Towards Abstraction and Control for Large Groups of Robots

Calin Belta and Vijay Kumar

University of Pennsylvania, GRASP Laboratory, 3401 Walnut St., Philadelphia, PA 19104, USA

Abstract. This paper addresses the problem of controlling a large number of robots required to accomplish a task as a group. We propose an abstraction based on the definition of a map from the configuration space of the robots to a lower dimensional manifold, whose dimension does not scale with the number of robots. The task to be accomplished by the team suggests a natural feedback control system on the group manifold. We show that, if mean and covariance matrix are chosen as group variables for fully actuated robots, it is possible to design decoupling control laws, *i.e.,* the feedback control for a robot is only dependent on the state of the robot and the state of the group, therefore the communication necessary to accomplish the task is kept to a minimum.

1 Introduction

There has been a lot of interest in cooperative robotics in the last few years, triggered mainly by the technological advances in control techniques for single vehicles and the explosion in computation and communication capabilities. The research in the field of control and coordination for multiple robots is currently progressing in areas like automated highway systems, formation flight control, unmanned underwater vehicles, satellite clustering, exploration, surveillance, search and rescue, mapping of unknown or partially known environments, distributed manipulation, and transportation of large objects.

In this paper, we consider the problem of controlling a large number of robots required to accomplish a task as a group. For instance, consider the problem of moving 100 planar robots with arbitrary initial positions through a tunnel while staying grouped so that the distance between each pair does not exceed a certain value. The simplest solution, generating reference trajectories and control laws for each robot to stay on the designed trajectory, is obviously not feasible from a computational viewpoint. It is desired to have a certain level of abstraction: the motion generation/control problem should be solved in a lower dimensional space which captures the behavior of the group and the nature of the task.

For example, the robots can be required to form a *virtual structure*. In this case, the motion planning problem is reduced to a left invariant control system on $SE(3)$ (or $SE(2)$ in the planar case), and the individual trajectories are

A. Bicchi, H.I. Christensen, D. Prattichizzo (Eds.): Control Problems in Robotics, STAR 4, pp. 169–182, 2003
© Springer-Verlag Berlin Heidelberg 2003

$SE(3)$ ($SE(2)$) - orbits [1]. The literature on stabilization and control of virtual structures is rather extensive. Most of the recent works model formations using *formation graphs*, which are graphs whose nodes capture the individual agent kinematics or dynamics, and whose edges represent inter-agent constraints that must be satisfied [2,8,9,7]. Characterizations of rigid formations can be found in [3,1]. The controllers guaranteeing local asymptotic stability of a given rigid formation are derived using Lyapunov energy-type functions [7]. More flexibility is added to the formation if *virtual leaders* are defined [5]. It is interesting to note that in all these works on rigid formations local asymptotic stability can be achieved by decentralized controllers using local information. Moreover, the equilibria exhibit $SE(l)$, $l = 1, 2, 3$ symmetry and also expansion/contraction symmetries. These symmetries can be used to decouple the mission control problem into a formation keeping subproblem and a formation maneuver subproblem as in [6].

In many applications, like swarming, the virtual structure constraint might be too much, or simply not appropriate. The abstraction we propose in this paper involves the definition of a map from the configuration space of the robots to a lower dimensional *group manifold*. We require that the dimension of the group manifold do not scale with the number of robots and an arbitrary element of the group manifold captures the behaviour of the ensemble as a group, *i.e.*, it has a *behavioral* significance. The next step is to equip the group manifold with a vector field, which is built based on two types of restrictions. First, its flow lines should define the desired time evolution of the group, in accordance with a given task. Second, the robots should be able to move as a group in the desired fashion given the possible underactuation / nonholonomy constraints. If these two problems are solved , we propose that the possible remaining degrees of freedom from the individual control laws be used to achieve two more goals. First, it is desired to keep the amount of inter - robot communication in the overall control architecture to a minimum by use of *partial state feedback*. Ideally, we want to achieve *decoupled* architectures, *i.e.,* the control law of a robot only depends on its own state and the low dimensional state of the team from the group manifold. Second, the energy spent by the group to achieve the given task should be kept to a minimum.

