Menu-Dependent Self-Control*

Jawwad Noor *Norio Takeoka*

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Abstract

The literature on self-control problems has typically put forth models that imply behavior that is consistent with the weak axiom of revealed preference (WARP). We argue that when choice is the outcome of some underlying internal conflict, the resulting choices may not be perfectly consistent across choice problems: an agent's ability to resist temptation may well depend on what alternatives are available to him. We generalize Gul and Pesendorfer [17] so that self-control weakens in the presence of temptation. To model choices from menus explicitly, we consider a choice correspondence as well as a preference over menus and relax both the Independence axiom for the preference and the WARP condition for the choice correspondence. The model is shown to unify a range of well-known findings in the experimental literature on choice under risk and over time within a single specification.

1 Introduction

Decision-making under temptation involves a compromise between two potentially conflicting underlying preferences: a normative preference reflecting his perspective on what he "should" choose, and a temptation preference reflecting his desires. This paper proposes that a plausible outcome of this internal conflict is that *choice behavior may be inconsistent across choice problems*, in the sense of violating the Weak Axiom of Revealed Preference (WARP). Specifically, we hold that an agent's ability to resist temptation may well depend on what is available in the menu: self-control may be menu-dependent. For instance,

*Noor is at the Dept of Economics, Boston University, 270 Bay State Road, Boston MA 02215; Email: jnoor@bu.edu. Takeoka is at the Faculty of Economics, Yokohama National University, 79-3 Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan; Email: takeoka@ynu.ac.jp. We thank the audiences at Boston, Keio, Hitotsubashi Universities, Econometric Society Summer Meeting (Minnesota), Canadian Economic Theory Conference (Toronto), the 2005 JEA Spring Meeting (Tokyo), and Econometric Society World Congress (Shanghai) for helpful comments. We would like to thank an anonymous referee for his/her useful comments and suggestions. We also owe a special thanks to Larry Epstein and Bart Lipman for valuable discussions. The usual disclaimer applies. Takeoka gratefully acknowledges the financial support by Grant-in-Aid for Young Scientists (B).

the extent to which an agent deviates from his diet may depend on the strength of his sugar craving, and the strength of this craving may in turn depend on what desserts are available on the menu. Clearly, when self-control is menu-dependent, the agent's choices may not satisfy WARP. The dieter may resist temptation and choose to have no dessert from the menu {no dessert, small piece}, but the presence of a large piece of cake in {no dessert, small piece, large piece} may *trigger* a strong sugar craving, which he responds to by choosing the small piece, the compromise between his strong craving and his normative preference.

Examples suggestive of temptation-driven violations of WARP are available in the experimental literature on social preferences (List [26], Bardsley [5]). Consider the following experiment involving the dictator game [26]. 'Dictators' and 'recipients' were given an endowment ($10,$5), where $10 denotes the dictator's endowment and $5 the recipient's. Each dictator was given the option of sharing any part of $5 from his endowment with a recipient, that is, they were offered the menu:

$$x_1 = \{(10 - x, 5 + x) : 0 \leq x \leq 5\}.$$

The mean offer among dictators was $1. However, when given also the option of taking exactly $1 from the recipients,

$$x_2 = x_1 \cup \{(11, 4)\},$$

few dictators took the new option but the rate of giving substantially declined, and the mean offer fell to $0. In the context of social preferences, it is natural to hypothesize a normative desire to share but a temptation to be selfish. The above finding suggests that greater temptation may cause choice to become more closely aligned with temptation preferences.

WARP (or a probabilistic version of it) is a peculiar feature of most models put forth in the literature on temptation since the seminal work of Gul and Pesendorfer [17] (henceforth GP). In this paper we introduce a generalization of GP that permits violations of WARP. GP's model takes the form of a representation for an ex ante preference over menus that describes behavior in the anticipation of (and prior to the experience of) temptation. Foundations for any assertion about the nature of ex post choice in these models are absent: ex post choice in these models is derived as an interpretation of a functional form. While this is common for the literature, it is problematic for this paper since our emphasis is explicitly on the nature of ex post choice.[1] Therefore our analysis considers an ex ante preference $\succsim$ over menus (the set Z of nonempty compact subsets of a mixture space Δ) as in the literature, but also a choice correspondence $\mathcal{C} : Z \rightsquigarrow \Delta$ that captures ex post choice from menus.

[1] There are several papers in the literature that provide axiomatizations taking as primitives both an ex ante preference over menus and an ex post choice correspondence. The earliest ones are: Gul and Pesendorfer [19], who study the Strotz model, and Noor [31] and Kopylov [24] who study models that generalize GP. In contrast to the current paper, these papers explicitly assume that the choice correspondence satisfies WARP.

We seek behavioral conditions on $(\succsim, \mathcal{C})$ that admit the following representation: the ex ante preference $\succsim$ is represented by a *Menu-Dependent Self-Control (MDSC)* representation, defined as

$$W(x) = \max_{\mu \in x} \left\{ u(\mu) - \psi\Big(\max_{\eta \in x} v(\eta)\Big) \left(\max_{\eta \in x} v(\eta) - v(\mu)\right) \right\}, \quad x \in Z,$$

where $\psi(\cdot) \geq 0$ is increasing. The non-negative term $(\max_{\eta \in x} v(\eta) - v(\mu))$ is the self-control cost of choosing μ from x: this is strictly positive whenever the choice μ is different from the most tempting item in x. The function $\psi(\cdot)$ bumps up or down the self-control costs from menu to menu, as the size of the temptation varies. Thus the utility $W(x)$ of a menu x is the maximum of normative utility net of self-control costs, and this value function represents $\succsim$. When $\psi(\cdot)$ is a constant, the model reduces to GP's model. As will be evident below, $\psi(\cdot)$ parametrizes the agent's self-control, or lack of it thereof, at the moment of choice.

The choice correspondence implied by the MDSC representation is described as:

$$\begin{aligned} \mathcal{C}[u, v, \psi](x) &= \arg\max_{\mu \in x} \left\{ u(\mu) - \psi\Big(\max_{\eta \in x} v(\eta)\Big) \left(\max_{\eta \in x} v(\eta) - v(\mu)\right) \right\} \\ &= \arg\max_{\mu \in x} \left\{ u(\mu) + \psi\Big(\max_{\eta \in x} v(\eta)\Big) v(\mu) \right\}. \end{aligned} \tag{1}$$

Menu-dependence obtains through $\psi(\cdot)$, specifically via the degree of temptation in the menu, given by '$\max_x v$', which gives rise to the feature that self-control may weaken in the presence of temptation, which in turn can give rise to violations of WARP.

We say that a preference over menus and choice from menus pair $(\succsim, \mathcal{C})$ is rationalized by the MDSC model if $\succsim$ admits a MDSC representation (u, v, ψ) and $\mathcal{C}$ coincides with the choice correspondence induced by the MDSC representation, that is, $\mathcal{C} = \mathcal{C}[u, v, \psi]$. Our axiomatization is obtained by relaxing both the Independence axiom for the ex ante preference and the WARP condition for the ex post choice.

We explore implications of the model for ex post choice $\mathcal{C}$. As an application we use the model to unify disparate evidence from experiments on choice under risk and over time within a single specification. We assume that in the case of static choice under risk, the temptation utility v is more risk averse than the normative utility u. Furthermore, in the case of choice over time, both utilities are assumed to be additively separable over time but the temptation utility exhibits greater impatience than the normative utility. A similar application has been done by Fudenberg-Levine [15, 16], Noor-Takeoka [32] and Takeoka [36] for the Convex Self-Control model, which is a model that exhibits convex costs of self-control. We note that the MDSC model can accommodate more experimental findings that the Convex model cannot.

The remainder of the paper is organized as follows. Section 2 describes GP's model and identifies conditions that lead to the existence of vNM temptation preferences. Section 3 presents our model – it provides axioms and representation theorems. Section 4 discusses related research on convex self-control models. Section 5 demonstrates how the ex post choice generated by the model can accommodate various findings (from experiments on choice under risk and over time) within a single specification. All proofs are relegated to appendices.

2 Preliminaries

2.1 Domain

For any compact metric space X, $\Delta(X)$ denotes the set of all probability measures on the Borel σ-algebra of X, endowed with the weak convergence topology; $\Delta(X)$ is compact and metrizable [2], and we often write it simply as Δ. Let $Z = \mathcal{K}(\Delta)$ denote the set of all nonempty compact subsets of Δ. When endowed with the Hausdorff topology, Z is a compact metric space (see Theorems 3.93 and 3.95 of [2]). An element $x \in Z$ is referred to as a menu. Generic elements of Z are x, y, z whereas generic elements of Δ are μ, η, ν. For $\alpha \in [0,1]$, $\alpha\mu + (1-\alpha)\eta \in \Delta$ is the α-mixture that assigns $\alpha\mu(A) + (1-\alpha)\eta(A)$ to each A in the Borel $\sigma-$algebra of X. Similarly, $\alpha x + (1-\alpha)y \equiv \{\alpha\mu + (1-\alpha)\eta : \mu \in x, \eta \in y\} \in Z$ is an α-mixture of menus x and y.[2]

We will consider a preference defined over Z, and subsequently a choice correspondence over Z as well.

2.2 GP Model

GP model an agent who struggles with temptation when choosing from a menu, and foresees this in an ex-ante stage where he selects a menu. This ex ante preference $\succsim$ over Z is the primitive of the model (ex-post choice is unmodelled). GP adopt the following axioms.

Axiom 1 (Order) *$\succsim$ is complete and transitive.*

Axiom 2 (Continuity) *The sets $\{y \in Z : y \succsim x\}$ and $\{y \in Z : x \succsim y\}$ are closed for each $x \in Z$.*

Axiom 3 (Set-Betweenness) *For all $x, y \in Z$,*

$$x \succsim y \Longrightarrow x \succsim x \cup y \succsim y.$$

Order and Continuity are standard. The anticipation of a struggle with temptation is reflected in Set-Betweenness. A preference for commitment,

$$x \succ x \cup y,$$

reveals temptation by some alternative in y. Anticipated behavior is revealed as follows. Suppose $\{\mu\} \succ \{\eta\}$. When $\{\mu, \eta\} \sim \{\eta\}$ holds, the indifference suggests that the agent

[2]In the appendices, we often use the notation $\mu\alpha\eta$ and $x\alpha y$ instead of $\alpha\mu + (1-\alpha)\eta$ and $\alpha x + (1-\alpha)y$ for abbreviation.

would choose the same item when faced with $\{\mu, \eta\}$ or $\{\eta\}$. The ranking $\{\mu, \eta\} \succ \{\eta\}$ suggests that μ is chosen from $\{\mu, \eta\}$. Observe that if $\{\mu\} \succ \{\mu, \eta\}$, that is, if η is tempting, then the preceding rankings reveal whether the agent anticipates successfully exerting self-control.

GP's fourth and last axiom is the standard vNM Independence condition adapted to the menus setting:

Axiom (Independence) *For any $x, y, z \in Z$ and $\alpha \in (0,1)$,*

$$x \succ y \Longrightarrow \alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z.$$

GP prove the following representation theorem.

Theorem 1 (Gul-Pesendorfer (2001)) *A preference $\succsim$ satisfies Order, Continuity, Set-Betweenness, and Independence if and only if there exist continuous and linear utilities $u, v : \Delta \to \mathbb{R}$ such that $\succsim$ is represented by a function $W : Z \to \mathbb{R}$ defined by:*

$$W(x) = \max_{\mu \in x} \left\{ u(\mu) - \left(\max_{\eta \in x} v(\eta) - v(\mu) \right) \right\}, \quad x \in Z.$$

2.3 Derivation of Temptation Utilities

Independence clearly plays the key role in guaranteeing that temptation preferences are represented by vNM utility in the GP model, but it is also clear that the full force of Independence is not required for this. Because of its relevance for generalizations of the GP model, it is of interest to inquire into minimalistic conditions that guarantee a vNM temptation preference. In this subsection, we present these below. The result in this subsection will be one important component of the proof of our main theorem.

In order to obtain vNM temptation preference we impose the following Independence-type conditions.

Axiom 4 (Commitment Independence) *For all $\mu, \eta, \nu \in \Delta$ and $\alpha \in (0,1)$,*

$$\{\mu\} \succ \{\eta\} \Longrightarrow \{\alpha\mu + (1-\alpha)\nu\} \succ \{\alpha\eta + (1-\alpha)\nu\}.$$

This is the vNM Independence axiom imposed on commitment preference. Given Order and Continuity, the existence of a vNM representation for commitment preference is thus guaranteed.

Axiom 5 (Temptation Independence) *For any μ, η, ν and $\alpha \in (0,1)$ s.t. $\{\mu\} \succ \{\eta\}$,*

$$\{\mu\} \succ \{\mu, \eta\} \Longleftrightarrow \{\alpha\mu + (1-\alpha)\nu\} \succ \{\alpha\mu + (1-\alpha)\nu, \alpha\eta + (1-\alpha)\nu\}.$$

The axiom states that if η tempts μ then $\alpha\eta + (1-\alpha)\nu$ tempts $\alpha\mu + (1-\alpha)\nu$ as well. This is clearly consistent with the existence of a vNM temptation preferences. However, while Order imposes completeness and transitivity on commitment preferences, none of our axioms impose these basic properties on temptation preferences. Neither does the following (at least not on its own), but it ensures that temptation preferences are minimally consistent with a vNM structure.

Axiom 6 (Temptation Convexity) *For any μ, η, η' and $\alpha \in (0,1)$ s.t. $\{\mu\} \succ \{\eta\}, \{\eta'\}$,*

$$\{\mu\} \succ \{\mu, \eta\} \text{ and } \{\mu\} \succ \{\mu, \eta'\} \Longrightarrow \{\mu\} \succ \{\mu, \alpha\eta + (1-\alpha)\eta'\}$$
$$\{\mu\} \sim \{\mu, \eta\} \text{ and } \{\mu\} \sim \{\mu, \eta'\} \Longrightarrow \{\mu\} \sim \{\mu, \alpha\eta + (1-\alpha)\eta'\}.$$

The axiom says simply that a mixture of two tempting items is tempting, and a mixture of two non-tempting items is non-tempting as well.

To derive a vNM temptation utility, we do not need the full force of the Continuity and Set Betweenness axioms. The following are weaker versions of them.

Axiom (Semi-Continuity) *The following sets are closed for each $x \in Z$:*

$$\{y \in Z : y \succsim x\} \text{ and } \{\{\eta\} \in Z : x \succsim \{\eta\}\}.$$

The first claim states that upper contour sets are closed, while the second states that the set of *singletons* in the lower contour set of a menu is closed. Note that Semi-Continuity implies that commitment preference must be continuous in the sense that the sets $\{\eta : \{\eta\} \succsim \{\mu\}\}$ and $\{\eta : \{\mu\} \succsim \{\eta\}\}$ are closed for each $\mu \in \Delta$. The next axiom is a restriction of Set-Betweenness to singleton menus.

Axiom (Binary Set-Betweenness) *For all $\mu, \eta \in \Delta$,*

$$\{\mu\} \succsim \{\eta\} \Longrightarrow \{\mu\} \succsim \{\mu, \eta\} \succsim \{\eta\}.$$

Binary Set-Betweenness does not restrict the nature of menu-dependence of self-control in any way. Self-control is not relevant for singleton menu, and although it may be relevant for binary menus, the comparison of a binary menu with a singleton menu speaks nothing of the nature of menu-dependence of self-control.

We can now state:

Theorem 2 *Suppose that $\succsim$ satisfies Order, Semi-Continuity, Binary Set-Betweenness and Commitment Independence. Then the following statements are equivalent:*

(i) $\succsim$ satisfies Temptation Independence and Temptation Convexity.

(ii) There exists a continuous linear function $v : \Delta \to \mathbb{R}$ such that if $\{\mu\} \succ \{\eta\}$ then

$$\{\mu\} \succ \{\mu, \eta\} \Longleftrightarrow v(\eta) > v(\mu). \tag{2}$$

Moreover, if there exists μ, η s.t. $\{\mu\} \succ \{\mu, \eta\}$ and if there are two such v, v', then v' is a positive affine transformation of v.

The result identifies the extent to which the Independence axiom is responsible for the existence of a vNM temptation preference. It tells us that Independence-type restrictions that go over and above Temptation Independence and Temptation Convexity require justification in terms of features other than linearity of temptation preference.

The proof of this result is inspired by the literature on vNM extensions of preorders (Aumann [4], Fishburn [14], Dubra et al [11]). In this literature, conditions on a preorder (reflexive and transitive binary relation) are sought such that there exists a compatible extension that admits a vNM representation. In our setting, the preference $\succsim$ defines a set $\mathcal{T}$ of (not necessarily complete or transitive) temptation preferences over Δ by the condition that each $T \in \mathcal{T}$ satisfies:

$$\{\eta\} \succ \{\eta, \mu\} \Longrightarrow \mu T \eta \text{ and } \neg \eta T \mu$$
$$\{\mu\} \sim \{\mu, \eta\} \succ \{\eta\} \Longrightarrow \mu T \eta.$$

Our theorem identifies conditions that guarantee the existence of a temptation preference in $\mathcal{T}$ that admits a continuous vNM representation. As in the literature, the proof identifies such a preference in the form of a hyperplane that supports an appropriately defined closed convex cone at the origin. Because our axioms do not guarantee transitivity we cannot simply invoke results from the literature. Nevertheless similar mathematical tools (in particular see Dubra et al [11]) are applicable in our setting.

Other related literature includes Abe [1] and Chatterjee and Krishna [7]. Abe [1] also uses an extension of vNM preorders to obtain a vNM temptation preference with assuming the Independence axiom as in GP.[3] In contrast, due to the demands placed by our objectives, our proof relies on considerably less structure on preference. We exactly identify the weakest set of axioms that ensures a unique vNM temptation preference.

Chatterjee and Krishna [7, Lemma 4.0.4] derive a vNM temptation preference using a different approach. They define a temptation relation and verify that it satisfies the vNM axioms. Their proof is constructive but also more involved.[4] They also maintain the Independence axiom, though it is evident from their proof that they do not require the full force of the axiom: their proof utilizes a 'Translation Invariance' condition and the counterpart of our Temptation Convexity axiom.

3 Menu-Dependent Self-Control

3.1 The Model

We consider an ex ante preference $\succsim$ over menus like GP, but also a choice correspondence that captures ex post choice from menus.

[3]We learned of his paper while writing ours, but the ideas were conceived independently, as is reflected in the different proof strategies.

[4]A part of their proof makes use of a separating hyperplane argument to show that the temptation preference they define is complete and transitive. However, they do not derive a temptation preference in the form of a separating hyperplane.

Definition 1 *$\mathcal{C} : Z \rightsquigarrow \Delta$ is a choice correspondence if it is upper hemicontinuous, non-empty and closed-valued and satisfies $\mathcal{C}(x) \subset x$ for all $x \in Z$.*

The primitive of our model is a pair $(\succsim, \mathcal{C})$ of an ex ante preference and a choice correspondence.

In this section we present a model that expresses our theory of menu-dependent self-control, namely that self-control weakens in the presence of temptation.

Definition 2 (Menu-dependent self-control preference) *A preference $\succsim$ is said to be a menu-dependent self-control (MDSC) preference if it admits a representation $W : Z \to \mathbb{R}$ for $\succsim$ defined by:*

$$W(x) = \max_{\mu \in x} \left\{ u(\mu) - \psi\Big(\max_{\eta \in x} v(\eta)\Big) \left(\max_{\eta \in x} v(\eta) - v(\mu)\right) \right\}, \tag{3}$$

for continuous linear functions $u, v : \Delta \to \mathbb{R}$ and some continuous and weakly increasing function $\psi : v(\Delta) \to \mathbb{R}_+$ such that $\psi(l) > 0$ for all $l > \min v(\Delta)$.

Given a MDSC representation (u, v, ψ), we can define the induced choice correspondence by

$$\mathcal{C}[u, v, \psi](x) = \arg\max_{\mu \in x} \left\{ u(\mu) + \psi\Big(\max_{\eta \in x} v(\eta)\Big) v(\mu) \right\},$$

for all $x \in Z$. Note that this choice correspondence exhibits context effects through varying temptation intensity $\max_x v$. That is, this model is consistent with the hypothesis that self-control weakens in the presence of temptation and can capture menu-dependent self-control.

The following definition requires that the ex ante preference admits a MDSC representation and the choice correspondence coincides with the choice induced by the MDSC representation.

Definition 3 (Rationalization) *A pair of an ex ante preference and a choice correspondence $(\succsim, \mathcal{C})$ is said to be rationalized by a MDSC model if $\succsim$ admits a MDSC representation (u, v, ψ) of the form (3) and $\mathcal{C} = \mathcal{C}[u, v, \psi]$.*

Since $\mathcal{C}[u, v, \psi]$ is derived from the ex ante preference, it captures the agent's expectation about the ex post choice, while the choice correspondence $\mathcal{C}$ is the agent's actual choice at the ex post stage. Thus, Definition 3 requires that the agent is sophisticated in the sense that he correctly anticipates his future behavior.

3.2 Axioms for the MDSC Representation

We first introduce the axioms for the MDSC representation. As preliminaries, we define two notions as below: For all $\mu, \eta \in \Delta$, say that η *weakly tempts* μ, denoted by

$$\eta \succsim_T \mu,$$

if and only if either $\{\mu\} \succ \{\mu, \eta\}$ or $\{\eta\} \sim \{\mu, \eta\} \succ \{\mu\}$ or $\mu = \eta$ holds.

For all menus $x \in Z$, define its *singleton equivalent* $e_x \in \Delta$ by [5]

$$x \sim \{e_x\}.$$

In particular, when $x = \{\mu, \eta\}$, its singleton equivalent is denoted by $e_{\mu\eta}$.

The following axiom is introduced by Noor and Takeoka [32] and is arguably uncontentious.

Axiom 7 (Temptation Aversion) *If $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ for some μ, η, then for any ν,*

$$\nu \precsim_T \eta \Longrightarrow \{\mu, \nu\} \succsim \{\mu, \eta\}.$$

Suppose that the menu $\{\mu, \eta\}$ is such that η is tempting but resisted. The axiom makes the simple and intuitive claim that if η is replaced with something less tempting, then the menu becomes more attractive. This is intuitive particularly considering that we are modelling an agent whose self-control improves in the presence of lower temptation.

Now we seek a weaker version of Independence which is consistent with menu-dependent self-control. For all $x, y \in Z$, if $\succsim$ satisfies the Independence axiom, we will have

$$\alpha x + (1 - \alpha) y \sim \alpha\{e_x\} + (1 - \alpha)\{e_y\}. \tag{4}$$

However, under the hypothesis that self-control weakens in the presence of temptation, this ranking may not hold. When $\{\alpha e_x + (1 - \alpha) e_y\}$ is given to the agent, he makes a commitment to this lottery, and does not have to exert self-control. On the other hand, when $\alpha x + (1 - \alpha) y$ is given, he may exercise self-control at this menu, and hence concerns about the most tempting option in it. The level of temptation in the menu may well depend on how the two menus are mixed.

