong overreaction, it is possible that forecasts will settle on an incorrect limit. This follows because the positive drift of the above mentioned submartingale property on  $\mu_t(\theta^*)$  may be compensated by sufficiently strong volatility which permits that, with positive probability,  $\mu_t(\theta^*)$  converges to zero.

## 2.3. Sample-Bias

In this section, we consider a bias akin to the *hot-hand fallacy* - the tendency to over-predict the continuation of recent observations (see Kahneman and Tversky [18], Camerer [6] and Rabin [17]). Suppose that there are  $K$  possible states in each period,  $S = \{s^1, \dots, s^K\}$  and that  $\ell(s^k | \theta) = \theta_k$  for each parameter  $\theta = (\theta_1, \dots, \theta_K)$  in  $\Theta$ , where  $\Theta$  is the set of points  $\theta = (\theta_1, \dots, \theta_K)$  in the interior of the  $K$ -simplex having rational coordinates. Define

$$\psi_{t+1}(\theta) = \begin{cases} 1 & \text{if the empirical frequency of } s^k \text{ is } \theta_k, 1 \leq k \leq K, \\ 0 & \text{otherwise.} \end{cases}$$

The law of motion now takes the form

$$\mu_{t+1} = (1 - \gamma_t) BU(\mu_t; s_{t+1}) + \gamma_t \psi_{t+1}, \quad (2.2)$$

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<sup>5</sup>We conjecture that beliefs  $\mu_t(\theta^*)$  may not converge in some examples when the weight  $\gamma_{t+1}$  is  $\mathcal{S}_{t+1}$ -measurable. In our example, it does converge, but to an incorrect limit.

where  $0 \leq \gamma_t \leq 1$ .

So, if  $\gamma_t = 1$  then probability one is assigned to the parameter that coincides with the observed past frequencies. If  $\gamma_t = 0$  then the model reduces to the Bayesian model. If  $0 < \gamma_t < 1$  the posterior beliefs are linear combinations of the Bayesian posterior and the observed frequencies. We have the following partial counterpart of part (a) of Theorem 2.2.

**Theorem 2.3.** *Suppose that  $(\mu_t)$  evolve according to (2.2), where  $0 < \underline{\gamma} \leq \gamma_t \leq 1$ . Then forecasts are eventually accurate  $\mathbb{P}^* - a.s.$*

The positive lower bound  $\underline{\gamma}$  excludes the Bayesian case. The result does hold in the Bayesian case  $\gamma_{t+1} = 0$ . However, unlike the proof of Theorem 2.2, the proof of Theorem 2.3 is in some ways significantly different from the standard proof used in the Bayesian case. We suspect that the differences in the approach make the lower bound assumption technically convenient but ultimately disposable. We also conjecture (but cannot yet prove) that just as in part (d) of Theorem 2.2, convergence to the truth fails in general if the weights  $\gamma_{t+1}$  are allowed to be  $\mathcal{S}_{t+1}$ -measurable, instead of being  $\mathcal{S}_t$ -measurable as in Theorem 2.3.

### 3. Conclusion

The long-run implications of biased revisions of beliefs may differ. Multiple revision of beliefs with biases such as underreaction and the sample bias may, like multiple revisions of Bayes' Rule, eventually transform the historical record into a near perfect guide for the future. The case of overreaction to data is different. Beliefs also converge after multiple overreactions to observations, but possibly to incorrect forecasts.

### 4. Proofs

*Proof of Theorem 2.2:* (a) Given our measurability assumption, we can replace the weights  $\gamma_{t+1}$  with  $\gamma_t$  in parts (a) and (b). First we show that  $\log \mu_t(\theta^*)$  is a submartingale under  $\mathbb{P}^*$ . Because

$$\log \mu_{t+1}(\theta^*) - \log \mu_t(\theta^*) = \log \left( (1 - \gamma_t) \frac{\ell(s_{t+1}|\theta^*)}{m_t(s_{t+1})} + \gamma_t \right), \quad (4.1)$$

it suffices to show that

$$E^* \left[ \log \left( (1 - \gamma_t) \frac{\ell(s_{t+1}|\theta^*)}{m_t(s_{t+1})} + \gamma_t \right) \mid \mathcal{S}_t \right] \geq 0, \quad (4.2)$$



where  $E^*$  denotes expectation with respect to  $\mathbb{P}^*$ . By assumption,  $\gamma_t$  is constant given  $\mathcal{S}_t$ . Thus the expectation equals

