Intertemporal Choice and the Magnitude Effect*

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Abstract

A robust finding in experiments on time preference is the magnitude effect: subjects tend to be more patient towards larger rewards. Using a calibration theorem, we argue against standard curvature-based explanations for the finding. We axiomatize a model of preferences over dated rewards that generalizes the standard exponential discounting model by permitting the discount factor to depend on the reward being discounted. The model is shown to behaviorally subsume the hyperbolic discounting model as a special case. When embedded in a sequential bargaining game the model gives rise to multiple stationary subgame perfect equilibria. There may exist equilibria in which the first mover gets a smaller share despite also being the more patient player.

Keywords: Discounting, magnitude effect, hyperbolic discounting, preference reversals, calibration theorem, bargaining.

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1. Introduction

Most existing models of intertemporal choice hypothesize that agents evaluate time and money separately when making intertemporal trade-offs. Such models posit that preferences $\succsim$ over *dated rewards* (alternatives of the form (m, t) that specify a reward $\$m$ to be received at time t) admit a representation of the form:

$$U(m, t) = D(t)u(m),$$

where time t is evaluated according to the discount function $D(t)$ and money m according to the instantaneous utility index $u(m)$. We refer to this model as the *Separable Discounted Utility* (SDU) model. Exponential discounting is the special case where $D(t) = \delta^t$ for a discount factor $\delta \in (0, 1)$. Hyperbolic discounting (Ainslie [1], Loewenstein and Prelec [6]) is the special case where $D(t) = \frac{1}{1+t}$.

Motivation. A highly robust finding in the experimental literature is that estimated discount functions exhibit a *magnitude effect*: subjects appear to exhibit greater patience toward larger rewards. For instance, Thaler [10] finds that subjects were on average indifferent between \$15 now and \$60 in a year, \$250 now and \$350 in a year, and \$3000 now and \$4000 in a year, suggesting a (yearly) discount factor of 0.25, 0.71 and 0.75 respectively (see [4] for a review of the evidence). Observe that the magnitude effect is a property of how money (as opposed to utility) is discounted. Researchers since Loewenstein and Prelec [6] have argued that it may simply be a reflection of the curvature of utility for money.[1] Indeed, Loewenstein and Prelec [6] explain the magnitude effect in terms of a u that satisfies a 'subproportionality' property.

However, observed money-discount functions exhibit very extreme magnitude effects, giving rise to extreme money-discounting of small m, such as the fact that a \$60 reward is discounted by a factor of 0.25 in the evidence cited above. This suggests that the curvature of u required to accommodate the evidence may be extreme. In Appendix A we present a simple calibration theorem that expresses

[1] To see this formally, suppose an agent reveals that $\$\psi(m, t)$ received now is as good as receiving (m, t). Under the SDU model, $\psi(m, t)$ satisfies $u(\psi(m, t)) = D(t)u(m)$, and consequently the money-discount function is given by

$$\phi(m, t) := \frac{\psi(m, t)}{m} = \frac{u^{-1}(D(t)u(m))}{m}.$$

Clearly, if u is not linear, ϕ will generically be magnitude-dependent.

that this is indeed the case.[2] To illustrate, suppose that subjects in experiments respect the SDU model with concave u extended to consumption streams. Then the above cited evidence implies the following: if a subject's next-year wealth is increased by \$60, then the agent will reject an opportunity for investing *any* amount $\$x$ today that pays $\$4x$ next year. Thus, for instance, the agents in the noted evidence would not pay \$3000 now for $\$12,000$ in a year.[3] Such behavior is arguably implausible, which leads us to consider a model of the magnitude effect that does not explain it in terms of curvature of utility.

This Paper. There are several reasons why it may be intuitive that subjects are inherently more patient toward larger rewards. If discounting arises due to diminished visibility of the future (Pigou [8]), agents may find it optimal to exert greater effort thinking about and paying attention to larger future rewards. Alternatively, impatience may be a temptation, and an agent may be more inclined to exert self-control when dealing with larger stakes. Interpreting the magnitude effect as reflecting a property of underlying discount functions, we introduce and study the *Magnitude Effect Discounting (MED) model*:

$$U(m,t) = \delta(m)^t \cdot u(m),$$

where the discount factor $\delta(m) \in (0,1)$ is an increasing function of rewards. Thus, the more desirable a reward, the more patient is the agent towards it. The MED model is a nonseparable generalization of the standard exponential discounting model that allows for the discount factor to depend on the magnitude of the reward.

We explore how the behavioral properties of the MED model relate to those of hyperbolic discounting. Hyperbolic discounting relaxes exponential discounting while maintaining magnitude-independent discounting. We find that any property of preferences $\succsim$ that is explicable by this is also explicable by relaxing magnitude-independent impatience while maintaining exponential discounting. Formally, the MED model behaviorally subsumes the hyperbolic discounting model. To illustrate, consider the behavioral pattern known as *preference reversals*, which express

[2] We are grateful to Yusufcan Masatlioglu and a referee for suggesting that we prove a calibration theorem. The result we present in the Appendix holds for whatever given time path of wealth levels the agent anticipates, and in particular does not rely on assumptions about choices at different wealth levels as in Rabin [9].

[3] Note that the agent rejects investing at 400% rate of return although (by the magnitude effect) he exhibits substantial patience toward a \$12,000 reward: while his observed money-discount factor for \$4000 is 0.75 it would be even higher for \$12,000.

lower impatience in more distant trade-offs (see Section 3). Such behavior has received much attention in economics and is commonly attributed to hyperbolic discounting – indeed it seems unrelated to the magnitude effect. Yet such behavior is characteristic of the MED model. Intuitively, while an agent may prefer a small immediate reward to a larger later reward simply due to impatience, pushing both rewards into the future will make the larger later reward more relatively attractive since it will be discounted at a lower rate than the smaller earlier reward. Thus with sufficient delay, the larger later reward will be preferred, giving rise to a preference reversal and causing the agent to appear more impatient in closer trade-offs and less impatient in distant ones.

We also consider an application of the MED model that embeds it in a sequential bargaining game. Under the SDU model, there is a unique stationary subgame perfect equilibrium and a first mover advantage. In contrast, the MED model gives rise to multiple stationary subgame perfect equilibria, and there may exist equilibria in which the first mover gets a smaller share despite also being the more patient player. The intuition is that the effective degree of impatience of a player is now dependent on what offer he expects to receive from the other player in the next period. Therefore, even if one player is more patient than the other for any given future reward, expectations of low offers may make him effectively more impatient than the other.

Related Literature. Various theoretical studies have generalized the standard exponential discounting model in different ways in order to accommodate the experimental findings on intertemporal choice. Loewenstein and Prelec [6] and Harvey [5] axiomatize the hyperbolic discounting model to accommodate preference reversals and dynamic inconsistency. Ok and Masatlioglu [7] generalize the SDU model to accommodate evidence of intransitivity. A common feature of these studies is that they maintain the separability between the evaluation of time and money. Motivated by the evidence of the magnitude effect, this paper departs precisely from this feature. There are few papers in the theoretical literature that study the magnitude effect. As noted earlier, Loewenstein and Prelec [6] explain the magnitude effect in terms of the curvature of utility. Recently, Baucells and Heukamp [2] consider a model of preferences over 'probabilistic dated rewards' that encodes time is the form of probability. An important aspect of the model is the fact that the rate at which time is exchanged with probability can depend on the outcome – this feature can accommodate greater patience toward larger rewards. With the probability dimension shut down, their model has a more general structure than ours: $w(\delta(m)^t)u(m)$, where w is an increasing continuous function.

Beside this, there is no overlap between the set of formal results presented in the papers. Their paper seeks to enrich expected discounted utility in order to capture experimental findings on time and risk, while ours is focused on the implications of the magnitude effect for intertemporal choice behavior.

The remainder of the paper is organized as follows. Section 2 studies the foundations of the MED model. Section 3 demonstrates the intimate connection between the MED model and preference reversals, and establishes the relationship with the hyperbolic discounting model. Section 4 considers an application to sequential bargaining theory. Section 5 offers concluding remarks. All proofs are relegated to appendices.

2. The Magnitude Effect Discounting Model

In this section we present the axiomatic underpinnings of the MED model. Time is continuous and given by $\mathcal{T} = \mathbb{R}_+$, with generic elements t, t'. The set of all possible rewards is $\mathcal{M} = \mathbb{R}_+$, with generic elements m, m', s, l. Both $\mathcal{T}$ and $\mathcal{M}$ are endowed with the usual topology. The set of *dated rewards* is $X = \mathcal{M} \times \mathcal{T}$, and is endowed with the product topology. The primitive is a preference relation $\succsim$ over X. Such preference data forms the basis for the majority of experiments on time preference.

2.1. Axioms and Definitions

We impose standard regularity properties on $\succsim$.

Definition 2.1 (Regularity). *A preference $\succsim$ over X is regular if it satisfies:*

*1- **Order**: $\succsim$ is complete and transitive.*

*2- **Continuity**: For each (m,t), the sets $\{(m',t') : (m',t') \succsim (m,t)\}$ and $\{(m',t') : (m,t) \succsim (m',t')\}$ are closed.*

*3- **Impatience**:*

(i) For all $m > 0$ and $t < t'$, $(0,t) \sim (0,t')$ and $(m,t) \succ (m,t')$.

(ii) For each m, m' such that $m' > m > 0$, there is t such that $(m,0) \succ (m',t)$.