The above intuition also suggests that the implication of Independence given as (4) will hold if two menus x and y contain the same most tempting alternative, in which case mixing does not change the level of maximal temptation. The following axiom is motivated by such an intuition.

Axiom 8 (Linear Self-Control) *For any $\mu, \nu, \eta \in \Delta$ and $\alpha \in (0, 1)$, if $\eta \succsim_T \mu, \nu$ then*

$$\alpha\{\mu, \eta\} + (1 - \alpha)\{\nu, \eta\} \sim \alpha\{e_{\mu\eta}\} + (1 - \alpha)\{e_{\nu\eta}\}.$$

By assumption, the two menus $\{\mu, \eta\}$ and $\{\nu, \eta\}$ have the same most-tempting alternative, η. The axiom imposes the implication of Independence only on these menus.

The last axiom is also motivated by our hypothesis that self-control weakens in the presence of temptation. Suppose that for some $x \in Z$, $\nu, \bar{\nu} \in \Delta$, and $\alpha \in (0, 1)$,

$$\alpha x + (1 - \alpha)\{\nu\} \sim \alpha\{\overline{\nu}\} + (1 - \alpha)\{\nu\}.$$

[5] Under Continuity and Set Betweenness, a singleton equivalent exists for all menus $x \in Z$.

If $\succsim$ satisfies Independence, this indifference still holds after replacing ν with any other lottery ν'. However, the intensity of temptation from ν' may affect self-control at $\alpha x+(1-\alpha)\nu'$, while there is no such effect under commitment. Thus, we may have $\alpha x+(1-\alpha)\{\nu'\} \not\sim \alpha\{\overline{\nu}\}+(1-\alpha)\{\nu'\}$ as below:

Axiom 9 (Decreasing Self-Control) *For any $\mu, \eta, \nu, \bar{\nu} \in \Delta$ and $\alpha \in (0,1)$, if $\alpha\{\mu,\eta\}+(1-\alpha)\{\nu\} \sim \alpha\{\overline{\nu}\}+(1-\alpha)\{\nu\}$, then,*

$$\nu \succsim_T \nu' \implies \alpha\{\mu,\eta\}+(1-\alpha)\{\nu'\} \succsim \alpha\{\overline{\nu}\}+(1-\alpha)\{\nu'\},$$

$$\nu \precsim_T \nu' \implies \alpha\{\mu,\eta\}+(1-\alpha)\{\nu'\} \precsim \alpha\{\overline{\nu}\}+(1-\alpha)\{\nu'\}.$$

The interesting case is where $\{\mu\} \succ \{\mu,\eta\}$. Then Linear Self-Control (with $\nu=\eta$) tells us the hypothesis of Decreasing Self-Control is satisfied with

$$\alpha\{\mu,\eta\}+(1-\alpha)\{\eta\} \sim \alpha\{e_{\mu\eta}\}+(1-\alpha)\{\eta\}. \tag{5}$$

This ranking suggests that the agent is indifferent between exercising self-control at the menu $\alpha\{\mu,\eta\}+(1-\alpha)\{\eta\}$ and making a commitment to $\alpha e_{\mu\eta}+(1-\alpha)\eta$. Suppose η is replaced with a less tempting alternative ν. This will reduce temptation on $\alpha\{\mu,\eta\}+(1-\alpha)\{\nu\}$, and hence the agent's self-control is enhanced with the reduction of temptation, while there is no such effect on the agent, who has already made a commitment. Thus, the agent will exhibit the ranking stated as in Axiom 9. On the other hand, if η is replaced with a more tempting alternative ν, this will increase temptation on $\alpha\{\mu,\eta\}+(1-\alpha)\{\nu\}$, while there is no such effect on the agent making a commitment. Thus, the agent will exhibit the reversed ranking stated as in Axiom 9.

Now we are ready to state a representation theorem for the ex ante preference.

Theorem 3 *A preference $\succsim$ satisfies Axioms 1-9 if and only if it is a MDSC preference.*

An outline of the proof is presented in section 4.1, while a formal proof is given in Appendix B.

The next corollary is an axiomatization of the GP model as a special case of our MDSC representation. Obviously, if ψ is a constant function, the model is reduced to the GP model.

Corollary 1 *A MDSC preference $\succsim$ admits a MDSC representation with a constant function $\psi : v(\Delta) \to \mathbb{R}_{++}$ if and only if for all μ, η, ν and $\alpha \in (0,1)$,*

$$\{\mu\} \succ \{\mu,\eta\} \implies \alpha\{\mu,\eta\}+(1-\alpha)\{\nu\} \sim \alpha\{e_{\mu\eta}\}+(1-\alpha)\{\nu\}.$$

The result reveals the various implications of assuming that a preference satisfying Order, Continuity and Set-Betweenness also satisfies Independence, thereby highlighting what is bought with Independence and how precisely it interacts with preferences to characterize the model.

Finally, consider the uniqueness properties of the MDSC representation. Given a representation (u, v, ψ), the *self-control subdomain* is defined as follows:

$$L = \{l \in \mathbb{R}_+ \,|\, l = v(\eta) \text{ for some } \mu, \eta \text{ s.t. } \{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}\}.$$

This is the set of all values of $v(\eta)$ for which $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ holds for some μ. Observe that if $v(\eta) \notin L$, then for any μ such that $\{\mu\} \succ \{\eta\}$, it must be that either there is no temptation in $\{\mu, \eta\}$ or there is overwhelming temptation. In either case, the precise shape of ψ is immaterial for the description of choice behavior; for instance it could be increased without affecting behavior. Thus, the precise shape of ψ is meaningful only on L. Notice also that L is an interval with $\inf L = \min_\Delta v$.[6] Say that preference $\succsim$ is nondegenerate if there exists μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$.

Theorem 4 *Suppose that (u, v, ψ) and (u', v', ψ') are both representations of a nondegenerate MDSC preference. Then there exist constants $\alpha_u, \alpha_v > 0$ and $\beta_u, \beta_v \in \mathbb{R}$ such that*

$$u' = \alpha_u u + \beta_u,\ v' = \alpha_v v + \beta_v.$$

Moreover, if L and L' are the self-control subdomains associated with the two representations, then

$$L' = \alpha_v L + \beta_v, \text{ and } \psi'(\alpha_v l + \beta_v) = \frac{\alpha_u}{\alpha_v}\psi(l) \text{ for all } l \in L.$$

Note that $\alpha_v L + \beta_v$ is standard notation for the set $\{\alpha_v l + \beta_v \,|\, l \in L\}$. See Appendix D for the proof. A corollary of the uniqueness result is that if the functions $W', W : Z \to \mathbb{R}$ represent the same nondegenerate MDSC preference, then there exist constants $\alpha_u > 0$ and β_u such that for all x,

$$W'(x) = \alpha_u W(x) + \beta_u.$$

That is, MDSC utility functions are unique up to an affine transformation.

3.3 Rationalization of the MDSC Model

We turn to the rationalization for $(\succsim, \mathcal{C})$. The purpose is threefold. First, like Gul and Pesendorfer [19], Noor [31] and Kopylov [24], we seek to identify behavioral conditions that reveal that the agent is sophisticated, in that his ex post choice behavior is correctly anticipated ex ante. Second, unlike these papers, our agent violates WARP, and consequently

[6] To see this, take any $l \in L$. By definition, there exists μ, η such that $l = v(\eta)$ and $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$. Moreover, let ν be a minimal lottery with respect to v. By the representation, for all $\alpha \in (0, 1)$,

$$\begin{aligned} & u(\mu) - \psi(v(\eta))(v(\eta) - v(\mu)) > u(\eta) \\ \Rightarrow\ & u(\alpha\mu + (1-\alpha)\nu) - \psi(v(\eta))(v(\alpha\eta + (1-\alpha)\nu) - v(\alpha\mu + (1-\alpha)\nu)) > u(\alpha\eta + (1-\alpha)\nu) \\ \Rightarrow\ & u(\alpha\mu + (1-\alpha)\nu) - \psi(v(\alpha\eta + (1-\alpha)\nu))(v(\alpha\eta + (1-\alpha)\nu) - v(\alpha\mu + (1-\alpha)\nu)) > u(\alpha\eta + (1-\alpha)\nu), \end{aligned}$$

which implies that $\{\alpha\mu + (1-\alpha)\nu\} \succ \{\alpha\mu + (1-\alpha)\nu, \alpha\eta + (1-\alpha)\nu\} \succ \{\alpha\eta + (1-\alpha)\nu\}$ and $\alpha l + (1-\alpha)\min_\Delta v \in L$.

we are interested in studying the properties of ex post choice implied by the MDSC model. Lastly, these properties will serve as a means of testing a possible hypothesis that we provide in Section 5. There, we show that the findings of several experiments can be generated by the choice correspondence induced by the MDSC model:

$$\mathcal{C}[u, v, \phi](x) = \arg\max_{\mu \in x} \left\{ u(\mu) + \psi\Big(\max_{\eta \in x} v(\eta)\Big) v(\mu) \right\} \tag{6}$$

However, the choice correspondence (6) can arise from other models as well. Consequently in order for future research to test whether the experimental evidence is indeed arising due to temptation as our model suggests, the testable implications of the MDSC model on the joint primitive $(\succsim, \mathcal{C})$ is needed. This is precisely what our rationalization for $(\succsim, \mathcal{C})$ yields.

Note first that WARP requires the following consistency condition for a choice correspondence: for all $x, y \in Z$ and $\mu, \eta \in x \cap y$,

$$\mu \in \mathcal{C}(x) \text{ and } \eta \in \mathcal{C}(y) \Longrightarrow \mu \in \mathcal{C}(y).$$

As stated in the Introduction, WARP may not be satisfied when choices are made under temptation. Recall the example. Denote n =no dessert, s =small piece of cake, and l =large piece of cake. Then, the WARP violation driven by temptation is illustrated as

$$\mathcal{C}(\{n, s\}) = \{n\}, \text{ and } \mathcal{C}(\{n, s, l\}) = \{s\}.$$

Although the choice under temptation may violate WARP, it may still satisfy some consistency condition. Note that the above violation of WARP arises only because temptation levels change between two menus. It is therefore intuitive to hypothesize that *WARP will hold if two menus have the same level of temptation.* Though not obvious at first, this motivates the next axiom which requires WARP only for pairs of menus that are indifferent in the ex ante preference.

Axiom E1 (Weak WARP) *For all $x, y \in \mathcal{Z}$ with $x \sim y$ and $\mu, \eta \in x \cap y$,*

$$\mu \in \mathcal{C}(x) \textit{ and } \eta \in \mathcal{C}(y) \Longrightarrow \mu \in \mathcal{C}(y).$$

Given the hypothesis, $x \sim y$ implies that choosing μ in x is just as good as choosing η in y, from the ex ante perspective. Since μ is (weakly) chosen over η in x, it follows that choosing η in x is (weakly) worse than choosing η in y, which is only possible if the temptation in x is (weakly) greater. On the other hand, since η is (weakly) chosen over μ in y then it follows that choosing μ in y is (weakly) worse than choosing μ in x, which is only possible if the temptation in y is (weakly) greater. Therefore, we can conclude that given the hypothesis, x and y have the same level of temptation, and so WARP should hold between these menus.

Axiom E2 (ex post Decreasing Self-Control) *If $\{\mu\} \succ \{\mu, \eta\}$ then*

$$\mu \in \mathcal{C}(\{\mu, \eta\}) \Longrightarrow \mathcal{C}(\{\mu, \alpha\mu + (1-\alpha)\eta\}) = \{\mu\} \textit{ for all } \alpha \in (0, 1).$$

This restriction says that if the agent normatively prefers μ over η but is tempted by the latter, then his self-control can only increase if η is replaced by something less tempting (for instance, $\alpha\mu + (1-\alpha)\eta$ as in the axiom). Thus, if he can pick μ from $\{\mu, \eta\}$ – albeit not uniquely if he is on the margin between exerting self-control or not – then he can pick μ uniquely in $\{\mu, \alpha\mu + (1-\alpha)\eta\}$. The idea that a reduction in temptation increases self-control is the heart of the MDSC model, and the restriction is its expression in ex post choice.

Our final axiom expresses the agent's sophistication: the choice from a menu he anticipates ex ante is the one he makes ex post. The axiom requires a way to express that one menu contains more temptation that another. When $\{\mu\} \not\sim \{\eta\}$, then we can directly infer if μ tempts η or the converse. However, if $\{\mu\} \sim \{\eta\}$ then the only way to infer which is more tempting is to see whether rewards in the neighborhood of μ tempt those in the neighborhood of η, or conversely. Indeed, we can infer that η is temptation-ranked weakly higher than μ if there exists a sequence $(\mu_n, \eta_n) \to (\mu, \eta)$ s.t. $\eta_n \succsim_T \mu_n$ for all n. Say that a menu x *temptation-dominates* a lottery μ if there is $\eta \in x$ that is revealed to be temptation-ranked weakly higher than μ in this way.

Axiom E3 (Sophistication) *For any x and μ,*

$$x \cup \{\mu\} \succ x \Longrightarrow \mathcal{C}(x \cup \{\mu\}) = \{\mu\}.$$

Moreover, the converse holds if x temptation-dominates μ.

If adding μ to x makes the menu strictly more attractive then it should be because μ is chosen in the new menu – otherwise μ would only potentially add to temptation thereby making the menu weakly less attractive. The restriction requires that choice behavior respect this suggested implication of the ranking of menus. For the converse, if μ is uniquely chosen from $x \cup \{\mu\}$ then in general it is not obvious that $x \cup \{\mu\} \succ x$, because μ could be a unique overwhelmingly tempting alternative in $x \cup \{\mu\}$ in which case we could have $x \cup \{\mu\} \not\succ x$. However, when x temptation-dominates μ, then we can be sure that μ is not a unique overwhelmingly tempting alternative, and the only possibility that remains is $x \cup \{\mu\} \succ x$.

Say that $\succsim$ is *nondegenerate** if there exists $\mu^*, \eta^*, \mu', \eta'$ s.t. $\{\mu^*\} \succ \{\mu^*, \eta^*\} \succ \{\eta^*\}$ and $\{\mu'\} \sim \{\mu', \eta'\} \succ \{\eta'\}$.

Theorem 5 *Suppose that $\succsim$ is a nondegenerate* preference satisfying the axioms of the MDSC representation. Then the following statements are equivalent:*

(a) The pair $(\succsim, \mathcal{C})$ satisfies Weak WARP, ex post Decreasing Self-Control, and Sophistication.

(b) The pair $(\succsim, \mathcal{C})$ is rationalized by the MDSC model.

The theorem specifies that the three axioms of this section are the key joint implications of the MDSC model for ex ante preference $\succsim$ and ex post choice $\mathcal{C}$. In other words, the only observable implications of sophisticated behavior are these three axioms in the MDSC model.

4 Proof Outline and Related Literature

4.1 Proof Outline for Theorem 3

The MDSC representation is constructed as follows. First of all, Theorem 2 ensures that there exist continuous mixture linear functions $u, v : \Delta \to \mathbb{R}$, where u represents the commitment ranking of $\succsim$, and v satisfies

$$\{\mu\} \succ \{\mu, \eta\} \ \Rightarrow \ v(\eta) > v(\mu), \ \text{and} \ \{\eta\} \sim \{\mu, \eta\} \succ \{\mu\} \ \Rightarrow \ v(\eta) \geq v(\mu).$$

Since u is continuous on Δ, there exists a maximal and a minimal lottery $\mu^{\Delta}, \mu_{\Delta} \in \Delta$ with respect to u. Given Continuity and Set Betweenness, we can show that for each $x \in Z$ it must be that $\{\mu^{\Delta}\} \succsim x \succsim \{\mu_{\Delta}\}$ and there exists a unique number $\alpha(x) \in [0, 1]$ such that $x \sim \{\alpha(x)\mu^{\Delta} + (1 - \alpha(x))\mu_{\Delta}\}$. Thus,

$$W(x) \equiv u(\alpha(x)\mu^{\Delta} + (1 - \alpha(x))\mu_{\Delta})$$

is a representation of $\succsim$ such that $W(\{\mu\}) = u(\mu)$ for all $\mu \in \Delta$.

The next question is how to define $\psi : v(\Delta) \to \mathbb{R}_+$ and show that W has the desired form. As a first step, we define the self-control cost function. Take any lotteries μ, η with $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$. This ranking suggests that self-control is exerted in $\{\mu, \eta\}$. Thus, the difference $u(\mu) - W(\{\mu, \eta\})$ should express the self-control cost when the absolute level of temptation is $l = v(\eta)$ and the temptation frustration is $w = v(\eta) - v(\mu)$. Thus, define the self-control cost function $\varphi(l, w)$ as

$$\varphi(v(\eta), v(\eta) - v(\mu)) = u(\mu) - W(\{\mu, \eta\}).$$

To show that this definition is indeed well-defined, we need to show that for any other μ', η' with $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$,

$$v(\eta) = v(\eta') \text{ and } v(\mu) = v(\mu') \implies u(\mu) - W(\{\mu, \eta\}) = u(\mu') - W(\{\mu', \eta'\}).^7 \qquad (7)$$

This key result comes from the following observations. Intuitively, Decreasing Self-Control implies that if $v(\nu) = v(\nu')$ then

$$\begin{aligned} &\alpha\{\mu, \eta\} + (1 - \alpha)\{\nu\} \sim \alpha\{\overline{\nu}\} + (1 - \alpha)\{\nu\} \\ &\iff \alpha\{\mu, \eta\} + (1 - \alpha)\{\nu'\} \sim \alpha\{\overline{\nu}\} + (1 - \alpha)\{\nu'\}. \end{aligned} \qquad (8)$$

That is, the ranking of $\{\mu, \eta\}$ and $\{\overline{\nu}\}$ when mixed with a common singleton $\{\nu\}$ is unchanged when the singleton is replaced with one containing an equally tempting lottery. This reflects a 'translation invariance' property that states that if a common 'translation' is applied to the elements of both the menus $\{\mu, \eta\}$ and $\{\overline{\nu}\}$, then the ranking of the menus

[7] By Continuity and the fact that the set of all lotteries with finite supports is dense in Δ under the weak convergence topology, we can assume also that μ, μ', η, η' have finite supports, and hence can be viewed as vectors in a finite dimensional space $\mathbb{R}^n$.

is unaffected. More formally, a translation is a vector $\theta \in \mathbb{R}^n$ such that $\sum_{i=1}^n \theta(i) = 0$. We say that a translation θ is admissible for a menu x if $\mu + \theta \in \Delta$ for all $\mu \in x$. The counterpart of (8) for the representation is the following "translation linearity" property: if θ is admissible for $\{\mu, \eta\}$, then [8]

$$v(\theta) = 0 \implies W(\{\mu + \theta, \eta + \theta\}) = W(\{\mu, \eta\}) + u(\theta). \tag{9}$$

This is used to show (7). So suppose $v(\eta) = v(\eta')$ and $v(\mu) = v(\mu')$. To consider the simplest case, assume that the translation $\theta \equiv \mu' - \mu$ is admissible for $\{\mu, \eta\}$.[9] Note that $v(\theta) = 0$. By (9), $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ implies $\{\mu + \theta\} \succ \{\mu + \theta, \eta + \theta\} \succ \{\eta + \theta\}$, that is, $\{\mu'\} \succ \{\mu', \eta + \theta\} \succ \{\eta + \theta\}$. Since $v(\eta') = v(\eta + \theta)$ by assumption, we have $W(\{\mu', \eta'\}) = W(\{\mu', \eta + \theta\})$. Again, by (9),

$$W(\{\mu', \eta'\}) = W(\{\mu', \eta + \theta\}) = W(\{\mu, \eta\}) + u(\theta) = W(\{\mu, \eta\}) + u(\mu') - u(\mu),$$

as desired.

The next step is to show that $\varphi(l, w)$ is homogeneous of degree one with respect to w. This property comes from Linear Self-Control. Take any μ, η with $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$. Let $l = v(\eta)$ and $w = v(\eta) - v(\mu)$. By Linear Self-Control, for all $\alpha \in (0, 1)$, $\alpha\{\mu, \eta\} + (1 - \alpha)\{\eta\} \sim \alpha\{e_{\mu\eta}\} + (1 - \alpha)\{\eta\}$. By definition of φ,

$$u(\alpha\mu + (1 - \alpha)\eta) - \varphi(l, \alpha w) = \alpha(u(\mu) - \varphi(l, w)) + (1 - \alpha)u(\eta).$$

Thus, $\varphi(l, \alpha w) = \alpha\varphi(l, w)$. From this property, for a fixed $\overline{w}$ and for all $w < \overline{w}$,

$$\varphi(l, w) = \varphi(l, \frac{w}{\overline{w}}\overline{w}) = \frac{w}{\overline{w}}\varphi(l, \overline{w}).$$

That is, on the self-control subdomain L, ψ can be defined as

$$\psi(l) \equiv \frac{\varphi(l, \overline{w})}{\overline{w}}.$$

By using (u, v, ψ) defined as above, we first show that the desired representation is possible for binary menus. In this step, ψ is extended appropriately to the whole domain $v(\Delta)$ in an increasing way. The remaining argument is more or less the same as in Gul and Pesendorfer [17]. Since $\succsim$ satisfies Set Betweenness, the representation can be extended to the set of all finite menus. Finally, by Continuity and a property of the Hausdorff metric, the representation can be extended to Z as desired.

[8] Let $u(\theta)$ and $v(\theta)$ denote $\sum_i u(c_i)\theta(i)$ and $\sum_i v(c_i)\theta(i)$, respectively.

[9] In general, $\theta \equiv \mu' - \mu$ is not necessarily admissible for $\{\mu, \eta\}$. Then, we need more elaborated arguments. See Lemma 17 in Appendix B for details.

4.2 Related Axiomatic Literature and the Convex Model

There are several related studies that investigate generalizations of the temptation model by relaxing the Independence axiom. In Noor and Takeoka [32] we relax Independence in order to study non-linear self-control costs - the paper will be discussed below. Epstein and Kopylov [12] provide a theory of "cold feet" using a GP-style model of agents who are tempted, post-choice, to change beliefs over a state space. Their model violates Independence. However, their's is not a generalization of GP [17] because their choice domain is specialized to the set of menus of Anscombe-Aumann acts. Two models in a discrete settings (where Independence has no meaning) that allow for WARP violations are Dillenberger and Sadowski [10] and Masatlioglu, Nakajima, and Ozdenoren [28].

