Delay Functions as the Foundation of Time Preference: Testing for Separable Discounted Utility*

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January 11, 2022

Abstract

Delay functions elicit the form of discount functions with minimal assumptions by varying rewards' timing while fixing amounts. We provide conditions to test for separable discounted utility (SDU) using delay function. After eliciting individual delay functions from a representative U.S. sample, we focus on impatient participants whose discounting can be observed. Assuming SDU, the majority are classified as exponential discounters. However, we reject the SDU assumption for 68% in favor of magnitude-dependent discounting with time distortion. Thus, the behavior of impatient participants in small-stakes experiments is not informative about large-stakes market behavior, given the rejection of the SDU assumption.

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1. Introduction

This paper examines time preference using a novel behavioral object we refer to as a "delay function". While the empirical literature has debated over individuals are exponential or hyperbolic discounters,[1] both exponential and hyperbolic models share the common assumption that the agent's preferences can be represented by a *Separable Discounted Utility* (SDU) function of the form $U(m,t) = D(t) \cdot u(m)$.

Here, separability refers to the fact that the evaluation of time and money are unrelated. While SDU is attractive because of its parsimony, it matters for interpreting the results of experiments: separability implies that the level of discounting shown in small stakes behavior (as often used in experiments) is informative about more important decisions, such as saving for retirement. Our delay functions approach can test the validity of the SDU assumption as well as provide information about the structure of discounting even if SDU assumptions are violated. Moreover, it offers a means of quantifying the departure from SDU.

We conduct an experiment measuring delay functions on a sample of the American population and find that the SDU assumption is rejected for 68% of our analysis sample. The take-away lessons from this result are that (i) non-separable models may be important for understanding economic behavior, and (ii) experimental estimates of time preference from low-stakes decisions may not easily translate into high-stakes contexts outside the lab.[2]

Empirical discounting studies often work with "money earlier or later" decisions (Cohen et al. 2020). For instance, in one type of "multiple price list" experiment, the dates of potential payments are fixed and an amount $\$l$ is elicited that makes the participant indifferent between a smaller amount $\$s$ at time t_1 and larger amount $\$l$ at time t_2. Other studies present participants with a series of binary choices between earlier and later payments. In both cases, researchers then typically assume an SDU

[1] See e.g. Andreoni and Sprenger [2012], Augenblick, Niederle, and Sprenger [2015].

[2] This result is consistent with recent research of Ericson et al. [2015], showing that a non-SDU heuristic model outperforms all common SDU models in a cross-validated prediction exercise. Their heuristic model, however, is only defined for binary choices, not multiple price list or multiple delay list elicitation methods.

model and attempt to estimate D and u.

Delay functions take a different approach and instead ask, what *delay* $\Phi_l(s,t)$ makes the subject indifferent between \$$s$ at time t and \$$l$ at time $\Phi_l(s,t)$? Delay functions thus fix the money dimension for both earlier and later rewards, and vary only the time dimension for the later reward. Specifically, for a fixed small reward s and large reward l, a delay function elicits for each t the delay from time zero $\Phi_l(s,t)$ such that

$$(s,t) \sim (l, \Phi_l(s,t)). \tag{1.1}$$

That is, the subject reveals that s at time t is as good as l at time $\Phi_l(s,t)$. Delay functions can be obtained in an experimental setting by using a "multiple delay list" rather than a "multiple price list" procedure, as we do in our experiment, or alternatively by the Becker-DeGroot-Marschak mechanism.

Delay functions allow researchers to estimate more general discounting models that allow participants to discount different size rewards differently (magnitude-dependent discounting). Existing evidence suggests that this may be the case: experiments often find a *magnitude effect* in choice, with participants seeming to exhibit more patience towards larger rewards (Thaler [1981], Frederick et al [2002], Noor [2011], Andersen et al. [2013]). Whether this observed fact rejects SDU is controversial, because experiments must typically estimate both $D(t)$ and the curvature of $u(m)$. The delay function approach does not require us to estimate u.

This paper derives an explicit formula for computing the subject's discount function on the basis of the data Φ. This is done for preferences with very little structure: complete, transitive, continuous preferences that satisfy monotonicity and impatience, admitting *General Discounted Utility* representations of the form

$$U(m,t) = D(m,t) \cdot u(m),$$

where $D(\cdot, 0) = 1$. We provide the general expression for D in terms of Φ, and then specialize it for the purpose of experimental applications. In particular, we are interested in a simple test for the existence of an SDU representation. While SDU is an attractive theoretical assumption (parsimony, appealing axioms, etc.), its descriptive validity is of interest here. We provide a method for data to speak on whether separability adequately describes observed behavior.

Our main specialization restricts attention to a flexible class of delay functions:

$$\Phi(m,t) = (a(m)t^{\gamma} + b(m)^{\gamma})^{1/\gamma}.$$

SDU models (whether exponential, generalized hyperbolic, etc.) correspond to delay functions that lie in this class. In SDU models, delay functions take the specific form

$$\Phi(m,t) = [1 + kb(m)]\, t + b(m),$$

with $k \geq 0$ and b decreasing in m. Our flexible functional form permits nonlinearity in the delay function and magnitude dependence. For this class of delay functions, the test for SDU discounting is simply that there is some $k \geq 0$ such that $a(m) = 1 + kb(m)$ for all m.

We can quantify deviations from SDU by estimating a more general discount function. We show that participants who fail the SDU test can nevertheless be attributed magnitude-dependent discounting with time-distortion of the form: $D(m,t) = e^{-a(m)\cdot t^{\gamma}}$ where a is decreasing in m. In this model, the subject is more patient towards larger rewards, as suggested by the magnitude effect, and moreover, has a distorted perception of time. Time distortions are motivated by a literature suggesting non-linear perception of time (e.g. Zauberman et al. [2009], Takahashi [2005]). Magnitude-dependent discounting could arise from a decision process where the agent's current self optimally uses limited cognitive resources to overcome selfishness and put herself in the shoes of her future selves (Noor and Takeoka [1]). It may also relate to models of mental accounting in which larger rewards are assigned to different accounts, or models of self-control and temptation preferences (e.g. Gul and Pesendorfer [2001], Fudenberg and Levine [2006]) in which the degree of self-control implemented may vary with the magnitude of reward at stake.

We elicit delay functions in an experiment on a representative sample of the American population recruited from a professional sampling service. We focus on our analysis sample: the 40% of individuals who are impatient enough for us to reliably observe their discounting. In particular, we drop the 26% who ever make a non-monotonic choice (not rationalized by any discounting model), and the 34% who are too patient for us to reliably estimate their delay function. We therefore view our results as characterizing the behavior of relatively impatient participants in experiments, a population of particular interest.

We conduct all our estimation at the individual level, allowing for heterogeneity across individuals in preferences and models. We first assume that delay functions are linear ($\gamma = 1$ in the above class) and that the individuals have SDU ($a(m) = 1 + kb(m)$ for all m). Within this subclass, the individual is exponential if $k = 0$ and hyperbolic if $k > 0$. We find that the median k in our analysis sample is virtually zero, suggesting that more than half of our sample were exponential discounters if we limited consideration to SDU models with linear delay functions only.[3] However, we then directly test the SDU assumption. For 68% of our analysis sample, we reject SDU (with or without time distortion), finding strong evidence of non-SDU discounting. Using model selection criteria, only 18% of our sample has a best-fit model that is consistent with SDU. Our results suggest that if researchers only examine SDU models, they may mistakenly conclude that individuals are exponential discounters.

The rejection of separability has a significant implication for the extrapolation of experimental results. We find magnitude-dependent discounting in a relatively modest range of stakes ($50 v. $90), indicating magnitude-dependence may be more important than previously thought. Assuming separability is necessary for behavior in experiments involving small stakes to be informative about decisions outside the lab with large stakes. The results also reveal that there is a possibly significant benefit to researching economic explanations outside the class of SDU models. A

[3] Using a related elicitation method, Attema et al [2010] restrict consideration to SDU models and find that 58% of classifiable subjects were exponential discounters. They do not, however, test the SDU assumption. Other literature has also assumed SDU, and focused on testing exponential v. present-biased discounting (hyperbolic or quasi-hyperbolic). Using a quite different elicitation method (convex time budgets), Andreoni and Sprenger [2012] do not find present-bias for money. Augenblick, Niederle, and Sprenger [2015] also do not find substantial present-bias for money, but do find present-bias for effort.

worthwhile avenue for future research is to model magnitude-dependent discounting of the type revealed in our experiment and to study its implications in economic settings.

Our theoretical analysis applies to any type of dated reward (money, food, work, etc.). Our experiment uses Amazon.com gift certificates, which is similar to money but perhaps less fungible. Much of the literature uses money (see Cohen et al. 2020). However, there is a dispute over whether money from experiments is consumed when received. While many models assume that individuals will smooth their consumption across time, models of mental accounting or dual-self models (e.g. Fudenberg and Levine [2006]) predict that income from experiments may be consumed when received. Andersen et al. [2008] have data on risk questions and time-money tradeoff questions, and jointly estimate the curvature of utility, discount function, and degree of consumption smoothing. They estimate that payments are consumed when received; see also Booij and van Praag [2009]. However, in recent work, Augenblick, Niederle, and Sprenger [2015] find substantial differences in the degree of preference reversals for money versus real effort tasks, suggesting choices over money may not reveal the discount function. However, if individuals do smooth consumption over time, taking advantage of outside-the-lab borrowing and lending, then participants' choices should reveal the interest rate they face (Cubitt and Read [2007]). If the interest rate faced is constant in the range of dollar amounts considered (in our experiment, \$50 to \$100), then their choices should appear as though they were SDU discounters. We in fact reject SDU for most participants. Our results therefore reject, at the very least, exponential discounting of income (as opposed to consumption).

While our delay function approach is defined for the analysis of time preference, it can be adapted easily to other domains as well. For instance, experiments on risk often offer participants lotteries that have one nonzero payoff. Such lotteries can be written as (m, p), where p is the probability of the nonzero outcome. By defining 'time' as $t = \frac{1}{p} - 1$ our procedure becomes immediately applicable to the study of risk preference, where the general representation takes the form $U(m, p) = f(m, p)u(m)$ and where f is the decision weight.

The remainder of the paper proceeds as follows. We close the introduction with related literature. Section 2 presents the main theoretical results and Section 3 presents specializations. Section 4 presents our experiment and results and Section 5 concludes. All proofs are contained in appendices.

Related literature

We term the common method of eliciting discount functions as the "present value" approach: participants choose between dated rewards (m, t) and give indifference points of the form $(s, t) \sim (l, t')$, where, for any dates t,t' either the future reward l is fixed and the participants' present value s is obtained, or the present reward s is fixed and the future value l is obtained.[4] In recording how present/future value of a reward changes with t, the data reflects behavior when both the money and time dimensions are changed. This is reflected in how conclusions are drawn about the discount function. Presuming the SDU model, the discount function $D(t)$ is elicited by computing

$$D(t) = \frac{u(s)}{u(l)},$$

and so eliciting the discount function requires an assumption on u. The early literature assumed that u is linear. Since this assumption typically yields implausibly high discount rates, the literature has sought methods of eliciting discount functions and the curvature of u simultaneously. Andersen et al [2008] replace the linearity assumption with the expected utility assumption, and they use both risk preferences and time preferences to jointly estimate several specifications of u and D. Andreoni and Sprenger [2012] replace the linearity assumption with the assumption that preferences over consumption streams are represented by a time-additive SDU model $\sum D(t)u(m_t)$ with CRRA u. Participants are asked to choose their allocation of an endowment over two periods for different interest rates and endowments, and thus their intertemporal demand curves are obtained, to which u and D are jointly fit.

The theoretical literatures on multi-attribute utility and conjoint measurement (Fishburn [1967], Krantz et al [1971]) introduce the "sawtooth method" to behaviorally isolate the components of any separable representation, and this is built on

[4]See Fredrick et al [2002] for a review of the experimental literature, and later experimental work by Coller and Williams [1999] and Harrison et al [2002].

the idea of varying only one dimension while fixing others.[5] Attema et al [2010]'s study of discount functions and Wakker and Deneffe [1996]'s study of probability weighting implement the sawtooth method experimentally. The delay function approach differs from the sawtooth method, most importantly, because we established our approach for non-SDU preferences.[6]

Other work has examined time preference by varying the delay instead of reward amount. Laury, McInnes, and Swarthout [2012] also show a procedure for eliciting discount rates using variation in the probability a payment will be made, rather than the amount or the delay (as in our proposed approach). Their approach also assumes SDU, as well as either expected utility theory or a particular probability weighting function. Similarly, Olea and Strzalecki [2014] provide a method using "annuity compensations" to estimate quasi-hyperbolic discounting without estimating the utility function u. Their approach assumes a quasi- or semi-hyperbolic discounting model– members of the SDU class– while our approach can test the underlying SDU assumption. Takeuchi [2011] also uses "equivalent delays" to test for present bias without needing to estimate the utility function, but does not test for SDU; in an additional result, he shows how to estimate the discount function independently of the utility function u under the assumption of SDU. Finally, Olivola and Wang (2016) compare discount functions elicited via auctions using either delays (similar to our approach) or money under the assumption of linear utility for money.

[5]These papers consider a preference over binary attributes (x, y). Fix any y, y' and x_0 and suppose that x_1 is a quantity such that the agent exhibits $(x_1, y) \sim (x_0, y')$. Furthermore, suppose it is determined that, iteratively for $i = 2, .., n$, that $(x_i, y) \sim (x_{i-1}, y')$. If the preference has a multiplicative representation, $U(x, y) = v(x) \cdot u(y)$, then each indifference point satisfies $\frac{v(x_i)}{v(x_{i-1})} = k$ for all $i = 1, .., n$, for some constant $k := \frac{u(y')}{u(y)}$. Since v is unique up to an affine transformation, $v(x_0)$ and $v(x_1)$ can be normalized, and consequently v is pinned down on $\{x_0, .., x_n\}$. The grid can be made arbitrarily finer. This procedure is referred to as the 'saw-tooth method' (Fishburn [1967]) and the noted sequence is an example of a 'standard sequence' (Krantz et al [1971]).

[6]The theoretical derivation of the discount function is very different – we find a solution to a functional equation rather than doing a direct construction with the sawtooth method. We also avoid incentive incompatibility issues that exist in the experimental application of the sawtooth method, as discussed in Harrison and Ruström [2009]. In the sawtooth method, a subject's answer to one question becomes an input into the next question. Therefore by misstating preferences it is possible for subjects to affect the sequence of questions they face in a way that improves their expect outcome.

2. Theoretical Framework

2.1. General Discounted Utility

Consider a preference relation $\succsim$ over the set of dated rewards $X = \mathcal{M} \times \mathcal{T}$, where time is continuous and given by $\mathcal{T} = \mathbb{R}_+$, with generic elements t, t'. The set of rewards (e.g. money) is a bounded interval $\mathcal{M} = [0, \overline{m}]$ with generic elements m, m', s, l.

We provide a general result for preferences $\succsim$ over X that have very minimal structure. Say that a preference $\succsim$ is *regular* if it satisfies the following familiar basic restrictions:

1- **Order**: $\succsim$ is complete and transitive.

2- **Continuity**: For each (m,t), the sets $\{(m',t') : (m',t') \succsim (m,t)\}$ and $\{(m',t') : (m,t) \succsim (m',t')\}$ are closed.

3- **Impatience**:

(i) For all $m > 0$ and $t < t'$, $(0,t) \sim (0,t')$ and $(m,t) \succ (m,t')$.

(ii) For each m, m' such that $m' > m > 0$, there is t such that $(m, 0) \succ (m', t)$.

4- **Monotonicity**: For all t, if $m < m'$ then $(m', t) \succ (m, t)$.

Monotonicity states that more is better at any given time. Impatience (i) states that the agent does not care when she receives \$0 but otherwise strictly prefers earlier rewards. Impatience (ii) states that with sufficient delay, any large reward m' can be made worse than an immediate small reward m.

Using standard arguments[7], one can show that a preference $\succsim$ is regular if and only if it admits a *General Discounted Utility* (GDU) representation:

$$U(m,t) = D(m,t) \cdot u(m),$$

where $u : \mathcal{M} \to \mathbb{R}_+$ is a *utility index* (a strictly increasing, continuous function

[7] See Lemma A.1 in the appendix.. By a standard result (see for instance Fishburn and Rubinstein [1982]), for any utility index u there exists a unique representation U for a regular preference $\succsim$. However, any representation U can be uniquely written in the form of a GDU representation (D, u) as follows: for any representation U, the utility index u in any GDU functional form is uniquely defined by $u(m) = U(m, 0)$, and D is uniquely defined by $D(m,t) = \frac{U(m,t)}{u(m)}$ for all $m > 0$.

satisfying $u(0) = 0$) and $D : \mathcal{M} \times \mathcal{T} \to (0, 1)$ is a magnitude-dependent *discount function* (a continuous, strictly decreasing function satisfying $D(m, 0) = 1$ and $\lim_{t\to\infty} D(m, t) = 0$ for all $m > 0$) such that $D(m, t)u(m)$ is strictly increasing in m. We will often refer to the tuple (D, u) as the GDU representation.

At this level of generality, a given regular preference $\succsim$ will admit uncountably many GDU representations, and so one would wish to define a unique canonical representation that can be estimated. Below, we first establish a result for the full GDU class of representations and subsequently specialize the model and identify a canonical representation for the purpose of experimental application (Section 2.4).

2.2. Delay Functions

The choice data needed for our analysis is the *delay function* $\Phi : \mathcal{M} \times \mathcal{T} \to \mathcal{T}$, which is obtained via the indifference:

$$(m, t) \sim (\overline{m}, \Phi(m, t)),$$

for all $0 < m \leq \overline{m}$ and each t. That is, $\Phi(m, t)$ is defined as the date such that m at t is just as good as $\overline{m}$ at $\Phi(m, t)$.[8,9] Varying t leads to a variation in the desirability of (m, t), and this is measured by variation in the delay $\Phi(m, \cdot)$. The simplest example is a linear delay function, $\Phi(m, t) = a(m)t + b(m)$.