*4- **Monotonicity**: For all t, if $s < l$ then $(l,t) \succ (s,t)$.*

Order and Continuity are standard. Impatience states that earlier receipt of a $m > 0$ reward is always better, the receipt of the 0 reward is a matter of indifference, and moreover any reward can be made arbitrarily unattractive with sufficient delay. Monotonicity states that more is better at any t.

The axiomatization of the exponential discounting model by Fishburn and Rubinstein [3, Thm 3] imposes a *Stationarity axiom*:[4] for all s, l, τ, t,

$$(s, 0) \sim (l, \tau) \Longrightarrow (s, t) \sim (l, t + \tau).$$

Recall that Impatience states that a dated reward loses its attractiveness with delay. According to Stationarity, if $(s, 0) \sim (l, \tau)$, then both the rewards lose attractiveness at the same rate as they are delayed by a common number of periods t. This embodies what may be referred to as *date-independent impatience*: each given reward regardless of its date of receipt loses the same total attractiveness when delayed by an additional t periods.

However, Stationarity implies a lot more than date-independent impatience. For instance, it implies the following property as well: if $(s, 0) \sim (l, \tau)$ and $(s', 0) \sim (l', \tau)$ then

$$(s, t) \sim (l, T) \Longrightarrow (s', t) \sim (l', T). \tag{2.1}$$

This property is an implication of what may be referred to as *magnitude-independent impatience*: the agent discounts both small and large rewards the same way. To illustrate, suppose the agent exhibits $(800, 0) \sim (1000, 1)$ and $(5, 0) \sim (10, 1)$, that is, a 1 week delay induces indifference in both the low-magnitude pair and high-magnitude pair of rewards. The above property requires that if both the \$800 and \$5 rewards are pushed into the future by t periods, the \$1000 and \$10 rewards must be pushed into the future by *the same number of periods* T in order to maintain indifference. Evidently, the way the agent trades-off money and time when dealing with large rewards (\$800 and \$1000) is precisely the same as when dealing with small rewards (\$5 and \$10).

We seek to weaken Stationarity in a way that retains the notion of date-independent impatience but expunges any form of magnitude-independence. The following axiom restricts the nature of the dependence of impatience on delay in the desired manner, but without placing any restriction on how impatience may depend on the rewards.

Definition 2.2 (Weak Stationarity). *A preference $\succsim$ over X exhibits Weak Stationarity if for any $0 < s \leq l$, any τ, T (that may depend on s, l) and any*

[4] To be precise, Fishburn and Rubinstein [3, Thm 3] use a slightly stronger version that states $[(s, t) \sim (l, t + \tau) \Longrightarrow (s, t') \sim (l, t' + \tau)]$. However, our Impatience axiom is slightly stronger, and under it the two version of Stationarity are equivalent.

$t, \lambda > 0$,

$$(s,0) \sim (l,\tau) \text{ and } (s,t) \sim (l, T+\tau) \Longrightarrow (s, \lambda t) \sim (l, \lambda T + \tau).$$

To illustrate, suppose that time is discrete and that an agent exhibits:

$$(80,0) \sim (100,\tau) \text{ and } (80,1) \sim (100, 3+\tau).$$

That is, while $(80,0)$ and $(100,\tau)$ initially have the same value for the agent, the total loss of attractiveness due to a 1 period delay in the immediate \$80 reward equals that of a 3 period delay in the later \$100 reward. If the rate of loss of attractiveness of any given reward is constant, then the loss due to another k period delay in the \$80 reward must equal the loss due to another $3k$ periods delay in the \$100 reward:

$$(80,k) \sim (100, 3k+\tau).$$

This is the content of Weak Stationarity.[5] It requires that the rate at which any *given* reward loses attractiveness with delay be constant.

Note that the axiom expunges the magnitude-independence property (2.1) implied by Stationarity, since it permits T to depend on the rewards s, l. Also, the axiom also weakens Stationarity by allowing $T \neq t$.

Finally, we define the Magnitude Effect Discounting representation.

Definition 2.3 (MED Representation). *A Magnitude Effect Discounting (MED) representation of a preference* $\succsim$ *is a representation* $U : X \to \mathbb{R}_+$ *such that, for all* $(m,t) \in X$,

$$U(m,t) = \delta(m)^t \cdot u(m),$$

where $u : \mathcal{M} \to \mathbb{R}_+$ *is a strictly increasing continuous utility index satisfying* $u(0) = 0$*, and* $\delta : \mathcal{M} \to (0,1)$ *is a weakly increasing magnitude-dependent discount factor that is continuous on* $\mathcal{M} \setminus \{0\}$.

2.2. Representation Results

The main result of this section is a representation theorem for regular preferences that satisfy only Weak Stationarity. The result generalizes the exponential discounting model (Fishburn and Rubinstein [3]).

[5] We will see in the next section that Weak Stationarity is also consistent with a knife-edge case of date-dependent impatience (Theorem 3.1).

Theorem 2.4. *A preference $\succsim$ over X satisfies Regularity and Weak Stationarity if and only if it admits an MED representation.*

Therefore, by relaxing Stationarity to Weak Stationarity, we get a representation with a potentially magnitude-dependent exponential discount function. A notable aspect of the representation theorem is that the discount function in the MED representation must necessarily exhibit "greater patience towards larger rewards", a property reflected in the fact that $\delta(\cdot)$ is nondecreasing in any MED representation. The *behavioral* expression is that the MED model implies the following property: for any s, l, τ such that $(s, 0) \sim (l, \tau)$, and any t, T,

$$(s, t) \sim (l, T + \tau) \Longrightarrow T \geq t.$$

To see that this reflects greater patience towards larger rewards, suppose $(80, 0) \sim (100, 2)$. If the \$80 reward is delayed by t periods, and if the agent is more patient toward larger rewards, the \$100 reward must require a delay by more than t periods in order to restore indifference; for instance, $(80, 1) \sim (100, 3 + 2)$. It is because the agent is more patient toward larger rewards that more periods of delay are required in order to match the loss of attractiveness in the smaller reward. This property of the MED model follows from a theorem presented shortly (see footnote 6).

Thus without any explicit restriction on how impatience may depend on rewards, the model gives rise to a very particular kind of magnitude-dependence. The proof of the theorem reveals that, while Weak Stationarity helps give rise to a representation $U(m, t) = \delta(m)^t \cdot u(m)$ with some $\delta(\cdot)$, *it is Monotonicity that then implies that $\delta(\cdot)$ must be weakly increasing.* To see this, suppose by way of contradiction that $\delta(s) > \delta(l)$ for some $s < l$. Monotonicity requires that $\delta(l)^t \cdot u(l) > \delta(s)^t \cdot u(s)$ for all t, and thus for all t,

$$\frac{u(l)}{u(s)} > \left(\frac{\delta(s)}{\delta(l)}\right)^t.$$

However, since $\frac{\delta(s)}{\delta(l)} > 1$, this inequality cannot hold for all t, a contradiction.

For completeness, we identify the uniqueness properties of the MED representation. Identify any MED representation with the tuple (u, δ) that defines it.

Theorem 2.5. *If* (u, δ) *and* $(\widehat{u}, \widehat{\delta})$ *are two MED representations for some preference* $\succsim$ *over* X, *then there exists* $\theta, \lambda > 0$ *such that* $\widehat{u}(\cdot) = \lambda u(\cdot)^\theta$ *and* $\widehat{\delta}(\cdot) = \delta(\cdot)^\theta$ *on* $\mathcal{M} \setminus \{0\}$. *Indeed,* $\frac{\ln \delta(s)}{\ln \delta(l)} = \frac{\ln \widehat{\delta}(s)}{\ln \widehat{\delta}(l)}$ *for all* s, l, *and moreover,*

$$(s, 0) \sim (l, \tau) \text{ and } (s, t) \sim (l, T + \tau) \Longrightarrow \frac{T}{t} = \frac{\ln \delta(s)}{\ln \delta(l)}.$$

The uniqueness properties of the SDU model extend to the MED model in large part – the exception being that the discount factor $\delta(0)$ for the 0 reward is unrestricted by preferences. The reason is that in the MED representation, $\delta(0)u(0) = 0$ regardless of the value of $\delta(0)$. Although the discount factor $\delta(\cdot)$ is not unique, the result tells us that the ratio of the log of the discount factor for different rewards is uniquely pinned down by preferences, that is, this ratio is constant regardless of the particular MED representation chosen. In the last expression, the result shows how this ratio can be calculated directly from preferences.[6]

We conclude by pointing out how characterizations of special cases of the MED model can be obtained as simple corollaries of Theorem 2.5. As an example, consider the following proposition: A preference $\succsim$ over X admits an MED representation (u, δ) for which there exists $\delta \in (0, 1)$ and $\alpha > 0$ such that for all m,

$$\delta(m) = \delta^{\frac{1}{m^\alpha}},$$

if and only if $\succsim$ satisfies regularity, Weak Stationarity, and for all s, l,

$$(s, 0) \sim (l, \tau) \text{ and } (s, t) \sim (l, T + \tau) \Longrightarrow \frac{T}{t} = \left(\frac{l}{s}\right)^\alpha.$$

This can be proved as follows. Note that $\frac{\ln \delta(s)}{\ln \delta(l)} = \left(\frac{l}{s}\right)^\alpha$ and thus $\delta(s)^{s^\alpha} = \delta(l)^{l^\alpha}$ for all s, l. By regularity (specifically Lemma B.1(a) in the appendix), fixing $l = M$ and varying τ tells us that in particular, $\delta(m)^{m^\alpha}$ is constant for all m. Denote this constant by $\delta \in (0, 1)$ and deduce that $\delta(m) = \delta^{\frac{1}{m^\alpha}}$.