In this paper we only consider fully-actuated planar robots abstracted to mean and covariance of the positions with respect to some reference frame. We prove that in this case the *decoupling* control vectors are also *minimum energy* controls. Illustrative examples of trajectory tracking on the group manifold and globally asymptotic stabilization to a point on the group manifols are included.

2 Definitions and Problem Formulation

In this section, we reformulate the ideas presented in Section 1 in a mathematical form.

Consider N robots with states q_i belonging to manifold Q_i. q_i will be interpreted as both a generic element on an abstract manifold or as coordinates of the element, depending on the context. The kinematics of each robot are defined by drift free control distributions on Q_i:

$$\Delta_i : Q_i \times U_i \to TQ_i \tag{1}$$

where U_i is the control space and TQ_i is the tangent bundle of Q_i.

Collect all robot states q_i on a single manifold $Q = q = [q_1, \ldots, q_N]^T$ and equip it with a control distribution Δ obtained from the individual control distributions through direct sum:

$$Q = \prod_{i=1}^{N} Q_i, \;\; \Delta : Q \times U \to TQ, \;\; \Delta = \bigoplus_{i=1}^{N} \Delta_i \tag{2}$$

where U is the Cartesian product of the control spaces $U = \prod_{i=1}^{N} U_i$. We will refer to (2) as the *product control system.* Allow to recover the states and the control distributions of the individual agents by use of the canonical projection on the ith agent:

$$\pi_i : Q \to Q_i, \; \pi_i(q) = q_i, \; d\pi_i : TQ \to TQ_i, \; d\pi_i(\Delta) = \Delta_i \tag{3}$$

Note that relations of type (1) are general enough to accomodate individual underactuation constraints, which are all captured in (2). Also, the notation $d\pi_i$ from (3) does not necessarily stand for the differential of π_i, it just represents an operator to recover individual control distributions.

We need the following definitions before formulating the problem:

Definition 1 (Group Abstraction). Any surjective submersion

$$\phi : Q \to G, \; \phi(q) = g$$

so that the dimension n of the group manifold G is not dependent on the number of the robots N is callled a group abstraction.

Definition 2 (Group Behavior). Any vector field X_G on G is called a group behavior.

Given a set of N robots with control systems (1) and corresponding product control system (2), we want to find solutions to the following problems:

Problem 1 (Group Abstraction). Determine a physically meaningful group abstraction which captures the behaviour of the ensamble of robots as a group, *i.e.*, it has a *behavioral* significance.

If the map ϕ is determined, the next step is to solve the motion generation (control) problem on the small dimensional manifold G so that a given task is accomplished by the robots as a team:

Problem 2 (Group Behavior). Design a group behavior X_G on G, so that the flow lines of X_G define the desired time evolution of the group and there exist $X_Q \in \Delta$ on TQ which are pushed forward to X_G through the map ϕ.

Note that Problems 1 and 2 can actually be seen as an input - output linearization problem [4] for the control system (2) with output $g = \phi(q)$. The total relative degree is $\dim(Q) - n$ since each robot is kinematically controlled. The vector field X_G guarantees some desired behavior of the output (which we call group variable) g, which will, of course, guarantee its boundness. Now the hardest problem, as usual in input - output linearization, is calculating and stabilizing the internal dynamics. This would imply, in general, finding the appropriate coordinate transformation separating the internal dynamics from output dynamics, calculating the corresponding zero dynamics and studying its stability. To avoid this, we try to define the output map so that bounds on output would easily imply bounds on the state, so it will not be necessary to explicitly calculate the internal dynamics.