The delay function can be measured in practice by using the Becker-DeGroot-Marschak mechanism[10] or by adapting the Multiple Price List (MPL) popularized

[8] Compared to the notation $\Phi_l(s, \cdot)$ in the Introduction for any pair of rewards $s \leq l$, here we fix the largest reward l at $\overline{m}$, and suppress it in the notation.

[9] Since Φ is built from indifferences, the preference $\succsim$ will reflect itself in Φ. For completeness, we note (see lemma A.3 in the appendix) that the regularity of $\succsim$ is equivalent to the following restrictions on Φ:

(i) $\Phi(m, t)$ is continuous.

(ii) For any t, $\Phi(\cdot, t)$ is strictly decreasing and $\lim_{m\to 0} \Phi(m, t) = \infty$.

(iii) For $m > 0$, $\Phi(m, \cdot)$ is strictly increasing and $\Phi(\overline{m}, t) = t$ for all t.

The second condition expresses Monotonicity and the third expresses Impatience. Order is expressed in the fact that Φ is a function.

[10] With the BDM mechanism, a participant would state their maximum acceptable delay such that they would take the larger later payment. A random number would be drawn. If the number was lower than the maximum acceptable delay, they would get the larger later payment, and if

by Coller and Williams [1999] and Harrison et al [2002]. An MPL asks questions of the form "Do you prefer \$100 now or \$$x$ in 6 months?" where x varies over a grid $x_1, ..., x_{N+1}$ of dollar amounts. The implied interest rate associated with x increases monotonically moving down the list, and the point at which the subject switches from preferring the earlier reward to the later reward determines an interval $[x_i, x_{i+1}]$ within which an indifference point '(\$100,now) $\sim$ (\$$z$,6 months)' lies. A 'Multiple Delay List' asks a sequence of questions of the form "Do you prefer \$50 in 1 month or \$100 in t months?" where t varies over a range of time periods $t_1, .., t_{N+1}$ in a way that the implied interest rate decreases monotonically moving down the list.

We prepare to present a general result that provides the foundations for our experimental procedure. Readers that are more interested in the experimental application can proceed directly to Section 2.4.

2.3. General Framework

Given a regular preference $\succsim$, we show how to compute any GDU representation (D, u) from its delay function Φ. Say that a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a *restricted transformation* if it is continuous, strictly increasing, unbounded and satisfies $g(0) = 0$.

Theorem 2.1. *Consider a regular preference $\succsim$ and its delay function Φ. Then $\succsim$ admits the GDU representation (D, u) if and only if there is a restricted transformation g and some scalar $u(\overline{m}) > 0$ such that for all $m > 0$ and t,*

$$D(m, t) = e^{-[g(\Phi(m,t)) - g(\Phi(m,0))]},$$

and for all $m \geq 0$,

$$u(m) = e^{-g(\Phi(m,0))} \cdot u(\overline{m}).$$

The result characterizes the discount functions and corresponding utility indices that can be attributed to the preference $\succsim$. The functional forms involve an increasing

higher, they would get the smaller sooner payment.

transformation g of Φ. For each g, there is a discount function that can be computed in terms of the difference $g(\Phi(m,t)) - g(\Phi(m,0))$, whereas utility indices can be computed in terms of $g(\Phi(m,0))$. While u is characterized in terms of Φ, it essentially only reflects the information contained in present values: by definition, $(m,0) \sim (\overline{m}, \Phi(m,0))$. In contrast, D requires information on how Φ changes as a function of t. The result reveals that obtaining a functional form for Φ is all that is necessary to obtain all the discount functions attributable to the subject, and it spells out explicitly how this can be done.[11]

While Theorem 2.1 characterizes all the possible representations for $\succsim$ in terms of Φ, its value for practical applications lies in the fact that it allows one to posit tractable functional forms for Φ to estimate, and then to find a GDU representation that can be attributed to it. It is natural to focus on *canonical* representations that generalize the exponential and hyperbolic discount functions. We take this route in Section 2.4. Table 1 lists some possibilities as an illustration.

Φ-Function	Discount Function D	Generated by transformatio
$\Phi(m,t) = g^{-1}(g(t) + g(\Phi(m,0)))$	$D(t) = e^{-rg(t)}$	$a(m) = 1$, any g
$\Phi(m,t) = (1+\alpha\Phi(m,0))t + \Phi(m,0)$	$D(t) = (1+\alpha t)^{-1}$	$a(m) = 1, \ g(t) = \ln(1+\alpha t)$
$\Phi(m,t) = [a(m) \cdot t^{\alpha} + \Phi(m,0)^{\alpha}]^{\frac{1}{\alpha}}$	$D(m,t) = e^{-ra(m) \cdot t^{\alpha}}$	$g(t) = t^{\alpha}$
$\Phi(m,t) = \frac{1}{\alpha}[(1+\alpha t)^{a(m)}(1+\alpha\Phi(m,0)) - 1]$	$D(m,t) = (1+\alpha t)^{-\varphi(m)}$	$g(t) = \ln(1+\alpha t)$

Table 1: Φ-functions and associated D.

The simple idea behind the proof of Theorem 2.1 is as follows. Note that the two indifference points

$$(s,0) \sim (\overline{m}, \Phi(s,0)) \text{ and } (s,t) \sim (\overline{m}, \Phi(s,t))$$

[11] Theorem 2.1 not withstanding, the set of compatible discount functions can be computed through present value data as well: it is readily seen that (D,u) is a GDU representation for a regular preference $\succsim$ if and only if u is a utility index and the discount function satisfies $D(m,t) = \frac{u(p(m,t))}{u(m)}$, where $p(m,t)$ is the present value of (m,t). However, in practical settings, where data is necessarily limited, Φ-data better expresses the discount function than present value data. We demonstrate this claim in Appendix C.

reveal that the loss of attractiveness (due to discounting) in (s,t) relative to $(s,0)$ must equal the loss in $(\overline{m},\Phi(s,t))$ relative to $(\overline{m},\Phi(s,0))$. This translates into the statement that any discount function D attributable to the preference $\succsim$ must satisfy the equality $\frac{D(s,t)}{D(s,0)} = \frac{D(\overline{m},\Phi(s,t))}{D(\overline{m},\Phi(s,0))}$. By definition, $D(s,0)=1$, and so this inequality can be rewritten as:

$$D(s,t) \cdot D(\overline{m},\Phi(s,0)) = D(\overline{m},\Phi(s,t)).$$

But this is a functional equation where D is the unknown function and Φ is the known function. The proof verifies that a discount function D is a solution to this functional equation if and only if there exists a utility index u for which (D,u) is a GDU representation for the preference $\succsim$. The general solution of the functional equation leads to the statement of the theorem.

2.4. A Tractable Class of Delay Functions

In order to impose some structure on preferences $\succsim$, we introduce a class of delay functions:

$$\Phi(m,t) = \left(a(m)t^{\gamma} + b(m)^{\gamma}\right)^{\frac{1}{\gamma}}, \tag{2.1}$$

for $\gamma > 0$. Setting $t=0$ yields that the function $b(\cdot)$ is identified by $b(m) = \Phi(m,0)$, and thus must be strictly decreasing and satisfy $b(\overline{m}) = 0$. Since $\Phi(\overline{m},t) = t$, the function $a(\cdot)$ must satisfy $a(\overline{m}) = 1$. Regularity necessitates, in addition, that $a(\cdot)$ is weakly decreasing.[12] Indeed, this is an *ordinal* restriction. Put together, we see that the curves $\Phi(m,\cdot)$ are upward sloping, non-intersecting, and the curves move down for higher m. The curvature of the delay function with respect to t is captured by γ. The delay function is linear if $\gamma = 1$, concave if $\gamma > 1$, and convex if $\gamma < 1$.[13]

The class of delay functions (2.1) is attractive due its tractability for empirical

[12] To see this, suppose $m > m'$ and $a(m) > a(m')$. Then, by (2.1), for large t we would obtain $\Phi(m,t) > \Phi(m',t)$. But then $(m,t) \sim (\overline{m},\Phi(m,t)) \prec (\overline{m},\Phi(m',t)) \sim (m',t)$. But then we have $m > m'$ and $(m,t) \prec (m',t)$, contradicting regularity (specifically Monotonicity).

[13] The expression for the second derivative of $\Phi(m,t)$ with respect to t is $(1-\gamma)(a(m)t^{\gamma} + b(m)^{\gamma})^{\frac{1-2\gamma}{\gamma}} t^{2(\gamma-1)}[1 - (\frac{\Phi(m,t)}{t})^{\gamma}]$, and the term in the square brackets is always negative since $\frac{\Phi(m,t)}{t} > 1$.

work, the tractability of the tests it yields (see below), and the fact that it can accommodate magnitude-dependence and nonlinearities in the data. As noted, this class, in conjunction with regularity, requires that $a(\cdot)$ is weakly decreasing. However, in our experiment we will not impose this property, but rather let the data speak on whether it holds or not.

Applying Theorem 2.1 with the restricted transformation $g(t) = t^\gamma$ immediately yields that:

Proposition 2.2. *The delay function* Φ *has the form (2.1) if and only if the following general exponential discount function can be attributed to it:*

$$D(m,t) = e^{-ra(m)\cdot t^\gamma}, \qquad r > 0$$

where $a(\cdot)$ *is weakly decreasing in the size of the reward.*

The functional form generalizes exponential discounting in the two key dimensions studied in the literature. The first is time distortion (non-linear perception of time, as in Zauberman et al. [2009]). Indeed, the time distortion $\gamma \neq 1$ in the discount function is a reflection of the nonlinearity in the delay function. The second is magnitude-dependence. It is worth noting that since $a(\cdot)$ is weakly decreasing, $D(m,t)$ in fact exhibits the *magnitude effect*, thereby exhibiting greater patience towards larger rewards (Frederick et al [2002], Noor and Takeoka [1]). Indeed, if we maintain regularity and (2.1), checking whether $a(\cdot)$ is in fact weakly decreasing in the data constitutes a novel test of the magnitude effect.

The free parameter r needs richer data to be pinned down, but for our purposes in the sequel it will suffice to restrict attention to the representation obtained by setting $r = 1$. An attractive feature of this canonical representation is that all its parameters are fixed by Φ.

Next, we identify the conditions under which there exists an SDU representation. Refer to a magnitude–independent discount function $D(t)$ as a *separable* discount function.

Proposition 2.3. *Consider a regular preference* $\succsim$ *and with a delay function* Φ *that has the nonlinear form (2.1). A separable discount function* $D(t)$ *can be attributed if and only if there exists* $k \geq 0$ *such that, for all* m*,*

$$a(m) = 1 + kb(m)^{\gamma}. \tag{2.2}$$

If $k = 0$*, then the only attributable separable discount function is exponential discounting with time distortion,*

$$D(t) = e^{-rt^{\gamma}}, \qquad r > 0.$$

If $k > 0$*, then the only attributable separable discount function is hyperbolic discounting with time distortion,*

$$D(t) = \left(1 + kt^{\gamma}\right)^{-r}, \quad r > 0.$$

Thus, the test for the existence of an SDU representation is simply that the slopes $a(m)$ must be a linear function of $b(m)^{\gamma}$ in (2.1). The slope $k \geq 0$ of this function determines the shape of the separable discount function, which can either be exponential or hyperbolic, but with the possibility of time distortion when $\gamma \neq 1$.

A noteworthy observation is that (magnitude-independent) hyperbolic discounting is *behaviorally a special case* of the magnitude-dependent general exponential discount function on the domain of dated rewards. Therefore the analysis reveals that magnitude-dependent discounting could be an alternative explanation for preference reversals attributed to hyperbolic discounting. The two forms of discounting are substantially different in spirit. Hyperbolic discounting is suggestive of a self-control problem, whereas the magnitude effect is suggestive of bounded cognition: the former suggests a passion for the present Laibson ([1997]) whereas the latter suggests that participants pay greater attention to larger rewards (Noor and Takeoka [1]).

3. Empirical Application

3.1. Experiment Design

For an empirical application of the method, we recruited 100 participants aged 18-65 from an online sampling service (Qualtrics) designed to produce an approximately representative sample of the U.S. based on age, gender, and income.[14] Prior to collecting this data, we ran a pilot using participants recruited from an online labor market (Amazon MTurk); results are quite similar.[15]

Participants were paid a flat participation fee and received incentives to respond truthfully, as 10% of participants were randomly selected to be paid for one of their choices. This level of incentives is in line with previous work, e.g. Andersen et al. [2008] pay 10% of their sample. Payments were made via Amazon.com gift certificate and emailed to the participants.

Participants faced a series of multiple delay list decisions. On each list, they were asked to choose between (m, t)– smaller amount m at time t– and a series of options $(100, t+\Delta)$, with $\Delta \in \{1, 2, 3, 5, 7, 9, 13, 17, 22, 28, 35, 43, 68\}$. Participants saw lists for 5 smaller-sooner amounts $m \in \{\$50, \$60, \$70, \$80, \$90\}$ available at 6 different time horizons for the smaller-sooner amount $t \in \{0, 1, 3, 5, 12, 24\}$ weeks, for a total of 30 lists. As required by the theory, the larger-later amount was constant on all lists (here, $100). Thus, on one particular list, participants choose between $50 at time 0 versus $100 at time 1, then between $50 at time 0 versus $100 at time 2, etc. (See Online Empirical Appendix for screenshots).

Our objective is to find each participant's delay function $\Phi_i(m, t)$ such that $(m, t) \sim (\overline{m}, \Phi_i(m, t))$. For each participant i on each multiple delay list, the de-

[14]The sampling service screens out inattentive participants who fail basic attention checks, such as answering questions too quickly or answering illogically or inconsistently. 11 such participants were screened out. Additionally, 20 subjects began the experiment but did not complete it. These participants do not count toward our 100 completed participants.

[15]The Mturk pilot was unincentivized. The procedure was similar, but differed in the set of smaller, sooner magnitudes considered: $m \in \{\$25, \$50, \$75, \$90\}$.The pilot analysis sample (dropping non-monotonic choices and subjects who ever have a list on which they always take the larger-later option) was 44 of 118 participants; (many subjects always chose $100 over $25). In pilot analysis sample, we find more evidence of decreasing impatience. We get similar results testing model restrictions.

lay at which the participant switches from choosing the larger-later reward to the smaller-sooner reward places a bound on their indifference point. That is, if a participant switches from preferring the later payment to the sooner payment when the delay is 6 v. 7 weeks, the indifference point could be anywhere between 6 and 7 weeks. We run a set of interval regressions (a generalization of tobit models) to explicitly account for these bounds on the indifference point. However, these regressions produce results that are nearly identical to those obtained by assuming that the indifference point is the midpoint on the interval . As a result, we place the indifference point at the midpoint of the interval: if the participant chooses larger-later at $t + \Delta_j$ and smaller-sooner at $t + \Delta_{j+1}$ then $\Phi_i(m,t) = \frac{1}{2}\left(\Delta_j + \Delta_{j+1}\right)$. If a participant always chooses the earlier option on a list, $\Phi_i(m,t) = \frac{1}{2}(1+0) = \frac{1}{2}$. So, for example, a participant who chose the larger-later option in "$50 in 1 week v. $100 in 2 weeks", but the smaller-sooner option in "$50 in 1 week v. $100 in 3 weeks", has revealed they are willing to wait 2 weeks but not 3 weeks for $100. Hence, $\Phi_i(50,1) = 2.5$.

We exclude from our sample the 26/100 subjects who ever make a non-monotonic choice (they do not have a unique switching point on some list). These non-monotonic choices are clearly not able to be rationalized by SDU functions. This rate of non-monotonic choice is not out of line with previous work. For instance, 11% of participants in Meier and Sprenger (2010) make such a non-monotonic choice. Their subjects saw 2 multiple price lists (i.e. measured two indifference points). In contrast, our subjects saw a total of 30 multiple delay lists, so had more opportunity to make non-monotonic choices.

We next identify a relatively patient subsample: the 34 out of the 74 remaining participants who have at least one multiple delay list on which they always choose the larger-later option. Of these patient participants, 26% always make the more patient choice on every question asked, and 59% always make the more patient choice on the majority of multiple delay lists.[16] We cannot reject SDU for a participant who

[16]The maximum available delay offered was constrained for the purposes of payment reliability and feasibility. The latest available payment in this experiment was 92 weeks later, or almost 2 years. The preferences of patient participants could be better captured with lists that use higher smaller-sooner amounts (e.g. $m = 99$). However, we wanted to limit the total number of decisions made to avoid taxing participants' attention and cognitive ability. Incentive-compatible dynamic

always chooses the larger-later option, as they are too patient for us to estimate characteristics of their delay function. If a participant always chooses the later reward, we only observe a lower bound on their discount function. Consequently, patient exponential and patient hyperbolic subjects are indistinguishable. This issue is not unique to our method. Our method– like other multiple price list methods– is not very discerning for patient participants. In studies that impose the SDU assumption, bounds place the discount factor of maximally patient participants near 1 (no discounting). Our focus, though, is on the characteristics of the delay function.

Our final Analysis Sample is comprised of the 40 out of 74 remaining participants who never have a multiple delay list on which they always choose the larger-later option. In robustness checks, we broaden this sample to include those participants who always choose the larger-later option on less than 10 m, t pairs and impute a value of the delay function for those choices; the fraction of participants consistent with an SDU representation is quite similar. As a result, we focus the application of our method to this sample because it guarantees we observe the delay function very well for it.

3.2. The Data

Our Analysis Sample has broad demographic coverage and is similar to the U.S. population. It is 57% male, with a median age of 48, and 50% are married. By comparison, the 18-65 U.S. population has a median age of 43 and a marriage rate of about 50% (2011-2013 American Community Survey). Our sample has a range of income levels: 33% have household incomes below $35,000 and 30% have household incomes above $75,000. By comparison, about 35% of U.S. households have incomes below $35,000 and about 33% have incomes about 75,000. The full sample also has very similar characteristics.