2.3. General Outcomes

If outcomes are not necessarily monetary, then an intuitive formulation of the model would involve the functional form

$$U(m, t) = \delta(u(m))^t \cdot u(m),$$

[6] Note that $\delta(s) \leq \delta(l)$ implies $\frac{\ln \delta(s)}{\ln \delta(l)} \geq 1$ and thus $T \geq t$.

where the discount factor is an increasing function of the utility of an outcome. Thus, for instance, if outcomes are lotteries or bundles, then the agent is less inclined to discount more attractive lotteries or bundles, where attractiveness is determined by u. An attractive feature of this formulation is that, for instance, risk attitude remains independent of time:

$$(p, 0) \succsim (q, 0) \Longleftrightarrow (p, t) \succsim (q, t),$$

for any pair of lotteries p, q and all t. In particular, risk preferences are independent of the magnitude effect.

2.4. Non-Concavity

Suppose u is concave. Then, as a dated reward (m, t) increases in magnitude, the agent experiences a reduction in the marginal utility from m. However, this will also be accompanied with a reduction in the agent's impatience, as the increased magnitude also makes the reward more visible to him. Indeed, the magnitude effect naturally gives rise to possibly nondecreasing marginal utility for future rewards even if immediate rewards are subject to diminishing marginal utility. In Appendix D we confirm that nonconcavity always arises with sufficient delay.

Nonconcavity with respect to money does not necessarily generate undesirable behavior here. For instance, in a risk setting it does not imply that the agent could be risk loving toward distant lotteries – as we observed above, when properly formulated the model produces date-independent risk attitudes. Non-concavity may give rise to non-convex upper contour sets.[7] However, in Appendix D we identify necessary and sufficient restrictions on the representation for the preference to be convex, and thus confirm that nonconcavity with respect to money does not necessitate nonconvexity – intuitively, the factors affecting marginal utility for money also impact the marginal disutility of waiting, and under suitable restrictions both combine to produce convex upper contour sets. For instance, the MED model

$$U(m, t) = \delta^{\frac{t}{m^{\alpha}}} \cdot u(m),$$

where the utility of money is CRRA $u(m) = m^{\beta}$, $\beta \in (0, 1)$, describes a convex preference iff $\alpha \leq \frac{1}{2}$, while $U(m, t)$ remains nonconcave in m for large t regardless of α.

[7] We are grateful to an anonymous referee for first bringing our attention to the possibility of non-convex upper contour sets in the model.

3. MED and Hyperbolic Discounting

Magnitude effect discounting and hyperbolic discounting describe distinct psychological stories, but it remains to study how distinct are the behaviors generated by either of these stories. This is pursued next.

The main result of this section is an observational equivalence result.

Theorem 3.1. *Consider a regular preference $\succsim$ over X that violates Stationarity. Then $\succsim$ admits an MED representation if and only if there exists a strictly increasing continuous utility function $u : \mathcal{M} \to \mathbb{R}_+$ satisfying $u(0) = 0$, an arbitrary $\varphi(0)$, and a continuous function $\varphi : \mathcal{M}\backslash\{0\} \to (0, 1)$ that is either constant or satisfies $\frac{\varphi(s)}{\varphi(l)} \geq \frac{u(s)}{u(l)}$ for all $s < l$ and, moreover, $\succsim$ is represented by the Magnitude-Dependent Hyperbolic Discounting (MDHD) representation defined by:*

$$U(m, t) = \frac{1}{1 + \varphi(m)t} \cdot u(m).$$

The result establishes that the MED model can replicate *any* behavior explicable by the MDHD model. The MDHD representation departs from both the date-independence and magnitude-independence of impatience embodied in the exponential discounting representation. The result shows that behaviorally there is just as much explanatory power obtained by departing from magnitude-independence only, at least in the domain of dated rewards.

When φ is constant, the MDHD model reduces to the *Hyperbolic Discounting* (HD) model introduced by Ainslie [1] and axiomatized by Loewenstein and Prelec [6]:

$$U(m, t) = \frac{1}{1 + \alpha t} \cdot u(m), \quad \alpha > 0.$$

While it is obvious from the representation that the HD model is a special case of the SDU model, what is less obvious from an inspection of functional forms is that the HD class is wholly contained in the MED class – this is an immediate corollary of the above result. Thus, all choice patterns that are explicable by the HD model are also explicable by the MED model – this includes the behaviors known as *preference reversals* and *dynamic inconsistency*, which have attracted considerable attention in economics.[8] Preference reversals can be defined by the following property: for some $s \leq l$ and $t, d \geq 0$,

[8] An example of a preference reversal is: (100,now) $\succ$ (150, 6 months) and (100,1 yr) $\prec$ (150,1 yr 6 months).That is, there is greater patience underlying money-time trade-offs when rewards

(i) $(s,t) \precsim (l,t+d) \Longrightarrow (s,t') \precsim (l,t'+d)$ for all $t' > t$.

(ii) $(s,0) \succsim (l,d)$ and $(s,t) \prec (l,t+d) \Longrightarrow (s,t') \prec (l,t'+d)$ for all $t' > t$.

That is, delaying a pair of rewards can lead to no more than one reversal in preference. The MED model exhibits this property for all $s \leq l$ and $t, d \geq 0$.[9] See the concluding section of this paper for further comments.

The next result clarifies the relationship between the MED, HD and the SDU model, where the latter is formally defined by the representation

$$U(m,t) = D(t)u(m)$$

for a preference $\succsim$ over X where $u : \mathcal{M} \to \mathbb{R}_+$ is a strictly increasing continuous utility function with $u(0) = 0$, and $D : \mathcal{T} \to (0,1)$ is a continuous strictly decreasing discount function with $D(0) = 1$ and $\lim_{t\to\infty} D(t) = 0$.

Theorem 3.2. *For a preference $\succsim$ over X that violates Stationarity, the following statements are equivalent:*

(a) $\succsim$ admits an SDU representation and satisfies Weak Stationarity.

(b) $\succsim$ admits an HD representation.

(c) $\succsim$ admits an MED representation (u, δ) where $u(\cdot) < 1$ and $\delta(\cdot) = u(\cdot)^\alpha$ on $\mathcal{M} \setminus \{0\}$ for $\alpha > 0$.

The equivalence of (a) and (b) reveals that not only is the HD class contained in both the SDU and MED classes, *it is also the only one that lies in both classes.* That is, the HD class is precisely the intersection of the MED and SDU classes.

In the literature, the connection between the HD model and a variant of Weak Stationarity has already been noted by Loewenstein and Prelec [6] and Harvey [5]. Specifically, these authors impose a variant of Weak Stationarity on the SDU model and obtain the HD model. We conclude with an alternative axiomatization of the HD model that does not explicitly presume the existence of an SDU representation for $\succsim$.

are pushed into the future by a common number of periods. Dynamic inconsistency is a dynamic version of preference reversals: (100, 1yr) $\prec_0$ (150,1yr 6 months) and (100, now) $\succ_1$ (150,6 months), where $\prec_0$ denotes the preference today and $\succ_1$ denotes the preference after 1 yr. It is clear that if preferences are time-invariant (that is, preferences over X at each point in time are identical), then preference reversals implies dynamic inconsistency.

[9] This can be shown by noting simply that $(s,t) \succsim (l,t+d) \Longleftrightarrow \delta(s)^t \cdot u(s) \geq \delta(l)^{t+d} \cdot u(l) \Longleftrightarrow \left(\frac{\delta(s)}{\delta(l)}\right)^t \geq \frac{\delta(l)^d \cdot u(l)}{u(s)}$. Since $\frac{\delta(s)}{\delta(l)} \leq 1$, preferences reverse only in favor of the larger reward, if at all.

Theorem 3.3. *A preference $\succsim$ over X admits an HD representation if and only if it is regular and there exists $\alpha \geq 0$ such that for all s, l, τ, t, T it satisfies*

$$(s,0) \sim (l,\tau) \text{ and } (s,t) \sim (l, T+\tau) \Longrightarrow \frac{T}{t} = 1 + \alpha\tau.$$

4. Application: Sequential Bargaining

Consider a Stahl-Rubinstein bargaining game with alternating offers involving two players A and B, where player A is the first mover. Time is discrete and the horizon infinite. The size of the cake is 1, and if no agreement is reached then each player gets 0. A division proposed by any player is denoted $(x, 1-x)$ where x denotes the share for player A and $1-x$ denotes the share for player B. At every point in time, the preference of player $i = A, B$ is denoted $\succsim_i$, and we assume that the agent is sophisticated in the sense of knowing future preferences. We assume further that $\succsim_i$ admits a linear (or risk neutral) MED representation:[10]

$$U(m,t) = \delta_i(m)^t \cdot m.$$

It is known that if each agent follows an SDU model $D_i(t) \cdot m$ and is sophisticated, then the game gives rise to a unique stationary subgame perfect equilibrium [7]. In contrast we find that uniqueness is lost under magnitude-dependent impatience. Moreover, being the first mover while also being more patient does not guarantee a larger share of the cake.