$$\begin{aligned} \sum_{s_{t+1}} \ell(s_{t+1} | \theta^*) \log \left( (1 - \gamma_t) \frac{\ell(s_{t+1} | \theta^*)}{m_t(s_{t+1})} + \gamma_t \right) &\geq \\ \sum_{s_{t+1}} \ell(s_{t+1} | \theta^*) (1 - \gamma_t) \log \left( \frac{\ell(s_{t+1} | \theta^*)}{m_t(s_{t+1})} \right) &= \\ (1 - \gamma_t) \sum_{s_{t+1}} \ell(s_{t+1} | \theta^*) \log \left( \frac{\ell(s_{t+1} | \theta^*)}{m_t(s_{t+1})} \right) &\geq 0 \end{aligned}$$

as claimed, where both inequalities are due to concavity of  $\log(\cdot)$ . (The second is the well-known entropy inequality.)

Clearly  $\log \mu_t(\theta^*)$  is bounded above by zero. Therefore, by the martingale convergence theorem, it converges  $\mathbb{P}^* - a.s.$  From (4.1),

$$\log \mu_{t+1}(\theta^*) - \log \mu_t(\theta^*) = \log \left( (1 - \gamma_t) \frac{\ell(s_{t+1} | \theta^*)}{m_t(s_{t+1})} + \gamma_t \right) \longrightarrow 0$$

and hence  $\frac{\ell(s_{t+1} | \theta^*)}{m_t(s_{t+1})} \longrightarrow 1 \mathbb{P}^* - a.s.$  ■

*Part (b)*

$$\begin{aligned} E^* \left[ \left( (1 - \gamma_t) \frac{\ell(s_{t+1} | \theta^*)}{m_t(s_{t+1})} + \gamma_t \right) \mid \mathcal{S}_t \right] &= (1 - \gamma_t) E^* \left[ \frac{\ell(s_{t+1} | \theta^*)}{m_t(s_{t+1})} \mid \mathcal{S}_t \right] + \gamma_t \\ &\geq (1 - \gamma_t) + \gamma_t = 1. \end{aligned}$$

The last inequality is implied by the fact that

$$\min_X \left\{ E^* \left[ \frac{1}{X(s_{t+1})} \mid \mathcal{S}_t \right] : E^* [X(s_{t+1}) \mid \mathcal{S}_t] = 1 \right\} = 1.$$

The minimization is over random variable  $X$ 's,  $X : S_{t+1} \longrightarrow \mathbb{R}_{++}^1$ , and it is achieved at  $X(\cdot) = 1$  because  $\frac{1}{x}$  is a convex function on  $(0, \infty)$ . Deduce that  $E^* \left[ \frac{\mu_{t+1}(\theta^*)}{\mu_t(\theta^*)} \mid \mathcal{S}_t \right] \geq 1$  and hence that  $\mu_t(\theta^*)$  is a submartingale. By the martingale convergence theorem,

$$\mu_\infty(\theta^*) \equiv \lim \mu_t(\theta^*) \text{ exists } \mathbb{P}^* - a.s.$$

*Claim:*  $\mu_\infty(\theta^*) > 0$  on a set with positive  $\mathbb{P}^*$ -probability.

By the bounded convergence theorem,

$$E^* \mu_t(\theta^*) \longrightarrow E^* \mu_\infty(\theta^*);$$

and  $E^* \mu_t(\theta^*) \nearrow$  because  $\mu_t(\theta^*)$  is a submartingale. Thus  $\mu_0(\theta^*) > 0$  implies that  $E^* \mu_\infty(\theta^*) > 0$ , which proves the claim.

It suffices now to show that if  $\mu_\infty(\theta^*) > 0$  along a sample path  $s^\infty \in \Omega$ , then forecasts are eventually accurate along  $s^\infty$ . But along such a path,  $\frac{\mu_{t+1}(\theta^*)}{\mu_t(\theta^*)} \longrightarrow 1$  and hence

$$(1 - \gamma_t) \left( \frac{\ell(s_{t+1}|\theta^*)}{m_t(s_{t+1})} - 1 \right) \longrightarrow 0.$$

By assumption,  $(1 - \gamma_t)$  is bounded away from zero. Therefore,

$$\left( \frac{\ell(s_{t+1}|\theta^*)}{m_t(s_{t+1})} - 1 \right) \longrightarrow 0. \quad \blacksquare$$

*Part (c):* Convergence to wrong forecasts may occur with  $\mathbb{P}^*$ -positive probability when  $\gamma_{t+1} < 0$ , even where  $\gamma_{t+1}$  is  $\mathcal{S}_t$ -measurable (overreaction); in fact, we take the weight  $\gamma_t = \gamma < 0$  to be constant.