It is also desired to keep the amount of inter - robot communication in the overall control architecture to a minimum, by use of *partial state feedback*. Ideally, we want to achieve a *decoupled* control architecture, *i.e.,* the control law of a robot only depends on its own state and the low dimensional state of the team from the group manifold:

Problem 3 (Decoupling). From the set of vector fields $X_Q \in \Delta$ on Q that are pushed forward to X_G, we want to identify a subset X_Q^* whose projection on TQ_i, $d\pi_i(X_Q^*)$ $i = 1, \ldots, N$ is only dependent on q_i and g.

Pictorially, the desired control architecture combining abstraction and partial state feedback features is given in Figure 1.

On the solution of Problem 2, the vector fields X_Q and X_G are related by

$$d\phi X_Q = X_G \tag{4}$$

where $d\phi$ is the tangent (differential) of the map ϕ. First, note that, for the problem to be well defined, $\phi : Q \to G$ should be a surjective submersion. This condition, together with (4) guarantees that any vector field X_Q satisfying (4) is ϕ - related to X_G. There are two slightly different approaches to finding a solution to Problem 2. If all the robots are fully actuated, *i.e.,* $\dim(\Delta_i) = \dim(Q_i)$, then any group vector field can be implemented by the individual robots. The general solution of (4) is the affine space

$$\mathcal{A} = \mathrm{Ker} d\phi + X_Q^p \tag{5}$$

where the vector field X_Q^p is a particular solution to (4). The degrees of freedom from $\mathrm{Ker} d\phi$ can be used to solve the decoupling Problem 3.

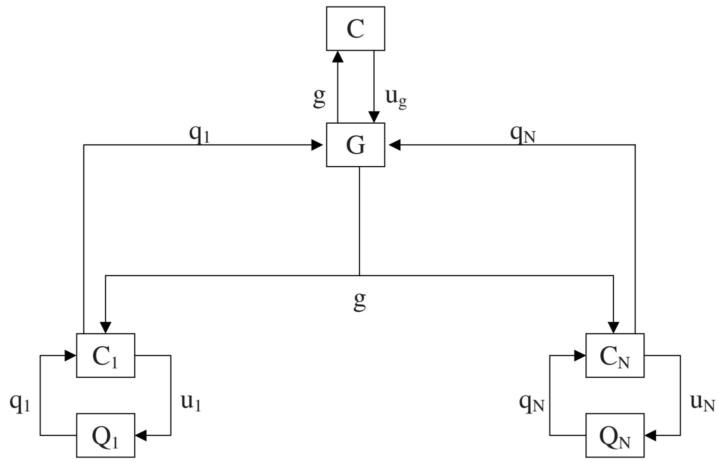

Fig. 1. A decoupled control architecture: the group is controlled on the "abstract" group manifold G; the control law of each robot is only dependent on its own state q_i and the state of the group g.

If some of the robots in the team are underactuated, it is more convenient to start with parameterizing the individual control distributions Δ_i, lift them to Δ and then push forward to G through the map ϕ. The obtained control vector field on G is conveniently parameterized in the robot control variables and exhaustively covers all the possible motions of the group. The individual control parameters can then be used to solve the decoupling Problem 3.

In both fully and under-actuated case, if after decoupling, there are still degrees of freedom left, one can use the minimum energy criterion or some other criteria, depending on the specific application, to choose among the several options.

3 Mean and Covariance Control for Fully Actuated Planar Robots

Assume N fully actuated robots free to move in the plane with position vectors $q_i = [x_i \; y_i]^T \in \mathbb{R}^2$, $i = 1, \ldots, N$ with respect to some reference frame. We will identify all the elements defined in Section 2 and then provide solutions to Problems 1, 2, 3.