To state these results, define:

$$D_i(x) = 1 - \delta_i(x)x,$$

for $x \in [0,1]$ and $i = A, B$. If (say) player B expects to receive x in period $t+1$ then $D_B(x)$ is the largest share of the cake that player A can extract in period t. Write $D_B D_A(x)$ for $D_B(D_A(x))$. If player A is expects to receive x in period $t+2$, then $D_B D_A(x)$ is the most player A can receive in period t if player B extracts the most he can in period $t+1$. The function $D_B D_A(\cdot)$ is continuous, strictly increasing and satisfies $D_B D_A(0) > 0$ and $D_B D_A(1) < 1$.

[10] To justify the 'risk neutrality' terminology, imagine that the model is a special case of one where rewards are lotteries, and where the utility of a dated lottery (p,t) is $\delta_i(EV(p))^t \cdot EV(p)$. It is clear that $(EV(p), t) \sim_i (p,t)$ and thus the 'date t certainty equivalent' of (p,t) is $EV(p)$. See the remarks in Section 2.3.

Theorem 4.1. *Suppose $\succsim_A$ and $\succsim_B$ admit linear MED representations (u, δ_A) and (u, δ_B) resp. Then a bargaining game of alternating offers with players $\succsim_A$ and $\succsim_B$ has a (possibly nonunique) stationary subgame perfect equilibrium. Moreover, even if $\delta_A \geq \delta_B$, there may exist such an equilibrium in which the first mover gets less than $\frac{1}{2}$. Specifically:*

(i) For every fixed point x^* of $D_B D_A(\cdot)$ there exists a stationary subgame perfect equilibrium, the outcome of which is that player A proposes $(x^*, 1 - x^*)$ in period 0, and player B immediately accepts this offer.

(ii) The set of fixed points of $D_B D_A(\cdot)$ is nonempty and possibly a nonsingleton set.

(iii) If $D_B D_A(\frac{1}{2}) < \frac{1}{2}$, then there exists an equilibrium in which player A's payoff is $x^* < \frac{1}{2}$. The condition can hold even if $\delta_A \geq \delta_B$.

Statement (i) is familiar from the standard analysis of the bargaining game. Statement (ii) concerns existence and multiplicity of stationary subgame perfect equilibria. Technically, multiplicity is possible since $\delta_i(x)x$ may be a nonconcave function of x, despite the agent's risk neutrality. Statement (iii) states a sufficient condition for the existence of equilibria in which the first mover gets strictly less than half the share of the cake. This condition can be satisfied even if player A is more patient than player B.[11] For a concrete example, suppose $\delta_A(x) = \delta_B(x) = 0.9x^{0.5}$. Then $D_B D_A(\cdot)$ has three fixed points at $0.24, 0.6$ and 0.9 approximately.

To see the intuition for the results, recall that in the standard bargaining game (with the exponential discounting model) the relatively more patient player receives a higher share in equilibrium. With the MED model, the relative patience of the players is endogenous. The smaller the share of the cake player A expects to receive in the future, the more impatient he becomes relative to player B, and conversely. This feature allows for the possibility of more than one equilibrium. Indeed, it also allows for the possibility that player B gets more than half the share although player A is the first mover and the more 'patient' player. Thus we see that when impatience is magnitude-dependent, then a given player's payoff does not depend crucially on whether s/he is the first mover or the more patient player, but rather on the beliefs that each player holds about the other's strategy.

[11] This interpretation of the restriction $\delta_A \geq \delta_B$ can be justified by the following (straightforward) proposition: if $\succsim_A$ and $\succsim_B$ admit linear MED representations (u, δ_A) and (u, δ_B) resp, then $\delta_A \geq \delta_B$ if and only if for all $s < l$ and t,

$$(s, 0) \succsim_A (l, t) \Longrightarrow (s, 0) \succsim_B (l, t),$$

that is, whenever $\succsim_A$ rejects waiting for a larger reward, then so does $\succsim_B$.

Finally, we look at the payoffs in the limit of finite horizon bargaining games. Let $[D_B D_A]^0(x) = x$ and inductively for each $k > 0$ define $[D_B D_A]^k(x) = D_B D_A([D_B D_A]^{k-1}(x))$. For any sequence $\{A_n\}_{n=0}^{\infty}$ refer to the subsequence corresponding to odd n (resp. even n) as the odd subsequence (resp. even subsequence).

Theorem 4.2. *In the bargaining game of alternating offers with finite horizon n, the first mover, player A, has a unique subgame perfect equilibrium payoff given by:*

$$A_n = \begin{cases} [D_B D_A]^{\frac{n-1}{2}}(1) & \text{if } n \text{ is odd} \\ [D_B D_A]^{\frac{n-2}{2}}(D_B(1)) & \text{if } n \text{ is even.} \end{cases}$$

The odd subsequence of $\{A_n\}_{n=0}^{\infty}$ converges to the greatest fixed point of $D_B D_A(\cdot)$ and the even subsequence of $\{A_n\}_{n=0}^{\infty}$ converges to the least fixed point of $D_B D_A(\cdot)$.

5. Concluding Remarks

The theoretical literature on time-preference has focused on exploring how impatience depends on delay. In particular, the idea that impatience decreases with delay (as embodied by hyperbolic discounting) has received much attention. This paper explores the dependence of impatience on magnitude. We showed that magnitude-dependent impatience can behaviorally subsume particular models of delay-dependent impatience, specifically the HD model.

To see intuitively how the MED model subsumes the HD model, consider the thought experiment in Thaler [10] that illustrates preference reversals: an apple today is generally preferred to two apples tomorrow, but two apples after a year and a day are generally preferred to an apple after a year. In the HD model, the preference for the immediate apple is explained by a desire for immediate gratification, which is irrelevant for the preference for the distant two apples. In the MED model, the preference for the immediate apple over tomorrow's two apples just reflects the agent's impatience – earlier rewards are always more attractive. The preference for the distant two apples arises due to greater patience toward larger rewards: since two apples are more attractive than one, the agent pays relatively more attention to the larger reward, and this effect begins to dominate in distant intertemporal trade-offs, thereby creating a preference reversal. Thus behaviors attributable to the existence of a desire for immediate gratification can also arise from a less viscerally-charged process that involves a tendency to pay more attention to larger rewards.

Under the immediate gratification story, observing preference reversals for small stakes implies the existence of preference reversals for large stakes. In contrast, the magnitude-dependent impatience story allows for the possibility that preference reversals disappear for large stakes – for instance, think of an MED model where $\delta(\cdot)$ is strictly increasing for rewards over an interval $[0, m^*]$ and then constant for any $m \geq m^*$. These observations have implications for inferences made from experimental findings. Due to practical considerations, experiments on preference reversals have predominantly used small stakes. It is the assumption of magnitude-*independent* impatience that leads researchers to infer that findings in these experiments would hold also for large stakes. This potentially leads to an exaggerated sense of relevance of the findings for behavior in the market.[12] If the mechanism generating the results is in part magnitude-dependent impatience, then it is possible that the results are relevant mainly for the 'small' decisions that economic agents make.

A. Appendix: Calibration Result

For each dated reward (m, t), let $\psi(m, t)$ (the '*present equivalent*' of (m, t)) denote the amount received immediately such that

$$(\psi(m, t), 0) \sim (m, t).$$

The money-discount function is defined by

$$\varphi(m, t) := \frac{\psi(m, t)}{m}.$$

Suppose that the agent respects the SDU model defined for consumption streams $(c_{0,} c_1, ...)$:

$$U(c_{0,} c_1, ...) = \sum D(t) u(c_t).$$

Note that present equivalents must then satisfy the restriction

$$u(w_0 + \psi(m)) - u(w_0) = D(t)[u(w_t + m) - u(w_t)], \tag{A.1}$$

where w_t refers to the agent's base consumption in period t.

[12] Similarly, it would lead to an exaggerated view of how impatient people are, given that impatience decreases with magnitudes.

Proposition A.1. *If u is concave then for any $t, m, \varepsilon > 0$,*

$$u(w_0) + D(t)u(w_t + m) \geq u(w_0 - \varepsilon) + D(t)u(w_t + m + \frac{\varepsilon}{\varphi(m,t)})$$

Proof. To ease notation, write $\psi(m,t)$ as $\psi(m)$ and let $\delta := D(t)$. Given the assumption in the proposition and this notation, we can write (A.1) as:

$$u(w_0 + \psi(m)) - u(w) = \delta[u(w + m) - u(w)].$$

By concavity, for any $m, \varepsilon > 0$,

$$\frac{u(w_0) - u(w_0 - \varepsilon)}{\varepsilon} \geq \frac{u(w_0 + \psi(m)) - u(w_0))}{\psi(m)}.$$

Therefore,
$u(w_0) - u(w_0 - \varepsilon) \geq \frac{\varepsilon}{\psi(m)}[u(w_0 + \psi(m)) - u(w_0))]$
$= \frac{\varepsilon\delta}{\psi(m)}[u(w_t + m) - u(w_t)]$ by (A.1)
$= \delta \frac{m\varepsilon}{\psi(m)} \frac{[u(w_t+m)-u(w_t)]}{m}$
$\geq \delta \frac{m\varepsilon}{\psi(m)} \frac{[u(w_t+m+\frac{m\varepsilon}{\psi(m)})-u(w_t+m)]}{\frac{m\varepsilon}{\psi(m)}}$ by concavity
$= \delta[u(w_t + m + \frac{m\varepsilon}{\psi(m)}) - u(w_t + m)]$
$= \delta[u(w_t + m + \frac{\varepsilon}{\varphi(m,t)}) - u(w_t + m)]$ by definition of $\varphi(m,t)$. This completes the proof. ■

The result states that if time t base consumption is increased by $\$m$, then fixing consumption in all other periods, the agent would not forgo any ε today that yields $\frac{\varepsilon}{\varphi(m,t)}$ at time t. The evidence cited in the Introduction gives an example where $\varphi(60, 1yr) = 0.25$. Therefore, if next yr consumption is higher by \$60, then the agent would not forgo any ε today for 4ε next year. Note that the magnitude effect implies that for amounts smaller than \$60 the discount factor φ would be even lower. Thus, even more extreme quantitative results will be gained for smaller rewards.