In Noor and Takeoka [32] we show that a preference $\succsim$ over Z that satisfies Axioms 1-7 admits a *General Self-Control* representation,

$$W(x) = \max_{\mu \in x} \left\{ u(\mu) - c(\mu, \max_{\eta \in x} v(\eta)) \right\}, \tag{10}$$

which generalizes GP so as to retain its basic features while expunging linearity and imposing minimal structure on the nature of the cost of self-control, captured by the cost function $c : \Delta \times v(\Delta) \to \mathbb{R}_+$.

There are three main differences between Noor and Takeoka [32] and the present paper. First, Noor and Takeoka [32] consider only the ex ante preference $\succsim$ on Z and the ex post choice is unmodeled, while the choice correspondence is also considered as a primitive in the present paper, whereby we can identify the testable implication of the ex post choice and to what extent WARP can be relaxed. Second, the representation theorem for the General Self-Control representation relies in a fundamental way on the key result obtained in Theorem 2 of this paper, that is, the existence of a vNM temptation utility. Third, the proof of Theorem 3 does not start with the general representation (10) and impose additional axioms in order to specialize the self-control costs into the desired form. Rather, we use a different construction altogether, which achieves a separation between the "temptation frustration" (that is, $\max_x v - v(\mu)$) and the menu-dependent effect $\psi(\max_x v)$.

Noor and Takeoka [32] axiomatize also a special class of representation, called the Convex Self-Control model, where $c(\mu, \max_x v)$ is specialized as

$$\varphi(\max_x v - v(\mu)) \tag{11}$$

for some increasing convex function φ. As can be seen from the corresponding axiomatizations in terms of ex ante preference over menus, the Convex and MDSC models are not nested within each other. The properties of ex post choice in the Convex model have not been studied, but one clear difference is that the MDSC model satisfies the following "indifference to randomization" property while the Convex model routinely violates it: $\mathcal{C}(\{\mu, \eta\}) = \{\mu, \eta\} \Longrightarrow \mathcal{C}(co\{\mu, \eta\}) = \{\mu, \eta\}$, where $co\{\mu, \eta\}$ is the convex hull of $\{\mu, \eta\}$.

We see the two models as telling different stories about the nature of temptation and self-control. The Convex model embodies the intuitive idea that the exertion of self-control is met with increasing marginal self-control costs. The MDSC model highlights the idea that the intensity of temptation is dependent on cues which *trigger* cravings. A given dessert may appear more or less attractive depending on how strong the agent's sugar craving is. The strength of the craving in turn is cued by the kind of desserts she is faced with. In the MDSC model the temptation ranking of alternatives is fixed but the intensity of temptation, and consequently the agent's ability to resist it, changes with the menu. An alternative way to look at this is that at the time of choice the agent has a particular propensity of self-control and this propensity may be affected by cravings: for instance, a craving may weaken the agent's resolve to stick to her diet as she starts thinking to herself that "well, a little indulgence won't kill me."

The proof of the MDSC and Convex models consists of common important steps: (a) construction of a representation W, and derivation of a temptation utility v, (b) by using self-control behavior $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$, a self control cost is measured by $u(\mu) - W(\{\mu, \eta\})$, and is converted to the desired form, (c) construction of the desired representation over binary menus, (d) extension of the representation to finite menus by Set Betweenness, and (e) extension to general menus by Continuity.

The argument in steps (d) and (e) are the same between the MDSC and Convex models. Step (a) is also the same, but, as stated above, the derivation of v is established in the present paper. We do this step differently than GP and in a more general way. Regarding step (b), notice that the Convex model satisfies the translation invariance property, that is, for all admissible θ, $W(\{\mu+\theta, \eta+\theta\}) = W(\{\mu, \eta\}) + u(\theta)$, which makes easier to show the well-definedness of the self-control cost function, $\varphi(v(\eta)-v(\mu)) = u(\mu)-W(\{\mu, \eta\})$. On the other hand, the MDSC model satisfies the translation invariance only when $v(\theta) = 0$, and hence, a more elaborated argument is required to establish well-definedness of $\varphi(v(\eta), v(\eta)-v(\mu)) = u(\mu) - W(\{\mu, \eta\})$. Moreover, it has to be converted to $\psi(v(\eta))(v(\eta) - v(\mu))$, which is a peculiar feature of the MDSC model.

5 Implications for Risk and Time Preference

Experiments uncover several properties of choice under risk that are inconsistent with expected utility, and also several properties of choice over time that are inconsistent with exponential discounting. There are several nonexpected utility models and nonexponential discounting models that respectively account for these. Furthermore there are relationships between choice under risk and over time. Keren and Roelofsma [23] hypothesize that the findings arise because the future is inherently risky.[10] We show below that menu-dependent self-control, coupled with a temptation to be risk averse and impatient, produces these three sets of findings.[11]

[10] Thus, delaying a sure reward or making an immediate reward uncertain give rise to the same effect on decisions. See Halevy [20] for a model that formalizes this idea.

[11] Several of these findings can also be produced by the Convex models (Noor and Takeoka [32], Fudenberg and Levine [15, 16]) and differences will be noted below.

Consider an agent whose choices from any menu x maximizes a menu-dependent utility:

$$\mathcal{C}(x) = \arg\max_{\mu \in x} \left\{ u(\mu) + \psi\Big(\max_{\eta \in x} v(\eta)\Big) v(\mu) \right\}. \tag{12}$$

Throughout we assume that X is a closed interval of money.

5.1 Risk

We adopt the following specification:

Assumption 1 *u and v are increasing and concave on X. Moreover, v is more risk averse than u.*[12]

This assumption generates the following behaviors.

Common Ratio Effect. Subjects in experiments are typically observed to choose \$3000 over a 0.8 chance of \$4000, while also choosing a 0.2 chance of \$4000 over a 0.25 chance of \$3000. This violates the vNM Independence axiom. Letting r denote a non-degenerate lottery, s a degenerate nonzero lottery, and 0 the degenerate zero lottery, the Common Ratio Effect is given by

$$\mathcal{C}(\{r, s\}) = \{s\} \text{ and } \mathcal{C}(\{\alpha r + (1-\alpha)0, \alpha s + (1-\alpha)0\}) = \{\alpha r + (1-\alpha)0\}.$$

If u prefers r to s and v has the opposite preference, then the above choices arise in our model. Intuitively, by mixing all alternatives with 0, the weight $\psi(\max_x v)$ on v reduces, thus causing a decrease in risk aversion.

Common Consequence Effect. This is the popular form of the Allais Paradox. Subjects prefer \$1m to a lottery that yields a 0.1 chance for \$5m, 0.89 chance of \$1m and 0.01 chance of 0, but they prefer 0.1 chance of 5m to a 0.11 chance of 1m. Letting r denote a non-degenerate lottery, s is a degenerate nonzero lottery, and 0 the degenerate zero lottery, the Common Consequence Effect is given by

$$\begin{aligned} &\mathcal{C}(\{\alpha s + (1-\alpha)s, \alpha r + (1-\alpha)s\}) = \{\alpha s + (1-\alpha)s\}, \quad \text{and} \\ &\mathcal{C}(\{\alpha s + (1-\alpha)0, \alpha r + (1-\alpha)0\}) = \{\alpha r + (1-\alpha)0\}. \end{aligned}$$

These choices can arise for the same reason as in the Common Ratio Effect.

While the Convex Self-Control model can accommodate several findings described in this section, it cannot accommodate the Common Consequence Effect since

$$\varphi\left(v(\alpha\mu' + (1-\alpha)\eta) - v(\alpha\mu + (1-\alpha)\eta)\right) = \varphi\left(\alpha[v(\mu') - v(\mu)]\right),$$

[12] The axiom on ex ante preference $\succsim$ that characterizes this is that for any degenerate lottery c and any lottery μ, $\{c\} \succ \{\mu\} \Longrightarrow \{c\} \sim \{c, \mu\} \succ \{\mu\}$. We omit the straightforward proof.

that is, the self-control cost of choosing $\alpha\mu + (1-\alpha)\eta$ over $\alpha\mu' + (1-\alpha)\eta$ is independent of η.[13]

Risk Aversion and Stakes Size. Estimates both from the field and the lab show that risk aversion tends to increase with stake size (see Holt and Laury [21] and the reference cited therein). In our model, increase in stake size is associated with a reduction in self-control, and thus an increase in risk aversion.

5.2 Risk and Time

Let $X = \Delta^\infty$ and endow it with the product topology. A generic element is denoted $c = (\mu_0, \mu_1, ...)$. The MDSC model can readily be extended to an infinite horizon in the style of GP [18] where there is temptation by immediate consumption. We omit an axiomatization of the extension because it involves no new ideas. The following choice correspondence would be implied by such an extension:

$$\mathcal{C}(x) = \arg\max_{c \in x} \left\{ U(c) + \psi\Big(\max_{c' \in x} V(c')\Big) V(c) \right\},$$

where

$$U(c) = U(\mu_0, \mu_1, \cdots) = \sum_{t=0}^{\infty} \delta^t u(\mu_t), \text{ and } V(c) = V(\mu_0, \mu_1, \cdots) = \sum_{t=0}^{\infty} \gamma^t v(\mu_t).$$

Thus, the agent's normative and temptation perspectives evaluate a stream $c = (\mu_0, \mu_1, ...)$ by the expected discounted utilities. As in the previous subsection, assume that v is more risk averse than u. We also assume that the temptation utility exhibits greater impatience than the normative utility [30, 31].

Assumption 2 *$\gamma \leq \delta$.*

We note below that $\gamma > \delta$ is also consistent with the behaviors we discuss if we place restrictions on ψ.

It will be convenient to let $u(0) = v(0) = 0$ for some degenerate lottery 0, which is naturally interpreted as the reward that yields zero. Below we also abuse notation and use 0 to denote the degenerate stream $(0, 0, ...)$ yielding no reward in every period. For any stream c define $c^{+0} = c$ and inductively $c^{+(t+1)} = (0, c^{+t})$.

[13] Fudenberg-Levine show that in their intertemporal setting the Common Consequence Effect can arise if the short-run self is tempted to consume all the winnings of a lottery but chooses to save part of the winnings of a lottery. It is also assumed that the lotteries are unanticipated. In addition, the model can accommodate Rabin's paradox: winnings from small-stakes lotteries are completely consumed but those from large-stakes lotteries may be saved, and the smoothing consequently reduces risk aversion. Some of the predictions for risk aversion arising from intertemporal effects run counter to the evidence. For instance, the model implies a reduction in risk aversion as stake size exceeds a threshold, whereas the evidence reveals that risk aversion tends to increase with stake size.

Preference Reversals. A preference reversal is:

$$\mathcal{C}(\{c^{+1}, d\}) = \{d\} \text{ and } \mathcal{C}(\{c^{+(T+1)}, d^{+T}\}) = \{c^{+(T+1)}\}, \tag{13}$$

that is, an agent may choose a later reward $c^{+(T+1)}$ over a sooner reward d^{+T} when both are in the future ($T > 0$), but may choose the immediate reward d over the later reward c^{+1}. GP's model generates preference reversals: if normative and temptation utilities are specified as above, then preference reversals arise if and only if temptation preference discounts the future more steeply than normative preference, $\gamma < \delta$ (GP [18] assume specifically that $\gamma = 0$). Intuitively, due to the differing discount rates, normative preferences dominate the choice of more distant rewards. Thus, if temptation sways choice when the rewards are close, a reversal will take place as the rewards are delayed. This mechanism can exist also in our model, but it should be noted that we get preference reversals even if this mechanism is shut down (by taking $\gamma = \delta$), in which case they are driven purely by menu-dependence of self-control. That is, delaying the rewards also reduces the maximum temptation the agent is faced with, which in turn increases self-control and thus if temptation drives the choice $\mathcal{C}(\{c^{+1}, d\}) = \{d\}$, with sufficient delay this temptation will be resisted and a preference reversal will arise.

More precisely and in more general terms, it is straightforward to show that for any pair of streams a and b where b is more tempting, and for any delay T, $\mathcal{C}(\{a^{+T}, b^{+T}\}) = \{a^{+T}\}$ if and only if

$$\begin{aligned} &\delta^T U(a) + \psi(\gamma^T V(b))\gamma^T V(a) > \delta^T U(b) + \psi(\gamma^T V(b))\gamma^T V(b) \\ \implies &U(a) + \psi(\gamma^T V(b)) \left(\frac{\gamma}{\delta}\right)^T V(a) > U(b) + \psi(\gamma^T V(b)) \left(\frac{\gamma}{\delta}\right)^T V(b). \end{aligned}$$

Observe that as T grows, the weight $\psi(\gamma^T V(b))$ on temptation utility reduces. Temptation utility further loses importance relative to normative utility because it discounts the future relatively more steeply – this acts via the factor $\left(\frac{\gamma}{\delta}\right)^T$. Thus, even if temptation sways choice when $T = 0$, normative preferences begin to dominate choice for large T, thus causing a preference reversal.

Finally, we observe that the assumption $\gamma \leq \delta$ is not necessary for our model to generate preference reversals. For instance, if the weighting function takes the form $\psi(l) = l^\theta$, $\theta \geq 0$, then preference reversals arise if $\gamma^{\theta+1} < \delta$. Thus, menu-dependent self-control can give rise to preference reversals even if $\gamma > \delta$. If $\theta = 0$, then $\psi(l) = 1$ for all l, and the above condition for preference reversal is reduced to $\gamma < \delta$, which corresponds to GP's assumption.

The subsequent experimental findings concern interactions between risk and time. The MDSC model accommodates these findings, whereas GP [18] do not.

Preference Reversals and Risk. It is observed by Keren and Roelofsma [23] and Weber and Chapman [38] that preference reversals tend to disappear when all rewards are uniformly made probabilistic. This corresponds to observing (13) in conjunction with:

$$\mathcal{C}(\{(\alpha c+(1-\alpha)0)^{+1}, \alpha d+(1-\alpha)0\}) = \{(\alpha c+(1-\alpha)0)^{+1}\}, \text{ and}$$
$$\mathcal{C}(\{(\alpha c+(1-\alpha)0)^{+(T+1)}, (\alpha d+(1-\alpha)0)^{+T}\}) = \{(\alpha c+(1-\alpha)0)^{+(T+1)}\}.$$

It is clear why our model would exhibit this: Since both of delaying rewards and mixing rewards with 0 reduce the weight

$$\psi\Big(\max_{\{(\alpha c+(1-\alpha)0)^{+(T+1)},(\alpha d+(1-\alpha)0)^{+T}\}} V\Big) = \psi(\gamma^T \alpha V(d))$$

on V, choice from menus will tend to be determined by U.

The Allais Paradoxes and Time. Weber and Chapman [38] find that the Common Consequence Effect tends to disappear when the lotteries are delayed, and Baucells and Heukamp [6] show that the same is true also for the Common Ratio Effect. In particular, choices over lotteries were less inconsistent with expected utility theory when the lotteries were to be played out in the future. This is exhibited in our model because future lotteries tend to be determined by U, as above.

Risk Attitude and the 'Moment of Truth'. Some studies suggest that people tend to be less risk averse towards gambles that are played out in the future than those in the present. According to Liberman et al [25] subjects focus on rewards when evaluating distant gambles and on probabilities when evaluating current gambles.[14] Loewenstein et al [27] and Savitsky et al [34] hypothesize that the emotional response to risk is that of aversion and dread, and thus there is increased risk aversion as the 'moment of truth' approaches. In our model, the agent's preference over current gambles is determined by a convex combination of two utilities, one more risk averse than the other, but those over delayed gambles is determined by the less risk averse utility. Thus our model generates this finding.

5.3 Further Experiments

We showed that menu-dependent self-control, coupled with a temptation to be risk averse and impatient, is consistent with various experimental findings. However, further experiments are required in order to support or reject the possibility that temptation might serve as an *explanation* for the findings. There seem to be at least two avenues to explore:

1- Dynamically inconsistent choice behavior is one possible indication of temptation. Therefore an avenue to explore is whether such reversals are associated also with dynamic inconsistency. That is, would subjects' preferences over lotteries played at time $t+1$ depend on whether preferences are elicited at t or $t+1$? The evidence finds that from the perspective of one point it time, risk preferences reverse with delay, and so it is conceivable that dynamic inconsistency would be found. If it is indeed found, it may be explored next

[14] In [25], subjects actually tended to prefer mean-preserving spreads for future gambles.

whether the agent at time t would seek to commit to her choices. (However, the latter would presume sophistication on the part of agents).

2- A key implication of the idea of menu-dependent self-control for ex post choice is the existence of violations of WARP. A sizable experimental literature documents that choice under risk violates WARP. Much of the literature focusses on establishing the counterparts of violations observed in riskless settings, which seem unrelated to issues of self-control.[15] An avenue to explore is whether the presence of unchosen safe lotteries can cause an increase in risk aversion.

6 Concluding Comments

Most generalizations of GP's model (Chatterjee and Krishna [7], Dekel, Lipman and Rustichini [9], Kopylov [24], Stovall [35]) have focused on relaxing Set-Betweenness – which is clearly a substantive axiom for temptation – while maintaining Independence. It has not been well understood how exactly Independence 'works' – for instance, it has not been known what is the weakest form of Independence that yields the existence of a vNM temptation preference and whether the full force of Independence is required to get linearity of self-control costs. This paper gets a handle on such questions, and more broadly it clarifies the price of adopting the Independence axiom by highlighting that there are substantive stories about self-control that are inconsistent with Independence. Our MDSC representation is obtained by relaxing Independence, not Set-Betweenness.

A Appendix: Proof of Theorem 2

For all $\mu, \eta \in \Delta$ and $\alpha \in [0, 1]$, we use the notation $\mu\alpha\eta \equiv \alpha\mu + (1 - \alpha)\eta$ for abbreviation. Similarly, for all $x, y \in Z$ and $\alpha \in [0, 1]$, $x\alpha y \equiv \alpha x + (1 - \alpha)y$.

If there is no μ, η s.t. $\{\mu\} \succ \{\mu, \eta\}$, then the Theorem holds with any constant function v. Henceforth assume that:

A 1 (Nondegeneracy) *There exists μ, η s.t. $\{\mu\} \succ \{\mu, \eta\}$.*

Given the compact metric space X, consider the space $ca(X)$ of finite Borel signed measures on X, normed by the total variation norm. Note that

$$ca(X) := span(\Delta).$$

Denote the space of continuous functions on X by $C(X)$. Since X is compact, $ca(X)$ is isometrically isomorphic to $C(X)^*$, the topological dual of (ie, the space of continuous linear

[15] For instance, the experiments find the attraction effect (an alternative is more likely to be chosen after the introduction of an inferior version of it) and the compromise effect (an alternative is more likely to be chosen if it serves as a compromise between two other alternatives).

functionals on) $C(X)$ normed by the sup-norm (Aliprantis and Border, Corollary 13.15). Given this duality, the *weak* topology* $\sigma(C(X)^*, C(X))$ on $ca(X)$ induces the topology of weak convergence on Δ. Thus, when $v : ca(X) \to \mathbb{R}$ is w*-continuous linear function, the restriction $v|_\Delta$ is continuous in the appropriate sense. The space $ca(X)$ is a locally convex Hausdorff topological linear space under the weak* topology.

Begin with some notation: say that

- $\mu \succcurlyeq_T \eta$ if $\{\mu\} \sim \{\mu, \eta\} \succ \{\eta\}$,
- $\mu \succ_T \eta$ if $\{\eta\} \succ \{\eta, \mu\}$.

Say that μ *tempts* η if $\mu \succcurlyeq_T \eta$ or $\mu \succ_T \eta$ and consider:

A 2 (Strong Temptation Convexity) *For any μ, η, μ', η' and $\alpha \in (0,1)$, if μ tempts η and μ' tempts η' then $\eta\alpha\eta'$ does not tempt $\mu\alpha\mu'$.*

The following lemma allows us to adopt this stronger axiom in place of Temptation Convexity.

Lemma 1 *A preference $\succsim$ that satisfies Temptation Independence satisfies Strong Temptation Convexity iff it satisfies Temptation Convexity.*

Proof. The 'only if' part is immediate. To prove the 'if' part, suppose Temptation Convexity is satisfied.

Case (a): $\{\mu\} \succ \{\mu, \eta\}$ and $\{\mu'\} \succ \{\mu', \eta'\}$.

Take any $\alpha \in (0,1)$. By Temptation Independence,

$$\{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\mu'\} \text{ and } \{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \mu\alpha\eta'\}$$

hold. Observe that $\frac{1}{2}\eta\alpha\mu' + \frac{1}{2}\mu\alpha\eta' = \frac{1}{2}\eta\alpha\eta' + \frac{1}{2}\mu\alpha\mu'$. Therefore, by Temptation Convexity,

$$\{\frac{1}{2}\mu\alpha\mu' + \frac{1}{2}\mu\alpha\mu'\} \succ \{\frac{1}{2}\mu\alpha\mu' + \frac{1}{2}\mu\alpha\mu', \frac{1}{2}\eta\alpha\mu' + \frac{1}{2}\mu\alpha\eta'\} = \{\frac{1}{2}\mu\alpha\mu' + \frac{1}{2}\mu\alpha\mu', \frac{1}{2}\eta\alpha\eta' + \frac{1}{2}\mu\alpha\mu'\},$$

and by Temptation Independence, $\{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\eta'\}$, as desired.

Case (b): $\{\mu\} \sim \{\mu, \eta\} \succ \{\eta\}$ and $\{\mu'\} \sim \{\mu', \eta'\} \succ \{\eta'\}$.

Same argument as in the previous case.

Case (c): $\{\eta\} \succ \{\eta, \mu\}$ and $\{\mu'\} \sim \{\mu', \eta'\} \succ \{\eta'\}$.

Take any $\alpha \in (0,1)$. The result holds trivially if $\{\mu\alpha\mu'\} \sim \{\eta\alpha\eta'\}$. So first let $\{\mu\alpha\mu'\} \succ \{\eta\alpha\eta'\}$ and suppose by way of contradiction that

$$\{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\eta'\}.$$

By Temptation Independence, $\{\eta\} \succ \{\eta, \mu\}$ implies $\{\eta\alpha\eta'\} \succ \{\eta\alpha\eta', \mu\alpha\eta'\}$. By our above result for Case (a),

$$\{(\mu\alpha\mu')\frac{1}{2}(\eta\alpha\eta')\} \succ \{(\mu\alpha\mu')\frac{1}{2}(\eta\alpha\eta'), (\eta\alpha\eta')\frac{1}{2}(\mu\alpha\eta')\}$$

and by Temptation Independence, $\{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \mu\alpha\eta'\}$ and in particular, $\{\mu'\} \succ \{\mu', \eta'\}$, a contradiction.