Let e_i denote a vector of the standard Euclidean base in some dimension, which will be obvious from the context. Then, each agent is described by the following control system

$$Q_i = \mathbb{R}^2, \; \Delta_i = \text{span}\{e_1, e_2\} = \mathbb{R}^2$$

or, in coordinates,

$$\dot{q}_i = u_i = e_1 u_i^1 + e_2 u_i^2$$

Collecting all the robot states together, we get a $2N$-dimensional product control system

$$Q = \mathbb{R}^{2N},\ \Delta = \mathbb{R}^{2N} = \text{span}\{e_1, \ldots, e_{2N}\}$$

which in coordinates can be written as:

$$\begin{aligned} &Q = \{q = [q_1^T, \ldots q_N^T]^T,\ q_i \in \mathbb{R}^2, i = 1, \ldots, N\}, \\ &\Delta = \{u = [u_1^T, \ldots u_N^T]^T, u_i \in \mathbb{R}^2\} = \text{span}\{e_1, \ldots, e_{2N}\} \end{aligned}$$

with the corresponding projections:

$$\pi_i(q) = q_i,\ d\pi_i(u) = u_i$$

3.1 Group Abstraction

To provide an answer to Problem 1, the group variables that we choose in this paper are sample mean $\mu \in \mathbb{R}^2$ and covariance matrix $\Sigma \in \mathbb{R}^{2\times 2}$:

$$\mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} = \frac{1}{N}\sum_{i=1}^{N} q_i \tag{6}$$

$$\Sigma = \frac{1}{N}\sum_{i=1}^{N}(q_i - \mu)(q_i - \mu)^T = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} \tag{7}$$

i.e., we define the map $\phi : Q \to G \equiv \mathbb{R}^5$ by

$$\phi(q) = g = [\mu^T\ \sigma_1\ \sigma_2\ \sigma_3]^T, \tag{8}$$

where μ is given by (6) and σ_1, σ_2, σ_3 are the entries in matrix Σ as in (7). Explicitly,

$$\sigma_1 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_x)^2 \tag{9}$$

$$\sigma_2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_x)(y_i - \mu_y) \tag{10}$$

$$\sigma_3 = \frac{1}{N}\sum_{i=1}^{N}(y_i - \mu_y)^2 \tag{11}$$

Note that, by the Cauchy-Schwartz inequality, $\sigma_1\sigma_3 \geq \sigma_2^2$. The equality holds if and only if all the points q_i are on a line passing through μ. We will call this the *degenerate* case. This includes the situation when all the agents have the same position: $q_i = \mu$, $\sigma_1 = \sigma_2 = \sigma_3 = 0$. Also, non-degeneracy implies $\sigma_1 > 0$ and $\sigma_3 > 0$, because otherwise the agents would be on a line parallel with the x or y axis. We will assume the non-degenerate case throughout this paper, which will guarantee that the map ϕ is a submersion, as seen from the proof of Proposition 1 in Section 3.2.

An interesting physical interpretation in terms of a subset of the group variable g is given in Section 4 together with some illustrative examples. If the full vector g is available, one can diffeomorphically map G to another 5 - dimensional manifold parameterized by, let's say μ, θ, a, b, where μ is the center, θ the orientation, and a and b the semiaxes of some spanning ellipsoid. Then, the group manifold would have a product structure $G = SE(2) \times S$, where $SE(2)$ is the Euclidean group in two dimensions parameterized by (μ, θ) and S is a *shape* space parameterized by (a, b). The motion planning problem on the group manifold can then be decomposed into motion planning on the Lie group SE(2) (and benefit from the results in this area [1]) and the shape space S. This approach will be presented in a future paper.

3.2 Decoupling

Since we assumed fully actuated robots, as suggested in Section 2, we can design arbitrary group vector fields describing the time evolution of the group. In this section we study the decoupling Problem 3, for an arbitrary given vector field X_G on the group manifold G. The following proposition is the key result of this section:

Proposition 1. *Let X_G be an arbitrary vector field on G. Then the minimum length solution X_Q^* of (4) is also a decoupling control vector as defined in Problem 3,* i.e., *$d\pi_i(X_Q^*)$ is only dependent on q_i and g for $i = 1, \ldots, N$.*