B. Appendix: Preliminary Lemmas

Lemma B.1. *If $\succsim$ is regular then*

(a) For every l, t and d there exists $s \leq l$ such that $(s,t) \sim (l, t+d)$. Moreover, for every $s \leq l$ and t there exists d such that $(s,t) \sim (l, t+d)$.

(b) For any $s \leq l$ and τ such that $(s,0) \sim (l,\tau)$, and for every $t' \geq \tau$ there exists t such that $(s,t) \sim (l,t')$.

(c) For any continuous increasing $u : \mathcal{M} \to \mathbb{R}$, the preference $\succsim$ admits a representation $U : \mathcal{M} \times \mathcal{T} \to \mathbb{R}$ such that $U(\cdot,t)$ is continuous and strictly increasing, $U(m,\cdot)$ is continuous and strictly decreasing if $m > 0$ and constant if $m = 0$, and $U(m,0) = u(m)$.

(d) For each (m,t) there exists a unique $\psi(m,t)$ (the 'present equivalent' of (m,t)) satisfying

$$(\psi(m,t),0) \sim (m,t).$$

Moreover, $\psi(0,\cdot) = 0$, $\psi(m,\cdot)$ is strictly decreasing for any $m > 0$, $\lim_{t\to\infty} \psi(m,t) = 0$ for all m, and $\psi(m,\cdot)$ is continuous.

(e) If $(s,0) \sim (l,\tau)$ and $(s,t) \sim (l,T+\tau)$, then $T+\tau \geq t$.

Proof. Part (a) follows from Impatience, Monotonicity and Continuity; we omit the proof. The t in part (b) exists by Impatience, Monotonicity and Continuity: By Monotonicity, $(s,t') \precsim (l,t')$. By Impatience and the fact that $(s,0) \sim (l,\tau)$ and $t' \geq \tau$, it follows that $(s,0) \succsim (l,t')$. Thus, by Continuity, there is t such that $(s,t) \sim (l,t')$, as desired.

Part (c) is established in [3, Thm 1]. Turn to part (d). Part (a) establishes the existence of present equivalents, and Impatience implies that $\psi(m,\cdot)$ is strictly decreasing for any $m > 0$. To see that $\lim_{t\to\infty} \psi(m,t) = 0$ for all m, suppose not. Then there exists m and $s > 0$ such that $(s,0) \prec (\psi(m,t),0) \sim (m,t)$ for all t. But this contradicts Impatience. Finally, to see that $\psi(m,\cdot)$ must be continuous, take any strictly increasing homeomorphism and consider the representation U delivered in part (c). Since $u(\psi(m,t)) = U(m,t)$ and in particular, $\psi(m,t) = u^{-1}(U(m,t))$, continuity of u^{-1} implies that of $\psi(m,\cdot)$.

Part (e) note that if $T+\tau < t$ then $(s,T+\tau) \succ (l,T+\tau)$ by Impatience, which then violates Monotonicity. ■

Lemma B.2. *If a preference $\succsim$ is regular then there exists a function $\Phi(s,l,t)$ that satisfies*

$$(s,0) \sim (l,\tau) \Longrightarrow (s,t) \sim (l,\Phi(s,l,t)+\tau),$$

for every s,l and t. Moreover, $\Phi(s,l,\cdot)$ is strictly increasing and continuous.

Proof. The existence of a function $\Phi(s,l,t)$ defined for every s,l and t by

$$(s,0) \sim (l,\tau) \Longrightarrow (s,t) \sim (l,\Phi(s,l,t)+\tau),$$

follows from Impatience and Continuity. To see that Φ is strictly increasing in t, note that as t increases, Impatience implies that (s,t) becomes less desirable. Thus in order to maintain indifference with $(l, \Phi(s,l,t)+\tau)$, there should also be an increase in $\Phi(s,l,t)$.

To see that Φ is continuous in t, note that Lemma B.1(b) shows that for any $s \leq l$ and τ such that $(s,0) \sim (l,\tau)$, and for every $t' \geq \tau$ there exists t such that

$$\Phi(s,l,t) + \tau = t'.$$

Now observe that if Φ is not continuous, then given that $\Phi(s,l,\cdot)$ is strictly increasing, the image of $\mathbb{R}_+$ in $\Phi(s,l,\cdot)$ for any s,l cannot be an interval. However, the above displayed result shows that for any $k \geq 0$, there exists t^* such that $\Phi(s,l,t^*) = k$, a contradiction. ■

Lemma B.3. *If a regular preference $\succsim$ satisfies Weak Stationarity then for any s,l there exists $a_{sl} > 0$ s.t. for all t,*

$$\Phi(s,l,t) = a_{sl}t.$$

Proof. By Weak Stationarity, $\Phi(s,l,\lambda t) = \lambda\Phi(s,l,t)$. Observe that $\Phi(s,l,1) = \frac{1}{t}\Phi(s,l,t)$ and thus by letting $a_{sl} := \Phi(s,l,1)$ we have $\Phi(s,l,t) = a_{sl}t$. Since $\Phi(s,l,\cdot)$ is strictly increasing (by previous lemma), it must be that $a_{sl} > 0$. ■

Lemma B.4. *If a preference $\succsim$ admits an SDU representation then there exists a function $F(\tau,t)$ that is strictly increasing and continuous, and for each s,l,τ and all t, $F(\tau,t)$ satisfies*

$$(s,0) \sim (l,\tau) \Longrightarrow (s,t) \sim (l, F(\tau,t)+\tau).$$

Moreover, for any τ,t,

$$F(\tau,t) + \tau = F(t,\tau) + t.$$

Proof. We need to show that Φ depends on s,l only through the τ that satisfies $(s,0) \sim (l,\tau)$. Take any τ, s, s', l, l' such that $(s,0) \sim (l,\tau)$ and $(s',0) \sim (l',\tau)$. Then for any t,t', the SDU representation implies $(s,t) \sim (l,t') \Longleftrightarrow (s',t) \sim (l',t')$. In particular, $\Phi(s,l,t)+\tau = t' = \Phi(s',l',t)+\tau)$ and so $\Phi(s,l,t) = \Phi(s',l',t)$. Therefore, we can write $\Phi(s,l,t) = F(\tau,t)$ where τ is such that $(s,0) \sim (l,\tau)$.

To establish the second part of the lemma, take any t,τ and define s,l,M by $(s,0) \sim (M,\tau)$ and $(l,0) \sim (M,t)$. By the SDU representation, $(s,t) \sim (l,\tau)$ and in particular

$$(M, F(\tau,t)+\tau) \sim (s,t) \sim (l,\tau) \sim (M, F(t,\tau)+t).$$

Since $M > 0$, Impatience implies $F(\tau,t) + \tau = F(t,\tau) + t$, as desired. ■

C. Appendix: Proof of Theorems 2.4 and 2.5

Proof of '$\Longrightarrow$': To see that Weak Stationarity is necessary, note that for $s > 0$, $(s, 0) \sim (l, \tau)$ and $(s, t) \sim (l, \Phi(s, l, t) + \tau)$ imply $l > 0$ and

$$\delta_l^\tau = \frac{u(s)}{u(l)} = \frac{\delta_l^{\tau+\Phi(s,l,t)}}{\delta_s^t},$$

and thus $\Phi(s, l, t) = a_{s,l}t$ where $a_{s,l} = \frac{\ln \delta_s}{\ln \delta_l} > 0$. Thus, $\Phi(s, l, t)$ is a linear function of t, or equivalently, Weak Stationarity holds. Verifying the necessity of all other axioms is routine.

Proof of '$\Longleftarrow$': Begin with some observations. By Lemmas B.2 and B.3, we know that for each $s \leq l$ and t, $(s, t) \sim (l, H(s, l, t))$ where

$$H(s, l, t) = a(s, l)t + \tau(s, l),$$

and where $\tau(s, l)$ is defined by

$$(s, 0) \sim (l, \tau(s, l)).$$

Impatience and Continuity ensure that $\tau(s, l)$ always exists. We claim that $a(M, \cdot)$ is weakly increasing:

$$M \leq s < l \Longrightarrow a(M, s) \leq a(M, l).$$

This follows from Monotonicity. Suppose by contradiction that $a(M, s) > a(M, l)$. Since $M \leq s < l$, Monotonicity and Impatience imply $\tau(M, s) < \tau(M, l)$. But then there exists t^* such that $a(M, s)t^* + \tau(M, s) = a(M, l)t^* + \tau(M, l) = T$, which implies $(l, T) \sim (M, t^*) \sim (s, T)$, while Monotonicity requires that $(s, T) \not\sim (l, T)$, a contradiction.