For each participant i, magnitude m of the smaller-sooner payment, and time horizon t to the earlier payment, we calculate $\Delta t_{i,m,t} = \Phi_i(m,t) - t$, which is the maximum *additional* delay the participant is willing to accept while still choosing

designs would be complicated to explain and implement. Pre-testing indicated that our chosen range of magnitudes and delays captured the preferences of the largest fraction of participants.

the larger-later payment. The larger $\Delta t_{i,m,t}$ is, the more patient the choice. Figure 1 plots the average $\Delta t_{i,m,t}$ across participants, broken out separately by magnitude of and time to the smaller-sooner payment. In the upper left panel, the average maximum additional delay participants are willing to take to choose \$100 over \$50 is 10.7 weeks when the smaller-sooner reward was available today, and 13.1 weeks when the smaller-sooner payment was available at 24 weeks. In the lower right panel, we find that on average, participants are only willing to wait 3-4 additional weeks for \$100 over \$90, regardless of when the \$90 became available.

Constant impatience (i.e. exponential discounting) implies that the maximum additional delay $\Delta t_{i,m,t}$ should not vary by the horizon t to the smaller-sooner reward. In contrast, decreasing impatience (e.g. hyperbolic discounting) implies that $\Delta t_{i,m,t}$ should increase with t. Visually, Figure 1 shows only very limited evidence of decreasing impatience at the aggregate level– only the \$50 v. \$100 choice displays a noticeable increasing pattern.[17] However, individual heterogeneity is important, and is not visible in the Figure.

We fist conduct a strong test: do all participants display constant impatience, even if their level of impatience differs? To do so, run the regression $\Delta t_{i,m,t} = \theta_{im} + \xi_{it}$, where $\Delta t_{i,m,t}$ is the maximum additional delay for each participant i, amount m, and horizon t, θ_{im} is participant-specific fixed effects for the smaller-sooner amount, and ξ_{it} is participant-specific fixed effects for the horizon to the smaller-sooner payment. While $\Delta t_{i,m,t}$ can vary across participants and amounts in an arbitrary way (captured in θ_{im}), the test for constant impatience requires that $\Delta t_{i,m,t}$ not depend on the horizon t to the smaller-sooner reward. Thus, to test the hypothesis that all participants have constant impatience, we test the restriction that all $\xi_{it} = 0$. We reject this restriction (and thus constant impatience) via a likelihood ratio test at $p < 0.001$.[18]

[17] This is consistent with a stream of literature that has not found aggregate decreasing impatience for monetary rewards, including McClure et al. [2004], who use the same payment method as we do, and Andreoni and Sprenger [2012], who take care to remove many confounds in their estimation of preferences. Other work, such as Olea and Strzalecki [2014], has found evidence of both decreasing impatience and increasing impatience for monetary rewards.

[18] We can also test separately, for each participant, whether $\xi_{it} = 0$ for that particular i, but doing so has much less power. For 25% of the participants in the Analysis Sample, we reject that

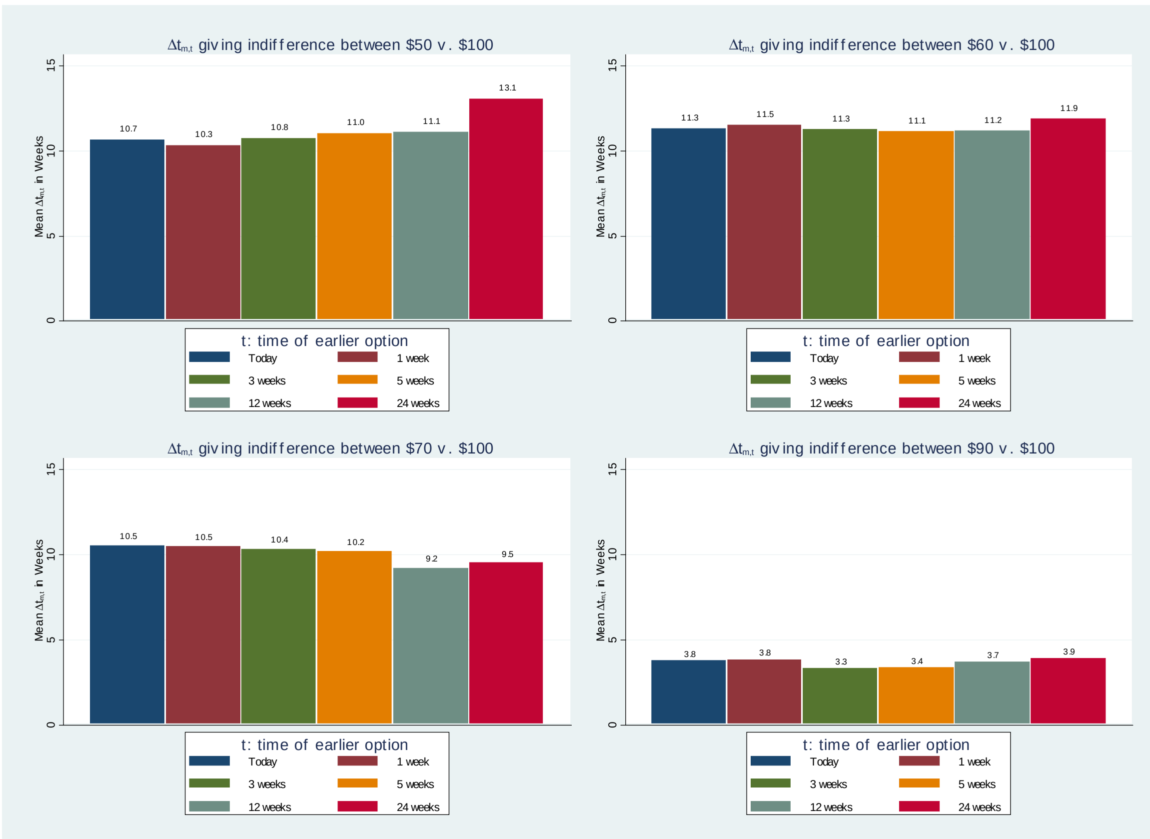


Figure 1: Average maximum *additional* delay accepted for larger-later reward. Sample: Analysis Sample.

3.3. Results

Having calculated the values of the delay function $\Phi_i(m,t)$ from participants' switching points, we then seek to characterize its structure. From the observed values of $\Phi_i(m,t)$ for each m,t pair, we estimate four delay functions via non-linear least squares, separately for each individual. These delay functions vary in whether they impose SDU and the absence of time distortion (i.e. linear delay function), and are as follows:

- SDU-Linear: $\Phi_i(m,t) = \beta_{im} + (1 + k_i\beta_{im})\,t$, where β_{im} can vary across participants and magnitudes. The degree of hyperbolicity k_i can vary across participants, and following Proposition 2.2, we constrain $k_i \geq 0$.
- SDU-NonLinear: $\Phi_i(m,t) = [\beta_{im}^{\gamma_i} + (1 + k_i\beta_{im}^{\gamma_i})\,t^{\gamma_i}]^{\frac{1}{\gamma_i}}$. Relative to SDU-Linear, this adds a time distortion parameter γ_i that can vary across participants.
- NonSDU-Linear: $\Phi_i(m,t) = \beta_{im} + \alpha_{im}t$, where α_{im} and β_{im} can vary across participants and magnitudes.
- NonSDU-NonLinear: $\Phi_i(m,t) = [\beta_{im}^{\gamma_i} + \alpha_{im}t^{\gamma_i}]^{1/\gamma_i}$. Relative to NonSDU-Linear, this adds a time distortion parameter γ_i that can vary across participants.

The SDU-Linear model is of particular interest, since most economic applications of discounting assume SDU and do not allow for time distortion. When we estimate the SDU-Linear model, we find the distribution of k_i seen in Figure 2: the mean k_i is 0.028, and the median k_i is virtually zero (2.13×10^{-16}), suggesting that the majority of our sample discounts exponentially. In fact, for 75% of our sample, we cannot reject the restriction that $k_i = 0$ at the $p < 0.05$ level. (We find similarly small median k_i in the SDU-NonLinear model as well, as shown in Table 2)

However, what appears to be evidence of exponential discounting is instead an artifact of imposing the SDU model. We reject the SDU restrictions, indicating participants cannot be exponential discounters. Because each model is more general

$\xi_{it} = 0$ at $p < 0.05$.

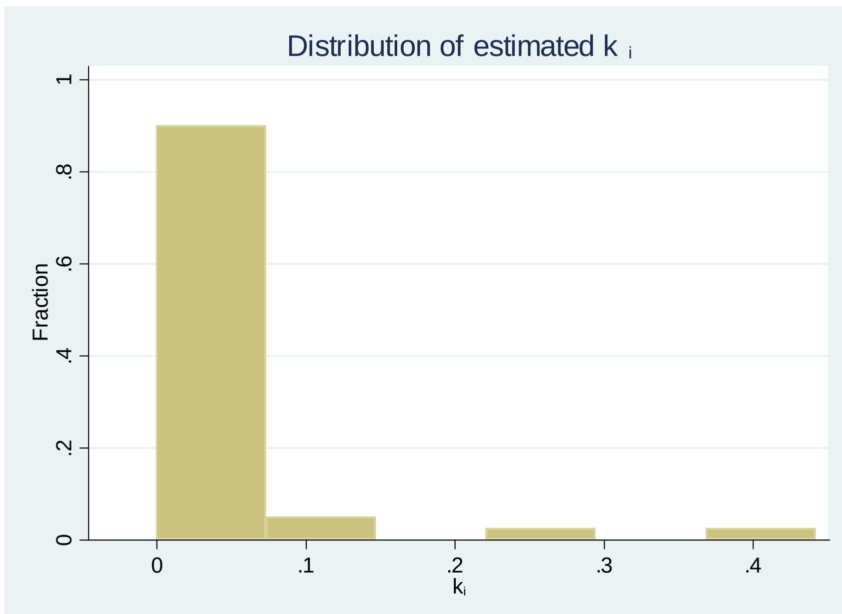


Figure 2: Distribution of Estimated k_i Under the Assumption of the SDU-Linear Model. Sample: Analysis Sample.

than the previous ones, we can use a likelihood ratio test to assess the restrictions implied SDU and linearity (no time distortion). The SDU model is simply the NonSDU-NonLinear model with the constraints that $\gamma_{im} = 1$ and $\alpha_{im} = 1 + k_i\beta_{im.}$ The parameter estimates in the Non-Linear Non-SDU then tell us how choices deviate from the SDU model.

Table 2 shows that we strongly reject the restrictions implied by the SDU model using our likelihood ratio tests.[19] For 68% of the sample we can reject at $p < 0.05$ the restrictions implied by the SDU-Linear model in favor of the NonSDU-NonLinear model; we reject the SDU-NonLinear in favor of the NonSDU-NonLinear model for 70% of the sample.[20]

Time distortion is not the crucial source of our deviation from the SDU-Linear model, as we only reject the NonSDU-Linear model in favor of NonSDU-NonLinear for 38% of the sample. In additional to the results in the table, we have also compared the SDU-Linear model directly to the NonSDU-Linear model. Doing so simply fixes $\gamma_i = 1$ and tests the restriction that $\alpha_{im} = 1 + k_i\beta_{im.}$ For 58% of the sample, we can reject at $p < 0.05$ the restriction implied by the SDU model in favor of the Linear Non-SDU model.

We then use a commonly used model selection criterion, the Akaike Information Criterion (AIC), to determine the "best-fit" model, without privileging any model as the null hypothesis. Of course, a model with more parameters will of course be able to capture more variation in the data. However, the AIC trades off a penalty for additional parameters against the improved fit to select among models. (Note that

[19] The likelihood ratio test is valid under the maximum likelihood interpretation on non-linear least squares (assuming normally distributed errors). Alternatively, we can conduct an F-test of the parameter restrictions implied by SDU-Linear, which does not require the assumption of normally distributed errors. The results of the F-test are similar to that of the likelihood ratio test: we reject SDU-Linear in favor of NonSDU-NonLinear for 68% of the participants.

[20] While it is slightly surprising that we reject the more general SDU model for a larger fraction of the population, this can occur because the null model being tested against the unrestricted model differs. For the participant for which we reject the SDU-NonLinear model but not the SDU-Linear model at p<0.05, we have verified that while the SDU-NonLinear fits slightly better than the SDU-Linear model, the test rejects SDU-NonLinear at p=0.032 and SDU-Linear at p=0.054 because the degrees of freedom for the SDU-NonLinear likelihood ratio test are 4 instead of 5.

the AIC is not a test statistic, merely a way of penalizing additional parameters. See e.g. Hey and Orme [1994]). Table 2 shows that one of the non-SDU models are preferred for 82% of participants. The most frequently selected model is the NonSDU-NonLinear model, chosen for 53% of participants. In addition to the results in the table, we directly compared the SDU-Linear model to the NonSDU-Linear model (thus removing any role for non-linear time perceptions). In this case, the AIC prefers the NonSDU-Linear model 73% of the time over the SDU-Linear model.

For each model, the row in Table 2 displays the mean and median estimated parameters for the all participants. Because each participant is described by a vector of parameters (e.g. β_{im} for $m = 50, 60, ..., 90$) we display a subset of the parameters–those for the lowest and highest m.

The estimated parameters differ when restrictions are placed on the model. Consider what the various $\beta_{m=50}$ parameters imply for the median participant making a decision between \$50 today and \$100 sometime in the future. In the NonSDU-Linear model, that participant would be willing to wait 6.76 weeks, but in the SDU-Linear model willingness to wait would be 7.55 weeks.[21] The lower $\beta_{m=90}$ parameter indicates that the median participant is willing to wait is only 2.5-3.2 weeks for \$100 over \$90 today. An mean $\alpha_{m=50}$ value of 1.36 means that for each additional week the smaller-sooner payment is delayed ($t = 1, 2, ...$) , on average participants would be willing to wait an additional 0.36 weeks (=1.36-1) after the time of the smaller-sooner payment. (Recall, the delay function is measured in absolute time, not time relative to the smaller-sooner payment.) While the median subject has an α near 1, close to constant impatience, there is substantial variation around the median. Finally, γ captures the non-linearity in the delay function; it is more clearly described in the figure.

[21] Note that while Figure 1 shows the mean additional delay accepted for \$50 v. \$100 when $t = 0$ is about 10 weeks, the median additional delay is approximately 8 weeks, near to what is implied by the $\beta_{m=50}$ estimates.

Table 2: Model Selection and Delay Function Parameters

	% of Sample For Which Model is		Parameters: Mean, [Median], (Std. Dev.)					
Model	Chosen by AIC:	Rejected by LR Test:	$\beta_{m=50}$	$\beta_{m=90}$	$\alpha_{m=50}$	$\alpha_{m=90}$	k_i	γ_i
SDU-Linear	13%	68%	10.41	3.43			2.84E-02	
			[7.55]	[2.46]			[2.13E-16]	
			(9.84)	(3.53)			(0.08229)	
SDU-NonLinear	5%	70%	10.72	3.90			4.95E-02	1.03
			[7.42]	[3.19]			[5.41E-12]	[1.01]
			(10.22)	(3.65)			(0.11284)	(0.06)
NonSDU-Linear	30%	38%	10.40	3.58	1.10	1.01		
			[6.76]	[3.13]	[1.04]	[1.00]		
			(9.59)	(3.22)	(0.38)	(0.19)		
NonSDU-NonLinear	53%	NA	9.80	3.54	1.36	1.05		1.35
			[7.39]	[3.19]	[1.05]	[1.01]		[1.14]
			(8.66)	(3.39)	(1.42)	(0.25)		(1.19)

Sample: Analysis Sample. Mean parameters displayed, with medians in brackets and standard deviations in parentheses below. For the NonSDU-NonLinear model, the means and standard deviations exclude one extreme outlier subject. Likelihood ratio tests conducted relative to NonSDU-NonLinear model.

To illustrate the different delay functions, we examine one particular participant in Figure 3, as the population average parameters mask substantial individual heterogeneity. The figure displays the participant's estimated delay functions for two different smaller-sooner amounts m for the SDU-Linear, NonSDU-Linear and NonSDU-NonLinear models (we omit the SDU-NonLinear model for readability). Each panel also includes a 45° line, which is the delay function that would be produced by SDU exponential discounting. First, note that on the right panel ($m = 90$), the lines virtually overlap and are close to the 45° line, showing the models don't make very different predictions for the choice between \$90 and \$100. However, the models substantially differ in the left panel, which describes the choice between \$50 and \$100. Note that the NonSDU-Linear model and SDU-Linear model lines cross– we find the SDU-Linear model predicts more patience when time to the smaller-sooner payment $t \approx 0$, but less patience when t is above 10. The NonSDU-NonLinear model predicts a similar willingness-to-wait to the other models when the smaller-sooner payment is available at short time horizons (t near zero). Yet it quickly diverges when the smaller-sooner payment is available at a longer delay. At about 1 month to the smaller-sooner payment ($t = 4$), the NonSDU-NonLinear delay function shows much more patience than the SDU-Linear model: a willingness-to-wait of about 36 weeks, 12 weeks more than the approximately 25 weeks predicted by the other models.

Finally, a quick glance at Table 2 suggests that the results satisfy the conditions that α_{im} and β_{im} decrease in m, as required by regularity. We more rigorously verify this for the most general NonSDU-NonLinear model by estimating a linear relationship between α_{im} and m and between β_{im} and m. Pooling all participants to address noise,[22] we find the expected statistically significant negative trend on both cases: $\alpha_{im} = -0.009m$, $\beta_{im} = -0.159m$, rejecting a zero coefficient with $p < 0.10$ in both cases. While our participant-specific estimates are subject to more noise, 68% of the participant-specific trends are negative, and only 1 out of 80 trends is positive and statistically significant.