Similarly, if $\{\eta\alpha\eta'\} \succ \{\mu\alpha\mu'\}$ then suppose by way of contradiction that

$$\{\eta\alpha\eta'\} \sim \{\eta\alpha\eta', \mu\alpha\mu'\} \succ \{\mu\alpha\mu'\}.$$

By Temptation and Commitment Independence, $\{\mu'\} \sim \{\mu', \eta'\} \succ \{\eta'\}$ implies $\{\eta\alpha\mu'\} \sim \{\eta\alpha\mu', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}$. By the above result for Case (b),

$$\{(\eta\alpha\mu')\frac{1}{2}(\eta\alpha\eta')\} \sim \{(\eta\alpha\mu')\frac{1}{2}(\eta\alpha\eta'), (\mu\alpha\mu')\frac{1}{2}(\eta\alpha\eta')\} \succ \{(\mu\alpha\mu')\frac{1}{2}(\eta\alpha\eta')\}$$

and by Temptation Independence, $\{\eta\alpha\mu'\} \sim \{\eta\alpha\mu', \mu\alpha\mu'\} \succ \{\mu\alpha\mu'\}$ and $\{\eta\} \sim \{\eta, \mu\} \succ \{\mu\}$, a contradiction. ■

Define the set:

$$\mathcal{T}_1 = \{\lambda(\mu - \eta) : \lambda > 0 \text{ and } \mu \succcurlyeq_T \eta \text{ or } \mu \succ_T \eta\}.$$

Lemma 2 *For any $\mu, \eta \in \Delta$ and $\lambda > 0$ such that $\{\mu\} \not\sim \{\eta\}$,*

$$\lambda(\mu - \eta) \in co(\mathcal{T}_1) \Longleftrightarrow \mu \succcurlyeq_T \eta \text{ or } \mu \succ_T \eta.$$

Proof. Define $\mu \rhd_T \eta$ if $\{\mu\} \sim \{\eta\}$ and there exist $\mu_1 \succcurlyeq_T \eta_1$ and $\mu_2 \succ_T \eta_2$ and $\gamma \in (0,1)$ s.t. $\mu = \mu_1\gamma\mu_2$ and $\eta = \eta_1\gamma\eta_2$.

Step 1: Show that

$$co(\mathcal{T}_1) = \{\lambda(\mu - \eta) : \lambda > 0 \text{ and } \mu \succcurlyeq_T \eta \text{ or } \mu \succ_T \eta \text{ or } \mu \rhd_T \eta\}.$$

The set inclusion "$\supset$" follows immediately from definitions. For the converse, take any $\nu \in co(\mathcal{T}_1)$. Then there exist $\lambda_i(\mu_i - \eta_i) \in \mathcal{T}_1$ and weights α_i, $i = 1, .., N$, such that $\mu_i \succcurlyeq_T \eta_i$ for $i = 1, .., n$ and $\mu_i \succ_T \eta_i$ for $i = n+1, .., N$, and

$$\sum_{i=1}^{N} \alpha_i\lambda_i(\mu_i - \eta_i) = \nu.$$

By Strong Temptation Convexity, a proof by induction yields that:

$$\begin{aligned} \mu \quad &:= \sum_{i=1}^{n} \frac{\alpha_i\lambda_i}{\sum_{j=1}^{n}\alpha_j\lambda_j}\mu_i \succcurlyeq_T \sum_{i=1}^{n} \frac{\alpha_i\lambda_i}{\sum_{j=1}^{n}\alpha_j\lambda_j}\eta_i =: \eta \\ \mu' \quad &:= \sum_{j=n+1}^{N} \frac{\alpha_i\lambda_i}{\sum_{j=n+1}^{N}\alpha_j\lambda_j}\mu_i \succ_T \sum_{j=n+1}^{N} \frac{\alpha_i\lambda_i}{\sum_{j=n+1}^{N}\alpha_j\lambda_j}\eta_i =: \eta'. \end{aligned}$$

Moreover,

$$\begin{aligned}\nu &= (\sum_{i=1}^{n} \alpha_i\lambda_i)(\mu-\eta) + (\sum_{i=n+1}^{N} \alpha_i\lambda_i)(\mu'-\eta') \\ &= (\sum_{i=1}^{N} \alpha_i\lambda_i)\left[(\frac{\sum_{i=1}^{n}\alpha_i\lambda_i}{\sum_{i=1}^{N}\alpha_i\lambda_i})(\mu-\eta) + (\frac{\sum_{i=n+1}^{N}\alpha_i\lambda_i}{\sum_{i=1}^{N}\alpha_i\lambda_i})(\mu'-\eta')\right] \\ &= (\sum_{i=1}^{N} \alpha_i\lambda_i)(\mu\gamma_1\mu' - \eta\gamma_1\eta'),\end{aligned}$$

where $\gamma_1 = \frac{\sum_{i=1}^{n}\alpha_i\lambda_i}{\sum_{i=1}^{N}\alpha_i\lambda_i} \in [0,1]$. By Strong Temptation Convexity and by definition of $\rhd_T$, either $\mu\gamma_1\mu' \succcurlyeq_T \eta\gamma_1\eta'$ or $\mu\gamma_1\mu' \succ_T \eta\gamma_1\eta'$ or $\mu\gamma_1\mu' \rhd_T \eta\gamma_1\eta'$ hold. Therefore ν belongs to the desired set. This completes the proof.

Step 2: Prove the result.

The 'if' part follows by definition. For the only if part, take any $\lambda(\mu-\eta) \in co(\mathcal{T}_1)$ and note that by Step 1, there exists $\mu', \eta' \in \Delta$ and $\lambda' > 0$ s.t. $\lambda(\mu-\eta) = \lambda'(\mu'-\eta')$ and $\mu' \succcurlyeq_T \eta'$ or $\mu' \succ_T \eta'$ or $\mu' \rhd_T \eta'$. If $\mu' \succcurlyeq_T \eta'$ holds, then by Temptation Independence applied twice we have

$$\frac{\lambda}{\lambda+\lambda'}\mu + \frac{\lambda'}{\lambda+\lambda'}\eta' = \frac{\lambda}{\lambda+\lambda'}\eta + \frac{\lambda'}{\lambda+\lambda'}\mu' \succcurlyeq_T \frac{\lambda}{\lambda+\lambda'}\eta + \frac{\lambda'}{\lambda+\lambda'}\eta',$$

and thus $\mu \succcurlyeq_T \eta$. A similar argument implies that if $\mu' \succ_T \eta'$ then $\mu \succ_T \eta$.

Finally, we show that $\mu' \rhd_T \eta'$ is not possible. If it is, then by definition $\{\mu'\} \sim \{\eta'\}$, and in a fashion analogous to the above argument, Commitment Independence applied twice yields $\{\mu\} \sim \{\eta\}$, contradicting the assumption that $\{\mu\} \not\sim \{\eta\}$. ■

Define:

$$\mathcal{T} = cl(co(\mathcal{T}_1)),$$

where $cl(co(\mathcal{T}_1))$ is the closure of $co(\mathcal{T}_1)$ in the weak* topology. Say that $\mathcal{T}$ is a *cone* if $\lambda\mathcal{T} \subset \mathcal{T}$ for all $\lambda \geq 0$.

Lemma 3 *$\mathcal{T}$ is a weak* closed convex cone.*

Proof. Since it is the closure of a convex set, $\mathcal{T}$ is convex. To see that $\mathcal{T}$ is a cone, take any $\nu \in \mathcal{T}$ and any net $\nu_\alpha \xrightarrow{w^*} \nu$ where $\nu_\alpha \in co(\mathcal{T}_1)$. For any $\lambda > 0$, we have $\lambda\nu_\alpha \in co(\mathcal{T}_1)$ (by definition of $co(\mathcal{T}_1)$) and $\lambda\nu_\alpha \xrightarrow{w^*} \lambda\nu$ and thus $\lambda\nu \in \mathcal{T}$. To see that $0 \in \mathcal{T}$ observe that by nondegeneracy there exists μ, η s.t. $\{\eta\} \succ \{\eta,\mu\}$ and by Temptation Independence $\{\eta\}\alpha\{\mu\} \succ \{\eta,\mu\}\alpha\{\mu\}$ for all α. Consequently, $\alpha(\mu-\eta) = (\mu - \eta\alpha\mu) \in co(\mathcal{T}_1)$ for all α and therefore $0 \in \mathcal{T}$. ■

Lemma 4 *If $\nu \in \mathcal{T}$ then there exists a norm-bounded net $\nu_\alpha \xrightarrow{w^*} \nu$ s.t. $\nu_\alpha \in co(\mathcal{T}_1)$.*

Proof. Since X is compact, $C(X)$ is separable and $ca(X)$ is isometrically isomorphic to the topological dual $C(X)^*$. Thus $\mathcal{T}$ is a weak* closed convex set in the dual of a separable normed space. By the Krein-Smulian theorem [29, Cor 2.7.13], $\mathcal{T}$ is sequentially weak* closed. In particular, $\mathcal{T}$ is the sequential weak* closure of $co(\mathcal{T}_1)$. Thus, for any $\nu \in \mathcal{T}$ there is a sequence $\nu_n \overset{w^*}{\to} \nu$ s.t. $\nu_n \in co(\mathcal{T}_1)$. To see that this sequence is norm-bounded, note that by definition of weak* convergence, $\nu_n \overset{w^*}{\to} \nu$ implies that $\int_X f \,\mathrm{d}\nu_n \to \int_X f \,\mathrm{d}\nu$ for any given $f \in C(X)$. This implies

$$\sup\left\{ \int_X f \,\mathrm{d}\nu_n \,\middle|\, n = 1, 2, .. \right\} < \infty.$$

By the Banach-Steinhaus theorem [29, Thm 1.6.9] there exists $K < \infty$ s.t. $\|\nu_n\| \leq K$ for each $n = 1, 2, ...$ This completes the proof. ■

Lemma 5 *If $\{\eta\} \succ \{\eta, \mu\}$ then $\eta - \mu \notin \mathcal{T}$.*

Proof. Take any μ^*, η^* such that $\{\eta^*\} \succ \{\eta^*, \mu^*\}$, and suppose by way of contradiction that $\nu := \eta^* - \mu^* \in \mathcal{T}$.

Step 1: *Show that there is a sequence $(\mu_n, \eta_n) \overset{w^*}{\to} (\mu, \eta)$ and $\lambda > 0$ s.t. $0 \neq \mu_n - \eta_n \in co(\mathcal{T}_1)$ for each n and $\lambda(\mu - \eta) = \nu$.*

As $\nu \in \mathcal{T}$, by the proof of the previous lemma there exists a sequence $\nu_n \overset{w^*}{\to} \nu$ where $\nu_n \in co(\mathcal{T}_1)$ and there is $K < \infty$ s.t.

$$\|\nu_n\| \leq K, \qquad n = 1, 2, ... \tag{14}$$

Since $\nu = \eta^* - \mu^* \neq 0$, we can assume wlog that $\nu_n \neq 0$ for all n. By the Jordan decomposition theorem, $\nu_n = \sigma_n^1 - \sigma_n^2$ for two mutually singular positive measures $\sigma_n^1, \sigma_n^2 \in ca(X)$. Since $\nu_n \neq 0$ and $\nu_n(X) = 0$,[16] we see that $\sigma_n^1 + \sigma_n^2 \neq 0$, $\sigma_n^1 \neq \sigma_n^2$ and $\lambda_n := \sigma_n^1(X) = \sigma_n^2(X) > 0$. Consequently, we can write

$$\nu_n = \lambda_n(\mu_n - \eta_n),$$

for two mutually singular $\mu_n, \eta_n \in \Delta$ and $\lambda_n > 0$. By mutual singularity $\|\mu_n - \eta_n\| = 2$. But then

$$\|\nu_n\| = \|\lambda_n(\mu_n - \eta_n)\| = \lambda_n\|(\mu_n - \eta_n)\| = 2\lambda_n.$$

By (14), λ_n is a real sequence in the compact interval $[0, \frac{K}{2}]$. Passing to a subsequence if necessary, we assume wlog that this sequence converges. Moreover, since Δ is a weak* compact subset of $ca(X)$, both μ_n and η_n have weak* convergent subsequences. Wlog we

[16] To see why $\nu_n(X) = 0$, note that since each $\sigma \in \mathcal{T}_0$ takes the form $\lambda(\mu - \eta)$, it must be that $\sigma(X) = \lambda(\mu(X) - \eta(X)) = \lambda(1 - 1) = 0$ for each such σ. Since any $\sigma \in \mathcal{T}$ is a limit of measures in $co(\mathcal{T}_1)$, it must be that $\sigma(X) = 0$.

may therefore assume that λ_n, μ_n and η_n are convergent. Write $\lambda_n \to \lambda, \mu_n \xrightarrow{w^*} \mu$ and $\eta_n \xrightarrow{w^*} \eta$. Then

$$\nu_n = \lambda_n(\mu_n - \eta_n) \xrightarrow{w^*} \lambda(\mu - \eta).$$

Since $\nu_n \xrightarrow{w^*} \nu$ in a Hausdorff space, $\nu = \lambda(\mu - \eta)$. Moreover, $\nu_n \in co(\mathcal{T}_1)$ implies $(\mu_n - \eta_n) = \frac{1}{\lambda_n}\nu_n \in co(\mathcal{T}_1)$. This completes the step.

Step 2: *Prove the result.*

Consider the sequences $\mu_n \xrightarrow{w^*} \mu$ and $\eta_n \xrightarrow{w^*} \eta$ constructed in step 1. First observe that since $\lambda(\mu - \eta) = \nu = \eta^* - \mu^*$ and $\mu^* \succ_T \eta^*$, arguing as in the proof of Lemma 2 implies $\eta \succ_T \mu$, that is, $\{\mu\} \succ \{\mu, \eta\}$. But since $\mu_n \xrightarrow{w^*} \mu$ and $\eta_n \xrightarrow{w^*} \eta$, Semi-Continuity implies that for all large n, $\{\mu_n\} \succ \{\mu_n, \eta_n\}$.[17] But by Lemma 2 this contradicts the fact that $\mu_n - \eta_n \in co(\mathcal{T}_1)$ for all n. ■

Lemma 6 *There exists a temptation utility. That is, there exists a weak* continuous linear v s.t. for all $\mu, \eta \in \Delta$,*

$$\begin{aligned} \{\eta\} &\succ \{\eta, \mu\} \Longrightarrow v(\mu) > v(\eta) \\ \{\mu\} &\sim \{\mu, \eta\} \succ \{\eta\} \Longrightarrow v(\mu) \geq v(\eta). \end{aligned}$$

Proof. Consider the set

$$S = \{w \in ca(X)^* : \ w(\mathcal{T}) \geq 0 \text{ and } w(\mathcal{T}) \neq 0\}.$$

That is, S is the set of nonzero weak* continuous linear functionals on $ca(X)$ that take positive values for each $\nu \in \mathcal{T}$ and a strictly positive value for some $\nu \in \mathcal{T}$.

Step 1: *S is nonempty.*

By nondegeneracy, there is μ, η s.t. $\{\eta\} \succ \{\eta, \mu\}$, and by the previous lemma, $\eta - \mu \notin \mathcal{T}$. Since $\{\eta - \mu\}$ is compact and $\mathcal{T}$ weak* closed and both are convex and disjoint, by a separating hyperplane theorem [29, Thm 2.2.28] there is a nonzero weak* continuous linear functional w such that

$$\inf\{w(\nu) : \nu \in \mathcal{T}\} > w(\eta - \mu). \tag{15}$$

We claim that the inf is achieved at exactly 0. Given that $0 \in \mathcal{T}$ and $w(0) = 0$, suppose there is $\nu' \in \mathcal{T}$ s.t. $w(\nu') < 0$. Since $\mathcal{T}$ is a cone and w is linear, $k\nu' \in \mathcal{T}$ and $w(k\nu') = kw(\nu')$ for

[17] The proof is as follows. First use Semi-Continuity and Binary Set-Betweenness to show that for any *binary* menu x there is a singleton equivalent: $\{e_x\} \sim x$. Note that by Binary Set-Betweenness, x is bounded above and below by singleton menus, and thus also by the u-best and u-worst alternatives μ^* and μ_* in Δ. By Semi-Continuity the sets $\{\alpha : \{\mu^* \alpha \mu_*\} \succsim x\}$ and $\{\alpha : \{\mu^* \alpha \mu_*\} \precsim x\}$ are closed subsets of $[0, 1]$. Given Order, connectedness of $[0, 1]$ implies that the two sets have a nonempty intersection. Now prove the result: Since $\{\mu\} \succ \{e_{\{\mu,\eta\}}\} \sim \{\mu, \eta\}$, Commitment Independence implies $\{\mu\} \succ \{\mu \frac{1}{2} e_{\{\mu,\eta\}}\} \succ \{\mu, \eta\}$. Then Semi-Continuity implies that $\{\mu_n\} \succ \{\mu_n, \eta_n\}$ for all large n.

each $k > 0$. But by selecting a large enough k it would follow that the infimum violates (15), a contradiction. Therefore the infimum is achieved at 0, and for each $\nu \in \mathcal{T}$,

$$w(\nu) \geq 0 > w(\eta - \mu).$$

Indeed, $w \in S$.

Step 2: *Any $v \in S$ serves as a temptation utility.*

By definition of S, we see that $\{\eta\} \succ \{\eta, \mu\}$ or $\{\mu\} \sim \{\mu, \eta\} \succ \{\eta\}$ implies $v(\mu) \geq v(\eta)$ for each $v \in S$. We need to show that $\{\eta\} \succ \{\eta, \mu\}$ must imply $v(\mu) > v(\eta)$ for any such v. Suppose it does not. Then there exists μ, η and v such that $\{\eta\} \succ \{\eta, \mu\}$ and $v(\mu) = v(\eta)$. But by step 1, there exists μ', η' s.t. $v(\mu') < v(\eta')$. By Semi-Continuity, for large enough α it must be that $\{\eta\alpha\eta'\} \succ \{\eta\alpha\eta', \mu\alpha\mu'\}$, and thus $\mu\alpha\mu' - \eta\alpha\eta' \in \mathcal{T}$. But by linearity it must also be that $v(\mu\alpha\mu' - \eta\alpha\eta') < 0$, contradicting the fact that $v \in S$. This concludes the proof. ■

Lemma 7 *$v|_\Delta$ is unique up to a positive affine transformation.*

Proof. Suppose v, v' are two temptation representations and there exists $\mu, \eta \in \Delta$ s.t. $v(\mu) \geq v(\eta)$ and $v'(\mu) < v'(\eta)$. If $\{\mu\} \succ \{\eta\}$, by Binary Set Betweenness, we have either $\{\mu\} \succ \{\mu, \eta\}$ or $\{\mu\} \sim \{\mu, \eta\} \succ \{\eta\}$. Since v and v' are temptation utility functions, the former implies $v(\eta) > v(\mu)$ and $v'(\eta) > v'(\mu)$, while the latter implies $v(\mu) \geq v(\eta)$ and $v'(\mu) \geq v'(\eta)$, which contradicts the hypothesis. Similarly, we have a contradiction when $\{\eta\} \succ \{\mu\}$. Thus, we must have $\{\mu\} \sim \{\eta\}$.

By nondegeneracy, there is μ', η' s.t. $\{\eta'\} \succ \{\eta', \mu'\}$. The temptation representations must satisfy $v(\mu') > v(\eta')$ and $v'(\mu') > v'(\eta')$. For small enough α, $v(\mu'\alpha\mu) > v(\eta'\alpha\eta)$ and $v'(\mu'\alpha\mu) < v'(\eta'\alpha\eta)$. On the other hand, since $\{\eta'\} \succ \{\mu'\}$ and $\{\eta\} \sim \{\mu\}$, by Commitment Independence, $\{\eta'\alpha\eta\} \succ \{\mu'\alpha\mu\}$ for all $\alpha \in (0, 1)$. Thus, by Binary Set Betweenness, either $\{\eta'\alpha\eta\} \succ \{\eta'\alpha\eta, \mu'\alpha\mu\}$ or $\{\eta'\alpha\eta\} \sim \{\eta'\alpha\eta, \mu'\alpha\mu\} \succ \{\mu'\alpha\mu\}$. The former contradicts $v'(\mu'\alpha\mu) < v'(\eta'\alpha\eta)$, and the latter contradicts $v(\mu'\alpha\mu) > v(\eta'\alpha\eta)$. It follows that v, v' are ordinally equivalent on Δ and therefore by linearity they are cardinally equivalent on Δ. ■

B Appendix: Proof of Theorem 3

For all $\mu, \eta \in \Delta$ and $\alpha \in [0, 1]$, we use the notation $\mu\alpha\eta \equiv \alpha\mu + (1 - \alpha)\eta$ for abbreviation. Similarly, for all $x, y \in Z$ and $\alpha \in [0, 1]$, $x\alpha y \equiv \alpha x + (1 - \alpha)y$.

B.1 Preliminary Lemmas

We first establish representations for commitment preference, temptation preference, and the whole preference $\succsim$, respectively.

Lemma 8 *(i) There exists a continuous linear function* $u : \Delta \to \mathbb{R}_+$ *such that*

$$\{\mu\} \succsim \{\eta\} \iff u(\mu) \geq u(\eta)$$

(ii) There exists a continuous function $W : Z \to \mathbb{R}_+$ *that represents* $\succsim$ *and satisfies* $W(\{\mu\}) = u(\mu)$ *for all* $\mu \in \Delta$.

(iii) There exists a continuous linear function $v : \Delta \to \mathbb{R}_+$ *such that if* $\{\mu\} \succ \{\eta\}$ *then*

$$\{\mu\} \succ \{\mu, \eta\} \iff v(\eta) > v(\mu).$$

Proof. (i) The first assertion follows from Order, Continuity, Commitment Independence, and the mixture space theorem.

(ii) Since u is continuous on Δ, there exist a maximal and a minimal lottery $\mu^{\Delta}, \mu_{\Delta} \in \Delta$ with respect to u. Without loss of generality, we can assume $u(\mu^{\Delta}) = 1$ and $u(\mu_{\Delta}) = 0$. From Continuity and Set Betweenness, $\{\mu^{\Delta}\} \succsim x \succsim \{\mu_{\Delta}\}$ for all $x \in Z$. By a standard argument, for all $x \in Z$, there exists a unique number $\alpha(x) \in [0, 1]$ such that $x \sim \{\mu^{\Delta}\alpha(x)\mu_{\Delta}\}$. Define

$$W(x) \equiv u(\mu^{\Delta}\alpha(x)\mu_{\Delta}) \in [0, 1].$$

Then W represents $\succsim$. Moreover, $W(\{\mu\}) = u(\mu)$ for all $\mu \in \Delta$.