Next we show that $a(M, \cdot)$ is continuous for all $m \geq M > 0$. We do this by first showing that $H(M, \cdot, t)$ is continuous on $[M, \infty)$. Take any sequence $M \leq m_n \to m$, and suppose by way of contradiction that $H(M, m_n, t)$ does not converge to $H(M, m, t)$. Then there is $\varepsilon > 0$ such that either there exists a subsequence s.t. $(m_{n(i)}, H(M, m_{n(i)}, t)) \succsim (m_{n(i)}, H(M, m, t) - \varepsilon)$ or one s.t. $(m_{n(i)}, H(M, m_{n(i)}, t)) \precsim (m_{n(i)}, H(M, m, t) + \varepsilon)$. In either case, we can assume wlog that the subsequence is convergent, since $H(M, m_{n(i)}, t)$ is bounded.[13] Taking limits, we then either have $(m, H(M, m, t)) \succsim (m, H(M, m, t) - \varepsilon)$ or $(m, H(M, m, t)) \precsim (m, H(M, m, t) + \varepsilon)$. But either cases violate Impatience, a contradiction. Thus $H(M, \cdot, t)$ is continuous. Note also that $\tau(M, \cdot) = H(M, \cdot, 0)$ is continuous. So, to conclude, since $H(M, m, t) = a(M, m)t + \tau(M, m)$, continuity of $H(M, \cdot, t)$ and $\tau(M, \cdot)$ implies that of $a(M, \cdot)$, as desired.

We now turn to establishing the existence of the desired representation. Fix $\delta \in (0, 1)$ and define u on $\mathcal{M} \backslash \{0\}$ by

$$u(m) = \begin{cases} \delta^{\tau(m,M)} \cdot M & \text{if } m \leq M \\ \delta^{-\frac{\tau(M,m)}{a(M,m)}} \cdot M & \text{if } m \geq M \end{cases}, \qquad \text{for all } m > 0.$$

Given the properties of present equivalents ψ in Lemma B.1(d), and the fact that by definition $\psi_M^{-1}(\cdot) = \tau(\cdot, M)$, we see that $\tau(\cdot, M)$ is well-defined, strictly decreasing and continuous and satisfies $\lim_{m \to 0} \tau(m, M) = \infty$. Therefore, on $(0, M]$, the function u is strictly increasing, continuous and satisfies $u(M) = M$, and moreover, satisfies $\lim_{m \to 0} u(m) = 0$. Thus the restriction of u to $(0, M]$ can be extended continuously to $[0, M]$ by setting $u(0) = 0$. As observed above, $a(M, \cdot)$ is weakly increasing and continuous. As $\tau(M, \cdot)$ is strictly decreasing and continuous, we see that u is strictly increasing and continuous on $[M, \infty)$. Put together, we we have a strictly increasing and continuous u over $\mathcal{M} = \mathbb{R}_+$.[14]

Given this u, denote the corresponding representation for $\succsim$ by U, which is guaranteed by Lemma B.1(c). Define $D : \mathcal{M} \times \mathcal{T} \to \mathbb{R}_+$ such that for all $(m, t) \in \mathcal{M} \backslash \{0\} \times \mathcal{T}$,

$$U(m, t) = D(m, t)u(m),$$

and $D(0, t)$ is arbitrary. We show that D has the desired form. By definition of u, letting $m = \psi(M, t)$, we have that for any t, $U(M, t) = u(\psi(M, t)) = \delta^t \cdot M$. Since $u(M) = M > 0$, it follows that

$$D(M, t) = \delta^t.$$

[13] To see this, suppose $H(M, m_n, t)$ is not bounded above. Then there is a subsequence where $H(M, m_{n(i)}, t) > H(M, m, t)$ tends to infinity and $m - \varepsilon \leq m_{n(i)} \leq m + \varepsilon$ for some $\varepsilon < m$. Then by Monotonicity and Impatience,

$$(m - \varepsilon, H(M, m, t)) \precsim (m_{n(i)}, H(M, m, t)) \precsim (m_{n(i)}, H(M, m_{n(i)}, t)) \precsim (m + \varepsilon, H(M, m_{n(i)}, t)).$$

However, by Impatience, $(m - \varepsilon, H(M, m, t)) \precsim (m + \varepsilon, H(M, m_{n(i)}, t))$ cannot hold as $H(M, m_{n(i)}, t)$ tends to infinity, a contradiction. Thus, $H(M, m_n, t)$ is bounded.

[14] This relies also on the fact that $a(M, M) = 1$, which follows from $(M, t) \sim (M, t)$ for all t.

By definition, for any $0 < m \leq M$, $\frac{u(m)}{u(M)} = \delta^{\tau}$, where the argument (m, M) in τ is suppressed here and below to ease notation (and similarly for a below). By Lemma B.3, we also have $\frac{u(m)}{u(M)} = \frac{\delta^{\tau+at}}{D(m,t)}$. These last two equalities put together imply that for all $0 < m \leq M$ and t,

$$D(m,t) = \frac{\delta^{\tau+at}}{\delta^{\tau}} = \delta_m^t,$$

where $\delta_m := \delta^a$. Note that in general a depends on m and M, and M is fixed in the entire construction. Hence δ_m is indeed a function of m. Since $0 < a < \infty$, we have $\delta_m \in (0,1)$.

Next we show that the representation has the desired form also for $m \geq M$. By definition of u, $\frac{u(M)}{u(m)} = \delta^{\frac{\tau}{a}}$. Given Lemma B.3, we also have $\frac{u(M)}{u(m)} = \frac{D(m,t)}{\delta^{\frac{t-\tau}{a}}}$. Thus for all $m \geq M$ and t,

$$D(m,t) = \delta^{\frac{\tau}{a}} \cdot \delta^{\frac{t-\tau}{a}} = \delta_m^t,$$

where $\delta_m := \delta^{\frac{1}{a}}$. Since $0 < a < \infty$, we have $\delta_m \in (0,1)$.

Finally, we check that δ_m is weakly increasing in m. Suppose by way of contradiction that $\delta_s > \delta_l$ for some $s < l$. Monotonicity requires that $\delta_l^t \cdot u(l) > \delta_s^t \cdot u(s)$ for all t, and thus for all t,

$$\frac{u(l)}{u(s)} > \left(\frac{\delta_s}{\delta_l}\right)^t.$$

However, since $\frac{\delta_s}{\delta_l} > 1$, this inequality cannot hold for all t, a contradiction. With this, we have established that D has the desired form and properties.

Proof of uniqueness: Consider two MED representations (u, δ) and (u', δ'). In the proof of necessity of Weak Stationarity, we saw that for any s, l the scalar $a > 0$ delivered by Reversal must satisfy $\frac{\ln \delta_s}{\ln \delta_l} = a = \frac{\ln \delta'_s}{\ln \delta'_l}$. Define $\theta := \frac{\ln \delta'_M}{\ln \delta_M}$ so that $\delta'_M = \delta_M^{\theta}$. Then, for any $m > 0$, $\frac{\ln \delta_m}{\ln \delta_M} = \frac{\ln \delta'_m}{\ln \delta'_M}$ and so $\theta \ln \delta_m = \ln \delta'_m$. Thus, $\delta'(\cdot) = \delta(\cdot)^{\theta}$ on $\mathcal{M} \setminus \{0\}$. Next we show that only monomial transformations of utility u preserve the representation. Since u and u' are increasing and $u(0) = u'(0) = 0$ there is an increasing transformation g with $g(0) = 0$ such that $u' = g(u)$. We see that

$\delta_m^{\theta t} g(u(m)) = \delta_m'^{t} u'(m) = U'(m,t) = u'(\psi(m,t))$
$= g(u(\psi(m,t))) = g(U(m,t)) = g(\delta_m^{\theta t} u(m))$, that is, for all m, t,

$$\delta_m^{\theta t} g(u(m)) = g(\delta_m^{\theta t} u(m)).$$

Observe that for every $z \in (0,1]$ and $r \in [0, u(M)]$ there is t and m s.t. $u(m) = r$ and $\delta_m^{\theta t} = z$. Therefore, for every $z \in (0,1]$ and $r \in [0, u(M)]$,

$$z^\theta g(r) = g(zr).$$

For any $r \in [0, u(M)]$, taking $z = \frac{r}{u(M)}$ yields $g(r) = g(zu(M)) = z^\theta g(u(M)) = \frac{r^\theta}{u(M)^\theta} u'(M)$. Setting $\lambda = \frac{u'(M)}{u(M)} > 0$ it follows that for any m, $u'(m) = g(u(m)) = \lambda u(m)^\theta$. This completes the proof for uniqueness.

To prove the last part of the theorem take any $s < l$. For some τ we have $(s, 0) \sim (l, \tau)$, which is equivalent to $\frac{u(s)}{u(l)} = \delta_l^\tau$, and for $t = 1$, we know by Weak Stationarity that there is $a_{s,l} > 0$ such that for all t, $(s, t) \sim (l, a_{s,l} t + \tau)$, that is, $\delta_s^t u(s) = \delta_l^{a_{s,l} t + \tau} u(l)$. But then

$$\delta_l^\tau = \frac{u(s)}{u(l)} = \left(\frac{\delta_l^{a_{s,l}}}{\delta_s} \right)^t \delta_l^\tau,$$

implying $\delta_l^{a_{s,l}} = \delta_s$, and thus $a_{s,l} = \frac{\ln \delta(s)}{\ln \delta(l)}$.

D. Appendix: Non-Concavity and Convexity

Write the MED representation as

$$U(m, t) = e^{-\alpha(m)t} \cdot u(m),$$

for some decreasing function $\alpha(\cdot) > 0$. Refer to the representation as *twice differentiable* if u and α are twice differentiable.