Proof. The proof is rather involved and only a sketch is presented. The tangent map of ϕ defined by (8), (6), (9), (10), and (11) is given at an arbitrary point $q \in Q$ by the following $5 \times (2N)$ matrix:

$$d\phi = \frac{1}{N} \begin{bmatrix} 1 & 0 & \ldots\ldots & 1 & 0 \\ 0 & 1 & \ldots\ldots & 0 & 1 \\ 2(x_1 - \mu_x) & 0 & \ldots\ldots & 2(x_N - \mu_x) & 0 \\ y_1 - \mu_y & x_1 - \mu_x & \ldots\ldots & y_N - \mu_y & x_N - \mu_x \\ 0 & 2(y_1 - \mu_y) & \ldots\ldots & 0 & 2(y_N - \mu_y) \end{bmatrix} \qquad (12)$$

It can be shown that

$$\det(d\phi\, d\phi^T) = \frac{16}{N^5}(\sigma_1 + \sigma_3)(-\sigma_2^2 + \sigma_1\sigma_3)$$

from which we conclude that, in the non - degenerate case, $d\phi$ is full row rank, therefore ϕ is a submersion as required in Problem 1. The minimum length solution X_Q^* of (4) can be computed in the form $X_Q^* = d\phi^T\,(d\phi\, d\phi^T)^{-1} X_G$.

After straightforward but rather tedious calculation, we get the projection $\dot{q}_i = u_i = d\pi_i(X_Q^*)$ along the control directions of the i-th robot in the form:

$$\dot{q}_i = \dot{\mu} + \begin{bmatrix} 2(b_{11}\dot{\sigma}_1 + b_{12}\dot{\sigma}_2 + b_{13}\dot{\sigma}_3) & b_{12}\dot{\sigma}_1 + b_{22}\dot{\sigma}_2 + b_{23}\dot{\sigma}_3 \\ b_{12}\dot{\sigma}_1 + b_{22}\dot{\sigma}_2 + b_{23}\dot{\sigma}_3 & 2(b_{13}\dot{\sigma}_1 + b_{23}\dot{\sigma}_2 + b_{33}\dot{\sigma}_3) \end{bmatrix} (q_i - \mu) \tag{13}$$

where

$$b_{11} = \frac{-\sigma_2^2 + \sigma_1\sigma_3 + \sigma_3^2}{4(\sigma_1 + \sigma_3)(-\sigma_2^2 + \sigma_1\sigma_3)} \tag{14}$$

$$b_{12} = \frac{-2\sigma_2\sigma_3}{4(\sigma_1 + \sigma_3)(-\sigma_2^2 + \sigma_1\sigma_3)} \tag{15}$$

$$b_{13} = \frac{\sigma_2^2}{4(\sigma_1 + \sigma_3)(-\sigma_2^2 + \sigma_1\sigma_3)} \tag{16}$$

$$b_{22} = \frac{4\sigma_1\sigma_3}{4(\sigma_1 + \sigma_3)(-\sigma_2^2 + \sigma_1\sigma_3)} \tag{17}$$

$$b_{23} = \frac{-\sigma_1\sigma_2}{4(\sigma_1 + \sigma_3)(-\sigma_2^2 + \sigma_1\sigma_3)} \tag{18}$$

$$b_{33} = \frac{\sigma_1^2 - \sigma_2^2 + \sigma_1\sigma_3}{4(\sigma_1 + \sigma_3)(-\sigma_2^2 + \sigma_1\sigma_3)} \tag{19}$$

It is easy to see that $d\pi_i(X_Q^*)$ is only dependent on q_i and g for $i = 1, \ldots, N$, so the decoupling Problem 3 is solved. In light of (5), X_Q^* can be seen as a particular solution which is orthogonal to the vector space Ker$d\phi$. Therefore, in this case, solving (4) consisted of finding a nice particular solution and putting to zero all the $2N - 5$ control variables that we had on Ker$d\phi$, *i.e.*, annihilate Ker$d\phi$.

Note that under the non-degeneracy assumption the common denominator of all b_{ij} is non-zero. Moreover, $b_{11} > 0$, $b_{22} > 0$, $b_{33} > 0$.