To show continuity of W, let $x^n \to x$. Since $u(\mu^{\Delta}) = 1$ and $u(\mu_{\Delta}) = 0$, $W(x) = \alpha(x)$. So we want to show $\alpha(x^n) \to \alpha(x)$. By contradiction, suppose otherwise. Then, there exists a neighborhood $B(\alpha(x))$ of $\alpha(x)$ such that $\alpha(x^m) \notin B(\alpha(x))$ for infinitely many m. Let $\{x^m\}$ denote the corresponding subsequence of $\{x^n\}$. Since $x^n \to x$, $\{x^m\}$ also converges to x. Since $\{\alpha(x^m)\}$ is a sequence in $[0, 1]$, there exists a convergent subsequence $\{\alpha(x^{\ell})\}$ with a limit $\alpha \neq \alpha(x)$. On the other hand, since $x^{\ell} \to x$ and $x^{\ell} \sim \{\mu^{\Delta}\alpha(x^{\ell})\mu_{\Delta}\}$, Continuity implies $x \sim \{\mu^{\Delta}\alpha\mu_{\Delta}\}$. Since $\alpha(x)$ is unique, $\alpha(x) = \alpha$, which is a contradiction.

(iii) By Theorem 2. ■

Without loss of generality, assume that $v(\Delta) = [0, 1]$. By construction, if $\{\mu\} \succ \{\mu, \eta\}$, then $v(\eta) > v(\mu)$. If $\{\mu\} \sim \{\mu, \eta\} \succ \{\eta\}$, then $v(\mu) \geq v(\eta)$.

Lemma 9 *For all* ν, ν', *if* $v(\nu) > v(\nu')$ *and* $u(\nu) \neq u(\nu')$, *then* $\nu \succsim_T \nu'$.

Proof. If $u(\nu') > u(\nu)$, by construction of v, we have $\{\nu'\} \succ \{\nu, \nu'\}$. Thus $\nu \succsim_T \nu'$. If $u(\nu) > u(\nu')$, Set Betweenness implies $\{\nu\} \succsim \{\nu, \nu'\} \succsim \{\nu'\}$. If $\{\nu\} \succ \{\nu, \nu'\}$, we have $v(\nu') > v(\nu)$, which contradicts the assumption. Thus we must have $\{\nu\} \sim \{\nu, \nu'\} \succ \{\nu'\}$. Thus $\nu \succsim_T \nu'$. ■

Lemma 10 *Assume that* μ, η, η' *satisfy* $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ *and* $v(\eta) \geq v(\eta')$. *Then, for all* $\alpha \in (0, 1)$,

(i) $\{\mu\alpha\eta', \eta\alpha\eta'\} \succsim \{e_{\mu\eta}\alpha\eta'\}$,

(ii) $\{\mu\alpha\eta'\} \succ \{\mu\alpha\eta', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}$.

Proof. (i) Step 1: If $v(\eta) > v(\eta')$, then the claim holds. If $\{\eta'\} \not\succ \{\eta\}$, by Lemma 9, $\eta \succsim_T \eta'$. Since $\{\mu,\eta\}\alpha\{\eta\} \sim \{e_{\mu\eta}\alpha\eta\}$ by Linear Self-Control, the result is implied by Decreasing Self-Control. Suppose instead that $\{\eta'\} \sim \{\eta\}$. Since $u(\mu) > u(\eta)$, for all $\beta \in (0,1)$ close to zero, $\{\mu\beta\eta'\} \succ \{\eta\}$ and $v(\mu\beta\eta') < v(\eta)$ and thus, $\{\mu\beta\eta'\} \succ \{\mu\beta\eta', \eta\}$. That is, $\eta \succsim_T \mu\beta\eta'$. Linear Self-Control and Decreasing Self-Control imply that $\{\mu,\eta\}\alpha\{\mu\beta\eta'\} \succsim \{e_{\mu\eta}\alpha(\mu\beta\eta')\}$, and by Continuity, the desired result holds as $\beta \to 0$.

Step 2: When $v(\eta) = v(\eta')$, the claim holds. Let η^+ and η^- be a maximal and a minimal lottery with respect to v. If $v(\eta') > v(\eta^-)$, $v(\eta) > v(\eta'\beta\eta^-)$ for all $\beta \in (0,1)$. By Step 1, $\{\mu\alpha(\eta'\beta\eta^-), \eta\alpha(\eta'\beta\eta^-)\} \succsim \{e_{\mu\eta}\alpha(\eta'\beta\eta^-)\}$. By Continuity, $\{\mu\alpha\eta', \eta\alpha\eta'\} \succsim \{e_{\mu\eta}\alpha\eta'\}$ as $\beta \to 1$.

If $v(\eta') = v(\eta^-)$, $v(\eta\beta\eta^+) > v(\eta')$, and, by Continuity, $\{\mu\} \succ \{\mu, \eta\beta\eta^+\} \succ \{\eta\beta\eta^+\}$ for all $\beta \in (0,1)$ sufficiently close to one. By Step 1, $\{\mu\alpha\eta', (\eta\beta\eta^+)\alpha\eta'\} \succsim \{e_{\{\mu,\eta\beta\eta^+\}}\alpha\eta'\}$. Recall the function $\alpha(x)$ defined in the proof of Lemma 8 (ii). By definition,

$$\{e_{\{\mu,\eta\beta\eta^+\}}\} \sim \{\mu^\Delta\alpha(\{\mu,\eta\beta\eta^+\})\mu_\Delta\}.$$

Thus, we have $\{\mu\alpha\eta', (\eta\beta\eta^+)\alpha\eta'\} \succsim \{(\mu^\Delta\alpha(\{\mu,\eta\beta\eta^+\})\mu_\Delta)\alpha\eta'\}$. Since $\alpha(x)$ is continuous as shown in the proof,

$$\{\mu\alpha\eta', \eta\alpha\eta'\} \succsim \{(\mu^\Delta\alpha(\{\mu,\eta\})\mu_\Delta)\alpha\eta'\} \sim \{e_{\mu\eta}\alpha\eta'\},$$

as $\beta \to 1$.

(ii) By Commitment Independence, Temptation Independence and part (i), $\{\mu\} \succ \{\mu,\eta\} \succ \{\eta\}$

$\implies \{\mu\} \succ \{e_{\mu\eta}\} \succ \{\eta\}$

$\implies \{\mu\alpha\eta'\} \succ \{e_{\mu\eta}\alpha\eta'\} \succ \{\eta\alpha\eta'\}$

$\implies \{\mu\alpha\eta'\} \succ \{\mu\alpha\eta', \eta\alpha\eta'\} \succsim \{e_{\mu\eta}\alpha\eta'\} \succ \{\eta\alpha\eta'\}$

$\implies \{\mu\alpha\eta'\} \succ \{\mu\alpha\eta', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}$. ■

Lemma 11 *For all $\alpha \in (0,1)$,*

$$\{\mu\} \succ \{\mu,\eta\} \succ \{\eta\},\ \{\mu'\} \succ \{\mu',\eta\} \succ \{\eta\} \implies \{\mu\alpha\mu'\} \succ \{\mu\alpha\mu',\eta\} \succ \{\eta\}.$$

Proof. Assume the hypothesis. Let $x = \alpha\{\mu,\eta\} + (1-\alpha)\{\mu',\eta\}$. By [17, Lemma 2], there are lotteries $a^*, b^* \in x$ s.t $\{a^*, b^*\} \sim x$ and (a^*, b^*) solves $\max_{a\in x}\min_{b\in x} W(\{a,b\})$ and (b^*, a^*) solves $\min_{b\in x}\max_{a\in x} W(\{a,b\})$.

Step 1: Show that $x \sim \{\mu\alpha\mu', \eta\}$.

We prove that $(a^*, b^*) = (\mu\alpha\mu', \eta)$. Observe that by Commitment Independence, Linear Self-Control and Decreasing Self-Control,

$$\begin{aligned}
\alpha\{\mu\} + (1-\alpha)\{\mu',\eta\} &\succsim \alpha\{\mu\} + (1-\alpha)\{e_{\mu'\eta}\} \succ x \succ \alpha\{\eta\} + (1-\alpha)\{e_{\mu'\eta}\} \\
&\sim \alpha\{\eta\} + (1-\alpha)\{\mu',\eta\}, \\
\alpha\{\mu,\eta\} + (1-\alpha)\{\mu'\} &\succsim \alpha\{e_{\mu\eta}\} + (1-\alpha)\{\mu'\} \succ x \succ \alpha\{e_{\mu\eta}\} + (1-\alpha)\{\eta\} \\
&\sim \alpha\{\mu,\eta\} + (1-\alpha)\{\eta\},
\end{aligned}$$

Therefore x can be indifferent only to $\{\mu\alpha\eta\}, \{\eta\alpha\mu'\}, \{\mu\alpha\mu', \eta\}$ or $\{\mu\alpha\eta, \eta\alpha\mu'\}$. To rule out the latter, suppose $(\mu\alpha\eta, \eta\alpha\mu')$ solves the maxmin problem. Then

$$x \sim \{\mu\alpha\eta, \eta\alpha\mu'\} \precsim \{\mu\alpha\eta, \eta\} \sim \alpha\{e_{\mu\eta}\} + (1-\alpha)\{\eta\} \prec \alpha\{e_{\mu\eta}\} + (1-\alpha)\{e_{\mu'\eta}\} \sim x,$$

a contradiction. Similarly, if $(\eta\alpha\mu', \mu\alpha\eta)$ solves the maxmin problem, then

$$x \sim \{\eta\alpha\mu', \mu\alpha\eta\} \precsim \{\eta\alpha\mu', \eta\} \sim \alpha\{\eta\} + (1-\alpha)\{e_{\mu'\eta}\} \prec \alpha\{e_{\mu\eta}\} + (1-\alpha)\{e_{\mu'\eta}\} \sim x,$$

a contradiction. Thus $x \not\sim \{\mu\alpha\eta, \eta\alpha\mu'\}$. An entirely similar argument establishes that x is not indifferent to $\{\mu\alpha\eta\}$ or $\{\eta\alpha\mu'\}$. Conclude that $x \sim \{\mu\alpha\mu', \eta\}$.

Step 2: Prove the result.

Observe that by Commitment Independence and Linear Self-Control,

$$x = \alpha\{\mu, \eta\} + (1-\alpha)\{\mu', \eta\} \sim \alpha\{e_{\mu\eta}\} + (1-\alpha)\{e_{\mu'\eta}\}.$$

Applying Step 1 and Commitment Independence proves the result. ■

Lemma 12 *If $v(\eta) = v(\eta')$, for all $\alpha \in (0,1)$,*

$$\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}\ , \{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\} \implies \{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}.$$

Proof. As a preliminary, we first show the following claim: For all μ, η and $\alpha \in (0,1)$, if $\{\mu\alpha\eta\} \succ \{\mu\alpha\eta, \eta\} \succ \{\eta\}$, then $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$. Since $\{\mu\alpha\eta\} \succ \{\mu\alpha\eta, \eta\}$, we have $v(\eta) > v(\mu\alpha\eta)$. Since v is mixture linear, $v(\eta) > v(\mu)$. Moreover, mixture linearity of u implies $u(\mu) > u(\eta)$. Hence, $\{\mu\} \succ \{\mu, \eta\}$. By Linear Self-Control and the assumption, $\{\mu\alpha\eta, \eta\} \sim \{e_{\mu\eta}\alpha\eta\} \succ \{\eta\}$. Hence, we have $\{\mu, \eta\} \sim \{e_{\mu\eta}\} \succ \{\eta\}$, as desired.

Next we show the result. By Lemma 10 (ii), $\{\mu\alpha\eta'\} \succ \{\mu\alpha\eta', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}$ and $\{\eta\alpha\mu'\} \succ \{\eta\alpha\mu', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}$. By Lemma 11,

$$\{(\mu\alpha\eta')\frac{1}{2}(\eta\alpha\mu')\} \succ \{(\mu\alpha\eta')\frac{1}{2}(\eta\alpha\mu'), \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}.$$

Since $(\mu\alpha\eta')\frac{1}{2}(\eta\alpha\mu') = (\mu\alpha\mu')\frac{1}{2}(\eta\alpha\eta')$,

$$\{(\mu\alpha\mu')\frac{1}{2}(\eta\alpha\eta')\} \succ \{(\mu\alpha\mu')\frac{1}{2}(\eta\alpha\eta'), \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}.$$

By the first claim, $\{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}$. ■

B.2 Construction of ψ on the Self-Control Subdomain

Define the *self-control subdomain* as

$$A \equiv \{(l, w) \in [0,1]^2 : \exists \mu, \eta \text{ s.t.} v(\eta) = l, v(\eta) - v(\mu) = w \text{ and} \{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}\}.$$

Let

$$\begin{aligned} L_A &\equiv \{l \in [0,1] \,|\, (l, w) \in A \text{ for some } w\}, \\ A(l) &\equiv \{w \,|\, (l, w) \in A\}. \end{aligned}$$

If $l \in L_A$, $A(l) \neq \emptyset$, and, by definition, $\sup A(l) \leq l$.

Lemma 13 *(i) If* $(l, w) \in A$, $(\alpha l, \alpha w) \in A$ *for all* $\alpha \in (0, 1)$.

(ii) L_A *is a non-degenerate interval with* $\inf L_A = 0$.

(iii) For all $l \in L_A$, $A(l)$ *is a non-degenerate interval with* $\inf A(l) = 0$.

(iv) If $\sup A(l) < l$, $A(l)$ *is open.*

(v) If $\sup A(l) \in A(l)$, $\sup A(l) = l$.

Proof. (i) Take $(l, w) \in A$. There exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$, $l = v(\eta)$, $w = v(\eta) - v(\mu)$. Let ν^- be a minimal lottery with respect to v. Since $v(\eta) = l > 0 = v(\nu^-)$, by Lemma 10 (ii), $\{\mu\alpha\nu^-\} \succ \{\mu\alpha\nu^-, \eta\alpha\nu^-\} \succ \{\eta\alpha\nu^-\}$ for all $\alpha \in (0, 1)$. Since $\alpha l = v(\eta\alpha\nu^-)$ and $\alpha w = v(\eta\alpha\nu^-) - v(\mu\alpha\nu^-)$, $(\alpha l, \alpha w) \in A$ as desired.

(ii) Take any $l \in L_A$. There exists w such that $(l, w) \in A$. By part (i), $(\alpha l, \alpha w) \in A$ for all $\alpha \in (0, 1)$. Hence, $\alpha l \in L_A$ as desired.

(iii) Take any $w \in A(l)$. It suffices to show that $\alpha w \in A$ for all $\alpha \in (0, 1)$. There exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$, $l = v(\eta)$ and $w = v(\eta) - v(\mu)$. By Lemma 10 (ii), $\{\mu\alpha\eta\} \succ \{\mu\alpha\eta, \eta\} \succ \{\eta\}$. Moreover, $\alpha w = v(\eta) - v(\mu\alpha\eta)$. Hence, $\alpha w \in A(l)$.

(iv) Since $A(l)$ is an interval such that $0 \notin A(l)$ and $\inf A(l) = 0$, it is enough to show that for all $w \in A(l)$, there exists $w' > w$ such that $w' \in A(l)$. There exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$, $v(\eta) = l$ and $v(\eta) - v(\mu) = w$. Continuity implies that there exists an open neighborhood $B(\mu)$ of μ such that $\{\mu'\} \succ \{\mu', \eta\} \succ \{\eta\}$ for all $\mu' \in B(\mu)$. Since $w \leq \sup A(l) < l$, we have $v(\mu) = l - w > 0 = \min_\Delta v$. Let ν^- satisfy $v(\nu^-) = \min_\Delta v$. Since $v(\mu) > v(\nu^-)$, $v(\eta) > v(\mu\alpha\nu^-)$ and $\mu\alpha\nu^- \in B(\mu)$ for all α close to 1. Thus $w < v(\eta) - v(\mu\alpha\nu^-) \in A(l)$ as desired.

(v) By definition, $\sup A(l) \leq l$. Suppose $\sup A(l) < l$. By part (iv), $A(l)$ is open. Since $\sup A(l) \in A(l)$, there exists a small $\varepsilon > 0$ such that $\sup A(l) + \varepsilon \in A(l)$, which is a contradiction. ■

Define $\varphi : A \to (0, 1]$ by

$$\varphi(l, w) \equiv u(\mu) - W(\{\mu, \eta\}),$$

s.t. $\mu, \eta \in \Delta$ satisfy $v(\eta) = l$, $v(\eta) - v(\mu) = w$ and $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$. The following lemmas ensure that φ is well-defined.

Take any finite subset $\mathbf{c} = \{c_1, \cdots, c_N\} \subset C$. Define

$$\Delta_{(N,\mathbf{c})} \equiv \left\{ \nu \in \mathbb{R}^N_+ \;\middle|\; \sum_{i=1}^N \nu(c_i) = 1 \right\} \subset \Delta, \; \Theta_{(N,\mathbf{c})} \equiv \left\{ \theta \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^N \theta(c_i) = 0 \right\}.$$

For all $\mu \in \Delta_{(N,\mathbf{c})}$ and $\theta \in \Theta_{(N,\mathbf{c})}$, if $\mu + \theta \in \Delta_{(N,\mathbf{c})}$, we can view $\mu + \theta$ as the lottery obtained by shifting μ toward θ. For all $\mu \in \Delta_{(N,\mathbf{c})}$, say that $\theta \in \Theta_{(N,\mathbf{c})}$ is *admissible for* μ if $\mu + \theta \in \Delta_{(N,\mathbf{c})}$. Moreover, $\theta \in \Theta_{(N,\mathbf{c})}$ is said to be *strictly admissible for* μ if $\mu + \theta$ belongs to the interior of $\Delta_{(N,\mathbf{c})}$.

For all $\theta \in \Theta_{(N,\mathbf{c})}$, let $u(\theta)$ and $v(\theta)$ denote $\sum_i u(c_i)\theta(c_i)$ and $\sum_i v(c_i)\theta(c_i)$, respectively.

Lemma 14 *Assume that* $\{\mu,\eta\} \sim \{\nu\}$ *and* $\theta \in \Theta_{(N,\mathbf{c})}$ *is admissible for* μ,η,ν. *Then,*

$$\begin{aligned} v(\theta) < 0 &\implies \{\mu,\eta\}+\theta \succsim \{\nu\}+\theta, \\ v(\theta) > 0 &\implies \{\nu\}+\theta \succsim \{\mu,\eta\}+\theta. \end{aligned}$$

Proof. We follow the proof of Ergin and Sarver [13, Lemma 3]. As in their proof, let $A \equiv \{\mu,\eta\}$ and $B \equiv \{\nu\}$. As shown in their proof, there exist $p,q \in \Delta_{(N,\mathbf{c})}$ and $0 < \kappa \leq 1$ such that $\theta = \kappa(p-q)$. Define menus A' and B' as in their proof. Then, by definition, $q\kappa A' = A$, $q\kappa B' = B$, $p\kappa A' = A+\theta$, and $p\kappa B' = B+\theta$.

First assume $v(\theta) < 0$. Then, $v(q) > v(p)$. If $\{p\} \not\sim \{q\}$, by Lemma 9, $q \succsim_T p$. Thus, by Decreasing Self-Control, $q\kappa A' = A \sim B = q\kappa B'$ implies

$$A+\theta = p\kappa A' \succsim p\kappa B' = B+\theta,$$

as desired. If $\{p\} \sim \{q\}$, there exists $r \in \Delta$ such that $\{p\beta r\} \not\sim \{q\}$ for all $\beta \in (0,1)$. Moreover, for all β sufficiently close to one, $v(p\beta r) < v(q)$ by continuity. Hence $q \succsim_T p\beta r$ for such β. By Decreasing Self-Control, we have $(p\beta r)\kappa A' \succsim (p\beta r)\kappa B'$. By Continuity, we have $A+\theta = p\kappa A' \succsim p\kappa B' = B+\theta$ as $\beta \to 1$. The symmetric argument can be applied when $v(\theta) > 0$. ■

Lemma 15 *Assume that* $\{\mu,\eta\} \sim \{\nu\}$ *and* $\theta \in \Theta_{(N,\mathbf{c})}$ *is strictly admissible for* μ,η,ν. *Then,*

$$\begin{aligned} v(\theta) \leq 0 &\implies \{\mu,\eta\}+\theta \succsim \{\nu\}+\theta, \\ v(\theta) \geq 0 &\implies \{\nu\}+\theta \succsim \{\mu,\eta\}+\theta. \end{aligned}$$

Proof. Since $\mu+\theta, \eta+\theta, \nu+\theta$ belong to the interior of $\Delta_{(N,\mathbf{c})}$, there exists a neighborhood $O(\theta)$ of θ such that $\mu+\theta', \eta+\theta', \nu+\theta' \in \Delta_{(N,\mathbf{c})}$ for all $\theta' \in O(\theta)$. Take any $\underline{\theta}$ such that $v(\underline{\theta}) < 0$. For all small $\beta > 0$, $\underline{\theta}\beta\theta \in O(\theta)$. Since $v(\underline{\theta}\beta\theta) < 0$, by Lemma 14,

$$\{\mu,\eta\}+\underline{\theta}\beta\theta \succsim \{\nu\}+\underline{\theta}\beta\theta.$$

By Continuity, $\{\mu,\eta\}+\theta \succsim \{\nu\}+\theta$ as $\beta \to 0$. Similarly, by taking any $\overline{\theta}$ such that $v(\overline{\theta}) > 0$, we can show that $\{\nu\}+\theta \succsim \{\mu,\eta\}+\theta$. ■

Lemma 16 *For all* $\mu,\mu' \in \Delta_{(N,\mathbf{c})}$ *and* $\theta \in \Theta_{(N,\mathbf{c})}$ *that is strictly admissible for* μ,μ',

$$W(\{\mu,\mu'\}) + u(\theta) - W(\{\mu+\theta,\mu'+\theta\}) \begin{cases} \leq 0 & \textit{if } v(\theta) \leq 0 \\ \geq 0 & \textit{if } v(\theta) \geq 0. \end{cases}$$

Proof. By Set Betweenness, assume that $\{\mu\} \succsim \{\mu,\mu'\} \succsim \{\mu'\}$. Since u is continuous, there exists $\alpha \in [0,1]$ such that $W(\{\mu,\mu'\}) = u(\mu\alpha\mu')$. Since $\mu\alpha\mu'+\theta = (\mu+\theta)\alpha(\mu'+\theta)$, $\mu\alpha\mu'+\theta$ also belongs to the interior of $\Delta_{(N,\mathbf{c})}$. If $v(\theta) \leq 0$, Lemma 15 implies

$$W(\{\mu+\theta,\mu'+\theta)\}) \geq u(\mu\alpha\mu'+\theta) = u(\mu\alpha\mu') + u(\theta) = W(\{\mu,\mu'\}) + u(\theta).$$

The same argument can be applicable when $v(\theta) \geq 0$. ■

Lemma 17 *Take all* $\mu, \mu', \eta, \eta' \in \Delta$ *with finite supports. Assume that* $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ *and* $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$*. Then,*

$$v(\eta) = v(\eta'),\ v(\mu) = v(\mu') \implies u(\mu) - W(\{\mu, \eta\}) = u(\mu') - W(\{\mu', \eta'\}).$$

Proof. Let $\mathbf{c} \equiv \{c_1, \cdots, c_N\} \subset C$ be the union of the supports of μ, μ', η, η'. Hence, these lotteries belong to $\Delta_{(N,\mathbf{c})}$. Take a lottery ν in the interior of $\Delta_{(N,\mathbf{c})}$. For all $\alpha \in (0, 1)$ sufficiently close to one, let $a \equiv \mu\alpha\nu$, $b \equiv \eta\alpha\nu$, $a' \equiv \mu'\alpha\nu$, $b' \equiv \eta'\alpha\nu \in \Delta_{(N,\mathbf{c})}$. Continuity implies $\{a\} \succ \{a, b\} \succ \{b\}$ and $\{a'\} \succ \{a', b'\} \succ \{b'\}$. Furthermore, $v(b) = \alpha v(\eta) + (1 - \alpha)v(\nu) = \alpha v(\eta') + (1-\alpha)v(\nu) = v(b')$ and $v(a) = \alpha v(\mu) + (1-\alpha)v(\nu) = \alpha v(\mu') + (1-\alpha)v(\nu) = v(a')$.