Proposition D.1. *Suppose that a preference $\succsim$ over X admits a twice differentiable MED representation (u, δ^α) with $\alpha(\cdot)$ strictly decreasing. Then for any $m > 0$ it must be that $\frac{\partial^2 U(m,t)}{\partial m^2} > 0$ for all large t.*

Proof. It may be verified that $\frac{\partial^2 U(m,t)}{\partial m^2} =$

$$-\alpha'(m) t e^{-\alpha(m)t} \left[2u'(m) + \frac{u''(m)}{\alpha'(m)t} + u(m) \left(\frac{\alpha''(m)}{\alpha'(m)} - \alpha'(m)t \right) \right].$$

Since $\alpha'(m) < 0$, it is evident from the expression in the parenthesis that $\frac{\partial^2 U(m,t)}{\partial m^2} > 0$ for large t and given $m > 0$. ■

Proposition D.2. *Suppose that a preference $\succsim$ over X admits a twice differentiable MED representation (u, δ^{α}) with u concave and $\alpha(\cdot)$ strictly decreasing. Then $\succsim$ has convex upper contour sets if and only if for all $m > 0$,*

$$\left[\frac{2\alpha'(m)}{\alpha(m)} - \frac{\alpha''(m)}{\alpha'(m)}\right] \geq 0 \text{ and } \left[\frac{2\alpha'(m)}{\alpha(m)} + \frac{u'(m)}{u(m)} - \frac{u''(m)}{u'(m)}\right] \geq 0.$$

Proof. Fix some utility level K and determine that on the indifference curve defined by $e^{-\alpha(m)t} \cdot u(m) = K$ it must be that $t = \frac{1}{\alpha(m)} \ln \frac{u(m)}{K}$ and thus:[15]

$$\begin{aligned}
\frac{\partial t}{\partial m} &= -\frac{\alpha'(m)}{\alpha(m)^2} \ln \frac{u(m)}{K} + \frac{u'(m)}{\alpha(m)u(m)} > 0 \\
\text{and } \frac{\partial^2 t}{\partial m^2} &= -\frac{\alpha''(m)}{\alpha(m)^2} \ln \frac{u(m)}{K} + \frac{2\alpha'(m)^2}{\alpha(m)^3} \ln \frac{u(m)}{K} \\
&\quad -\frac{\alpha'(m)}{\alpha(m)^2} \frac{u'(m)}{u(m)} - \frac{u'(m)}{\alpha(m)u(m)} \left[\frac{\alpha'(m)}{\alpha(m)} + \frac{u'(m)}{u(m)} - \frac{u''(m)}{u'(m)}\right] \\
&= \frac{\alpha'(m)}{\alpha(m)^2} \left[\frac{2\alpha'(m)}{\alpha(m)} - \frac{\alpha''(m)}{\alpha'(m)}\right] \ln \frac{u(m)}{K} - \frac{u'(m)}{\alpha(m)u(m)} \left[\frac{2\alpha'(m)}{\alpha(m)} + \frac{u'(m)}{u(m)} - \frac{u''(m)}{u'(m)}\right],
\end{aligned}$$

where u and α are assumed to be twice differentiable. Restricting attention to concave u, we see that convex upper contour sets ($\frac{\partial^2 t}{\partial m^2} \leq 0$) obtain if and only if the conditions in the statement of the lemma hold. Sufficiency is obvious (recall that $\alpha'(m) \leq 0$). Necessity is established readily using the fact that $\ln \frac{u(m)}{K} \geq 0$ can be made arbitarily large or small by varying K over $[0, u(m)]$. ∎

E. Appendix: Proof of Theorems 3.1-3.3

Proof of Theorem 3.1: To prove the sufficiency part, take a continuous increasing transformation $g : u(\mathcal{M}) \to \mathbb{R}_+$ defined by $g(0) = 0$ and, for all $r \in u(\mathcal{M})\backslash\{0\}$,

$$g(r) = \left(\frac{1}{-\ln r}\right)^{\frac{1}{\alpha}}.$$

This transforms an MED representation $\delta(m)^t v(m)$ into the MDHD representation:

[15] Note that $\frac{u(m)}{K} \geq 1$ on any indifference curve.

$$g(\delta(m)^t v(m)) = \frac{1}{-t\ln\delta(m) - \ln v(m)} = \frac{1}{1+t\frac{\ln\delta(m)}{\ln v(m)}}\frac{1}{-\ln v(m)} := \frac{1}{1+\varphi(m)t}\cdot u(m),$$

where $\varphi(m) := \frac{\ln\delta(m)}{\ln v(m)}$, $u(m) := \frac{1}{-\ln v(m)}$ for $m > 0$ and $u(0) := \lim_{m\to 0} u(m) = 0$. Observe that since $\succsim$ is regular by hypothesis, it satisfies Monotonicity. In particular, for any $s < l$, $\frac{1}{1+\varphi(l)t}\cdot u(l) > \frac{1}{1+\varphi(s)t}\cdot u(s)$ which is equivalent to

$$t\left[\varphi(l)u(s) - \varphi(s)u(l)\right] < u(l) - u(s).$$

However, this holds for all t iff $\varphi(l)u(s) - \varphi(s)u(l) \leq 0$ iff φ is constant or $\frac{\varphi(s)}{\varphi(l)} \geq \frac{u(s)}{u(l)}$.

Conversely, if an MDHD representation is given, then an appropriate MED representation obtains by applying the increasing transformation $h(r) = e^{-r^{-1}}$.

Proofs of Theorems 3.2 and 3.3: We prove more general results by not requiring that Stationarity necessarily be violated. To accommodate this, in what follows we extend the definition of the HD model as follows:

$$(m,t) \longmapsto \left[\lim_{r\to\alpha}\frac{1}{(1+rt)^{\frac{1}{r}}}\right]\cdot u(m)$$

for some $\alpha \geq 0$. Note that when $\alpha = 0$ the model reduces to the exponential discounting model. Consider the axiom stated in Theorem 3.3.

Axiom 1. *For each s, l, τ and all t,*

$$(s,0) \sim (l,\tau) \Longrightarrow (s,t) \sim (l,(1+\alpha\tau)t+\tau).$$

Theorems 3.2 and 3.3 follow from the Lemmas proved next.

Lemma E.1. *A preference $\succsim$ over X satisfies regularity and Axiom 1 if and only if there exists an MED representation (u,δ) such that $0 \leq u(\cdot) < 1$ and $\delta(\cdot)$ is either constant or $\delta(\cdot) = u(\cdot)^\alpha$ on $\mathcal{M}\setminus\{0\}$ for $\alpha > 0$.*

Proof. The existence of an MED representation (v,δ) follows from the fact that Axiom 1 is stronger than Weak Stationarity. For any $m > 0$ and corresponding

$\tau = \frac{\ln \frac{v(m)}{v(M)}}{\ln \delta(M)}$, we have $v(m) = \delta(M)^{\tau} v(M)$ and by Theorem 2.5, $\frac{\ln \delta(m)}{\ln \delta(M)} = (1+\alpha\tau) = 1+\alpha\frac{\ln \frac{v(m)}{v(M)}}{\ln \delta(M)}$. Rearranging yields $\delta(\cdot) = \frac{\delta(M)}{v(M)^{\alpha}} v(\cdot)^{\alpha}$. If $\alpha = 0$ then $\delta(\cdot)$ is a constant. If $\alpha > 0$ then use the uniqueness properties of the MED representation to replace the v with $u(\cdot) = \frac{\delta(M)^{\frac{1}{\alpha}}}{v(M)} v(\cdot)$ so that $\delta(\cdot) = u(\cdot)^{\alpha}$. Note that $0 \leq u(\cdot) < 1$.

Conversely, if $\delta(\cdot)$ is constant then the axiom holds with $\alpha = 0$. Otherwise, given the representation, note that for any s, l we have $\tau = \frac{\ln \frac{u(s)}{u(l)}}{\ln \delta(l)}$ and Theorem 2.5 tells us that the t, T in the statement of that Theorem must satisfy $\frac{T}{t} = \frac{\ln \delta(s)}{\ln \delta(l)}$. Thus,

$$\frac{T}{t} = \frac{1}{\ln \delta(l)} \ln \delta(s) = \frac{1}{\ln \delta(l)} \ln u(s)^{\alpha} = \frac{1}{\ln \delta(l)} \ln u(l)^{\alpha} \left(\frac{u(s)}{u(l)}\right)^{\alpha}$$
$$= \frac{1}{\ln \delta(l)}[\ln u(l)^{\alpha} + \alpha \ln \frac{u(s)}{u(l)}] = \frac{1}{\ln \delta(l)}[\ln \delta(l) + \alpha \ln \frac{u(s)}{u(l)}] = 1 + \alpha\tau, \text{ as desired.} \quad \blacksquare$$

Lemma E.2. *A preference $\succsim$ over X satisfies regularity and Axiom 1 if and only if it admits an HD representation (u, α) where α is as in Axiom 1.*

Proof. By Lemma E.1, the hypothesis is equivalent to the existence of a MED representation (u, δ) such that $0 \leq u(\cdot) < 1$ and for the $\alpha \geq 0$ in Axiom 1, $\delta(\cdot) = u(\cdot)^{\alpha}$ on $\mathcal{M} \setminus \{0\}$. First show that this representation implies the existence of a desired HD representation.