[22] We drop the one outlier participant with extreme parameters values (e.g. $\gamma = 27$, $\alpha_{50} = 4.97\text{e}{+}09$.) Including this participant still gives aggregate negative trends, significant for β but not for α.

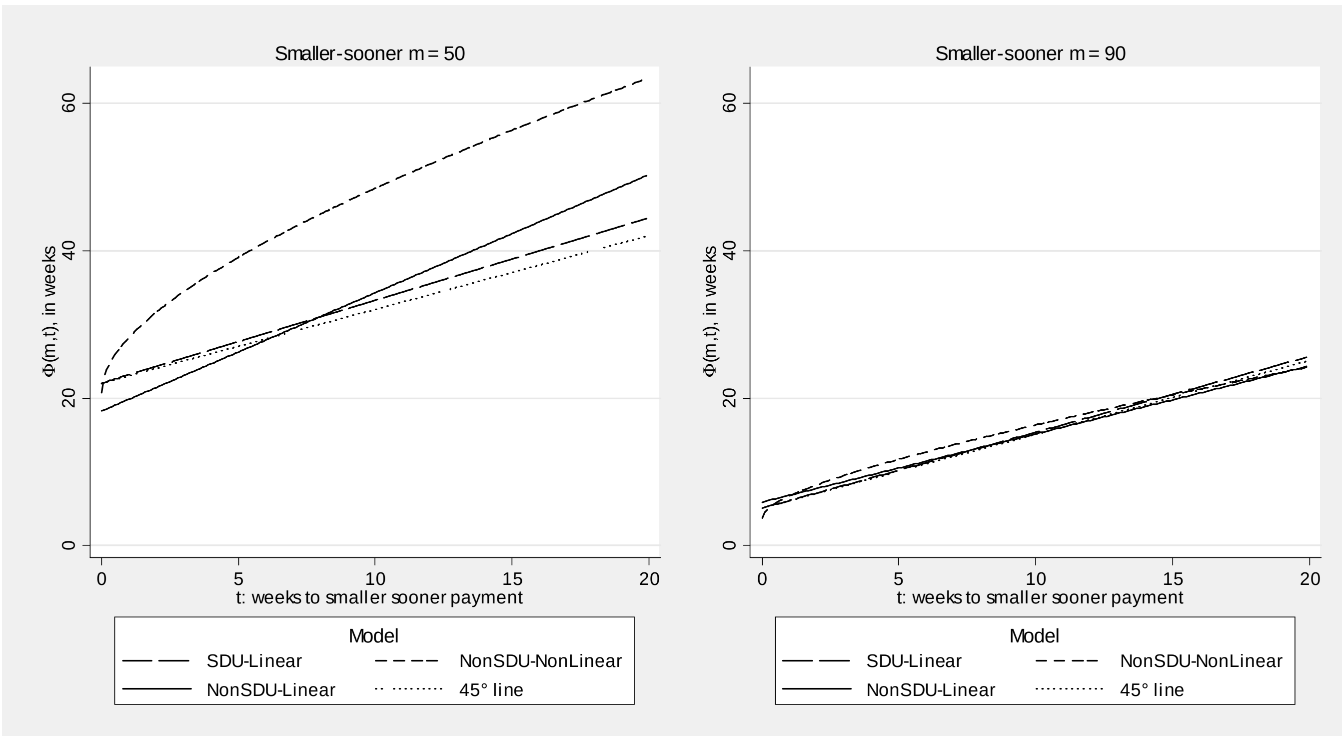


Figure 3: Delay Function Models Plotted Using an Example Participant's Estimates. Note: Parameters are as follows. SDU-Linear model: $k_i = 0.0055$, $\beta_{50} = 22.0$, $\beta_{90} = 5.1$. NonSDU-Linear: $\beta_{50} = 18.2$, $\alpha_{50} = 1.60$, $\beta_{90} = 5.9$, $\alpha_{90} = 0.92$. NonSDU-NonLinear: $\gamma = 0.50, \beta_{50} = 20.7$, $\alpha_{50} = 0.77$, $\beta_{90} = 3.7$, $\alpha_{90} = 0.67$.

3.4. Robustness

The common SDU discount functions (exponential, hyperbolic, generalized hyperbolic) produce a linear delay function and are thus appropriately tested for in our analysis above. However, some SDU discount functions treat $t = 0$ specially and produce a nonlinear delay function that is not of the form considered in Proposition 2.3. For instance, the quasi-hyperbolic $\beta - \delta$ model produces a linear delay function when no option is available at $t = 0$, but has a nonlinearity at $t = 0$. Similarly, the Benhabib, Bisin, and Schotter [2010] fixed-cost of delay model, in which payments delayed from $t = 0$ to $t > 0$ incur a fixed-cost, has a nonlinearity at $t = 0$; nonetheless, it produces a linear delay function when $t > 0$. Additionally, immediate $t = 0$ payments may be treated differently due to confounds, such as trust, uncertainty, or transactions costs.

To account for special treatment of $t = 0$, we repeat the above analysis excluding all choices in which the smaller-sooner option is available at $t = 0$. We compare the SDU-Linear model (which would be produced by the quasi-hyperbolic model and fixed-cost of delay model) to the NonSDU models. The results are very similar:[23] with a likelihood ratio test, the SDU-Linear model is rejected in favor of the NonSDU-Linear model for 64% of participants and rejected in favor of the NonSDU-NonLinear model for 69% of participants. Moreover, the AIC best-fit criterion only chooses the SDU-Linear model for 22% of participants.

We also explore robustness to sample selection. We examine an expanded sample that drops only participants who always choose the larger-later option for more than 10 of the m, t pairs or who ever make a non-monotonic choice.[24] For the resulting 51 participants, the AIC model selection criteria prefers the SDU-Linear model for only 24% of the sample, and we still reject the SDU-Linear model for 63% of the sample

[23] Dropping the $t = 0$ choices, we have difficult fitting the model for 4 participants, who we then exclude from this analysis.

[24] In this expanded sample, we need to impute the indifference point for lists on which the participant always chose the larger-later option. We impute their choices as though they choose the sooner option when the later option was available at an additional delay of 93 weeks, a 25 week increment over the last option they actually faced. We explored alternative imputation strategies, which gave similar results.

in favor of the Non-Linear Non-SDU model via a likelihood ratio test.

Finally, we explore the effect of our assumption that we have point-identified that the indifference point. We estimate two sets of parameters for the NonSDU-Linear model:[25]

- $\Phi_i(m,t) = \beta_{im} + \alpha_{im} t$ using our midpoint method, as before
- $\Phi_i(m,t) = \tilde{\beta}_{im} + \tilde{\alpha}_{im} t$ using interval regression that accounts for the unknown location of the indifference point between the upper and lower bounds provided by the participants' switching point.

The interval regression takes as inputs the upper and lower bounds on the delay function that is obtained from each multiple delay list: $\Phi_i^{Upper}(m,t)$ and $\Phi_i^{Lower}(m,t)$. The interval regression assumes that $\Phi_i(m,t) = \beta_{im} + \alpha_{im} t + \varepsilon$,where the error term ε is i.i.d. normal with variance σ^2. The likelihood for a given observation is then given by $\Pr(\Phi_i^{Lower}(m,t) < \Phi_i(m,t) < \Phi_i^{Upper}(m,t))$.

For each parameter, we correlate the midpoint and interval estimates across participants: e.g. we correlate β_{i10} with $\tilde{\beta}_{i10}$. We have 10 such pairs of parameters (α and β for 5 different magnitudes), giving 10 correlation coefficients between midpoint and interval regression parameters. The parameter estimates are virtually identical using the midpoint and interval regression methods, and all 10 correlation coefficients are above 0.99. We thus conclude that our method of assuming the indifference point is point identified is reliable.

4. Concluding Remarks

While separability is an assumption made for the sake of parsimony in economic models, we evaluate whether there is an empirical price for assuming it. In both our main experiment and pilot experiment, we find that many participants are not well-characterized by separable discounted utility. More than simply rejecting SDU, we

[25] We choose the NonSDU-Linear model because it is general and it can be estimated via Stata's interval regression command intreg. Intreg is unable to estimate models via non-linear least squares or impose the restrictions implied by the SDU-Linear model.

show that the form of our estimated delay functions is consistent with a magnitude-dependent discount function that discounts smaller rewards at higher rates. The general framework we examine invites further exploration. Our results suggest that exploring non-SDU preferences will be fruitful, and that empirical applications might benefit from searching for explanations for observed behavior that do not rely on SDU.

References

Aczel. J. 1966. *Lectures on Functional Equations and Their Applications*, NY Academic Press.

Ainslie, G. 1992. *Picoeconomics*, Cambridge University Press.

Andersen, S., G. Harrison, M. Lau and E. Rutström. 2008. 'Eliciting Risk and Time Preferences', *Econometrica* 76(3), pp 583-618.

Andersen, Steffen, Glenn W. Harrison, Morten I. Lau, and E. Elisabet Rutström. 2013. "Discounting Behaviour and the Magnitude Effect: Evidence from a Field Experiment in Denmark." *Economica* 80 (320): 670–97. doi:10.1111/ecca.12028.

Andreoni, J. and C. Sprenger. 2012. 'Estimating Time Preferences from Convex Budget Sets', *American Economic Review*, 102(7): 3333-56.

Attema, A., H. Bleichrodt, K. Rohde and P. Wakker. 2010. 'Time-Tradeoff Sequences for Analyzing Discounting and Time Inconsistency', *Management Science* 56:11, pp 2015-2030.

Augenblick, Ned, Muriel Niederle, and Charles Sprenger. 2015. 'Working Over Time: Dynamic Inconsistency in Real Effort Tasks.' *Quarterly Journal of Economics* 130 (3): 1067-1115. doi: 10.1093/qje/qjv020

Benhabib, Jess, Alberto Bisin, and Andrew Schotter. 2010. 'Present-Bias, Quasi-Hyperbolic Discounting, and Fixed Costs.' *Games and Economic Behavior* 69 (2): 205–23. doi:10.1016/j.geb.2009.11.003.

Booij, Adam S., and Bernard M. S. van Praag. 2009. 'A Simultaneous Approach to the Estimation of Risk Aversion and the Subjective Time Discount Rate.' *Journal of Economic Behavior & Organization* 70 (1–2): 374–88. doi:10.1016/j.jebo.2009.01.005.

Coller, M. and M. Williams 1999. 'Eliciting Individual Discount Rates,' *Experimental Economics* 2, pp 107-127.

Cubitt, Robin P., and Daniel Read. 2007. 'Can Intertemporal Choice Experiments Elicit Time Preferences for Consumption?' *Experimental Economics* 10 (4): 369–89. doi:10.1007/s10683-006-9140-2.

Cohen, Jonathan, Keith M. Ericson, David Laibson, and John Myles White. 2020. "Measuring Time Preferences." *Journal of Economic Literature* 58(2): 299-347.

Ericson, Keith M., John Myles White, David Laibson, and Jonathan D. Cohen. 2015. 'Money Earlier or Later? Simple Heuristics Explain Intertemporal Choices Better than Delay Discounting.' *Psychological Science* 26(6):826-833. doi: 10.1177/0956797615572232.

Fishburn, P. 1967. 'Methods of Estimating Additive Utilities', *Management Science* 13(7), pp 435-453.

Fishburn, P. and A. Rubinstein. 1982. 'Time Preference', *International Economic Review* 23:3, pp 677-94.

Frederick, S., G. Loewenstein and T. O'Donoghue. 2002. 'Time Discounting and Time Preference: A Critical Review', *Journal of Economic Literature* 40(2), pp 351-401.

Fudenberg, Drew, and David K. Levine. 2006. "A Dual-Self Model of Impulse Control." The American Economic Review 96 (5): 1449–76.

Gul, Faruk, and Wolfgang Pesendorfer. 2001. "Temptation and Self-Control." Econometrica 69 (6): 1403–35. doi:10.1111/1468-0262.00252.

Harrison, G., M. Lau, and M. Williams. 2002. 'Estimating Individual Discount Rates in Denmark: A Field experiment,' *American Economic Review* 92(5), pp 1606-1617.

Harrison, G. and E. Rutström. 2009. 'Risk Aversion in Experiments', *Research in Experimental Economics* 12, pp 41–196.

Hey, J. and C. Orme. 1994. 'Investigating Generalizations of Expected Utility Theory Using Experimental Data.' *Econometrica* 62(6), pp. 1291-1326.

Krantz, D.H., R. D. Luce, P. Suppes and A. Tversky. 1971. *Foundations of Measurement, Vol. I: Additive and Polynomial Representations*, NY Academic Press.

Laibson, D. 1997. 'Golden Eggs and Hyperbolic Discounting', *Quarterly Journal of Economics* 112, pp 443-77.

Laury, Susan K., Melayne Morgan McInnes, and J. Todd Swarthout. 2012. 'Avoiding the Curves: Direct Elicitation of Time Preferences.' *Journal of Risk and Uncertainty* 44 (3): 181–217. doi:10.1007/s11166-012-9144-6.

Mazur, J. E. 1987. 'An Adjustment Procedure for Studying Delayed Reinforcement' in M. L. Commons, J. E. Mazur, J. A. Nevin and H. Rachlin (Eds), *Quantitative Analyses of Behavior: Vol. 5. The Effect of Delay and Intervening Events on Reinforcement Value*, pp 55-73, Hillsdale, NJ: Erlbaum.

McClure, Samuel M., David I. Laibson, George Loewenstein, and Jonathan D. Cohen. 2004. 'Separate Neural Systems Value Immediate and Delayed Monetary Rewards.' Science 306 (5695): 503–7. doi:10.1126/science.1100907.

Noor, J. 2011. 'Intertemporal Choice and the Magnitude Effect', *Games and Economic Behavior* 72(1), pp 255-270.

Noor, J. and N. Takeoka 2021. 'Optimal Discounting', mimeo.

Olea, J. and T. Strzalecki. 2014. 'Axiomatization and measurement of Quasi-hyperbolic Discounting,' *Quarterly Journal of Economics* 129 (3), pp. 1449-1499.

Olivola, Christopher and Stephanie W. Wang. 'Patience auctions: the impact of time vs. money bidding on elicited discount rates.' *Experimental Economics* 19(4), pp 864-885

Rohde, K. 2010. 'The Hyperbolic Factor: A Measure of Time-Inconsistency', *Journal of Risk and Uncertainty* 41(2), pp. 125-140.

Takahashi, T. 2005. 'Loss of self-control in intertemporal choice may be attributable to logarithmic time-perception.' *Medical Hypotheses* 65(4): 691-3.

Takeuchi, K. 2011. 'Nonparametric Test of Time-Inconsistency: Present Bias and Future Bias', *Games and Economic Behavior* 71(2), pp 456–478.

Thaler, R. (1981). 'Some empirical evidence on dynamic inconsistency.' Economics Letters, 8, 201–207.

Wakker, P. and D. Deneffe. 1996. 'Eliciting von Neumann-Morgenstern Utilities when Probabilities are Distorted or Unknown', *Management Science* 42, pp 1131-1150.

Zauberman, Gal, B. Kyu Kim, Selin A Malkoc, and James R Bettman. 2009.'Discounting Time and Time Discounting: Subjective Time Perception and Intertemporal Preferences.' *Journal of Marketing Research* 46 (4): 543–56. doi:10.1509/jmkr.46.4.543.

A. Online Appendix: Regularity and GDU

We establish some basic results on regularity which are used later, though not always explicitly.

Lemma A.1. *For any continuous increasing $u : \mathcal{M} \to \mathbb{R}$, a regular preference $\succsim$ admits a representation $U : \mathcal{M} \times \mathcal{T} \to \mathbb{R}$ such that $U(\cdot, t)$ is continuous and strictly increasing, $U(m, \cdot)$ is continuous and strictly decreasing if $m > 0$ and constant if $m = 0$, and $U(m, 0) = u(m)$. Conversely, any preference that admits such a representation is regular. By defining $D(m,t) = \frac{U(m,t)}{u(m)}$ for any $m > 0$, any such representation can be written as a GDU representation (D, u).*

Proof. The first claim is established in [1982, Thm 1]. The remaining are trivial. ■

Lemma A.2. *If $\succsim$ is regular then*

(a) For every m, t and d there exists $m' \leq m$ such that $(m', t) \sim (m, t+d)$. Moreover, for every m, t and $m' \leq m$ there exists d such that $(m', t) \sim (m, t+d)$.

(b) For any $s \leq l$ and τ such that $(s, 0) \sim (l, \tau)$, and for every $t' \geq \tau$ there exists t such that $(s, t) \sim (l, t')$. Moreover, when $s > 0$ then for any $t \geq 0$ there is a unique $T \geq t$ such that $(s, t) \sim (l, T)$.

(c) For each (m, t) there exists a unique 'present value' $\psi(m, t)$ satisfying

$$(\psi(m,t), 0) \sim (m, t).$$

Moreover, $\psi(0, \cdot) = 0$, $\psi(m, \cdot)$ is strictly decreasing for any $m > 0$, $\lim_{t\to\infty} \psi(m, t) = 0$ for all m, and $\psi(m, \cdot)$ is continuous.

(d) If $(s, 0) \sim (l, \tau)$ and $(s, t) \sim (l, T + \tau)$, then $T + \tau \geq t$.

Proof. Part (a) follows from Impatience, Monotonicity and Continuity; we omit the proof. The t in part (b) exists by Impatience, Monotonicity and Continuity: By Monotonicity, $(s, t') \precsim (l, t')$. By Impatience and the fact that $(s, 0) \sim (l, \tau)$ and $t' \geq \tau$, it follows that $(s, 0) \succsim (l, t')$. Thus, by Continuity, $(s, 0) \succsim (l, t') \succsim (s, t')$ implies that there is t such that $(s, t) \sim (l, t')$, as desired. For the second claim in (b), the existence of T is established in a similar way. Impatience guarantees uniqueness when $s > 0$.