Think of repeatedly tossing an unbiased coin that is viewed at time 0 as being either unbiased or having probability of Heads equal to  $b$ ,  $0 < b < \frac{1}{2}$ . Thus take  $S = \{H, T\}$  and  $\ell(H|\theta) = \theta$  for  $\theta \in \Theta = \{b, \frac{1}{2}\}$ . Assume also that

$$1 < -\gamma < \frac{b}{\frac{1}{2} - b}. \quad (4.3)$$

The inequality  $\gamma < -1$  indicates a sufficient degree of overreaction.

The other inequality is motivated by the need of having measures non-negative valued in the choice-theoretic model of Epstein [7] and Epstein, Noor and Sandroni [8] that underlies these laws of motion.

We now show that if (4.3), then

$$m_t(\cdot) \longrightarrow \ell(\cdot | b) \quad \text{as } t \longrightarrow \infty,$$

with probability under  $\mathbb{P}^*$  at least  $\frac{1}{2}$ .

Abbreviate  $\mu_t(\frac{1}{2})$  by  $\mu_t^*$ .

Claim 1:  $\mu_\infty^* \equiv \lim \mu_t^*$  exists  $\mathbb{P}^* - a.s.$  and if  $\mu_\infty^* > 0$  for some sample realization  $s_1^\infty$ , then  $m_t(H) \rightarrow \frac{1}{2}$  and  $\mu_t^* \rightarrow 1$  along  $s_1^\infty$ . (The proof is analogous to that of part (b).) Deduce that

$$\mu_\infty^* \in \{0, 1\} \quad \mathbb{P}^* - a.s.$$

Claim 2:  $f(z) \equiv \left[ (1-\gamma) \frac{1}{z} + \gamma \right] \left[ (1-\gamma) \frac{1-\frac{1}{2}}{(1-z)} + \gamma \right] \leq 1$ , for all  $z \in [b, \frac{1}{2}]$ . Argue that  $f(z) \leq 1 \iff g(z) \equiv [(1-\gamma) + 2\gamma z][(1-\gamma) + 2\gamma(1-z)] - 4z(1-z) \leq 0$ . Compute that  $g(\frac{1}{2}) = 0$ ,  $g'(\frac{1}{2}) = 0$  and  $g$  is concave because  $\gamma < -1$ . Thus  $g(z) \leq g(0) = 0$ .

$$\begin{aligned} \text{Claim 3: } E^* \left[ \log \left( (1-\gamma) \frac{\ell(s_{t+1} | \frac{1}{2})}{m_t(s_{t+1})} + \gamma \right) \mid \mathcal{S}_t \right] \\ = \frac{1}{2} \log \left( (1-\gamma) \frac{\frac{1}{2}}{b + (\frac{1}{2}-b)\mu_t^*} + \gamma \right) + \frac{1}{2} \log \left( (1-\gamma) \frac{1-\frac{1}{2}}{(1-b - (\frac{1}{2}-b)\mu_t^*)} + \gamma \right) \\ = \frac{1}{2} \log \left( f \left( b + (\frac{1}{2}-b)\mu_t \left( \frac{1}{2} \right) \right) \right) \leq 0, \text{ by Claim 2.} \end{aligned}$$

By Claim 1, it suffices to prove that  $\mu_\infty^* = 1$   $\mathbb{P}^* - a.s.$  is impossible. Compute that

$$\begin{aligned} \mu_t^* &= \mu_0^* \left[ \prod_{k=0}^{t-1} \left( (1-\gamma) \frac{\ell(s_{k+1} | \frac{1}{2})}{m_k(s_{k+1})} + \gamma \right) \right], \\ \log \mu_t^* &= \log \mu_0^* + \sum_{k=0}^{t-1} \log \left( (1-\gamma) \frac{\ell(s_{k+1} | \frac{1}{2})}{m_k(s_{k+1})} + \gamma \right) \\ &= \log \mu_0^* + \sum_{k=0}^{t-1} (\log z_{k+1} - E[\log z_{k+1} \mid \mathcal{S}_k]) + \sum_{k=0}^{t-1} E[\log z_{k+1} \mid \mathcal{S}_k], \end{aligned}$$

where  $z_{k+1} = (1-\gamma) \frac{\ell(s_{k+1} | \frac{1}{2})}{m_k(s_{k+1})} + \gamma$ . Therefore,  $\log \mu_t^* \geq \frac{1}{2} \log \mu_0^*$  iff

$$\sum_{k=0}^{t-1} (\log z_{k+1} - E[\log z_{k+1} \mid \mathcal{S}_k]) \geq -\frac{1}{2} \log \mu_0^* - \sum_{k=0}^{t-1} E[\log z_{k+1} \mid \mathcal{S}_k] \equiv a_k.$$