Step 1: We claim that if $\theta \equiv a' - a \in \Theta_{(N,\mathbf{c})}$ is strictly admissible for b, then $u(a) - W(\{a, b\}) = u(a') - W(\{a', b'\})$. Since v is mixture linear, $v(\theta) = v(a') - v(a) = 0$, and $v(b + \theta) = v(b) + v(\theta) = v(b')$. Since $u(a) > W(\{a, b\}) > u(b)$ and $v(\theta) = 0$, Lemma 16 implies that

$$u(a + \theta) = u(a) + u(\theta) > W(\{a, b\}) + u(\theta) = W(\{a + \theta, b + \theta\}) > u(b) + u(\theta) = u(b + \theta),$$

that is, $\{a+\theta\} \succ \{a+\theta, b+\theta\} \succ \{b+\theta\}$. Equivalently, $\{a'\} \succ \{a', b+\theta\} \succ \{b+\theta\}$. Since $v(b + \theta) = v(b')$, by Lemma 2 of Noor and Takeoka [32], $\{a', b'\} \sim \{a', b + \theta\}$. Thus, from Lemma 16,

$$\begin{aligned} & W(\{a', b'\}) = W(\{a + \theta, b + \theta\}) = W(\{a, b\}) + u(\theta) = W(\{a, b\}) + u(a') - u(a) \\ \Leftrightarrow\ & u(a) - W(\{a, b\}) = u(a') - W(\{a', b'\}). \end{aligned}$$

Since $v(b) = v(b')$, from Lemma 12, for all $\beta \in [0, 1]$, $\{a\beta a'\} \succ \{a\beta a', b\beta b'\} \succ \{b\beta b'\}$. Notice also that $a\beta a'$, $b\beta b' \in \Delta_{(N,\mathbf{c})}$ for all $\beta \in [0, 1]$.

Step 2: We claim that for all $\beta \in [0, 1]$, there exists a relative open interval $O(\beta)$ containing β such that for all $\tilde{\beta} \in O(\beta)$,

$$u(a\tilde{\beta}a') - W(\{a\tilde{\beta}a', b\tilde{\beta}b'\}) = u(a\beta a') - W(\{a\beta a', b\beta b'\}). \tag{16}$$

Since $v(b) = v(b')$ and $v(a) = v(a')$, we have, for all $\tilde{\beta} \in [0, 1]$, $v(b\tilde{\beta}b') = v(b\beta b')$ and $v(a\tilde{\beta}a') = v(a\beta a')$. Let $\theta \equiv a\beta a' - a\tilde{\beta}a' \in \Theta_{(N,\mathbf{c})}$. Notice that

$$b\tilde{\beta}b' + \theta = (\eta\beta\eta')\alpha\nu + (\beta - \tilde{\beta})(a - a').$$

Since $(\eta\beta\eta')\alpha\nu$ is in the interior of $\Delta_{(N,\mathbf{c})}$, there exists a relative open interval $O(\beta)$ containing β such that $(\eta\tilde{\beta}\eta')\alpha\nu + (\beta - \tilde{\beta})(a - a') \in \Delta_{(N,\mathbf{c})}$ for all $\tilde{\beta} \in O(\beta)$. That is, for all $\tilde{\beta} \in O(\beta)$, θ is strictly admissible for $b\tilde{\beta}b'$. Thus, by Step 1, we have (16).

Step 3: We claim that $u(a) - W(\{a, b\}) = u(a') - W(\{a', b'\})$. Let $O(\beta)$ be an open interval containing $\beta \in [0, 1]$ guaranteed by Step 2. Since $\{O(\beta) | \beta \in [0, 1]\}$ is an open cover of $[0, 1]$, there exists a finite subcover, denoted by $\{O(\beta^i) | i = 1, \cdots, I\}$. Without loss of generality, assume $\beta^i \leq \beta^{i+1}$. Define $\beta^0 = 0$ and $\beta^{I+1} = 1$. Since $\beta^0 \in O(\beta^1)$ and $\beta^{I+1} \in O(\beta^I)$, from Step 2,

$$\begin{aligned} u(a') - W(\{a', b'\}) &= u(a\beta^1 a') - W(\{a\beta^1 a', b\beta^1 b'\}) = \\ \dots &= u(a\beta^I a') - W(\{a\beta^I a', b\beta^I b'\}) = u(a) - W(\{a, b\}). \end{aligned}$$

From Step 3, for all $\alpha \in (0,1)$ sufficiently close to one,

$$u(\mu\alpha\nu) - W(\{\mu\alpha\nu, \eta\alpha\nu\}) = u(\mu'\alpha\nu) - W(\{\mu'\alpha\nu, \eta'\alpha\nu\}).$$

Continuity ensures that $u(\mu) - W(\{\mu, \eta\}) = u(\mu') - W(\{\mu', \eta'\})$ as $\alpha \to 1$. ■

Lemma 18 *For all $\mu, \mu', \eta, \eta' \in \Delta$ such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ and $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$,*

$$v(\eta) = v(\eta'),\ v(\mu) = v(\mu') \implies u(\mu) - W(\{\mu, \eta\}) = u(\mu') - W(\{\mu', \eta'\}).$$

Proof. Define $\mu_n = \eta\frac{1}{n}\mu$, $\eta_n = \mu\frac{1}{n}\eta$, $\mu'_n = \eta'\frac{1}{n}\mu'$ and $\eta'_n = \mu'\frac{1}{n}\eta'$. Then, $\mu_n \to \mu$, $\eta_n \to \eta$, $\mu'_n \to \mu'$, and $\eta'_n \to \eta'$. For all sufficiently large n, by Continuity, $\{\mu_n\} \succ \{\mu_n, \eta_n\} \succ \{\eta_n\}$ and $\{\mu'_n\} \succ \{\mu'_n, \eta'_n\} \succ \{\eta'_n\}$. Moreover, $v(\mu_n) = v(\mu'_n)$ and $v(\eta_n) = v(\eta'_n)$.

For all sufficiently large n, let $O(\mu_n)$ be the $\frac{1}{n}$-neighborhood of μ_n. Since the set of lotteries with finite supports is dense in Δ, we can find $\mu_n^+, \mu_n^- \in O(\mu_n)$ with finite supports such that $v(\mu_n^-) < v(\mu_n) < v(\mu_n^+)$. Define

$$\tilde{\mu}_n \equiv \mu_n^+ \left(\frac{v(\mu_n) - v(\mu_n^-)}{v(\mu_n^+) - v(\mu_n^-)} \right) \mu_n^-.$$

Then, $v(\tilde{\mu}_n) = v(\mu_n)$ and $\tilde{\mu}_n$ has a finite support. Moreover, by the triangle inequality,

$$d(\tilde{\mu}_n, \mu) \leq d(\tilde{\mu}_n, \mu_n) + d(\mu_n, \mu) \to 0,$$

as $n \to \infty$. Similarly, we can construct $\tilde{\eta}_n$, $\tilde{\mu}'_n$, and $\tilde{\eta}'_n$ for all sufficiently large n. Then, $\tilde{\mu}_n$, $\tilde{\eta}_n$, $\tilde{\mu}'_n$, and $\tilde{\eta}'_n$ satisfy the assumptions of Lemma 17. For all such n, we have $u(\tilde{\mu}_n) - W(\{\tilde{\mu}_n, \tilde{\eta}_n\}) = u(\tilde{\mu}'_n) - W(\{\tilde{\mu}'_n, \tilde{\eta}'_n\})$. Continuity of u and W implies that $u(\mu) - W(\{\mu, \eta\}) = u(\mu') - W(\{\mu', \eta'\})$, as desired. ■

Lemma 19 *(i) φ is well-defined.*

(ii) For any $(l, w) \in A$ and $\alpha \in (0,1)$,

$$\varphi(l, \alpha w) = \alpha\varphi(l, w).$$

(iii) $\varphi(l, \cdot)$ is strictly increasing and continuous in the interior of $A(l)$.

Proof. (i) By Lemma 18, if $\mu, \eta, \mu', \eta' \in \Delta$ and $v(\eta) = v(\eta')$, $v(\eta) - v(\mu) = v(\eta') - v(\mu')$, $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ and $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$, then in particular $v(\mu') = v(\mu)$ and $v(\eta') = v(\eta)$, and thus $u(\mu) - W(\{\mu, \eta\}) = u(\mu') - W(\{\mu', \eta'\})$. Thus φ is well-defined.

(ii) Take any $(w, l) \in A$ and suppose μ, η are such that $v(\eta) = l$, $v(\eta) - v(\mu) = w$ and $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$. Suppose $\{\mu, \eta\} \sim \{\nu\}$ for some ν (assured by the argument of Lemma 8 (ii)). By Lemma 10 (ii), $\{\mu\alpha\eta\} \succ \{\mu\alpha\eta, \eta\} \succ \{\eta\}$. Moreover, $v(\eta) - v(\mu\alpha\eta) = \alpha w$. Now observe that

$$\begin{aligned} u(\nu\alpha\eta) &= W(\{\mu, \eta\})\alpha u(\eta) = u(\mu\alpha\eta) - \alpha\varphi(l, w) \\ W(\{\mu\alpha\eta, \eta\}) &= u(\mu\alpha\eta) - \varphi(l, \alpha w). \end{aligned}$$

Therefore, Linear Self-Control implies the result.

(iii) First we show that $\varphi(l, \cdot)$ is strictly increasing. Take any two points w_1, w_2 in the interior of $A(l)$ with $w_1 < w_2$. Since $A(l)$ is an interval, there exists $\tilde{w}$ in the interior of A such that $w_2 < \tilde{w}$. Let $\alpha_i \equiv \frac{w_i}{\tilde{w}}$ for $i = 1, 2$. By part (ii),

$$\varphi(l, w_1) = \alpha_1\varphi(l, \tilde{w}) < \alpha_2\varphi(l, \tilde{w}) = \varphi(l, w_2).$$

Hence $\varphi(l, \cdot)$ is strictly increasing.

To show continuity, let $w^n \to w$ in the interior of $A(l)$. Without loss of generality, assume that there exists $\tilde{w} \in A(l)$ such that $w^n, w \leq \tilde{w}$ for all n. Let $\alpha^n = \frac{w^n}{\tilde{w}}$ and $\alpha = \frac{w}{\tilde{w}}$. By part (ii),

$$\varphi(l, w^n) = \alpha^n\varphi(l, \tilde{w}) \to \alpha\varphi(l, \tilde{w}) = \varphi(l, w)$$

as desired. ■

Lemma 20 *For all* $(l, w) \in A$,

$$\varphi(l, w) = \psi(l) \cdot w$$

for some $\psi(l) > 0$.

Proof. Let $\overline{A(l)}$ denote the closure of $A(l)$. By Lemma 13 (iii), $\overline{A(l)}$ is a nondegenerate interval containing 0; let $\overline{A(l)} = [0, \overline{w}_l]$. Let $\overline{\varphi}(l, 0) = \inf_{w \in A(l)} \varphi(l, w)$ and $\overline{\varphi}(l, \overline{w}_l) = \sup_{w \in A(l)} \varphi(l, w)$. By Lemma 19 (ii), $\overline{\varphi}(l, 0) = 0$. By using $\overline{\varphi}(l, 0)$ and $\overline{\varphi}(l, \overline{w}_l)$, we can extend $\varphi(l, w)$ to $\overline{A(l)}$. Since $\varphi(l, \cdot)$ is strictly increasing and continuous, this extension, denoted by $\overline{\varphi}(l, \cdot)$, is a unique continuous and strictly increasing extension. Given continuity, the property in Lemma 19 (ii) is satisfied by $\overline{\varphi}(l, \cdot)$ as well. But then for any $(l, w) \in A$, we have $\varphi(l, w) = \overline{\varphi}(l, w) = \overline{\varphi}(l, \frac{w}{\overline{w}_l}\overline{w}_l) = \frac{w}{\overline{w}_l}\overline{\varphi}(l, \overline{w}_l)$. Indeed, $\varphi(l, w) = w \cdot \psi(l)$ where $\psi(l) \equiv \frac{1}{\overline{w}_l}\overline{\varphi}(l, \overline{w}_l) > 0$. ■

Lemma 20 states that $\psi(l)$ is a function defined on L_A.

B.3 Extension of ψ to $[0, 1]$

Lemma 21 *$\psi : L_A \to \mathbb{R}_{++}$ is (i) continuous, and (ii) increasing.*

Proof. (i) For all l such that $(l, w) \in A$ for some w, we will show that ψ is continuous at l. There exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$, $l = v(\eta)$ and $w = v(\eta) - v(\mu)$. By Continuity, there exists an open neighborhood $B(\eta)$ of η such that $\{\mu\} \succ \{\mu, \eta'\} \succ \{\eta'\}$ for all $\eta' \in B(\eta)$.

We first show that if $l < \max_\Delta v$, ψ is continuous at l. Since $0 < l < \max_\Delta v$, Continuity implies that there exist $\eta_i \in B(\eta)$, $i = 1, 2$, such that $l_1 \equiv v(\eta_1) < l < v(\eta_2) \equiv l_2$. Let $l^n \to l$. We want to show that $\psi(l^n) \to \psi(l)$. For all sufficiently large n, $l_1 < l^n < l_2$. Define

$$\eta^n \equiv \begin{cases} \eta \left(\frac{l^n - l_1}{l - l_1}\right) \eta_1 & \text{if } l_1 < l^n \leq l \\ \eta \left(\frac{l_2 - l^n}{l_2 - l}\right) \eta_2 & \text{if } l \leq l^n < l_2. \end{cases}$$

Since $l^n \to l$, we have $\eta^n \to \eta$. Especially, $\eta^n \in B(\eta)$ for all sufficiently large n, and, hence, $\{\mu\} \succ \{\mu, \eta^n\} \succ \{\eta^n\}$. Moreover, since v is mixture linear, $l^n = v(\eta^n)$. Let $w^n = v(\eta^n) - v(\mu)$. By Lemma 20,

$$\psi(l^n) w^n = \varphi(l^n, w^n) = u(\mu) - W(\{\mu, \eta^n\}).$$

Since W is continuous,

$$\lim_{n\to\infty} \psi(l^n) = \lim_{n\to\infty} \frac{u(\mu) - W(\{\mu, \eta^n\})}{w^n} = \frac{u(\mu) - W(\{\mu, \eta\})}{w} = \psi(l).$$

If $l = \max_\Delta v$, apply the above argument with assuming $l = l_2$.

(ii) Take $(l, w), (l', w') \in A$ s.t. $l > l'$. Let these correspond to μ, η and μ', η'. We have $v(\eta) = l > l' = v(\eta')$. By Lemma 10 (ii), $\{\mu\alpha\eta'\} \succ \{\mu\alpha\eta', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}$. Hence, by Lemma 20, $u(\mu\alpha\eta') - W(\{\mu\alpha\eta', \eta\alpha\eta'\}) = \psi(l\alpha l') \cdot \alpha w$. Moreover, by Lemma 10 (i), $\{\mu\alpha\eta', \eta\alpha\eta'\} \succsim \{e_{\mu\eta}\alpha\eta'\}$, and, hence,

$$u(\mu\alpha\eta') - \psi(l\alpha l') \cdot \alpha w \geq \alpha[u(\mu) - \psi(l) \cdot w] + (1 - \alpha) u(\eta').$$

Thus, we have $\psi(l\alpha l') \leq \psi(l)$. Since ψ is continuous, $\psi(l') \leq \psi(l)$ as $\alpha \to 0$. ■

Let $\bar{l} = \sup L_A$. Define $\psi(0) \equiv \inf\{\psi(l) \,|\, l \in L_A\}$ and $\psi(\bar{l}) \equiv \sup\{\psi(l) \,|\, l \in L_A\}$. By Lemma 21, $\psi : [0, \bar{l}] \to \mathbb{R}_+$ is a unique continuous increasing extension.

Lemma 22 *Let $\{\mu\} \succ \{\mu, \eta\} \sim \{\eta\}$. If $(v(\eta), v(\eta) - v(\mu)) \in A$, then $u(\eta) \geq u(\mu) - \varphi(v(\eta), v(\eta) - v(\mu))$.*

Proof. There exist μ', η' such that $\{\mu'\} \succ \{\mu', \eta'\} \succ \{\eta'\}$, $v(\eta') = v(\eta)$, and $v(\eta') - v(\mu') = v(\eta) - v(\mu)$. Since $\varphi(v(\eta), v(\eta) - v(\mu)) = \varphi(v(\eta'), v(\eta') - v(\mu')) = u(\mu') - W(\{\mu', \eta'\})$, it suffices to show that $u(\mu') - W(\{\mu', \eta'\}) \geq u(\mu) - u(\eta)$.

We will claim that $u(\mu') - u(\eta') > u(\mu) - u(\eta)$. Suppose otherwise, that is, $u(\mu) - u(\eta) \geq u(\mu') - u(\eta')$. Let

$$E \equiv \{\alpha \in [0,1] \,|\, \{\mu\alpha\mu'\} \succ \{\mu\alpha\mu', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}\}.$$

By assumption, $0 \in E$ and $1 \notin E$. Moreover, by Continuity, E is open in $[0,1]$. Let $\bar{\alpha} \equiv \sup E \in (0,1]$. By Continuity, $\bar{\alpha} \notin E$, and hence

$$\{\mu\bar{\alpha}\mu'\} \succ \{\mu\bar{\alpha}\mu', \eta\bar{\alpha}\eta'\} \sim \{\eta\bar{\alpha}\eta'\}. \tag{17}$$

Since $u(\mu) - u(\eta) \geq u(\mu') - u(\eta') > \varphi(v(\eta'), v(\eta') - v(\mu'))$, $v(\eta) = v(\eta')$, and $v(\eta') - v(\mu') = v(\eta) - v(\mu)$, we have

$$u(\mu\alpha\mu') - u(\eta\alpha\eta') > \varphi(v(\eta\alpha\eta'), v(\eta\alpha\eta') - v(\mu\alpha\mu')) \tag{18}$$

for all $\alpha \in [0,1]$. On the other hand, since $\bar{\alpha}$ is a supremum of E, there exists a sequence $\{\alpha^n\}$ in E converging to $\bar{\alpha}$. We have $\{\mu\alpha^n\mu'\} \succ \{\mu\alpha^n\mu', \eta\alpha^n\eta'\} \succ \{\eta\alpha^n\eta'\}$, and hence

$$\begin{aligned} u(\mu\alpha^n\mu') - u(\eta\alpha^n\eta') &> \varphi(v(\eta\alpha^n\eta'), v(\eta\alpha^n\eta') - v(\mu\alpha^n\mu')) \\ &= u(\mu\alpha^n\mu') - W(\{\mu\alpha^n\mu', \eta\alpha^n\eta'\}). \end{aligned}$$

From Continuity and (18),

$$u(\mu\bar{\alpha}\mu') - u(\eta\bar{\alpha}\eta') > u(\mu\bar{\alpha}\mu') - W(\{\mu\bar{\alpha}\mu', \eta\bar{\alpha}\eta'\}),$$

that is, $W(\{\mu\bar{\alpha}\mu', \eta\bar{\alpha}\eta'\}) > u(\eta\bar{\alpha}\eta')$, which contradicts (17).

Since $v(\eta') - v(\mu') = v(\eta\alpha\eta') - v(\mu\alpha\mu')$ for all $\alpha \in E$, by Lemma 19 (i), $u(\mu') - W(\{\mu', \eta'\}) = u(\mu\alpha\mu') - W(\{\mu\alpha\mu', \eta\alpha\eta'\})$. Thus taking Continuity and the above claims together,

$$\begin{aligned} & u(\mu') - W(\{\mu', \eta'\}) \\ = \; & u(\mu\bar{\alpha}\mu') - W(\{\mu\bar{\alpha}\mu', \eta\bar{\alpha}\eta'\}) = u(\mu\bar{\alpha}\mu') - u(\eta\bar{\alpha}\eta') \\ = \; & \bar{\alpha}(u(\mu) - u(\eta)) + (1-\bar{\alpha})(u(\mu') - u(\eta')) \geq u(\mu) - u(\eta), \end{aligned}$$

as desired. ■

Let

$$\begin{aligned} B &\equiv \{(l, w) \in [0,1]^2 \,|\, l = v(\eta),\ w = v(\eta) - v(\mu) \text{ for some } \{\mu\} \succ \{\mu, \eta\}\}, \\ L_B &\equiv \{l \in [0,1] \,|\, (l, w) \in B \text{ for some } w\}, \\ B(l) &\equiv \{w \in [0,1] \,|\, (l, w) \in B\}. \end{aligned}$$

Lemma 23 *(i) B is convex.*

(ii) If $(l, w) \in B$, $(\alpha l + (1-\alpha)l', \alpha w) \in B$ for all $l' \in [0,1]$ and $\alpha \in (0,1)$.

(iii) L_B is a non-degenerate interval satisfying $\inf L_B = 0$ and $\sup L_B = 1$.

(iv) For all $l \in L_B$, $B(l)$ is an interval satisfying $\inf B(l) = 0$.