Turning to part (c): part (a) establishes the existence of present values, and Impatience implies that $\psi(m, \cdot)$ is strictly decreasing for any $m > 0$. To see that $\lim_{t\to\infty} \psi(m,t) = 0$ for all m, suppose not. Then there exists m and $s > 0$ such that $(s, 0) \prec (\psi(m,t), 0) \sim (m,t)$ for all t. But this contradicts Impatience. Finally, to see that $\psi(m, \cdot)$ must be continuous, take any strictly increasing homeomorphism and consider the representation U delivered in Lemma A.1. Since $u(\psi(m,t)) = U(m,t)$ and in particular, $\psi(m,t) = u^{-1}(U(m,t))$, continuity of u^{-1} implies that of $\psi(m, \cdot)$.

For part (d), note that if $T + \tau < t$ then $(s, T+\tau) \succ (l, T+\tau)$ by Impatience, which then violates Monotonicity. ■

The next lemma characterizes regularity in terms of properties of Φ. Say that $\Phi : \mathcal{M} \times \mathcal{T} \to \mathcal{T}$ is *generated* by $\succsim$ if for any $0 < m \leq \overline{m}$ and each t, $(m,t) \sim (\overline{m}, \Phi(m,t))$.

Lemma A.3. *Φ is generated by a regular preference $\succsim$ if and only if:*

(i) $\Phi(m,t)$ is continuous.

(ii) For any t, $\Phi(\cdot, t)$ is strictly decreasing and $\lim_{m\to 0} \Phi(m,t) = \infty$.

(iii) For $m > 0$, $\Phi(m, \cdot)$ is strictly increasing and $\Phi(\overline{m}, t) = t$ for all t.

Proof. Prove the 'if' part. Let $\Phi_0^{-1}(t)$ be defined by $\Phi(\Phi_0^{-1}(t), 0) = t$. Define a function $U(m,t) = \Phi_0^{-1}(\Phi(m,t))$, where the inverse exists and is continuous by the monotonicity and continuity properties in (i)-(ii). Intuitively, $U(m,t)$ is the present value of (m,t), that is, if there was a regular preference generating Φ then $(x, 0) \sim (m,t) \sim (\overline{m}, \Phi(m,t))$ and $(x,0) \sim (\overline{m}, \Phi(x,0))$ would hold. Thus $\Phi(x,0) = \Phi(m,t)$, and in turn, $U(m,t) = x = \Phi_0^{-1}(\Phi(m,t))$.

We first verify that U represents a regular preference. By (ii), for fixed t, since $\Phi(m,t)$ strictly decreases in m and Φ_0^{-1} is also strictly decreasing, it follows that $\Phi_0^{-1}(\Phi(m,t))$ is strictly increasing in m. Therefore $U(m,t)$ is strictly increasing in m. Similarly, $U(m,t)$ is strictly decreasing in t if $m > 0$. By continuity of U and by (ii), $U(0,t) = \lim_{m\to 0} U(m,t) = \lim_{m\to 0} \Phi_0^{-1}(\Phi(m,t)) = 0$ for any t.

The other Impatience property follows from the fact that by (ii) and (iii), $0 \le \lim_{t\to\infty} U(m,t) = \lim_{t\to\infty} \Phi_0^{-1}(\Phi(m,t)) \le \lim_{t\to\infty} \Phi_0^{-1}(\Phi(\overline{m},t)) = \lim_{t\to\infty} \Phi_0^{-1}(t) = 0$, that is, $\lim_{t\to\infty} U(m,t) = 0$. As already noted, U is continuous.

Finally, we check that Φ is generated by the preference $\succsim$ represented by U, that is, $U(m,t) = U(\overline{m}, \Phi(m,t))$. Note that by definition and by (iii), $U(\overline{m},t) = \Phi_0^{-1}(\Phi(\overline{m},t)) = \Phi_0^{-1}(t)$. Thus, $U(\overline{m}, \Phi(m,t)) = \Phi_0^{-1}(\Phi(m,t)) = U(m,t)$, as desired. ■

B. Appendix: Proof of Theorem 2.1

B.1. Proof

We formally prove the result by taking $\mathcal{M} = \mathbb{R}_+$ and noting that the same argument establishes Proposition 2.1 as a corollary when $\mathcal{M} = [0, \overline{m}]$.

For a given preference $\succsim$ and any rewards $0 < s \le l$, define the function $\Phi_{s,l}(\cdot)$ by the indifference:

$$(s,t) \sim (l, \Phi_{s,l}(t)). \tag{B.1}$$

For $s = 0 < l$, let $\Phi_{s,l}(t) := \infty$.

We first clarify the exhaustive implications of regularity on Φ.

Lemma B.1. *Φ is generated by a regular $\succsim$ if and only if:*

(i) $\Phi(s,l,t)$ is continuous,

(ii) $\Phi(s,\cdot,t)$ is strictly increasing and $\Phi(\cdot,l,t)$ is strictly decreasing in s, and moreover $\lim_{s\to 0} \Phi(s,l,t) = \infty$ when $l > 0$,

(iii) $\Phi(s,l,\cdot)$ is strictly increasing if $s, l > 0$, and $\Phi(m,m,t) = t$ for all t,

(iv) $\Phi_{m_1,m_2}(\Phi_{m_0,m_1}(t)) = \Phi_{m_0,m_2}(t)$ for all t and $m_0 \le m_1 \le m_2$.

Proof. Prove the 'if' part. Define $\Phi^{-1}_{(l,0)}(r)$ by $\Phi(\Phi^{-1}_{(l,0)}(r), l, 0) = r$. Let $U(m,t) := \Phi^{-1}_{m,0}(t)$, where the inverse exists by the monotonicity and continuity properties in (i)-(ii). Intuitively, $U(m,t)$ is the present value of (m,t), that is, it is a small reward $s = U(m,t)$ that satisfies,

$$\Phi(s, m, 0) = t.$$

We first verify that U represents a regular preference.

To see that U is continuous, suppose $m_n \to m$, $t_n \to t$ and to ease notation write $s_n := U(m_n, t_n)$, that is, $\Phi(s_n, m_n, 0) = t_n$. We show that s_n converges. Since $m_n \to m$, there is some M and N such that $m_n \leq M$ for all $n \geq N$ (wlog let $N = 1$). Define $T_n := \Phi(m_n, M, t_n)$. By (i), T_n converges. Observe that $\Phi(s_n, M, 0) = T_n$ by (iv). Since $\Phi(\cdot, M, 0)$ is strictly monotone and continuous, it follows that $\Phi^{-1}_{M,0}(\cdot)$ is continuous. Therefore, since T_n converges to $T := \Phi(m, M, t)$, it must be that $s_n = \Phi^{-1}_{M,0}(T_n)$ converges to $s := \Phi^{-1}_{M,0}(T) = \Phi^{-1}_{M,0}(\Phi(m, M, t))$, and in particular,

$$\Phi(s, M, 0) = \Phi(m, M, t).$$

It remains to show that $\Phi(s, m, 0) = t$. By the displayed equality and (iv), $\Phi(m, M, \Phi(s, m, 0)) = \Phi(s, M, 0) = \Phi(m, M, t)$ and so by (iii), $\Phi(s, m, 0) = t$, as desired. Thus U is continuous.

Now show the remaining regularity properties. By (ii), for fixed t, the equation $\Phi(s, m, 0) = t$ implies that as m increases, s must also increase. Therefore $U(m, t)$ is strictly increasing in m. Similarly, $U(m, t)$ is strictly decreasing in t if $m > 0$. To show the second Impatience property, take any m, m' such that $m' > m > 0$. By (ii), there is a small enough $s' > 0$ s.t. $\Phi(s', m', 0) > 0 = \Phi(m, m, 0)$. Define $t := \Phi(s', m', 0)$. Then by (ii), $U(m', t) < U(m, 0)$, as desired. Finally we show the first Impatience property, that is, $U(0, t) = 0$. By (ii) and (iii), since $\Phi(m, m, t) = t$, it must be that for s that satisfies $\Phi(s, m, 0) = t$ it must be that $s \leq m$. That is, $0 \leq U(m, t) \leq m$. Then by continuity of U, $U(0, t) = \lim_{m \to 0} U(m, t) = 0$, as desired.

To conclude, we check that Φ is generated by the preference $\succsim$ represented by U. By definition, for any $s \leq l$, $s = U(l, \Phi(s, l, 0))$. Take any t and suppose $s'' = U(l, \Phi(s, l, t))$, that is,

$$\Phi(s'', l, 0) = \Phi(s, l, t).$$

Since (iii) implies $\Phi(s, l, t) \geq \Phi(s, l, 0)$, it follows that $s'' = U(l, \Phi(s, l, t)) \leq U(l, \Phi(s, l, 0)) =$

s. That is, $s'' \leq s \leq l$. By (iv), $\Phi(s, l, \Phi(s'', s, 0)) = \Phi(s'', l, 0)$ and so, by the displayed equality, $\Phi(s, l, \Phi(s'', s, 0)) = \Phi(s, l, t)$. By (iii), $\Phi(s'', s, 0) = t$, and this implies that $U(s, t) = s'' = U(l, \Phi(s, l, t))$, and thus Φ is generated by U, as desired.
■

Lemma B.2. *If D solves the functional equation (FE) below, then for any $0 < m_1 \leq m_2 \leq m_3$,*

$$D(m_2, \Phi_{m_1,m_2}(0)) \cdot D(m_3, \Phi_{m_2,m_3}(0)) = D(m_3, \Phi_{m_1,m_3}(0))$$

Proof. Suppose $m_1 \leq m_2 \leq m_3$ then the functional equation implies

$$D(m_2, \Phi_{m_1,m_2}(0)) \cdot D(m_3, \Phi_{m_2,m_3}(0)) = D(m_3, \Phi_{m_2,m_3}(\Phi_{m_1,m_2}(0)).$$

But transitivity of $\succsim$ implies $\Phi_{m_2,m_3}(\Phi_{m_1,m_2}(0)) = \Phi_{m_1,m_3}(0)$. The assertion follows.
■

Lemma B.3. *The following statements hold:*

(a) Consider any regular preference $\succsim$ and its Φ-function. Then D can be attributed to $\succsim$ if and only if D solves the functional equation:

$$D(s, t) \cdot D(l, \Phi_{s,l}(0)) = D(l, \Phi_{s,l}(t)), \tag{FE}$$

for all $0 < s \leq l$ and t.

(b) Suppose $\succsim$ is a regular preference with the function Φ, and that D is a solution to (FE). Then (D, u) represents $\succsim$ if and only if u is given by

$$u(m) = \begin{cases} D(\overline{m}, \Phi_{m,\overline{m}}(0)) \cdot u(\overline{m}) & \text{if } m \leq \overline{m} \\ [D(m, \Phi_{\overline{m},m}(0))]^{-1} \cdot u(\overline{m}) & \text{otherwise} \end{cases}, \qquad \text{for all } m,$$

where $\overline{m} > 0$ and $u(\overline{m}) > 0$ are arbitrary.

Proof. We prove (a), and part (b) follows as a corollary of the proof.

First show that any attributable D must satisfy the functional equation. By regularity, $\succsim$ admits a representation U. Any representation can be written as a GDU model with some D and u. By definition of the Φ-function (1.1), it must be that for all $s, l > 0$ and t, both $u(s) = D(l, \Phi_{s,l}(0))u(l)$ and $D(s,t)u(s) = D(l, \Phi_{s,l}(t))u(l)$ hold. Rearranging yields the functional equation. Observe that we have also determined that a solution must always exist if Φ comes from a regular preference $\succsim$.

For the converse, suppose D is a solution. Take any $\overline{m} > 0$ and assign it any utility $u(\overline{m}) > 0$. Define

$$u(m) = \begin{cases} D(\overline{m}, \Phi_{m,\overline{m}}(0))u(\overline{m}) & \text{if } m \leq \overline{m} \\ D(m, \Phi_{\overline{m},m}(0))^{-1}u(\overline{m}) & \text{otherwise} \end{cases}, \quad \text{for all } m.$$

By continuity of D, the utility u is continuous as well (monotonicity will be determined shortly). Next we show that, given transitivity of $\succsim$, the utility u is consistent with D in the sense that it satisfies

$$u(s) = D(l, \Phi_{s,l}(0))u(l) \tag{B.2}$$

for all s, l s.t. $s \leq l$. To see this, consider the following cases:

Case 1- $s, l \leq \overline{m}$.

Then $u(s) = D(\overline{m}, \Phi_{s,\overline{m}}(0))u(\overline{m})$ and $u(l) = D(\overline{m}, \Phi_{l,\overline{m}}(0))u(\overline{m})$, which implies

$$u(s) = \frac{D(\overline{m}, \Phi_{s,\overline{m}}(0))}{D(\overline{m}, \Phi_{l,\overline{m}}(0))}u(l).$$

By the Lemma, $D(l, \Phi_{s,l}(0)) \cdot D(\overline{m}, \Phi_{l,M}(0)) = D(\overline{m}, \Phi_{s,M}(0))$, that is, $\frac{D(\overline{m}, \Phi_{s,\overline{m}}(0))}{D(\overline{m}, \Phi_{l,\overline{m}}(0))} = D(l, \Phi_{s,l}(0))$. It follows that (B.2) holds.

Case 2- $s \leq \overline{m} \leq l$

Then $u(s) = D(\overline{m}, \Phi_{s,\overline{m}}(0))u(\overline{m})$ and $u(l) = \frac{u(\overline{m})}{D(l, \Phi_{\overline{m},l}(0))}$, which implies

$$u(s) = D(\overline{m}, \Phi_{s,\overline{m}}(0))D(l, \Phi_{\overline{m},1}(0))u(l).$$

By the Lemma, $D(\overline{m}, \Phi_{s,\overline{m}}(0)) \cdot D(l, \Phi_{\overline{m},l}(0)) = D(l, \Phi_{s,l}(0))$, and (B.2) follows.

Case 3- $\overline{m} \leq s \leq l$.

Then $u(s) = \frac{u(\overline{m})}{D(s,\Phi_{\overline{m},s}(0))}$ and $u(l) = \frac{u(\overline{m})}{D(l,\Phi_{\overline{m},l}(0))}$, which implies

$$u(s) = \frac{D(l, \Phi_{\overline{m},l}(0))}{D(s, \Phi_{\overline{m},s}(0))} u(l).$$

By the Lemma, $D(s, \Phi_{\overline{m},s}(0)) \cdot D(l, \Phi_{s,l}(0)) = D(l, \Phi_{\overline{m},l}(0))$, and (B.2) follows.

Thus u is consistent with D in the sense of (B.2). Observe that the equality also assures us that u must be strictly increasing: D is strictly increasing in its second argument and by Monotonicity and Impatience $\Phi_{s,l}(0)$ must be strictly increasing in s. To show that there is a GDU representation with D, define $U(m,t) := u(p(m,t))$, where $p(m,t)$ is the present value of (m,t). Since $p(m,t)$ is a representation for $\succsim$ and u is strictly increasing, it follows that $U(m,t)$ represents $\succsim$. But then $U(m,t) = u(p(m,t)) = D(m,t)u(m)$, as desired. ■

The next lemma determines how to check if D solves (FE) on the basis of information on $\Phi_{m,\overline{m}}$, $\Phi_{\overline{m},m}$ and the present value of (m,t) for all $m > \overline{m}$ and $t < \Phi_{\overline{m},m}(0)$. Write p_{mt} for the present value of (m,t), that is, $(p_{mt}, 0) \sim (m,t)$.

Lemma B.4. *Fix any $\overline{m} > 0$. Then D solves (FE) for all $0 < s \leq l$ and t if and only if:*

i) D solves (FE) for all s, l s.t. $0 < s \leq l$ for $s = \overline{m}$ or $l = \overline{m}$, and all t; and

ii) $D(m,t) = \frac{D(m,\Phi_{\overline{m},m}(0))}{D(p_{mt},\Phi_{\overline{m},p_{mt}}(0))}$ for all $m > \overline{m}$ and $t < \Phi_{\overline{m},m}(0)$.

Proof. The 'if' part is straightforward – note that part (ii) follows from lemma B.2. Turn to the 'only if' part. Suppose the hypothesis holds. Take any $0 < s \leq l$ and t. Consider the following cases. We make frequent use of the fact that if $m_1 \leq m_2 \leq m_3$ then transitivity implies $\Phi_{m_2,m_3}(\Phi_{m_1,m_2}(t)) = \Phi_{m_1,m_3}(t)$ for any t.

Case 1- $s, l \leq \overline{m}$.

By hypothesis, $D(s,t) \cdot D(\overline{m}, \Phi_{s,\overline{m}}(0)) = D(\overline{m}, \Phi_{s,\overline{m}}(t))$ and $D(l,t) \cdot D(\overline{m}, \Phi_{l,\overline{m}}(0)) = D(\overline{m}, \Phi_{l,\overline{m}}(t))$. Moreover, by transitivity, $\Phi_{l,\overline{m}}(\Phi_{s,l}(t)) = \Phi_{s,\overline{m}}(t)$. Observe that:

$D(l, \Phi_{s,l}(t))$

$= D(l, \Phi^{-1}_{l,\overline{m}}(\Phi_{s,\overline{m}}(t)))$ by transitivity

$= \frac{D(\overline{m},\Phi_{l,\overline{m}}[\Phi^{-1}_{l,\overline{m}}(\Phi_{s,\overline{m}}(t))])}{D(\overline{m},\Phi_{l,\overline{m}}(0))}$ by hypothesis

$= \frac{D(\overline{m},\Phi_{s,\overline{m}}(t))}{D(\overline{m},\Phi_{l,\overline{m}}(0))}$

$= \frac{D(s,t)D(\overline{m},\Phi_{s,\overline{m}}(0))}{D(\overline{m},\Phi_{l,\overline{m}}(0))}$ by hypothesis

$= D(s,t)[\frac{D(\overline{m},\Phi_{s,\overline{m}}(0))}{D(\overline{m},\Phi_{l,\overline{m}}(0))}]$. We are done if we show that $\frac{D(\overline{m},\Phi_{s,\overline{m}}(0))}{D(\overline{m},\Phi_{l,\overline{m}}(0))} = D(l,\Phi_{s,l}(0))$. But this follows

since

$D(l,\Phi_{s,l}(0))D(\overline{m},\Phi_{l,\overline{m}}(0))$

$= D(\overline{m},\Phi_{l,\overline{m}}(\Phi_{s,l}(0)))$ by hypothesis

$= D(\overline{m},\Phi_{s,\overline{m}}(0))$ by transitivity. This completes the argument.