By Claim 3,  $a_k > 0$ . The random variable  $\log z_{k+1} - E[\log z_{k+1} \mid \mathcal{S}_k]$  takes on two possible values, corresponding to  $s_{k+1} = H$  or  $T$ , and under the truth they are equally likely and average to zero. Thus

$$\mathbb{P}^*(\log z_{k+1} - E[\log z_{k+1} \mid \mathcal{S}_k] \geq a_k) \leq \frac{1}{2}.$$

Deduce that

$$\mathbb{P}^*(\log \mu_t^* \geq \frac{1}{2} \log \mu_0^*) \leq \frac{1}{2}$$

and hence that

$$\mathbb{P}^* (\log \mu_t^* \longrightarrow 0) \leq \frac{1}{2}. \quad \blacksquare$$

*Part (d):* Convergence to wrong forecasts may occur with  $\mathbb{P}^*$ -positive probability when  $\gamma_{t+1} > 0$ , if  $\gamma_{t+1}$  is only  $\mathcal{S}_{t+1}$ -measurable.

The coin is as before - it is unbiased, but the agent does not know that and is modeled via  $S = \{H, T\}$  and  $\ell(H | \theta) = \theta$  for  $\theta \in \Theta = \{b, \frac{1}{2}\}$ . Assume further that  $\alpha_{t+1}$  and  $\lambda_{t+1}$  are such that

$$\gamma_{t+1} \equiv \lambda_{t+1}(1 - \alpha_{t+1}) = \begin{cases} w & \text{if } s_{t+1} = H \\ 0 & \text{if } s_{t+1} = T, \end{cases}$$

where  $0 < w < 1$ . Thus, from (2.1), the agent updates by Bayes' Rule when observing  $T$  but attaches only the weight  $(1 - w)$  to last period's prior when observing  $H$ . Assume that

$$w > 1 - 2b.$$

Then

$$m_t(\cdot) \longrightarrow \ell(\cdot | b) \quad \text{as } t \longrightarrow \infty,$$

with probability under  $\mathbb{P}^*$  at least  $\frac{1}{2}$ .

The proof is similar to that of Example 1. The key is to observe that  $E^* \left[ \log \left( (1 - \gamma) \frac{\ell(s_{t+1} | \frac{1}{2})}{m_t(s_{t+1})} + \gamma \right) \mid \mathcal{S}_t \right] \leq 0$  under the stated assumptions.  $\blacksquare$

The proof of Theorem 2.3 requires the following lemmas:

**Lemma 4.1.** (*Freedman (1975)*) *Let  $\{z_t\}$  be a sequence of uniformly bounded  $\mathcal{S}_t$ -measurable random variables such that for every  $t \geq 1$ ,  $E^*(z_{t+1} | \mathcal{S}_t) = 0$ . Let  $V_t^* \equiv VAR(z_{t+1} | \mathcal{S}_t)$  where  $VAR$  is the variance operator associated with  $P^*$ . Then,*

$$\sum_{t=1}^n z_t \text{ converges to a finite limit as } n \rightarrow \infty, P^*\text{-a.s. on } \left\{ \sum_{t=1}^{\infty} V_t^* < \infty \right\}$$

and

$$\sup_n \sum_{t=1}^n z_t = \infty \text{ and } \inf_n \sum_{t=1}^n z_t = -\infty, P^*\text{-a.s. on } \left\{ \sum_{t=1}^{\infty} V_t^* = \infty \right\}.$$

**Definition 4.2.** A sequence of  $\{x_t\}$  of  $\mathcal{S}_t$ -measurable random variables is eventually a submartingale if,  $\mathbb{P}^* - a.s.$ ,  $E^*(x_{t+1}|\mathcal{S}_t) - x_t$  is strictly negative at most finitely many times.

**Lemma 4.3.** Let  $\{x_t\}$  be uniformly bounded and eventually a submartingale. Then,  $\mathbb{P}^* - a.s.$ ,  $x_t$  converges to a finite limit as  $t$  goes to infinity.