Proof. (i) Take $(l_i, w_i) \in B$, $i = 1, 2$ and $\alpha \in [0, 1]$. There exist μ_i, η_i such that $\{\mu_i\} \succ \{\mu_i, \eta_i\}$, $l_i = v(\eta_i)$, and $w_i = v(\eta_i) - v(\mu_i)$. Since we have $v(\eta_i) > v(\mu_i)$ and $u(\mu_i) > u(\eta_i)$ for $i = 1, 2$, mixture linearity of u and v implies $u(\mu_1 \alpha \mu_2) > u(\eta_1 \alpha \eta_2)$ and $v(\eta_1 \alpha \eta_2) > v(\mu_1 \alpha \mu_2)$, and hence $\{\mu_1 \alpha \mu_2\} \succ \{\mu_1 \alpha \mu_2, \eta_1 \alpha \eta_2\}$. Therefore, $\alpha(l_1, w_1) + (1 - \alpha)(l_2, w_2) \in B$.

(ii) Take any $(l, w) \in B$. There exist μ, η such that $\{\mu\} \succ \{\mu, \eta\}$, $l = v(\eta)$ and $w = v(\eta) - v(\mu)$. For all ν with $v(\nu) = l'$, by Temptation Independence, $\{\mu\alpha\nu\} \succ \{\mu\alpha\nu, \eta\alpha\nu\}$ for all $\alpha \in (0, 1)$. Hence, $(\alpha l + (1 - \alpha)l', \alpha w) = (v(\eta\alpha\nu), v(\eta\alpha\nu) - v(\mu\alpha\nu)) \in B$.

(iii) Take any $l \in L_B$. By part (ii), $\alpha l \in L_B$ and $\alpha l + (1 - \alpha) \in L_B$ for all $\alpha \in (0, 1)$ as desired.

(iv) Take $w \in B(l)$. By letting $l' = l$, part (ii) implies $(l, \alpha w) \in B$ for all $\alpha \in (0, 1)$. That is, $\alpha w \in B(l)$. ■

Define $F : B \to \mathbb{R}_+$ by

$$F(l, w) \equiv \sup\{u(\mu) - u(\eta) \,|\, l = v(\eta),\ w = v(\eta) - v(\mu) \text{ for some } \{\mu\} \succ \{\mu, \eta\}\}.$$

Lemma 24 *F is weakly concave.*

Proof. Take $(l_i, w_i) \in B$, $i = 1, 2$, and $\alpha \in (0, 1)$. There exist μ_i^n, η_i^n such that $\{\mu_i^n\} \succ \{\mu_i^n, \eta_i^n\}$, $l_i = v(\eta_i^n)$, $w_i = v(\eta_i^n) - v(\mu_i^n)$, and, $u(\mu_i^n) - u(\eta_i^n) \to F(w_i)$. Since $v(\eta_i^n) > v(\mu_i^n)$ and $u(\mu_i^n) > u(\eta_i^n)$, we have $v(\eta_1^n \alpha \eta_2^n) > v(\mu_1^n \alpha \mu_2^n)$ and $u(\mu_1^n \alpha \mu_2^n) > u(\eta_1^n \alpha \eta_2^n)$. Thus $\{\mu_1^n \alpha \mu_2^n\} \succ \{\mu_1^n \alpha \mu_2^n, \eta_1^n \alpha \eta_2^n\}$. Since $\alpha l_1 + (1 - \alpha)l_2 = v(\eta_1^n \alpha \eta_2^n)$ and $\alpha w_1 + (1 - \alpha)w_2 = v(\eta_1^n \alpha \eta_2^n) - v(\mu_1^n \alpha \mu_2^n)$,

$$\begin{aligned}
& F(\alpha l_1 + (1 - \alpha)l_2, \alpha w_1 + (1 - \alpha)w_2) \\
\geq\ & \limsup u(\mu_1^n \alpha \mu_2^n) - u(\eta_1^n \alpha \eta_2^n) \\
=\ & \limsup \alpha(u(\mu_1^n) - u(\eta_1^n)) + (1 - \alpha)(u(\mu_2^n) - u(\eta_2^n)) \\
=\ & \alpha F(l_1, w_1) + (1 - \alpha)F(l_2, w_2).
\end{aligned}$$

■

By Theorem 10.3 [33, p.85], $F(l, \cdot) : B(l) \to \mathbb{R}_+$ can be uniquely extended to the closure of $B(l)$ in a continuous and concave way.

By Lemma 23 (iii), $\inf L_B = 0$ and $\sup L_B = 1$. Take any $(l, w) \in B$. By Lemma 23 (ii), the interior of the convex hull of $\{(0, 0), (0, 1), (l, w)\}$ is a subset of B. Since this convex set is polyhedral, Theorem 10.3 [33, p.85] ensures that F can be uniquely extended to its closure in a continuous and concave way. Hence, for all $l \in L_B$, $F(l, 0)$ is defined by this extension and $F(\cdot, 0)$ is continuous and concave.

Denote $\sup A(l) = \overline{w}_l$. Notice that $\overline{w}_l \leq l$. Since $A(l) \subset B(l)$, $\overline{w}_l \leq \sup B(l)$.

Lemma 25 *Take all $l \in L_A$.*

(i) $F(l, w) > \psi(l)w$ for all $w \in A(l)$.

(ii) If $\overline{w}_l \notin A(l)$, $F(l, \overline{w}_l) = \psi(l)\overline{w}_l$.

Proof. (i) Since $(l, w) \in A$, there exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$, $l = v(\eta)$, and $w = v(\eta) - v(\mu)$. Then,

$$F(l, w) \geq u(\mu) - u(\eta) > u(\mu) - W(\{\mu, \eta\}) = \psi(l)w.$$

(ii) Since $F(l, \cdot) : \overline{B(l)} \to \mathbb{R}_+$ is continuous, part (i) implies $F(l, \overline{w}_l) \geq \psi(l)\overline{w}_l$. By contradiction, suppose $F(l, \overline{w}_l) > \psi(l)\overline{w}_l = \sup\{\varphi(l, w) | w \in A(l)\}$. By Lemma 22, for all $w \in A(l)$ and μ, η such that $l = v(\eta)$, $w = v(\eta) - v(\mu)$ and $\{\mu\} \succ \{\mu, \eta\} \sim \{\eta\}$, we have $\varphi(l, w) \geq u(\mu) - u(\eta)$. Thus there exist sequences $w^n \to \overline{w}$, $\{\mu^n\}_{n=1}^{\infty}$ and $\{\eta^n\}_{n=1}^{\infty}$ such that $w^n = v(b^n) - v(a^n) \in A(l)$, $\{\mu^n\} \succ \{\mu^n, \eta^n\} \succ \{\eta^n\}$, and $u(\mu^n) - u(\eta^n) > c > \sup\{\varphi(l, w) | w \in A(l)\}$, where $c > 0$ is a constant number. Since $\{\mu^n\}_{n=1}^{\infty}$ and $\{\eta^n\}_{n=1}^{\infty}$ are sequences in Δ, we can assume $\mu^n \to \mu^0$ and $\eta^n \to \eta^0$ without loss of generality. Since

$$u(\mu^n) - u(\eta^n) > c > \varphi(v(\eta^n), v(\eta^n) - v(\mu^n)) = u(\mu^n) - W(\{\mu^n, \eta^n\}),$$

continuity implies $u(\mu^0) - u(\eta^0) > u(\mu^0) - W(\{\mu^0, \eta^0\})$, that is, $W(\{\mu^0, \eta^0\}) > u(\eta^0)$. On the other hand, since $\overline{w}_l = v(\eta^0) - v(\mu^0) > 0$ and $u(\mu^0) > u(\eta^0)$, we have $\{\mu^0\} \succ \{\mu^0, \eta^0\}$. Hence $\{\mu^0\} \succ \{\mu^0, \eta^0\} \succ \{\eta^0\}$, which contradicts $\overline{w}_l \notin A(l)$. ■

Lemma 26 *Let $l \in L_B$.*
(i) $F(l, 0) > 0$ if and only if there exist μ, η such that $u(\mu) > u(\eta)$ and $v(\mu) = v(\eta) = l$.
(ii) If $F(l, 0) > 0$, then $l \in L_A$.
(iii) If $F(l, 0) > 0$, then $F(l', 0) > 0$ for all $l' \in (0, 1)$.

Proof. (i) First suppose $F(l, 0) > 0$. By definition, there exists a sequence $w^n \in B(l)$ such that $w^n \to 0$ and $F(l, w^n) > c > 0$ for some c. Let $\{\mu^n\}, \{\eta^n\}$ be corresponding sequences that satisfy $\{\mu^n\} \succ \{\mu^n, \eta^n\}$, $v(\eta^n) = l$, $w^n = v(\eta^n) - v(\mu^n)$, and $u(\mu^n) - u(\eta^n) > c$. Since $\{\mu^n\}$ and $\{\eta^n\}$ are sequences in Δ, without loss of generality, assume that $\mu^n \to \mu$ and $\eta^n \to \eta$ for some μ, η. Since $w^n \to 0$, $v(\mu) = v(\eta)$. Moreover, $u(\mu) - u(\eta) \geq c > 0$.

Next suppose that $u(\mu) > u(\eta)$ and $v(\mu) = v(\eta) = l$. Let ν^+ and ν^- be a maximal and a minimal lottery with respect to v. For all sufficiently large n, $u(\nu^- \frac{1}{n} \mu) > u(\nu^+ \frac{1}{n} \eta)$ and $v(\nu^+ \frac{1}{n} \eta) > v(\nu^- \frac{1}{n} \mu)$. Since $\{\nu^- \frac{1}{n} \mu\} \succ \{\nu^- \frac{1}{n} \mu, \nu^+ \frac{1}{n} \eta\}$, $(l^n, w^n) \equiv (v(\nu^+ \frac{1}{n} \eta), v(\nu^+ \frac{1}{n} \eta) - v(\nu^- \frac{1}{n} \mu)) \in B$. Thus,

$$F(l, 0) = \lim_{n \to \infty} F(l^n, w^n) \geq \lim_{n \to \infty} u(\nu^- \alpha \mu) - u(\nu^+ \alpha \eta) = u(\mu) - u(\eta) > 0.$$

(ii) By part (i), there exist μ, η such that $u(\mu) > u(\eta)$ and $v(\mu) = v(\eta) = l$. By Set Betweenness, $\{\mu\} \succsim \{\mu, \eta\} \succsim \{\eta\}$. If $\{\mu\} \succ \{\mu, \eta\}$, we have $v(\eta) > v(\mu)$, which is a contradiction. Hence, we have $\{\mu\} \sim \{\mu, \eta\} \succ \{\eta\}$. Let ν^- be a minimal lottery with respect to v. Since $v(\eta) = v(\mu) = l > 0 = v(\nu^-)$, for all small α, $u(\nu^- \alpha \mu) > u(\eta)$ and $v(\eta) > v(\nu^- \alpha \mu)$, and hence, $\{\nu^- \alpha \mu\} \succ \{\nu^- \alpha \mu, \eta\}$. Moreover, by Continuity, for all small α, $\{\nu^- \alpha \mu, \eta\} \succ \{\eta\}$. Hence, by definition, $l \in L_A$.

(iii) By contradiction, suppose $F(l', 0) = 0$ for some $l' \in (0, 1)$. Assume $l < l'$. There exists $l'' \in (0, 1)$ with $l' < l''$. Then, l' can be written as a convex combination between l

and l'', denoted by $\alpha l + (1-\alpha)l''$. Moreover, since L_B is an interval with $\inf L_B = 0$ and $\sup L_B = 1$, $l'' \in L_B$. Since $F(\cdot, 0)$ is concave and $F(\cdot, 0) \geq 0$,

$$F(l', 0) = F(\alpha l + (1-\alpha)l'', 0) \geq \alpha F(l, 0) + (1-\alpha)F(l'', 0) > 0,$$

which is a contradiction. The same argument can be applied when $l' < l$. ■

Lemma 27 *Let $l \in L_B$.*

(i) If $F(l, 0) = 0$, $F(l, w) = u(\mu) - u(\eta)$ for all μ, η such that $\{\mu\} \succ \{\mu, \eta\}$, $l = v(\eta)$, and $w = v(\eta) - v(\mu)$.

(ii) If $F(l, 0) = 0$, $F(l, \cdot) : B(l) \to \mathbb{R}_+$ is linear.

Proof. (i) Take all μ, μ', η, η' such that $\{\mu\} \succ \{\mu, \eta\}$, $\{\mu'\} \succ \{\mu', \eta'\}$, $l = v(\eta) = v(\eta')$ and $w = v(\eta) - v(\mu) = v(\eta') - v(\mu')$. Notice that $v(\mu) = v(\mu')$. By Lemma 26 (i) and (iii), since $v(\eta) = v(\eta')$, we must have $u(\eta) = u(\eta')$. Similarly, since $v(\mu) = v(\mu')$, we must have $u(\mu) = u(\mu')$. Hence, $u(\mu) - u(\eta) = u(\mu') - u(\eta')$. Therefore, we have $F(l, w) = u(\mu) - u(\eta)$.

(ii) Since $F(l, 0) = 0$, it is enough to show that $F(l, \cdot)$ is mixture linear on $B(l)$, that is, $F(l, \alpha w_1 + (1-\alpha)w_2) = \alpha F(l, w_1) + (1-\alpha)F(l, w_2)$ for all $w_1, w_2 \in B(l)$ and $\alpha \in (0, 1)$. There exist μ_i, η_i, $i = 1, 2$, such that $\{\mu_i\} \succ \{\mu_i, \eta_i\}$, $l = v(\eta_i)$ and $w_i = v(\eta_i) - v(\mu_i)$. Since $\{\mu_1 \alpha \mu_2\} \succ \{\mu_1 \alpha \mu_2, \eta_1 \alpha \eta_2\}$, part (i) implies that

$$F(l, \alpha w_1 + (1-\alpha)w_2) = u(\mu_1 \alpha \mu_2) - u(\eta_1 \alpha \eta_2) = \alpha F(l, w_1) + (1-\alpha)F(l, w_2).$$

■

Lemma 28 *Let $\bar{l} \equiv \sup L_A$. If $\bar{l} \notin L_A$ and $\bar{l} \in L_B$, then $\psi(\bar{l})w \geq F(\bar{l}, w)$ for all $w \in B(\bar{l})$.*

Proof. By contradiction, suppose that $\psi(\bar{l})w' < F(\bar{l}, w')$ for some $w' \in B(\bar{l})$. By Lemma 26 (ii), we must have $F(\bar{l}, 0) = 0$. By Lemma 27 (ii), $F(\bar{l}, \cdot)$ is linear. Thus, $\psi(\bar{l})w < F(\bar{l}, w)$ for all $w \in B(\bar{l})$.

Take an increasing sequence $l^n \to \bar{l}$. Notice that $l^n \in L_A$. For all $l \in L_A$, denote $w(l) \equiv \sup A(l)$. If $w(l^n) \in A(l^n)$, by Lemma 13 (v), $w(l^n) = l^n$. If $w(l^n) \notin A(l^n)$, by Lemma 25 (ii), $w(l^n)$ is a unique number satisfying $F(l^n, w(l^n)) = \psi(l^n)w(l^n)$. Since the sequence $\{w(l^n)\}_{n=1}^{\infty}$ belongs to $[0, 1]$, there exists a subsequence $\{w(l^m)\}_{m=1}^{\infty}$ converging to some point $w^* \in [0, 1]$.

Case 1: $w^* > 0$. Take any $w \in (0, w^*)$. Since $w(l^m) \to w^*$, $w < w(l^m)$ for all sufficiently large m. Since $w < w^* \leq \sup B(\bar{l})$, $(\bar{l}, w) \in B$. Hence, there exist μ, η such that $\{\mu\} \succ \{\mu, \eta\}$, $\bar{l} = v(\eta)$, and $w = v(\eta) - v(\mu)$. Let $\alpha^m \equiv \frac{l^m}{\bar{l}} \in (0, 1)$. Let $\mu^m \equiv \mu \alpha^m \nu^-$ and $\eta^m \equiv \eta \alpha^m \nu^-$, where ν^- is a minimal lottery with respect to v. Then, $l^m = v(\eta^m)$ and $w^m \equiv v(\eta^m) - v(\mu^m) = \alpha^m w < w^* < w(l^m)$. That is, $(l^m, w^m) \in A$. Moreover, by Temptation Independence, $\{\mu^m\} \succ \{\mu^m, \eta^m\}$. By Set Betweenness, $\{\mu^m\} \succ \{\mu^m, \eta^m\} \succsim \{\eta^m\}$ for all m.

Consider the following two sub-cases:

(a) For infinitely many m, $\{\mu^m\} \succ \{\mu^m, \eta^m\} \succ \{\eta^m\}$. Then, for all such m, $\psi(l^m)w^m = u(\mu^m) - W(\{\mu^m, \eta^m\})$. By passing through a corresponding subsequence, $\psi(\bar{l})w = u(\mu) - W(\{\mu, \eta\})$. On the other hand, by assumption, $\psi(\bar{l})w < u(\mu) - u(\eta)$. Hence, we have $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$, which contradicts $\bar{l} \notin L_A$.

(b) For all sufficiently large m, $\{\mu^m\} \succ \{\mu^m, \eta^m\} \sim \{\eta^m\}$. Since $(l^m, w^m) \in A$, by Lemma 22, we have $\psi(l^m)w^m \geq u(\mu^m) - u(\eta^m)$. Hence, $\psi(\bar{l})w \geq u(\mu) - u(\eta)$ as $m \to \infty$. Moreover, by Lemma 27 (i), $u(\mu) - u(\eta) = F(\bar{l}, w)$. Hence, $\psi(\bar{l})w \geq F(\bar{l}, w)$, which is a contradiction.

Case 2: $w^* = 0$. Take any $w \in B(\bar{l})$. Since $w > 0$ and $w(l^m) \to 0$, $w(l^m) < w$ for all sufficiently large m. Let μ^m, η^m be the sequence constructed as in Case 1. Since $w^m \equiv v(\eta^m) - v(\mu^m) \to w$ and $w(l^m) \to 0$, we have $w(l^m) < w^m$ for all sufficiently large m. Since $F(l^m, \cdot)$ is concave, by Lemma 25 (ii), $\psi(l^m)w^m \geq F(l^m, w^m)$. Thus, $\psi(\bar{l})w \geq F(\bar{l}, w)$ as $m \to \infty$. This is a contradiction. ■

If $\bar{l} \equiv \sup L_A = 1$, $\psi : [0,1] \to \mathbb{R}_+$ has been already defined. If $\bar{l} < 1$, we must have $F(\bar{l}, 0) = 0$ because of Lemma 26 (ii). Moreover, in this case, by Lemma 26 (iii), $F(l, 0) = 0$ for all $l \in (0,1)$. Moreover, by Lemma 27 (ii), $F(l, \cdot) : \overline{B(l)} \to \mathbb{R}_+$ is a linear function. Let $f(l) > 0$ be its slope, that is, $F(l, w) = f(l)w$. Since F is concave, so is $f : (0,1) \to \mathbb{R}_{++}$. By Theorem 10.3 [33, p.85], f admits a unique continuous concave extension to $[0,1]$. Abusing notation, denote this extension by f. By Lemma 28, $f(\bar{l})w = F(\bar{l}, w) \leq \psi(\bar{l})w$, that is, $\psi(\bar{l}) \geq f(\bar{l})$. Take any continuous increasing function $g : [\bar{l}, 1] \to \mathbb{R}_+$ such that $g(\bar{l}) = \psi(\bar{l})$ and $g(l) \geq f(l)$ for all $l \in [\bar{l}, 1]$. Now define $\overline{\psi} : [0,1] \to \mathbb{R}_+$ by

$$\overline{\psi}(l) \equiv \begin{cases} \psi(l) & \text{if } l \in [0, \bar{l}], \\ g(l) & \text{if } l \in (\bar{l}, 1]. \end{cases}$$

By construction, $\overline{\psi}$ is a continuous increasing extension of ψ to $[0,1]$.

B.4 Establishing the Representation

Lemma 29 *If $\{\mu\} \succ \{\mu, \eta\} \sim \{\eta\}$, then $\overline{\psi}(v(\eta))(v(\eta) - v(\mu)) \geq u(\mu) - u(\eta)$.*

Proof. Denote $l = v(\eta)$ and $w = v(\eta) - v(\mu)$. By assumption, $l \in L_B$. Let $\bar{l} \equiv \sup L_A$.

Case 0: If $l \in L_A$ and $w \in A(l)$, by Lemma 22, $\overline{\psi}(l)w = \psi(l)w \geq u(\mu) - u(\eta)$ as desired.

Case 1: $l \in (0, \bar{l})$. We have $l \in L_A$. If $\overline{w}_l \in A(l)$, we have $\overline{w}_l = l$, and, hence, $w \in A(l) = (0, l]$. Hence, Case 0 can be applied. If $\overline{w}_l \notin A(l)$, by Lemma 25 (ii), we have $\overline{\psi}(l)w = \psi(l)w \geq u(\mu) - u(\eta)$.

Case 2: $l = \bar{l}$. If $\bar{l} \in L_A$, L_A is not an open interval. Thus we must have $\bar{l} = 1$. That is, $L_A = (0, 1]$. The same argument as in Case 1 can be applied. If $\bar{l} \notin L_A$, by Lemma 28, we have $\overline{\psi}(l)w = \psi(l)w \geq u(\mu) - u(\eta)$.

Case 3: $l \in (\bar{l}, 1]$. By construction, we have $\overline{\psi}(l)w \geq f(l)w = F(l, w) = u(\mu) - u(\eta)$ as desired. ■

Lemma 30 *For all μ, η,*

$$W(\{\mu,\eta\}) = \max_{\nu\in\{\mu,\eta\}} \left\{ u(\nu) - \overline{\psi}\Big(\max_{\{\mu,\eta\}} v\Big) \left(\max_{\{\mu,\eta\}} v - v(\nu)\right) \right\}.$$

Proof. Without loss of generality, assume $\{\mu\} \succsim \{\eta\}$. By Set Betweenness, $\{\mu\} \succsim \{\mu,\eta\} \succsim \{\eta\}$. There are four cases:

Case (i): $\{\mu\} \succ \{\mu,\eta\} \succ \{\eta\}$. In this case, $v(\eta) > v(\mu)$. By definition of φ and Lemma 20,

$$\begin{aligned} W(\{\mu,\eta\}) &= u(\mu) - \varphi(v(\eta), v(\eta) - v(\mu)) \\ &= u(\mu) - \overline{\psi}(v(\eta))(v(\eta) - v(\mu)) > u(\eta). \end{aligned}$$

Thus $W(\{\mu,\eta\})$ can be expressed as the desired form.

Case (ii): $\{\mu\} \succ \{\mu,\eta\} \sim \{\eta\}$. We have $v(\eta) > v(\mu)$. By Lemma 29,

$$W(\{\mu,\eta\}) = u(\eta) \geq u(\mu) - \overline{\psi}(v(\eta))(v(\eta) - v(\mu)),$$

as desired.