Case 2- $s \leq \overline{m} \leq l$.

By hypothesis $D(s,t)\cdot D(\overline{m},\Phi_{s,\overline{m}}(0)) = D(\overline{m},\Phi_{s,\overline{m}}(t))$ and $D(\overline{m},t)\cdot D(l,\Phi_{\overline{m},l}(0)) = D(l,\Phi_{\overline{m},l}(t))$ and by transitivity, $\Phi_{\overline{m},l}(\Phi_{s,\overline{m}}(t)) = \Phi_{s,l}(t)$. Observe that

$D(l,\Phi_{s,l}(t))$

$= D(l,\Phi_{\overline{m},l}(\Phi_{s,\overline{m}}(t)))$ by transitivity

$= D(\overline{m},\Phi_{s,\overline{m}}(t)) \cdot D(l,\Phi_{\overline{m},l}(0))$ by hypothesis

$= [D(s,t) \cdot D(\overline{m},\Phi_{s,\overline{m}}(0))] \cdot D(l,\Phi_{\overline{m},l}(0))$ by hypothesis

$= D(s,t)\cdot[D(\overline{m},\Phi_{s,\overline{m}}(0))\cdot D(l,\Phi_{\overline{m},l}(0))]$. We are done if $D(\overline{m},\Phi_{s,\overline{m}}(0))\cdot D(l,\Phi_{\overline{m},l}(0)) = D(l,\Phi_{s,l}(0))$. But this follows since

$D(\overline{m},\Phi_{s,\overline{m}}(0)) \cdot D(l,\Phi_{\overline{m},l}(0))$

$= D(\overline{m},\Phi_{\overline{m},l}(\Phi_{s,\overline{m}}(0)))$ by hypothesis

$= D(\overline{m},\Phi_{s,l}(0))$ by transitivity. This completes the argument.

Case 3(i)- $\overline{m} < s \leq l$ and $t \geq \Phi_{\overline{m},s}(0)$.

By hypothesis $D(\overline{m},t)\cdot D(s,\Phi_{\overline{m},s}(0)) = D(s,\Phi_{\overline{m},s}(t))$ and $D(\overline{m},t)\cdot D(l,\Phi_{\overline{m},l}(0)) = D(l,\Phi_{\overline{m},l}(t))$ and by transitivity, $\Phi_{s,l}(\Phi_{\overline{m},s}(t)) = \Phi_{\overline{m},l}(t)$. The restriction $t \geq \Phi_{\overline{m},s}(0)$ implies that $\Phi^{-1}_{\overline{m},s}(t)$ exists. Observe that:

$D(l,\Phi_{s,l}(t))$

$= D(l,\Phi_{\overline{m},l}(\Phi^{-1}_{\overline{m},s}(t)))$ by transitivity

$= D(\overline{m},\Phi^{-1}_{\overline{m},s}(t)) \cdot D(l,\Phi_{\overline{m},l}(0))$

$= \frac{D(s,\Phi_{\overline{m},s}(\Phi^{-1}_{\overline{m},s}(t)))}{D(s,\Phi_{\overline{m},s}(0))} \cdot D(l,\Phi_{\overline{m},l}(0))$

$= D(s,t) \cdot \frac{D(l,\Phi_{\overline{m},l}(0))}{D(s,\Phi_{\overline{m},s}(0))}$. We are done if we show that $\frac{D(l,\Phi_{\overline{m},l}(0))}{D(s,\Phi_{\overline{m},s}(0))} = D(l,\Phi_{s,l}(0))$: observe that by hypothesis and transitivity, $D(s,\Phi_{\overline{m},s}(0))D(l,\Phi_{s,l}(0)) = D(l,\Phi_{s,l}(\Phi_{\overline{m},s}(0))) = D(l,\Phi_{\overline{m},l}(0))$, as desired.

Case 3(ii)- $\overline{m} < s \leq l$ and $t < \Phi_{\overline{m},s}(0)$.

By hypothesis, $D(m,t) = \frac{D(m,\Phi_{\overline{m},m}(0))}{D(p_{mt},\Phi_{\overline{m},p_{mt}}(0))}$ where p_{mt} satisfies $(p_{mt},0) \sim (m,t)$. Since $t < \Phi_{\overline{m},s}(0)$, it must be that $p_{mt} \geq \overline{m}$.

$D(s,t) \cdot D(l,\Phi_{s,l}(0))$

$= \frac{D(s,\Phi_{\overline{m},s}(0))}{D(p,\Phi_{\overline{m},p}(0))} \cdot \frac{D(l,\Phi_{\overline{m}l}(0))}{D(s,\Phi_{\overline{m},s}(0))}$ by hypothesis, where p is the present value of (s,t) (s is the present value of $(l,\Phi_{s,l}(0))$ by definition)

$= \frac{D(l,\Phi_{\overline{m}l}(0))}{D(p,\Phi_{\overline{m}p}(0))}$

$= D(l,\Phi_{s,l}(t))$ by hypothesis since by transitivity p must be the present value of $(l,\Phi_{s,l}(t))$ as well.

This completes the proof. ■

Fix any $\overline{m} > 0$. For any $m > 0$ and $t \geq 0$, define $\Phi(m,t)$ by:

$$\begin{array}{ll} (m,t) \sim (\overline{m},\Phi(m,t)) & \text{if } m \leq \overline{m} \\ (\overline{m},t) \sim (m,\Phi(m,t)) & \text{otherwise.} \end{array}$$

Lemma B.5. *D is attributable iff there is a continuous, strictly increasing and unbounded function g satisfying $g(0) = 0$ such that*

$$D(m,t) = \begin{cases} e^{-[g(\Phi(m,t))-g(\Phi(m,0))]} & \text{if } m \leq \overline{m} \\ e^{-[g(\Phi_m^{-1}(t))-g(\Phi(m,0))]} & \text{if } m \geq \overline{m} \text{ and } t \geq \Phi_{\overline{m},m}(0) \\ e^{-[g(m,0)-g(\Phi(p(m,t),0)]} & \text{if } m > \overline{m} \text{ and } t < \Phi_{\overline{m},m}(0) \end{cases}.$$

Proof. By lemma B.3, D is attributable if and only if it solves (FE). Suppose D solves (FE). Then by lemma B.3, (D,u) represents $\succsim$ for some u. Wlog, let $u(\overline{m}) = 1$. By regularity, $\Phi(m,0)$ is continuous and strictly decreasing in m for $m < \overline{m}$ and strictly increasing for $m > \overline{m}$. Since u is strictly increasing, there is a continuous strictly increasing function g satisfying $g(0) = 0$ such that[26]

$$u(m) = \begin{cases} e^{-g(\Phi(m,0))} & \text{if } m \leq \overline{m} \\ e^{g(\Phi(m,0))} & \text{if } m \geq \overline{m} \end{cases}.$$

We will see shortly that g must be unbounded.[27] Given that (D, u) represents $\succsim$ and $u(\overline{m}) = 1$, the definition of Φ implies $u(m) = D(\overline{m}, \Phi(m, 0))$ for $m \leq \overline{m}$, and $u(m) = D(m, \Phi(m, 0))^{-1}$ for $m > \overline{m}$. Therefore,

$$\begin{array}{ll} D(\overline{m}, \Phi(m, 0)) = e^{-g(\Phi(m,0))} & \text{for } m \leq \overline{m} \\ D(m, \Phi(m, 0)) = e^{-g(\Phi(m,0))} & \text{for } m \geq \overline{m} \end{array}.$$

We use this observation below. Another observation is that by regularity $\Phi(m, 0)$ ranges from 0 to ∞ as m varies over $[0, \overline{m}]$, and so we have

$$D(\overline{m}, t) = e^{-g(t)}.$$

Moreover, since D is a discount function it must satisfy the property $\lim_{t\to\infty} D(\overline{m}, t) = 0$, which implies that g must be unbounded.

To find the general solution of (FE), we first show that D has the desired form for $0 < m \leq \overline{m}$. By the previous lemma, D solves the functional equation for s, l s.t. $[0 < s \leq l = \overline{m}]$ and all t, and in particular, it solves

$$D(m, t) \cdot D(\overline{m}, \Phi(m, 0)) = D(\overline{m}, \Phi(m, t))$$

for any $0 < m \leq \overline{m}$ and all t. Since we have determined that $D(\overline{m}, t) = e^{-g(t)}$, this functional equation therefore implies

$$D(m, t) = e^{-[g(\Phi(m,t)) - g(\Phi(m,0))]} \quad \text{for all } m \leq \overline{m} \text{ and } t,$$

[26] By regularity $\Phi(\overline{m}, 0) = 0$ and so $u(\overline{m}) = e^{-g(\Phi(\overline{m},0))} = e^{-g(\Phi(\overline{m},0))} = 1$, consistent with our assumption that $u(\overline{m}) = 1$.

[27] This does not imply that u is unbounded: though $u(m) = e^{g(\Phi(m,0))}$ for $m \geq \overline{m}$ for unbounded g, regularity does not require $\Phi(\cdot, 0)$ to be unbounded.

as desired.

Next consider $m \geq \overline{m}$. By the previous lemma, D must satisfy

$$D(\overline{m}, t) \cdot D(m, \Phi(m, 0)) = D(m, \Phi(m, t)).$$

Then, given the earlier observations, $D(m, \Phi(m, t)) = D(\overline{m}, t) \cdot D(m, \Phi(m, 0)) = e^{-[g(t)+g(\Phi(m,0))]}$. Therefore,

$$D(m, t) = e^{-[g(\Phi_m^{-1}(t))+g(\Phi(m,0))]} \text{ for all } m > \overline{m} \text{ and } t \geq \Phi_{\overline{m},m}(0).$$

Finally, to consider the case $[m > \overline{m}$ and $t < \Phi_{\overline{m},m}(0)]$, we note that by the previous lemma $D(m, t) = \frac{D(m, \Phi_{\overline{m},m}(0))}{D(p_{mt}, \Phi_{\overline{m},p_{mt}}(0))}$ and therefore by our earlier observations,

$$D(m, t) = e^{-[g(\Phi(m,0))-g(\Phi(p(m,t),0))]} \text{ for } m > \overline{m} \text{ and } t < \Phi_{\overline{m},m}(0).$$

Thus, we have shown that if D is attributable to the preference then it must have the desired form.

To complete the proof, we need to check that the discount function solves (FE). This is straightforward to establish in light of the previous lemma. For instance, for the case where $l = \overline{m}$, we see that

$D(m, t) \cdot D(\overline{m}, \Phi(m, 0)) = D(\overline{m}, \Phi(m, t))$

$\iff e^{-[g(\Phi(m,t))-g(\Phi(m,0))]} \cdot e^{-[g(\Phi(\overline{m},\Phi(m,0)))-g(\Phi(\overline{m},0))]} = e^{-[g(\Phi(\overline{m},\Phi(m,t)))-g(\Phi(\overline{m},0))]}$

$\iff e^{-[g(\Phi(m,t))-g(\Phi(m,0))+g(\Phi(\overline{m},\Phi(m,0)))-g(\Phi(\overline{m},0))]} = e^{-[g(\Phi(\overline{m},\Phi(m,t)))-g(\Phi(\overline{m},0))]}$

$\iff g(\Phi(m, t)) - g(\Phi(m, 0)) + g(\Phi(\overline{m}, \Phi(m, 0))) = g(\Phi(\overline{m}, \Phi(m, t)))$. But $\Phi(\overline{m}, x) := \Phi_{\overline{m},\overline{m}}(x) = x$, and thus the last equation is an identity. ■

C. Appendix: The Efficiency of Delay Functions

Fix the set of periods and prizes and order them so that $0 = t_1 < t_2 < ... < t_J$ and $0 < m_1 < ... < m_I$. Write the corresponding finite space of dated rewards as $X_{IJ} := \{m_1, .., m_I\} \times \{0, t_2, .., t_J\}$. Suppose that the present value data is given by

p_{ij} such that

$$(p_{ij}, 0) \sim (m_i, t_j) \text{ for all } m_i \text{ and all } t_j > 0.$$

Assume that the analyst is interested in SDU representations. Say that the (magnitude-independent) discount function D is *attributable* to the present value data if there exists a utility index u such that $u(p_{ij}) = D(t_j)u(m_i)$ for all $(m_i, t_j) \in X_{IJ}$. Let the set of attributable D be denoted by $\mathcal{D}_I^p$. This is indexed by I since we will be varying I below.

Given X_{IJ}, let $p^*(= p_{I1})$ denote the present value of (m_I, t_1), the largest reward at the earliest future period. Suppose that Φ-data is obtained by determining τ_j such that

$$(p^*, \tau_j) \sim (m_I, t_j) \text{ for all } t_j > 0.$$

That is, we determine τ_j such that $\Phi(p^*, \tau_j) = t_j$. Say that the discount function D is attributable to the Φ-data if $D(t)D(\Phi(p^*, t_1)) = D(\Phi(p^*, t))$ for all these periods $\tau_1, .., \tau_J$.[28] Denote the set of attributable D by $\mathcal{D}^\Phi$.

The present value and Φ-data are related by a common time horizon t_J and also the indifference point $(p_{I1}, 0) \sim (m_I, t_1)$ which defines both p_{I1} and $\Phi(p_{I1}, 0)$. The proposition below reveals that the $J - 1$ data points for Φ are more discerning than the $I \cdot (J - 1)$ data points for present values, *regardless of the number I of rewards.*

The proof of the proposition is based on the following insight: limited present value data will at best put bounds on the participant's true delay function Φ and this is the *only* extent to which it restricts the range of possible D's. The remainder of the data, no matter how rich, will only help determine what u goes with any such D (observe that in Theorem 2.1 the utility index is determined by $\Phi(\cdot, 0)$, which essentially comes from money-time trade-off data). Limited direct data on Φ will speak more than data that just puts bounds on Φ.

The 'true' D is in both $\mathcal{D}^\Phi$ and $\mathcal{D}_I^p$. The proposition therefore tells us that there is greater efficiency achieved by using Φ, in that we can get closer to the true D

[28]This is equivalent to requiring that there are utilities $0 < u(p^*) < u(m_I)$ such that $D(t)u(p^*) = D(\Phi(p^*, t))u(m_I)$ for these t's. Observe that $u(p^*) = D(\Phi(p^*, t_1))u(m_I)$ must hold and so the utilities can be substituted out, yielding the original definition.

with fewer data points. Stated differently, the proposition reveals that the degree of potential misidentification is greater with present value data than it is with Φ-data. In this sense, limited Φ-data provides a better picture of the agent's entire preference than does present value data, which is the claim we set out to establish. This proposition thus provides further validation for our claim that Φ serves as a behavioral definition of discount functions.

Proposition C.1. *Suppose that $\succsim$ admits some SDU representation. Then for all I,*

$$\mathcal{D}^{\Phi} \subset \mathcal{D}^{p}_{I}.$$

Proof. It follows from the definition of admissibility that $D(t) = e^{-g(t)}$ is admissible for the data $\{\Phi(p_{I1}, t) : t = \tau_2, .., \tau_J\}$ if and only if g solves the functional equation

$$g(\Phi(p_{I1}, t)) = g(t) + g(\Phi(p_{I1}, 0)), \qquad \text{for all } t = \tau_2, .., \tau_J.$$

The set of admissible $D(t) = e^{-g(t)}$ is nonempty (the 'true' one is in the set). Take any admissible D, and corresponding g.

Below we extend the data $\{\Phi(p_{I1}, t) : t = \tau_2, .., \tau_J\}$ to some function Φ on a subset of $X = \mathbb{R}^2_+$ in a way that is consistent with the present value data, and then proceed to prove the theorem. Specifically, we inductively define Φ on $\{p_{ij}\} \times \mathbb{R}_+$. It will be convenient to define, for each $1 \leq \iota \leq I$, the set $S_\iota \subset \{p_{ij}\}$ of all observed present values of rewards $m_\iota, .., m_I$, that is, $S_\iota := \{p_{ij}\text{: } \iota \leq i \leq I \text{ and } j = 0, .., J\}$. Note that by regularity, $m_i = p_{i0}$.

First consider $\iota = I$. Define $\Phi(m_I, t) = t$ for all t. For all j, define $\Phi(p_{Ij}, 0) = t_j$ and moreover, $\Phi(p_{Ij}, t) = g^{-1}(g(t) + g(\Phi(p_{Ij}, 0))$.

Next suppose that, for $1 < \iota \leq I$,

(a) Φ is defined for $m \in S_\iota$ and all t,

(b) $\Phi(\cdot, 0)$ is strictly increasing on S_ι,

(c) for all $p_{ij} \in S_\iota$,

$$\Phi(p_{ij}, 0) = \Phi(m_i, t_j),$$

(d) for all $m_i \in S_\iota$,

$$g(t_j) = g(\Phi(m_i, t_j)) - g(\Phi(m_i, 0))$$

Observe that this is satisfied for the case $\iota = I$ that we just defined. We now extend Φ to $S_{\iota-1}$ and all t such that these conditions are satisfied.