**Proof.** Write

$$x_t = \sum_{j=1}^t (r_j - E^*(r_j|\mathcal{S}_{j-1})) + \sum_{j=1}^t E^*(r_j|\mathcal{S}_{j-1}) + x_0, \text{ where } r_j \equiv x_j - x_{j-1}.$$

By assumption,  $\mathbb{P}^* - a.s.$ ,  $E^*(r_j|\mathcal{S}_{j-1})$  is strictly negative at most finitely many times. Hence,  $\mathbb{P}^* - a.s.$ ,

$$\inf_t \sum_{j=1}^t E^*(r_j|\mathcal{S}_{j-1}) > -\infty.$$

Given that  $x_t$  is uniformly bounded,  $\mathbb{P}^* - a.s.$ ,

$$\sup_t \sum_{j=1}^t z_j < \infty, \text{ where } z_j \equiv r_j - E^*(r_j|\mathcal{S}_{j-1}).$$

It follows from Freedman's result that  $\mathbb{P}^* - a.s.$ ,

$$\sum_{j=1}^t z_j \text{ converges to a finite limit as } t \rightarrow \infty.$$

It now follows from  $x_t$  uniformly bounded that  $\sup_t \sum_{j=1}^t E^*(r_j|\mathcal{S}_{j-1}) < \infty$ . Because  $E^*(r_j|\mathcal{S}_{j-1})$  is strictly negative at most finitely many times,

$$\sum_{j=1}^t E^*(r_j|\mathcal{S}_{j-1}) \text{ converges to a finite limit as } t \rightarrow \infty.$$

Therefore,  $\mathbb{P}^* - a.s.$ ,  $x_t$  converges to a finite limit as  $t$  goes to infinity. ■

*Proof of Theorem 2.3:*

*Claim 1:* Define  $f(\theta, m) = \sum_k \theta_k^* \frac{\theta_k}{m_k}$  on the interior of the  $2K$ -simplex. There exists  $\delta' \in \mathbb{R}_{++}^K$  such that

$$|\theta_k - \theta_k^*| < \delta'_k \text{ for all } k \implies f(\theta, m) - 1 \geq -\underline{\gamma} K^{-1} \sum_k |m_k - \theta_k|.$$

Proof:  $f(\theta, \theta) = 1$ ,  $f(\theta, \cdot)$  is convex and hence

$$\begin{aligned} f(\theta, m) - 1 &\geq \sum_{k \neq K} \left( \frac{\partial f(\theta, m)}{\partial m_k} - \frac{\partial f(\theta, m)}{\partial m_K} \right) \Big|_{m=\theta} (m_k - \theta_k) \\ &= \sum_{k \neq K} \left( -\frac{\theta_k^*}{\theta_k} + \frac{\theta_K^*}{\theta_K} \right) (m_k - \theta_k). \end{aligned}$$

But the latter sum vanishes at  $\theta = \theta^*$ . Thus argue by continuity.

Given any  $\delta \in \mathbb{R}_{++}^K$ ,  $\delta \ll \delta'$ , define  $\Theta^* = (\theta^* - \delta, \theta^* + \delta) \equiv \Pi_{k=1}^K (\theta_k^* - \delta_k, \theta_k^* + \delta_k)$  and  $\mu_t^* = \sum_{\theta \in \Theta^*} \mu_t(\theta)$ .

*Claim 2:* Define  $m_t^*(s^k) = \sum_{\theta \in \Theta^*} \theta_k \mu_t(\theta) / \mu_t^*(s^k)$ . Then

$$|m_t(s^k) - m_t^*(s^k)| \leq 1 - \mu_t^*.$$

Proof:  $m_t(s^k) - m_t^*(s^k) = \frac{\sum_{\theta \in \Theta^*} \theta_k \mu_t(\theta)}{\mu_t^*} (\mu_t^* - 1) + \sum_{\theta \notin \Theta^*} \theta_k \mu_t(\theta)$ . Therefore,  $(\mu_t^* - 1) \leq m_t^*(s^k) (\mu_t^* - 1) = \frac{\sum_{\theta \in \Theta^*} \theta_k \mu_t(\theta)}{\mu_t^*} (\mu_t^* - 1) \leq m_t(s^k) - m_t^*(s^k) \leq \sum_{\theta \notin \Theta^*} \theta_k \mu_t(\theta) \leq 1 - \mu_t^*$ .