If $\alpha = 0$ then the claim is trivial. So suppose that $\alpha > 0$. Note that the MED representation is $U(m,t) = u(\cdot)^{1+\alpha t}$, and $u(\mathcal{M})$ is an interval in $[0,1)$ containing 0. Take a continuous increasing transformation $g : u(\mathcal{M}) \to \mathbb{R}_+$ defined by $g(0) = 0$ and, for all $r \in u(\mathcal{M})\setminus\{0\}$, $g(r) = \left(\frac{1}{-\ln r}\right)^{\frac{1}{\alpha}}$. Then $V(m,t) := g(U(m,t)) = g(u(m)^{1+\alpha t}) = \left(\frac{1}{-(1+\alpha t)\ln u(m)}\right)^{\frac{1}{\alpha}} = \frac{1}{(1+\alpha t)^{\frac{1}{\alpha}}} v(m)$ where $v(m) = \left(\frac{1}{-\ln u(m)}\right)^{\frac{1}{\alpha}}$ for $m > 0$ and $v(0) = \lim_{m \to 0} v(m) = 0$. Thus we have established the existence of an HD representation.

For the converse, consider the nontrivial case $\alpha > 0$, suppose there is an HD representation $V(m,t) = \frac{1}{(1+\alpha t)^{\frac{1}{\alpha}}} v(m)$, and then apply the increasing transformation $h(r) = e^{-r^{-\alpha}}$ to obtain the desired MED representation. $\blacksquare$

Lemma E.3. *A regular preference $\succsim$ over X satisfies Axiom 1 if and only if it is an SDU model that respects Weak Stationarity.*

Proof. $\Longrightarrow$: The previous Lemmas establish that regularity and Axiom 1 imply the existence of HD and MED representations. The HD representation is a special case of the SDU representation, and the MED representation implies Weak Stationarity.

$\Longleftarrow$: By Lemma B.4, the SDU model implies that there exists $F(\tau, t)$ that for each s, l, τ and all t satisfies

$$(s, 0) \sim (l, \tau) \Longrightarrow (s, t) \sim (l, F(\tau, t) + \tau).$$

By Weak Stationarity, $\frac{F(\tau,t)}{t}$ is constant for each τ. Thus, we can write $F(\tau, t) = b(\tau)t$ for some function b. By Theorem 2.5, $b(\tau) \geq 1$. By Lemma B.4, the SDU model implies also that

$$b(\tau)t + \tau = F(\tau, t) + \tau = F(t, \tau) + t = b(t)\tau + t.$$

Fixing $t = t^* > 0$, we have that $b(\tau) = 1 + \alpha\tau$, where $\alpha = \frac{b(t^*)-1}{t^*} \geq 0$. Thus Axiom 1 is satisfied. ■

F. Appendix: Proof of Theorems 4.1 and 4.2

Proof of Theorem 4.1(i): As in the standard analysis of the bargaining game, for any (x^*, y^*) that solves:

$$(1 - x^*, 0) \sim_B (1 - y^*, 1) \text{ and } (y^*, 0) \sim_A (x^*, 1), \tag{F.1}$$

there exists such an equilibrium in which player A proposes an agreement $(x^*, 1 - x^*)$ whenever it is her turn to make an offer, and accepts an offer $(y, 1 - y)$ of player B if and only if $y \geq y^*$; player B always proposes $(y^*, 1 - y^*)$, and accepts only those offers $(x, 1 - x)$ with $1 - x \geq 1 - x^*$. The outcome is that player A proposes $(x^*, 1 - x^*)$ in period 0, and player B immediately accepts this offer.

To see that the equilibrium outcome x^* is a fixed point of $D_B D_A$, note that the problem (F.1) is equivalent to finding (x^*, y^*) that solves

$$1 - x^* = \delta_B(1 - y^*)(1 - y^*) \text{ and } y^* = \delta_A(x^*)x^*.$$

For any x let $y(x)$ satisfy $y(x) = \delta_A(x)x$. Let

$$F(x) = 1 - x - \delta_B(1 - y(x))(1 - y(x)) = D_B D_A(x) - x.$$

Clearly, (F.1) is satisfied iff $F(x) = 0$ iff x is a fixed point of $D_B D_A$.

Proof of Theorem 4.1(ii): As in the proof of (i), x is a fixed point iff $F(x) = 0$. Note that F is continuous and satisfies $F(0) = u_B(1) - \delta_B(1)u_B(1) > 0$ and $F(1) = -\delta_B(1 - y(1))u_B(1 - y(x)) < 0$. Therefore by the intermediate value theorem there is z with $F(z) = 0$.

For an example of a case where $F(z) = 0$ for more than one z, suppose u is linear and $\delta_A(\cdot) = \lambda \in (0, 1)$. As in part (iii) proved below, suppose (F.2) holds:

$$\delta_B(1 - \frac{1}{2}\lambda) > \frac{\frac{1}{2}}{1 - \frac{1}{2}\lambda}.$$

This ensures that $F(\frac{1}{2}) < 0$ so that $F(z) = 0$ for some $z < \frac{1}{2}$. To get another solution that is larger that $\frac{1}{2}$ we show that it is possible that $F(\frac{3}{4}) > 0$. This is equivalent to

$$\delta_B(1 - \frac{3}{4}\lambda) < \frac{\frac{1}{4}}{1 - \frac{3}{4}\lambda}.$$

Observe that $1-\frac{3}{4}\lambda < 1-\frac{1}{2}\lambda$ and $\frac{\frac{1}{4}}{1-\frac{3}{4}\lambda} < \frac{\frac{1}{2}}{1-\frac{1}{2}\lambda}$. Thus the two displayed inequalities can be satisfied by δ_B that is increasing.

Proof of Theorem 4.1(iii): Since $F(0) > 0$, by the intermediate value theorem a sufficient condition for the existence of $x^* < \frac{1}{2}$ s.t. $F(x^*) = 0$ is that $F(\frac{1}{2}) < 0$. Given linear MED representations, compute that $F(\frac{1}{2}) < 0$ iff $D_B D_A(\frac{1}{2}) < \frac{1}{2}$ iff

$$\delta_B(1 - \frac{1}{2}\delta_A(\frac{1}{2})) > \frac{\frac{1}{2}}{1 - \frac{1}{2}\delta_A(\frac{1}{2})}. \tag{F.2}$$

Note that $1 - \delta_A(\frac{1}{2})\frac{1}{2} > \frac{1}{2}$. Condition (F.2) depends on the value of $\delta_A(\cdot)$ at $\frac{1}{2}$and of $\delta_B(\cdot)$ at $1 - \delta_A(\frac{1}{2})\frac{1}{2} > \frac{1}{2}$. Thus it can hold even if $\delta_A \geq \delta_B$. For instance, taking $\delta_A = \delta_B = \delta$ we can choose $\delta(\frac{1}{2})$ arbitrarily and set $\delta(1 - \delta(\frac{1}{2})\frac{1}{2}) > \max\{\frac{\frac{1}{2}}{1-\delta(\frac{1}{2})\frac{1}{2}}, \delta(\frac{1}{2})\}$. For a specific example, let $\delta_A(x) = \delta_B(x) = 0.5+0.45x$. Then $F(0.31) \simeq 0$.

Proof of Theorem 4.2: Denote the greatest fixed point of $D_B D_A(\cdot)$ by g. Since $D_B D_A(\cdot)$ is strictly increasing and $D_B D_A(1) < 1$, we see that $g < D_B D_A(x) < x$ on $(g, 1]$. Indeed, $[D_B D_A]^n(1)$ converges to some point $f \geq g$. By continuity, $f = \lim_{n\to\infty}[D_B D_A]^n(1) = \lim_{n\to\infty}[D_B D_A]([D_B D_A]^{n-1}(1))$
$= [D_B D_A](\lim_{n\to\infty}[D_B D_A]^{n-1}(1)) = [D_B D_A](f)$. Thus f is a fixed point of $D_B D_A(\cdot)$. Evidently then, $f = g$. This establishes that the odd subsequence

of $\{A_n\}$ converges to g. By a similar argument, the sequence $\{[D_A D_B]^n(1)\}$ converges to the greatest fixed point of $D_A D_B(\cdot)$.

Observe that

$$[D_B D_A]^n (D_B(1)) = D_B ([D_A D_B]^n(1)).$$

Therefore the even subsequence of $\{A_n\}$ is $\{D_B ([D_A D_B]^n(1))\}$. We just noted that $[D_A D_B]^n(1)$ converges to the greatest fixed point of $D_A D_B(\cdot)$. Denoting this greatest fixed point by h we therefore see that the even subsequence of $\{A_n\}$ converges to $D_B(h)$. To complete the proof, we need to show that $D_B(h)$ is in fact the least fixed point of $D_B D_A(\cdot)$. Denote this least fixed point by l.

First observe that $D_B(h)$ is a fixed point of $D_B D_A(\cdot)$: since $D_A D_B(h) = h$ it follows that $D_B(D_A D_B(h)) = D_B(h)$ and in particular, $[D_B D_A](D_B(h)) = D_B(h)$, as desired. By an identical argument, it follows that $D_A(l)$ is a fixed point of $D_A D_B(\cdot)$. To establish that $D_B(h) = l$, note that by definition of l and h,

$$l \leq D_B(h) \text{ and } D_A(l) \leq h.$$

Since $D_B(\cdot)$ is strictly decreasing, $D_A(l) \leq h$ and the definition of l implies $l = D_B D_A(l) \geq D_B(h)$, that is, $l \geq D_B(h)$. But $l \leq D_B(h)$ also holds, and thus we have proved that $D_B(h) = l$.

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