If $m_{\iota-1}(= p_{\iota-1,0})$ equals some $m \in S_\iota$ then define $\Phi(m_{\iota-1}, t) = \Phi(m, t)$ for all t. If $m_{\iota-1} < S_\iota$ then define $\Phi(m_{\iota-1}, 0)$ by taking any arbitrary number in $(\max_{m \in S_\iota} \Phi(m, 0), \infty)$ and let $\Phi(m_{\iota-1}, t) = g^{-1}(g(t) + g(\Phi(m_{\iota-1}, 0))$ for all t. If neither of these cases hold, then there exist $m^*, m_* \in S_\iota$ such that m^* is the smallest element in S_ι that is greater than m_ι and m_* is the largest element smaller than it. Define $\Phi(m_{\iota-1}, 0)$ by taking any number in the interval $(\Phi(m^*, 0), \Phi(m_*, 0))$ (regularity and the construction ensures that the interval is nonempty) and let $\Phi(m_{\iota-1}, t) = g^{-1}(g(t) + g(\Phi(m_{\iota-1}, 0))$ for all t. Next for all j, define $\Phi(p_{\iota-1,j}, 0) = \Phi(m_{\iota-1}, t_j)$ and moreover, $\Phi(p_{\iota-1,j}, t) = g^{-1}(g(t) + g(\Phi(p_{\iota-1,j}, 0))$. Then Φ is defined on $S_{\iota-1}$ and all t, and moreover, the analogues of (a)-(d) hold by construction. Continue this construction till we obtain Φ on $\{p_{ij}\} \times \mathbb{R}_+$. This satisfies the analogues of (a)-(d) for $\iota = 1$.

To prove the theorem, take the admissible discount function $D(t) = e^{-g(t)}$. By property (b), we can extend $\Phi(\cdot, 0)$ continuous and monotonically to all of $\mathbb{R}_+$ and define the utility index,

$$u(m) = e^{-g(\Phi(m,0))}.$$

Given properties (c) and (d), determine that, for all i, j,

$u(p_{ij}) = e^{-g(\Phi(p_{ij},0))}$

$= e^{-g(\Phi(m_i,t_j))} = e^{-[g(\Phi(m_i,t_j))-g(\Phi(m_i,0))]} \cdot e^{-g(\Phi(m_i,0))}$

$= e^{-g(t_j)} u(m_i) = D(t_j) u(m_i)$.

Thus, we have shown that any D that is attributable to Φ data is also attributable to the present value data. ■

D. Appendix: Proof of Proposition 2.3

Given (2.1), Theorem 2.1 yields that a separable discount function $D(t) = \delta^{g(t)}$ can be attributed if and only if g satisfies $\delta^{g(t)} = e^{-[g(\Phi(s,t))-g(\Phi(s,0))]}$ for all s, that is,

$$g((a(s)t^\gamma + b(s)^\gamma)^{\frac{1}{\gamma}}) = g(t) + g(b(s)),$$

for all s. First suppose a g that satisfies this equation exists. We show that $a(s)$ and $b(s)$ must be linearly related. Take any s'. Letting $t = \Phi(s', 0) = b(s')$ we see that

$$g((a(s)b(s')^\gamma + b(s)^\gamma)^{\frac{1}{\gamma}}) = g(b(s')) + g(b(s)) = g((a(s')b(s)^\gamma + b(s')^\gamma)^{\frac{1}{\gamma}}),$$

and since g is strictly increasing, $a(s)b(s')^\gamma + b(s)^\gamma = a(s')b(s)^\gamma + b(s')^\gamma$, which implies

$$\frac{a(s)-1}{b(s)^\gamma} = \frac{a(s')-1}{b(s')^\gamma}.$$

Thus if an g exists, then the ratio $\frac{a(s)-1}{b(s)^\gamma}$ must be a constant k for all s, and so the equation $a(s) = 1 + kb(s)^\gamma$ must hold, as desired. To see that $k \geq 0$, note when $\frac{a(s)-1}{b(s)^\gamma} = k$ for all s, then we have a functional equation $g(((1 + kb(s)^\gamma)t^\gamma + b(s)^\gamma)^{\frac{1}{\gamma}}) = g(t) + g(b(s))$. Denoting $x = t$ and $y = b(s)$ we can write this as:

$$g((x^\gamma + y^\gamma + kx^\gamma y^\gamma)^{\frac{1}{\gamma}}) = g(x) + g(y).$$

Suppose by way of contradiction that $k < 0$. Write $g((x^\gamma + y^\gamma(1 - |k|\, x^\gamma))^{\frac{1}{\gamma}}) = g(x) + g(y)$ and take x such that $(1 - |k|\, x^\gamma) < 0$. Then as y increases the LHS decreases and the RHS increases (since g is strictly increasing), a contradiction. Thus $k \geq 0$ must hold.

Conversely, suppose $a(s) = 1 + kb(s)^\gamma$ holds with $k \geq 0$. Consider the above displayed functional equation. Consider various cases:

(i) $k = 0$.

A solution is $g(x) = cx^\gamma$, $c > 0$. Then $D(t) = \delta^t$ is attributable for any $\delta \in (0, 1)$.

(ii) $k > 0$.

Then it is easily verified that $g(x) = \ln(1 + kx^{\gamma})$ is a solution, and so for any $r > 0$, an attributable discount function is $D(t) = e^{-r\ln(1+kt^{\gamma})} = (1 + kt^{\gamma})^{-r}$.

Supplemental Appendix For Online Publicaiton

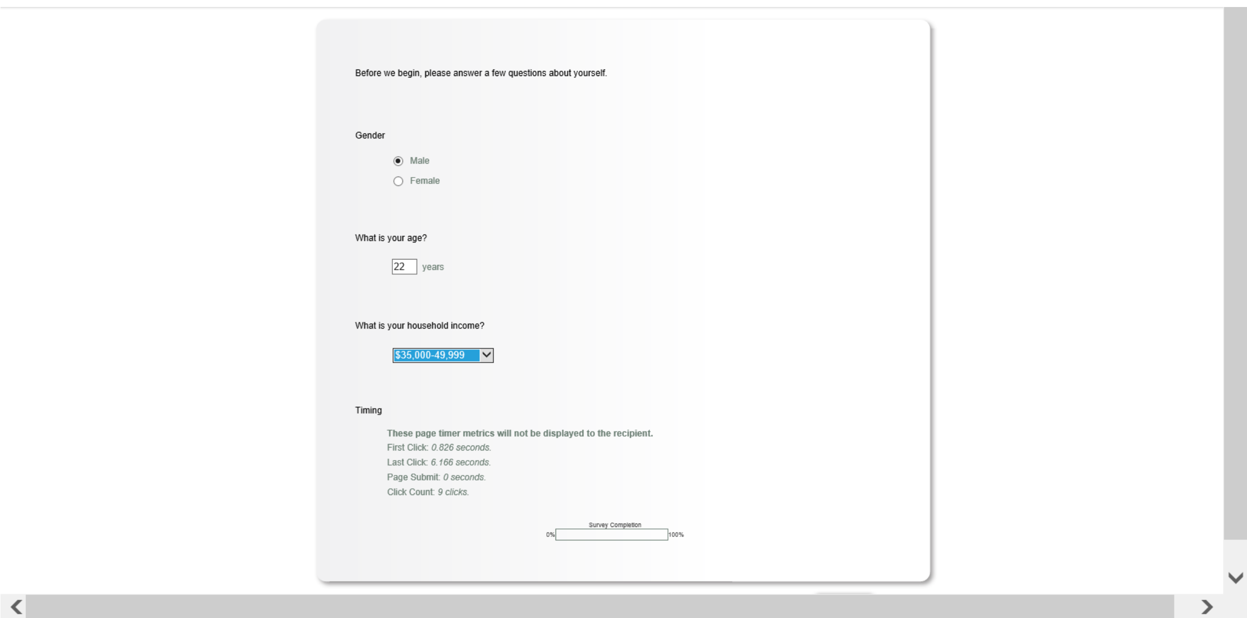
For "Delay Functions as the Foundation of Time Preference: Testing for Separable Discounted Utility"

Keith Marzilli Ericson and Jawwad Noor

1. Protocol

Experiments were conducted online using Qualtrics software. Subjects were recruited from Qualtrics Panels to be representative of the U.S. general population. The sampling service screens out inattentive participants who fail basic attention checks, such as answering questions too quickly or answering illogically or inconsistently. 11 such participants were screened out. Participants were paid a flat participation fee based on their relationship with the Qualtrics Panels. Choices were incentivized: 10% of participants were randomly selected to be paid for one of their choices. No deception was used. Mean duration in the experiment was 12.5 minutes, median duration was 11 minutes.

2. Screen Shots



(This page used by Qualtrics Panels to screen for eligibility)

Study Background and Purpose: This study examines decision-making. The results will be used to examine what people find important when making decisions. Your participation in this research will take approximately 10 minutes to complete.

What Happens in this Research Study: If you decide to participate, you will complete an online survey in which you will be asked to provide some information about yourself and to answer a series of questions about whether you'd prefer different economic options. You will also be asked questions about your participation in this study. The only anticipated risk to this study is possible discomfort at answering survey questions.

Payments: There are no known costs to you for participating in this research study except for your time. Your compensation will be determined following the standard procedures of the firm that administers your panel.

Confidentiality: Your data will be anonymous and will not be linked to your identity. You will never be asked to provide your name. Data will be stored on a password-protected computer and may be retained for future analysis.

Voluntary Participation: Participating in this research is voluntary. You can withdraw from the study at any time without penalty or loss of benefits to which you are otherwise entitled.

Contact: If you have questions regarding this research, either now or at any time in the future, please contact Keith Ericson at 617-353-4553 or kericson@bu.edu. You may obtain further information about your rights as a research subject by contacting the Boston University Institutional Review Board for Human Subjects Research at 617-358-6115 or irb@bu.edu.

Agreement to Participate
By clicking continue, you are indicating that you have read this consent form and that you voluntarily agree to participate in the study. You may now print a copy of the consent form to keep if you wish.

Timing

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Survey Completion
0% 100%

Survey Powered By Qualtrics

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In what state do you currently reside?

What best describes your marital status

- Married
- Single
- Separated
- Domestic partnership, not married

How many children under 18 are in your household?

- 0
- 1
- 2
- 3 or more

What is the highest level of education you've completed?

- Some high school
- Finished high school
- Some college
- Finished college
- Some graduate school
- Finished graduate school

Employment Status:

- Working full time
- Working part time
- Unemployed and looking for job
- Choose not to work

If you had a choice between the following two options, which would you prefer?

Option A: $20 for certain	Option B: 50% chance of $0 50% chance of $60
○	○

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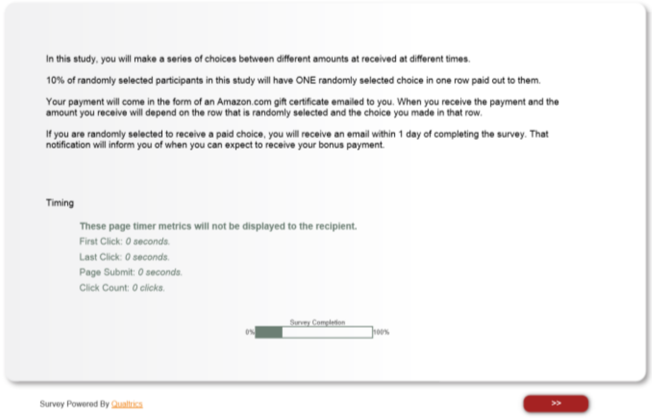

In this study, you will make a series of choices between different amounts at received at different times.

10% of randomly selected participants in this study will have ONE randomly selected choice in one row paid out to them.

Your payment will come in the form of an Amazon.com gift certificate emailed to you. When you receive the payment and the amount you receive will depend on the row that is randomly selected and the choice you made in that row.

If you are randomly selected to receive a paid choice, you will receive an email within 1 day of completing the survey. That notification will inform you of when you can expect to receive your bonus payment.

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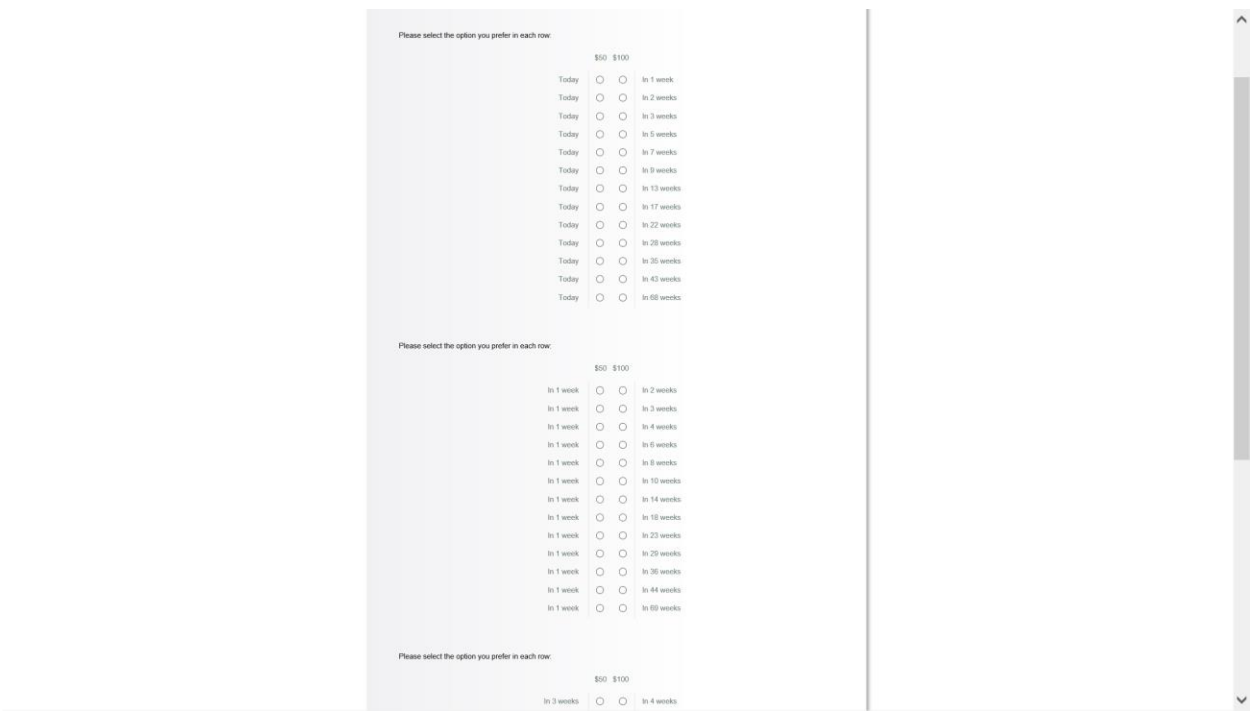

Please select the option you prefer in each row:

	$50	$100	
Today	○	○	In 1 week
Today	○	○	In 2 weeks
Today	○	○	In 3 weeks
Today	○	○	In 5 weeks
Today	○	○	In 7 weeks
Today	○	○	In 9 weeks
Today	○	○	In 13 weeks
Today	○	○	In 17 weeks
Today	○	○	In 22 weeks
Today	○	○	In 28 weeks
Today	○	○	In 35 weeks
Today	○	○	In 43 weeks
Today	○	○	In 68 weeks

Please select the option you prefer in each row:

	$50	$100	
In 1 week	○	○	In 2 weeks
In 1 week	○	○	In 3 weeks
In 1 week	○	○	In 4 weeks
In 1 week	○	○	In 6 weeks
In 1 week	○	○	In 8 weeks
In 1 week	○	○	In 10 weeks
In 1 week	○	○	In 14 weeks
In 1 week	○	○	In 18 weeks
In 1 week	○	○	In 23 weeks
In 1 week	○	○	In 29 weeks
In 1 week	○	○	In 36 weeks
In 1 week	○	○	In 44 weeks
In 1 week	○	○	In 69 weeks

Please select the option you prefer in each row:

	$50	$100	
In 3 weeks	○	○	In 4 weeks

Please select the option you prefer in each row:

	$50	$100	
In 5 weeks	○	○	In 6 weeks
In 5 weeks	○	○	In 7 weeks
In 5 weeks	○	○	In 8 weeks
In 5 weeks	○	○	In 10 weeks
In 5 weeks	○	○	In 12 weeks
In 5 weeks	○	○	In 14 weeks
In 5 weeks	○	○	In 18 weeks
In 5 weeks	○	○	In 22 weeks
In 5 weeks	○	○	In 27 weeks
In 5 weeks	○	○	In 33 weeks
In 5 weeks	○	○	In 40 weeks
In 5 weeks	○	○	In 48 weeks
In 5 weeks	○	○	In 73 weeks

Please select the option you prefer in each row:

	$50	$100	
In 12 weeks	○	○	In 13 weeks
In 12 weeks	○	○	In 14 weeks
In 12 weeks	○	○	In 15 weeks
In 12 weeks	○	○	In 17 weeks
In 12 weeks	○	○	In 19 weeks
In 12 weeks	○	○	In 21 weeks
In 12 weeks	○	○	In 25 weeks
In 12 weeks	○	○	In 29 weeks
In 12 weeks	○	○	In 34 weeks
In 12 weeks	○	○	In 40 weeks
In 12 weeks	○	○	In 47 weeks
In 12 weeks	○	○	In 55 weeks
In 12 weeks	○	○	In 80 weeks

Please select the option you prefer in each row:

	$50	$100	
In 24 weeks	○	○	In 25 weeks
In 24 weeks	○	○	In 26 weeks
In 24 weeks	○	○	In 27 weeks
In 24 weeks	○	○	In 29 weeks
In 24 weeks	○	○	In 31 weeks
In 24 weeks	○	○	In 33 weeks
In 24 weeks	○	○	In 37 weeks
In 24 weeks	○	○	In 41 weeks
In 24 weeks	○	○	In 46 weeks
In 24 weeks	○	○	In 52 weeks
In 24 weeks	○	○	In 59 weeks
In 24 weeks	○	○	In 67 weeks
In 24 weeks	○	○	In 92 weeks