*Claim 3:* For any  $\delta \ll \delta'$  as above,

$$\sum_k \theta_k^* \frac{m_t^*(s^k)}{m_t(s^k)} - 1 \geq -\underline{\gamma} (1 - \mu_t^*).$$

Proof: Because  $|m_t^*(s^k) - \theta_k^*| < \delta_k < \delta'_k$ , we have that

$$\sum_k \theta_k^* \frac{m_t^*(s^k)}{m_t(s^k)} - 1 \geq -\underline{\gamma} K^{-1} \sum_k |m_t(s^k) - m_t^*(s^k)|.$$

Now Claim 3 follows from Claim 2.

Compute that  $E^* [\mu_{t+1}(\theta) \mid \mathcal{S}_t] =$

$$(1 - \gamma_{t+1}) \left[ \sum_k \theta_k^* \frac{\theta_k}{m_t(s^k)} \right] \mu_t(\theta) + \gamma_{t+1} E^* [\psi_{t+1}(\theta) \mid \mathcal{S}_t], \quad (4.4)$$

where use has been made of the assumption that  $\gamma_{t+1}$  is  $\mathcal{S}_t$ -measurable. Therefore,

$$\begin{aligned} & E^* [\mu_{t+1}^*(\theta) \mid \mathcal{S}_t] - \mu_t^* = \\ & (1 - \gamma_{t+1}) \sum_k \left( \theta_k^* \frac{m_t^*(s^k)}{m_t(s^k)} \right) \mu_t^* + \gamma_{t+1} \sum_{\theta \in \Theta^*} E^* [\psi_{t+1}(\theta) \mid \mathcal{S}_t] - \mu_t^* \\ & = (1 - \gamma_{t+1}) \left[ \sum_k \left( \theta_k^* \frac{m_t^*(s^k)}{m_t(s^k)} \right) - 1 \right] \mu_t^* + \gamma_{t+1} \sum_{\theta \in \Theta^*} E^* [\psi_{t+1}(\theta) \mid \mathcal{S}_t] - \gamma_{t+1} \mu_t^*. \end{aligned}$$

By the law of large numbers,  $\mathbb{P}^* - a.s.$  for large enough  $t$  the frequency of  $s^k$  will eventually be  $\theta_k^*$  and

$$\sum_{\theta \in \Theta^*} E^* [\psi_{t+1}(\theta) \mid \mathcal{S}_t] = 1.$$

Eventually along any such path,

$$\begin{aligned} E^* [\mu_{t+1}^*(\theta) \mid \mathcal{S}_t] - \mu_t^* & = (1 - \gamma_{t+1}) \left[ \sum_k \left( \theta_k^* \frac{m_t^*(s^k)}{m_t(s^k)} \right) - 1 \right] \mu_t^* + \gamma_{t+1} (1 - \mu_t^*) \\ & \geq [-\underline{\gamma} (1 - \gamma_{t+1}) \mu_t^* + \gamma_{t+1}] (1 - \mu_t^*) \geq 0, \end{aligned}$$

where the last two inequalities follow from Claim 3 and the hypothesis  $\underline{\gamma} \leq \gamma_{t+1}$ .

Hence  $(\mu_t^*)$  is eventually a  $P^*$ -submartingale. By Lemma 4.3,  $\mu_\infty^* \equiv \lim \mu_t^*$  exists  $\mathbb{P}^* - a.s.$  Consequently,  $E^* [\mu_{t+1}^*(\theta) \mid \mathcal{S}_t] - \mu_t^* \longrightarrow 0$   $\mathbb{P}^* - a.s.$  and from the last displayed equation,  $[-\underline{\gamma} (1 - \gamma_{t+1}) \mu_t^* + \gamma_{t+1}] (1 - \mu_t^*) \longrightarrow 0$   $\mathbb{P}^* - a.s.$  It follows that  $\mu_\infty^* = 1$ . Finally,  $m_t(\cdot) = \int \ell(\cdot \mid \theta) d\mu_t$  eventually remains in  $\Theta^* = (\theta^* - \delta, \theta^* + \delta)$ .

Above  $\delta$  is arbitrary. Apply the preceding to  $\delta = \frac{1}{n}$  to derive a set  $\Omega_n$  such that  $\mathbb{P}^*(\Omega_n) = 1$  and such that for all paths in  $\Omega_n$ ,  $m_t$  eventually remains in  $(\theta^* - \frac{1}{n}, \theta^* + \frac{1}{n})$ . Let  $\Omega \equiv \bigcap_{n=1}^\infty \Omega_n$ . Then,  $\mathbb{P}^*(\Omega) = 1$  and for all paths in  $\Omega$ ,  $m_t$  converges to  $\theta^*$ . ■

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