Case (iii): $\{\mu\} \sim \{\mu,\eta\} \succ \{\eta\}$. By construction of v, $v(\mu) \geq v(\eta)$. Since $\overline{\psi}(v(\mu))(v(\mu) - v(\eta)) \geq 0$, we have

$$W(\{\mu,\eta\}) = u(\mu) > u(\eta) - \overline{\psi}(v(\mu))(v(\mu) - v(\eta)).$$

Thus, $W(\{\mu,\eta\})$ is represented by the desired form.

Case (iv): $\{\mu\} \sim \{\mu,\eta\} \sim \{\eta\}$. If $v(\eta) \geq v(\mu)$, $W(\{\mu,\eta\}) = u(\eta) \geq u(\mu) - \overline{\psi}(v(\eta))(v(\eta) - v(\mu))$. If $v(\mu) \geq v(\eta)$, we have $W(\{\mu,\eta\}) = u(\mu) \geq u(\eta) - \overline{\psi}(v(\mu))(v(\mu) - v(\eta))$. In either case, $W(\{\mu,\eta\})$ is represented by the desired form. ■

By using all binary menus $\{\mu,\eta\}$, define $c : \Delta \times v(\Delta) \to \mathbb{R}_+$ by

$$c\Big(\mu, \max_{\{\mu,\eta\}} v\Big) \equiv \overline{\psi}\Big(\max_{\{\mu,\eta\}} v\Big) \left(\max_{\{\mu,\eta\}} v - v(\mu)\right).$$

Since $\overline{\psi}$ is continuous and weakly increasing, $c(\mu, \cdot)$ is weakly increasing for all μ and c is continuous. Since $\succsim$ satisfies Set Betweenness, by Lemma 5 of Noor and Takeoka [32], the representation can be extended to the set of all finite menus. That is, for all finite menus $x \in Z$,

$$\begin{aligned} W(x) &= \max_{\nu\in x} \left\{ u(\nu) - c\Big(\nu, \max_{x} v\Big) \right\} \\ &= \max_{\nu\in x} \left\{ u(\nu) - \overline{\psi}\Big(\max_{x} v\Big) \left(\max_{x} v - v(\nu)\right) \right\}. \end{aligned}$$

Finally, by Continuity and Lemma 6 of Noor and Takeoka [32], the representation can be extended to Z as desired.

C Appendix: Proof of Corollary 1

Define $u, v : \Delta \to [0,1]$ and $\psi : L_A \to \mathbb{R}_{++}$ as in the proof of Theorem 3. By the same argument as in Lemma 21 (ii) together with assuming $\{\mu, \eta\}\alpha\{\nu\} \sim \{e_{\mu\eta}\}\alpha\{\nu\}$, we can show the next lemma. A proof is omitted.

Lemma 31 *$\psi : L_A \to \mathbb{R}_{++}$ is constant, that is, for all $l, l' \in L_A$, $\psi(l) = \psi(l')$.*

Recall the function $F : B \to \mathbb{R}_+$, defined as in the proof of Theorem 3. Let $\bar{l} \equiv \sup L_A$. By abusing notation, let ψ be a unique constant extension of ψ to $[0, \bar{l}]$. Define $\overline{\psi} : [0,1] \to \mathbb{R}_{++}$ as a unique extension of ψ that is constant on $[0,1]$. If $\bar{l} = 1$, $\overline{\psi} = \psi : [0,1] \to \mathbb{R}_{++}$. If $\bar{l} < 1$, we must have $F(\bar{l}, 0) = 0$ because of Lemma 26 (ii). Moreover, in this case, by Lemma 26 (iii), $F(l, 0) = 0$ for all $l \in (0,1)$. Moreover, by Lemma 27 (ii), $F(l, \cdot) : \overline{B(l)} \to \mathbb{R}_+$ is a linear function. Let $f(l) > 0$ be its slope, that is, $F(l, w) = f(l)w$. Since F is concave, so is $f : (0,1) \to \mathbb{R}_{++}$. By Theorem 10.3 [33, p.85], f admits a unique continuous concave extension to $[0,1]$. Abusing notation, denote this extension by f. By Lemma 28, $f(\bar{l})w = F(\bar{l}, w) \le \psi w$, that is, $\psi \ge f(\bar{l})$. By Lemma 25 (i), $f(l) > \psi$ for all $l < \bar{l}$. Since f is concave, we must have $\overline{\psi} \ge f(l)$ for all $l \in (\bar{l}, 1]$. That is, this constant function $\overline{\psi} : [0,1] \to \mathbb{R}_{++}$ satisfies

$$\overline{\psi} \begin{cases} = & \psi \quad \text{if } w \in [0, \bar{l}], \\ \ge & f(l) \quad \text{if } w \in (\bar{l}, 1]. \end{cases}$$

By Lemmas 29, 30 and the subsequent argument, the components $(u, v, \overline{\psi})$ is a MDSC representation of $\succsim$.

D Appendix: Proof of Theorem 4

Since u and u' represent the same commitment preference, they must be ordinally equivalent. Moreover, since both are also linear and continuous, there exist constants $\alpha_u > 0$ and β_u such that $u' = \alpha_u u + \beta_u$.

In case of temptation utility functions, Theorem 2 ensures that there exist constants $\alpha_v > 0$ and β_v such that $v' = \alpha_v v + \beta_v$.

Let $W : Z \to \mathbb{R}$ and $W' : Z \to \mathbb{R}$ be the representations associated with (u, v, ψ) and (u', v', ψ'), respectively. Now we show that following lemma:

Lemma 32 *For all x, $W'(x) = \alpha_u W(x) + \beta_u$.*

Proof. As shown in Lemma 8 (ii), there exists a unique function $\alpha : Z \to [0,1]$ such that $x \sim \{\mu^\Delta \alpha(x) \mu_\Delta\}$ for all x,where μ^Δ and μ_Δ are respectively the best and worst lotteries in Δ according to commitment preference. Thus, for all x,

$$W'(x) = u'(\mu^\Delta \alpha(x) \mu_\Delta) = \alpha_u u(\mu^\Delta \alpha(x) \mu_\Delta) + \beta_u = \alpha_u W(x) + \beta_u,$$

as was to be shown. ■

Now take any $l \in L$. By definition, there exists μ, η such that $v(\eta) = l$ and $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$. By Lemma 32 and the representations,

$$\begin{aligned}
& W'(\{\mu,\eta\}) = \alpha_u W(\{\mu,\eta\}) + \beta_u \\
\Rightarrow\ & u'(\mu) - \psi'(v'(\eta))(v'(\eta) - v'(\mu)) = \alpha_u(u(\mu) - \psi(v(\eta))(v(\eta) - v(\mu))) + \beta_u \\
\Rightarrow\ & \alpha_u u(\mu) + \beta_u - \alpha_v \psi'(\alpha_v l + \beta_v)(v(\eta) - v(\mu)) = \alpha_u(u(\mu) - \psi(v(\eta))(v(\eta) - v(\mu))) + \beta_u.
\end{aligned}$$

Thus, we have $\psi'(\alpha_v l + \beta_v) = \frac{\alpha_u}{\alpha_v}\psi(l)$ as desired.

E Appendix: Proof of Theorem 5

E.1 Sufficiency

By hypothesis, $\succsim$ admits a MDSC representation (u, v, ψ). We show that $\mathcal{C}$ must have the desired representation. To ease notation, let $w_x(\mu) := u(\mu) - c(\mu, \max_x v) := u(\mu) - \psi(\max_x v)(\max_x v - v(\mu))$.

Lemma 33 *If $\mu \in \arg\max_{\{\mu,\eta\}} w_{\{\mu,\eta\}}$ and $\eta \in \arg\max_{\{\mu,\eta\}} v$ then $\mu \in \mathcal{C}(\{\mu,\eta\})$.*

Proof. First we show that $\{\eta\} \succ \{\mu\}$ is not possible. Consider two cases: If $\{\eta\} \succsim \{\mu,\eta\} \succ \{\mu\}$, then the representation implies $u(\eta) - c(\eta, v(\eta)) > u(\mu) \geq u(\mu) - c(\mu, v(\eta))$, contradicting the definition of μ. If $\{\eta\} \succ \{\eta,\mu\} \sim \{\mu\}$, but then the representation implies $\eta \notin \arg\max_{\{\mu,\eta\}} v$, contradicting the definition of η. Conclude that $\{\mu\} \succsim \{\eta\}$ must hold.

Assume $\{\mu\} \succ \{\eta\}$. First of all, if $v(\eta) = v(\mu)$, the representation implies $\{\mu\} \sim \{\mu,\eta\} \succ \{\eta\}$, and hence, by Sophistication, $\mu \in \mathcal{C}(\{\mu,\eta\})$, as desired. From now on, assume $v(\eta) > v(\mu)$.

By assumption, $u(\mu) + \psi(v(\eta))v(\mu) \geq u(\eta) + \psi(v(\eta))v(\eta)$. If $u(\mu) + \psi(v(\eta))v(\mu) > u(\eta) + \psi(v(\eta))v(\eta)$, the representation implies $\{\mu,\eta\} \succ \{\eta\}$, and hence, by Sophistication, $\mu \in \mathcal{C}(\{\mu,\eta\})$.

Now consider the case that $u(\mu) + \psi(v(\eta))v(\mu) = u(\eta) + \psi(v(\eta))v(\eta)$. Let $\nu^* \in \arg\max_\Delta(u + \psi(v(\eta))v)$ and $\nu_* \in \min_\Delta(u + \psi(v(\eta))v)$. If $\mu \notin \arg\max_\Delta(u + \psi(v(\eta))v)$, for all $\alpha \in (0,1)$, $u(\mu\alpha\nu^*) + \psi(v(\eta))v(\mu\alpha\nu^*) > u(\eta) + \psi(v(\eta))v(\eta)$. Moreover, for all α sufficiently close to one, $v(\eta) > v(\mu\alpha\nu^*)$. Thus, by the representation, $\{\mu\alpha\nu^*, \eta\} \succ \{\eta\}$ for all such α. Sophistication in turn yields $\mu\alpha\nu^* \in \mathcal{C}(\{\mu\alpha\nu^*, \eta\})$. By upper hemicontinuity of $\mathcal{C}$, $\mu \in \mathcal{C}(\{\mu,\eta\})$ as $\alpha \to 1$.

If $\mu \in \arg\max_\Delta(u + \psi(v(\eta))v)$, let $\nu_\beta \equiv \mu\beta\nu_*$ for $\beta \in (0,1)$. Since $u(\mu) + \psi(v(\eta))v(\mu) = u(\eta) + \psi(v(\eta))v(\eta)$, we have $u(\eta) + \psi(v(\eta))v(\eta) > u(\nu_\beta) + \psi(v(\eta))v(\nu_\beta)$ for all β. Moreover, since $v(\eta) > v(\mu)$, for all β sufficiently close to one, $v(\eta) > v(\nu_\beta)$. Fix such a β arbitrarily. Then, for all $\alpha \in (0,1)$, $u(\mu) + \psi(v(\eta))v(\mu) > u(\eta\alpha\nu_\beta) + \psi(v(\eta))v(\eta\alpha\nu_\beta)$. Moreover, for all α sufficiently close to one, $v(\eta\alpha\nu_\beta) > v(\mu)$. Since ψ is non-decreasing,

$$\begin{aligned}
& u(\mu) - \psi(v(\eta\alpha\nu_\beta))(v(\eta\alpha\nu_\beta) - v(\mu)) \\
\geq\ & u(\mu) - \psi(v(\eta))(v(\eta\alpha\nu_\beta) - v(\mu)) \\
>\ & u(\eta\alpha\nu_\beta),
\end{aligned}$$

and thus, the representation implies $\{\mu, \eta\alpha\nu_\beta\} \succ \{\eta\alpha\nu_\beta\}$, and Sophistication in turn yields $\mu \in \mathcal{C}(\{\mu, \eta\alpha\nu_\beta\})$. By upper hemicontinuity of $\mathcal{C}$, $\mu \in \mathcal{C}(\{\mu, \eta\})$ as $\alpha \to 1$.

If $\{\mu\} \sim \{\eta\}$, then together with $v(\eta) \geq v(\mu)$ it must be that $u(\eta) \geq u(\mu) - c(\mu, v(\eta))$. However, by definition of μ the reverse inequality also holds, and thus $u(\eta) = u(\mu) - c(\mu, v(\eta))$. This in turn implies that $c(\mu, v(\eta)) = 0$ and so $v(\eta) = v(\mu)$. Consider the lotteries μ', η' s.t. $\{\mu'\} \sim \{\mu', \eta'\} \succ \{\eta'\}$ guaranteed by nondegeneracy*. Then $\{\mu\alpha\mu'\} \sim \{\mu\alpha\mu', \eta\alpha\eta'\} \succ \{\eta\alpha\eta'\}$ for all α, and by Sophistication, $\mathcal{C}(\{\mu\alpha\mu', \eta\alpha\eta'\}) = \{\mu\alpha\mu'\}$ for all α. By upper hemicontinuity of $\mathcal{C}$, $\mu \in \mathcal{C}(\{\mu, \eta\})$, as desired. ■

Lemma 34 *Result.*

Proof. Take any nonsingleton menu x.

Step 1: Show $\arg\max_x w_x \subset \mathcal{C}(x)$.

Let $\mu \in \arg\max_x w_x$. Take any $\nu \in \mathcal{C}(x)$ and $\eta \in \arg\max_x v$. By the previous lemma, $\mu \in \mathcal{C}(\{\mu, \eta\})$. Given that $w_{\{\mu,\nu,\eta\}}(\nu) = w_x(\nu) \leq w_x(\mu) = w_{\{\mu,\nu,\eta\}}(\mu)$ and $v(\eta) \geq v(\nu)$, the representation implies

$$\{\mu, \eta\} \sim \{\mu, \nu, \eta\} \sim x.$$

Observe that $\{\mu, \eta\}$ temptation-dominates ν: Since $\eta \in \arg\max_x v$, $\{\eta\} \succ \{\nu, \eta\}$ is ruled out. If $\{\nu, \eta\} \succ \{\nu\}$, then we have $\{\eta\} \sim \{\nu, \eta\} \succ \{\nu\}$ and so η weakly tempts ν. The remaining case is where $\{\nu\} \succsim \{\nu, \eta\} \succsim \{\eta\}$. Here for all $\alpha \in (0, 1)$,

$$\{\nu\alpha\mu^*\} \succ \{\nu\alpha\mu^*, \eta\alpha\eta^*\}.$$

In particular, η is temptation-ranked weakly higher than ν. Thus $\{\mu, \eta\}$ temptation-dominates ν.

Thus, by Sophistication, $\{\mu, \nu, \eta\} \sim \{\mu, \eta\}$ implies $\mathcal{C}(\{\mu, \nu, \eta\}) \neq \{\nu\}$. If $\mu \in \mathcal{C}(\{\mu, \nu, \eta\})$, then given that $\{\mu, \nu, \eta\} \sim x$ and $\nu \in \mathcal{C}(x)$, Weak WARP implies that $\mu \in \mathcal{C}(x)$, as desired. If on the other hand $\eta \in \mathcal{C}(\{\mu, \nu, \eta\})$, then a similar argument yields $\eta \in \mathcal{C}(x)$. However, given we established at the start of the proof that $\mu \in \mathcal{C}(\{\mu, \eta\})$, Weak WARP therefore implies $\mu \in \mathcal{C}(x)$, as desired.

Step 2: Show $\mathcal{C}(x) \subset \arg\max_x w_x$.

Let $\nu \in \mathcal{C}(x)$. Take any $\mu \in \arg\max_x w_x$ and $\eta \in \arg\max_x v$. Then by the representation

$$x \sim \{\mu, \eta\} \sim \{\mu, \nu, \eta\}.$$

Since clearly $\mu \in \arg\max_{\{\mu,\nu,\eta\}} w_{\{\mu,\nu,\eta\}}$, Step 1 implies that $\mu \in \mathcal{C}(\{\mu, \nu, \eta\})$. By Weak WARP, $\nu \in \mathcal{C}(\{\mu, \nu, \eta\})$ also holds. Therefore $\mathcal{C}(\{\mu, \nu, \eta\}) \neq \{\mu\}$, and so Sophistication implies $\{\nu, \eta\} \succsim \{\mu, \nu, \eta\}$. Also, since $\mu \in \arg\max_{\{\mu,\nu,\eta\}} w_{\{\mu,\nu,\eta\}}$, the representation implies $\{\mu, \nu, \eta\} \succsim \{\nu, \eta\}$. Therefore, we have determined that

$$\{\mu, \nu, \eta\} \sim \{\nu, \eta\},$$

and $\mu, \nu \in \mathcal{C}(\{\mu, \nu, \eta\})$. Note that by transitivity, $x \sim \{\nu, \eta\}$, and so Weak WARP also implies $\nu \in \mathcal{C}(\{\nu, \eta\})$.

Next we show that $W(\{\nu,\eta\}) = u(\nu) - c(\nu, v(\eta))$. Consider four cases:

1. $\{\eta\} \succ \{\nu\}$: Then $v(\eta) \geq v(\nu)$ and the representation implies $\{\eta\} \sim \{\eta,\nu\} \succ \{\nu\}$, which by Sophistication implies $\nu \notin \mathcal{C}(\{\nu,\eta\})$, a contradiction.

2. $\{\nu\} \succsim \{\nu,\eta\} \succ \{\eta\}$: Then the representation implies $W(\{\nu,\eta\}) = u(\nu) - c(\nu, v(\eta))$.

3. $\{\nu\} \succ \{\nu,\eta\} \sim \{\eta\}$: Then by Sophistication, $\eta \in \mathcal{C}(\{\nu,\eta\})$, and we also know from before that $\nu \in \mathcal{C}(\{\nu,\eta\})$. Therefore we have $\{\nu\} \succ \{\nu,\eta\}$ and $\mathcal{C}(\{\nu,\eta\}) = \{\nu,\eta\}$. By ex post Decreasing Self-Control, $\mathcal{C}(\{\nu,\nu\alpha\eta\}) = \{\nu\}$ for all $\alpha \in (0,1)$, which by Sophistication implies $\{\nu,\nu\alpha\eta\} \succ \{\nu\alpha\eta\}$. By the representation, we therefore see that $W(\{\nu,\nu\alpha\eta\}) = u(\nu) - c(\nu, v(\nu\alpha\eta))$ for all α, and thus by continuity of the representation, we see that $W(\{\nu,\eta\}) = u(\nu) - c(\nu, v(\eta))$.

4. $\{\nu\} \sim \{\nu,\eta\} \sim \{\eta\}$: Then by nondegeneracy* there are μ^*,η^* s.t. $\{\nu\alpha\mu^*\} \succ \{\nu\alpha\mu^*,\eta\alpha\eta^*\} \succsim \{\eta\alpha\eta^*\}$ for all α. As in the previous cases, it follows that $W(\{\nu\alpha\mu^*,\eta\alpha\eta^*\}) = u(\nu\alpha\mu^*) - c(\nu\alpha\mu^*, v(\eta\alpha\eta^*))$ for all α and thus by continuity of the representation, $W(\{\nu,\eta\}) = u(\nu) - c(\nu, v(\eta))$.

This establishes that $W(\{\nu,\eta\}) = u(\nu) - c(\nu, v(\eta))$. To conclude the proof, observe that by the representation, $\{\mu,\nu,\eta\} \sim \{\nu,\eta\}$ implies

$$u(\mu) - c(\mu, v(\eta)) = u(\nu) - c(\nu, v(\eta)).$$

Since $\mu \in \arg\max_x w_x$, it follows that $\nu \in \arg\max_x w_x$, as desired. ■

E.2 Necessity

Lemma 35 *Weak WARP is necessary.*

Proof. Suppose $\mu,\eta \in x \cap y$, $x \sim y$, $\mu \in \mathcal{C}(x)$ and $\eta \in \mathcal{C}(y)$ but $\mu \notin \mathcal{C}(y)$. Then

$$u(\mu) - c(\mu, \max_x v) = W(x) = W(y) = u(\eta) - c(\eta, \max_y v) > u(\mu) - c(\mu, \max_y v),$$

and in particular, $c(\mu, \max_x v) < c(\mu, \max_y v)$ and $\max_x v < \max_y v$. Then

$$u(\eta) - c(\eta, \max_x v) > u(\eta) - c(\eta, \max_y v) = u(\mu) - c(\mu, \max_x v),$$

where the equality is observed earlier. But then it must be that $\mu \notin \mathcal{C}(x)$, a contradiction. ■

Lemma 36 *ex post Decreasing Self-Control is necessary.*

Proof. Suppose $\{\mu\} \succ \{\mu,\eta\}$. Since $\mu \in \mathcal{C}(\{\mu,\eta\})$, $u(\mu) - \psi(v(\eta))(v(\eta) - v(\mu)) \geq u(\eta)$. Mixing both sides of the equality with $u(\mu)$ yields

$u(\eta\alpha\mu)$
$\leq \alpha\,[u(\mu) - \psi(v(\eta))(v(\eta) - v(\mu))] + (1-\alpha)u(\mu)$
$= u(\mu) - \alpha\psi(v(\eta))(v(\eta) - v(\mu))$
$= u(\mu) - \psi(v(\eta))(v(\eta\alpha\mu) - v(\mu))$
$< u(\mu) - \psi(v(\eta\alpha\mu))(v(\eta\alpha\mu) - v(\mu))$. Thus $\mathcal{C}(\{\mu,\mu\alpha\eta\}) = \{\mu\}$ for all $\alpha \in (0,1)$. ■

Lemma 37 *Sophistication is necessary.*

Proof. For any $\nu \in x$,

$$W(x) \geq u(\nu) - c(\nu, \max_x v) \geq u(\nu) - c(\nu, \max_{x \cup \{\mu\}} v).$$

Therefore, $W(x \cup \{\mu\}) > W(x) \geq u(\nu) - c(\nu, \max_{x \cup \{\mu\}} v)$ for all $\nu \in x$, from which it follows that the choice from $x \cup \{\mu\}$ cannot be in x. Thus, $\mathcal{C}(x \cup \{\mu\}) = \{\mu\}$. For the converse, suppose $\mathcal{C}(x \cup \{\mu\}) = \{\mu\}$. Since $v(\mu) \leq v(\eta)$, we have that for all $\nu \in x$

$$W(x \cup \{\mu\}) = u(\mu) - c(\mu, \max_{x \cup \{\mu\}} v) > u(\nu) - c(\nu, \max_{x \cup \{\mu\}} v) = u(\nu) - c(\nu, \max_x v),$$

and in particular, $W(x \cup \{\mu\}) > W(x)$, as desired. ■

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