Timing

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Survey Completion
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Please select the option you prefer in each row:

	$60	$100	
Today	○	○	In 1 week
Today	○	○	In 2 weeks
Today	○	○	In 3 weeks
Today	○	○	In 5 weeks
Today	○	○	In 7 weeks
Today	○	○	In 9 weeks
Today	○	○	In 13 weeks
Today	○	○	In 17 weeks
Today	○	○	In 22 weeks
Today	○	○	In 28 weeks
Today	○	○	In 35 weeks
Today	○	○	In 43 weeks
Today	○	○	In 68 weeks

Please select the option you prefer in each row:

	$60	$100	
In 1 week	○	○	In 2 weeks
In 1 week	○	○	In 3 weeks
In 1 week	○	○	In 4 weeks
In 1 week	○	○	In 6 weeks
In 1 week	○	○	In 8 weeks
In 1 week	○	○	In 10 weeks
In 1 week	○	○	In 14 weeks
In 1 week	○	○	In 18 weeks
In 1 week	○	○	In 23 weeks
In 1 week	○	○	In 29 weeks
In 1 week	○	○	In 36 weeks
In 1 week	○	○	In 44 weeks
In 1 week	○	○	In 69 weeks

Please select the option you prefer in each row:

	$60	$100	
In 3 weeks	○	○	In 4 weeks

In 1 week	○	○	In 6 weeks
In 1 week	○	○	In 8 weeks
In 1 week	○	○	In 10 weeks
In 1 week	○	○	In 14 weeks
In 1 week	○	○	In 18 weeks
In 1 week	○	○	In 23 weeks
In 1 week	○	○	In 29 weeks
In 1 week	○	○	In 36 weeks
In 1 week	○	○	In 44 weeks
In 1 week	○	○	In 69 weeks

Please select the option you prefer in each row:

	$60	$100	
In 3 weeks	○	○	In 4 weeks
In 3 weeks	○	○	In 5 weeks
In 3 weeks	○	○	In 6 weeks
In 3 weeks	○	○	In 8 weeks
In 3 weeks	○	○	In 10 weeks
In 3 weeks	○	○	In 12 weeks
In 3 weeks	○	○	In 16 weeks
In 3 weeks	○	○	In 20 weeks
In 3 weeks	○	○	In 25 weeks
In 3 weeks	○	○	In 31 weeks
In 3 weeks	○	○	In 38 weeks
In 3 weeks	○	○	In 46 weeks
In 3 weeks	○	○	In 71 weeks

Timing

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Survey Completion
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Please select the option you prefer in each row:

	$60	$100	
In 5 weeks	○	○	In 6 weeks
In 5 weeks	○	○	In 7 weeks
In 5 weeks	○	○	In 8 weeks
In 5 weeks	○	○	In 10 weeks
In 5 weeks	○	○	In 12 weeks
In 5 weeks	○	○	In 14 weeks
In 5 weeks	○	○	In 18 weeks
In 5 weeks	○	○	In 22 weeks
In 5 weeks	○	○	In 27 weeks
In 5 weeks	○	○	In 33 weeks
In 5 weeks	○	○	In 40 weeks
In 5 weeks	○	○	In 48 weeks
In 5 weeks	○	○	In 73 weeks

Please select the option you prefer in each row:

	$60	$100	
In 12 weeks	○	○	In 13 weeks
In 12 weeks	○	○	In 14 weeks
In 12 weeks	○	○	In 15 weeks
In 12 weeks	○	○	In 17 weeks
In 12 weeks	○	○	In 19 weeks
In 12 weeks	○	○	In 21 weeks
In 12 weeks	○	○	In 25 weeks
In 12 weeks	○	○	In 29 weeks
In 12 weeks	○	○	In 34 weeks
In 12 weeks	○	○	In 40 weeks
In 12 weeks	○	○	In 47 weeks
In 12 weeks	○	○	In 55 weeks
In 12 weeks	○	○	In 80 weeks

Please select the option you prefer in each row:

	$60	$100	
In 24 weeks	○	○	In 25 weeks
In 24 weeks	○	○	In 26 weeks
In 24 weeks	○	○	In 27 weeks
In 24 weeks	○	○	In 29 weeks
In 24 weeks	○	○	In 31 weeks
In 24 weeks	○	○	In 33 weeks
In 24 weeks	○	○	In 37 weeks
In 24 weeks	○	○	In 41 weeks
In 24 weeks	○	○	In 46 weeks
In 24 weeks	○	○	In 52 weeks
In 24 weeks	○	○	In 59 weeks
In 24 weeks	○	○	In 59 weeks
In 24 weeks	○	○	In 92 weeks

Timing

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Survey Completion
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Please select the option you prefer in each row:

	$70	$100	
Today	○	○	In 1 week
Today	○	○	In 2 weeks
Today	○	○	In 3 weeks
Today	○	○	In 5 weeks
Today	○	○	In 7 weeks
Today	○	○	In 9 weeks
Today	○	○	In 13 weeks
Today	○	○	In 17 weeks
Today	○	○	In 22 weeks
Today	○	○	In 28 weeks
Today	○	○	In 35 weeks
Today	○	○	In 43 weeks
Today	○	○	In 68 weeks

Please select the option you prefer in each row:

	$70	$100	
In 1 week	○	○	In 2 weeks
In 1 week	○	○	In 3 weeks
In 1 week	○	○	In 4 weeks
In 1 week	○	○	In 6 weeks
In 1 week	○	○	In 8 weeks
In 1 week	○	○	In 10 weeks
In 1 week	○	○	In 14 weeks
In 1 week	○	○	In 18 weeks
In 1 week	○	○	In 23 weeks
In 1 week	○	○	In 29 weeks
In 1 week	○	○	In 36 weeks
In 1 week	○	○	In 44 weeks
In 1 week	○	○	In 69 weeks

Please select the option you prefer in each row:

	$70	$100	
In 3 weeks	○	○	In 4 weeks
In 3 weeks	○	○	In 5 weeks
In 3 weeks	○	○	In 6 weeks
In 3 weeks	○	○	In 8 weeks
In 3 weeks	○	○	In 10 weeks
In 3 weeks	○	○	In 12 weeks
In 3 weeks	○	○	In 16 weeks
In 3 weeks	○	○	In 20 weeks
In 3 weeks	○	○	In 25 weeks
In 3 weeks	○	○	In 31 weeks
In 3 weeks	○	○	In 38 weeks
In 3 weeks	○	○	In 46 weeks
In 3 weeks	○	○	In 71 weeks

Timing

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Survey Completion

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Please select the option you prefer in each row:

	$70	$100	
In 5 weeks	○	○	In 6 weeks
In 5 weeks	○	○	In 7 weeks
In 5 weeks	○	○	In 8 weeks
In 5 weeks	○	○	In 10 weeks
In 5 weeks	○	○	In 12 weeks
In 5 weeks	○	○	In 14 weeks
In 5 weeks	○	○	In 18 weeks
In 5 weeks	○	○	In 22 weeks
In 5 weeks	○	○	In 27 weeks
In 5 weeks	○	○	In 33 weeks
In 5 weeks	○	○	In 40 weeks
In 5 weeks	○	○	In 48 weeks
In 5 weeks	○	○	In 73 weeks

Please select the option you prefer in each row:

	$70	$100	
In 12 weeks	○	○	In 13 weeks
In 12 weeks	○	○	In 14 weeks
In 12 weeks	○	○	In 15 weeks
In 12 weeks	○	○	In 17 weeks
In 12 weeks	○	○	In 19 weeks
In 12 weeks	○	○	In 21 weeks
In 12 weeks	○	○	In 25 weeks
In 12 weeks	○	○	In 29 weeks
In 12 weeks	○	○	In 34 weeks
In 12 weeks	○	○	In 40 weeks
In 12 weeks	○	○	In 47 weeks
In 12 weeks	○	○	In 55 weeks
In 12 weeks	○	○	In 80 weeks

Please select the option you prefer in each row:

	$70	$100	
In 24 weeks	○	○	In 25 weeks
In 24 weeks	○	○	In 26 weeks
In 24 weeks	○	○	In 27 weeks
In 24 weeks	○	○	In 29 weeks
In 24 weeks	○	○	In 31 weeks
In 24 weeks	○	○	In 33 weeks
In 24 weeks	○	○	In 37 weeks
In 24 weeks	○	○	In 41 weeks
In 24 weeks	○	○	In 46 weeks
In 24 weeks	○	○	In 52 weeks
In 24 weeks	○	○	In 59 weeks
In 24 weeks	○	○	In 67 weeks
In 24 weeks	○	○	In 92 weeks

Timing

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Survey Completion
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Please select the option you prefer in each row:

	$80	$100	
Today	○	○	In 1 week
Today	○	○	In 2 weeks
Today	○	○	In 3 weeks
Today	○	○	In 5 weeks
Today	○	○	In 7 weeks
Today	○	○	In 9 weeks
Today	○	○	In 13 weeks
Today	○	○	In 17 weeks
Today	○	○	In 22 weeks
Today	○	○	In 28 weeks
Today	○	○	In 35 weeks
Today	○	○	In 43 weeks
Today	○	○	In 68 weeks

Please select the option you prefer in each row:

	$80	$100	
In 1 week	○	○	In 2 weeks
In 1 week	○	○	In 3 weeks
In 1 week	○	○	In 4 weeks
In 1 week	○	○	In 6 weeks
In 1 week	○	○	In 8 weeks
In 1 week	○	○	In 10 weeks
In 1 week	○	○	In 14 weeks
In 1 week	○	○	In 18 weeks
In 1 week	○	○	In 23 weeks
In 1 week	○	○	In 29 weeks
In 1 week	○	○	In 36 weeks
In 1 week	○	○	In 44 weeks
In 1 week	○	○	In 69 weeks

Please select the option you prefer in each row:

	$80	$100	
In 3 weeks	○	○	In 4 weeks
In 3 weeks	○	○	In 5 weeks
In 3 weeks	○	○	In 6 weeks
In 3 weeks	○	○	In 8 weeks
In 3 weeks	○	○	In 10 weeks
In 3 weeks	○	○	In 12 weeks
In 3 weeks	○	○	In 16 weeks
In 3 weeks	○	○	In 20 weeks
In 3 weeks	○	○	In 25 weeks
In 3 weeks	○	○	In 31 weeks
In 3 weeks	○	○	In 38 weeks
In 3 weeks	○	○	In 46 weeks
In 3 weeks	○	○	In 71 weeks

Timing

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Survey Completion
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Please select the option you prefer in each row:

	$80	$100	
In 5 weeks	○	○	In 6 weeks
In 5 weeks	○	○	In 7 weeks
In 5 weeks	○	○	In 8 weeks
In 5 weeks	○	○	In 10 weeks
In 5 weeks	○	○	In 12 weeks
In 5 weeks	○	○	In 14 weeks
In 5 weeks	○	○	In 18 weeks
In 5 weeks	○	○	In 22 weeks
In 5 weeks	○	○	In 27 weeks
In 5 weeks	○	○	In 33 weeks
In 5 weeks	○	○	In 40 weeks
In 5 weeks	○	○	In 48 weeks
In 5 weeks	○	○	In 73 weeks

Please select the option you prefer in each row:

	$80	$100	
In 12 weeks	○	○	In 13 weeks
In 12 weeks	○	○	In 14 weeks
In 12 weeks	○	○	In 15 weeks
In 12 weeks	○	○	In 17 weeks
In 12 weeks	○	○	In 19 weeks
In 12 weeks	○	○	In 21 weeks
In 12 weeks	○	○	In 25 weeks
In 12 weeks	○	○	In 29 weeks
In 12 weeks	○	○	In 34 weeks
In 12 weeks	○	○	In 40 weeks
In 12 weeks	○	○	In 47 weeks
In 12 weeks	○	○	In 55 weeks
In 12 weeks	○	○	In 80 weeks

Please select the option you prefer in each row:

	$80	$100	
In 24 weeks	○	○	In 25 weeks
In 24 weeks	○	○	In 26 weeks
In 24 weeks	○	○	In 27 weeks
In 24 weeks	○	○	In 29 weeks
In 24 weeks	○	○	In 31 weeks
In 24 weeks	○	○	In 33 weeks
In 24 weeks	○	○	In 37 weeks
In 24 weeks	○	○	In 41 weeks
In 24 weeks	○	○	In 46 weeks
In 24 weeks	○	○	In 52 weeks
In 24 weeks	○	○	In 59 weeks
In 24 weeks	○	○	In 67 weeks
In 24 weeks	○	○	In 92 weeks

Timing

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Survey Completion
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Please select the option you prefer in each row:

	$90	$100	
Today	○	○	In 1 week
Today	○	○	In 2 weeks
Today	○	○	In 3 weeks
Today	○	○	In 5 weeks
Today	○	○	In 7 weeks
Today	○	○	In 9 weeks
Today	○	○	In 13 weeks
Today	○	○	In 17 weeks
Today	○	○	In 22 weeks
Today	○	○	In 28 weeks
Today	○	○	In 35 weeks
Today	○	○	In 43 weeks
Today	○	○	In 68 weeks

Please select the option you prefer in each row:

	$90	$100	
In 1 week	○	○	In 2 weeks
In 1 week	○	○	In 3 weeks
In 1 week	○	○	In 4 weeks
In 1 week	○	○	In 6 weeks
In 1 week	○	○	In 8 weeks
In 1 week	○	○	In 10 weeks
In 1 week	○	○	In 14 weeks
In 1 week	○	○	In 18 weeks
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In 1 week	○	○	In 29 weeks
In 1 week	○	○	In 36 weeks
In 1 week	○	○	In 44 weeks
In 1 week	○	○	In 69 weeks

Please select the option you prefer in each row:

	$90	$100	
In 3 weeks	○	○	In 4 weeks
In 3 weeks	○	○	In 5 weeks
In 3 weeks	○	○	In 6 weeks
In 3 weeks	○	○	In 8 weeks
In 3 weeks	○	○	In 10 weeks
In 3 weeks	○	○	In 12 weeks
In 3 weeks	○	○	In 16 weeks
In 3 weeks	○	○	In 20 weeks
In 3 weeks	○	○	In 25 weeks
In 3 weeks	○	○	In 31 weeks
In 3 weeks	○	○	In 38 weeks
In 3 weeks	○	○	In 46 weeks
In 3 weeks	○	○	In 71 weeks

Timing

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Please select the option you prefer in each row:

	$90	$100	
In 5 weeks	○	○	In 6 weeks
In 5 weeks	○	○	In 7 weeks
In 5 weeks	○	○	In 8 weeks
In 5 weeks	○	○	In 10 weeks
In 5 weeks	○	○	In 12 weeks
In 5 weeks	○	○	In 14 weeks
In 5 weeks	○	○	In 18 weeks
In 5 weeks	○	○	In 22 weeks
In 5 weeks	○	○	In 27 weeks
In 5 weeks	○	○	In 33 weeks
In 5 weeks	○	○	In 40 weeks
In 5 weeks	○	○	In 48 weeks
In 5 weeks	○	○	In 73 weeks

Please select the option you prefer in each row:

	$90	$100	
In 12 weeks	○	○	In 13 weeks
In 12 weeks	○	○	In 14 weeks
In 12 weeks	○	○	In 15 weeks
In 12 weeks	○	○	In 17 weeks
In 12 weeks	○	○	In 19 weeks
In 12 weeks	○	○	In 21 weeks
In 12 weeks	○	○	In 25 weeks
In 12 weeks	○	○	In 29 weeks
In 12 weeks	○	○	In 34 weeks
In 12 weeks	○	○	In 40 weeks
In 12 weeks	○	○	In 47 weeks
In 12 weeks	○	○	In 55 weeks
In 12 weeks	○	○	In 80 weeks

Please select the option you prefer in each row:

	$90	$100	
In 24 weeks	○	○	In 25 weeks
In 24 weeks	○	○	In 26 weeks
In 24 weeks	○	○	In 27 weeks
In 24 weeks	○	○	In 29 weeks
In 24 weeks	○	○	In 31 weeks
In 24 weeks	○	○	In 33 weeks
In 24 weeks	○	○	In 37 weeks
In 24 weeks	○	○	In 41 weeks
In 24 weeks	○	○	In 46 weeks
In 24 weeks	○	○	In 52 weeks
In 24 weeks	○	○	In 59 weeks
In 24 weeks	○	○	In 67 weeks
In 24 weeks	○	○	In 92 weeks

Timing

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Survey Completion
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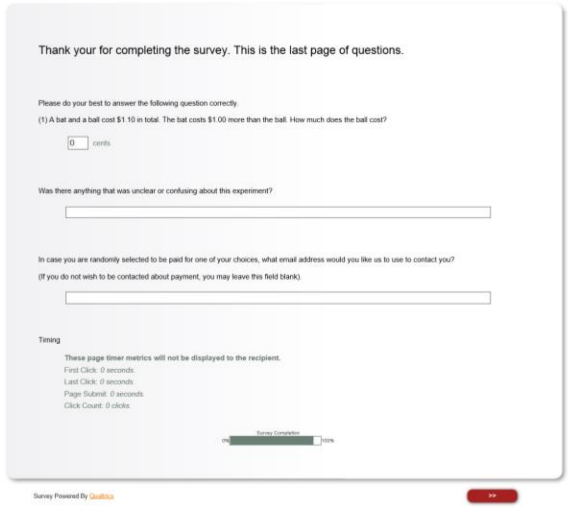
Thank your for completing the survey. This is the last page of questions.

Please do your best to answer the following question correctly.

(1) A bat and a ball cost $1.10 in total. The bat costs $1.00 more than the ball. How much does the ball cost?

0 cents

Was there anything that was unclear or confusing about this experiment?

In case you are randomly selected to be paid for one of your choices, what email address would you like us to use to contact you?

(If you do not wish to be contacted about payment, you may leave this field blank).

Timing

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Survey Completion

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