Equation (13) gives the control law which should be implemented by controller C_i as shown in Figure 1 if the output function ϕ is defined as in (8). At each time instant t, the control system on G acquires all the states q_i, updates its own state g in accordance to (6), (9), (10), (11), flows along its designed control vector field X_G and disseminates its state g to all the robots.

3.3 Group Behavior

As suggested in Section 2, in the absence of individual underactuation (non-holonomy) constraints, we can design arbitrary group vector fields X_G describing the time evolution of the group.

Assume the goal is to move the robots from arbitrary initial positions $q_i(0)$ to final rest positions of desired mean μ^d and covariance σ_1^d, σ_2^d, σ_3^d. Also, the control law for each agent should guarantee asymptotic stability of mean and covariance at the desired values under arbitrary position displacements.

An obvious choice of the control vector field $X_G = [\dot{\mu},\ \dot{\sigma}_1,\ \dot{\sigma}_2,\ \dot{\sigma}_3]$ on the group manifold G is

$$\dot{\mu} = K_\mu(\mu^d - \mu) \tag{20}$$

$$\dot{\sigma}_1 = k_{\sigma_1}(\sigma_1^d - \sigma_1) \tag{21}$$

$$\dot{\sigma}_2 = k_{\sigma_2}(\sigma_2^d - \sigma_2) \tag{22}$$

$$\dot{\sigma}_3 = k_{\sigma_3}(\sigma_3^d - \sigma_3) \tag{23}$$

where $K_\mu \in \mathbb{R}^{2\times 2}$ is a positive definite matrix and $k_{\sigma_{1,2,3}} > 0$. The explicit control law for each agent is obtained by substituting (20), (21), (22), (23) and (6), (9), (10), (11) into (13).

More generally, the task might require the robots to follow a desired trajectory $g^d(t) = [\mu^d(t),\ \sigma_1^d(t),\ \sigma_2^d(t),\ \sigma_3^d(t)]$ on the group manifold G. A control vector field on G can be of the form:

$$\dot{\mu} = K_\mu(\mu^d(t) - \mu(t)) + \dot{\mu}^d(t) \tag{24}$$

$$\dot{\sigma}_1 = k_{\sigma_1}(\sigma_1^d(t) - \sigma_1(t)) + \dot{\sigma_1}^d(t) \tag{25}$$

$$\dot{\sigma}_2 = k_{\sigma_2}(\sigma_2^d(t) - \sigma_2(t)) + \dot{\sigma_2}^d(t) \tag{26}$$

$$\dot{\sigma}_3 = k_{\sigma_3}(\sigma_3^d(t) - \sigma_3(t)) + \dot{\sigma_3}^d(t) \tag{27}$$

Note that (20), (21), (22), (23) (in the stabilization case) or (24), (25), (26), (27) (in the trajectory tracking case) only guarantee the desired behavior on the group manifold G. If the imposed trajectory $g^d(t)$ is bounded at all times, it is easy to see that $g(t)$ is bounded. For the problem to be well defined, we still need to make sure that the internal states are bounded. We have:

Proposition 2. *If g is bounded, then so are q_i, $i = 1, \ldots, N$.*

Proof. It is enough to assume boundness of μ, σ_1, and σ_3 to prove boundness of q_i. Assume

$$\|\mu - \mu^d\| \leq M_\mu, \tag{28}$$

$$\|\sigma_1 - \sigma_1^d\| \leq M_{\sigma_1}, \tag{29}$$

$$\|\sigma_3 - \sigma_3^d\| \leq M_{\sigma_3}. \tag{30}$$

Then, from (29) and (9), we have

$$\sum_{i=1}^{N}(x_i - \mu_x)^2 \leq N(\sigma_1^d + M_{\sigma_1})$$

Similarly, from (30) and (11), we derive

$$\sum_{i=1}^{N}(y_i-\mu_y)^2 \leq N(\sigma_3^d+M_{\sigma_3})$$

from which we conclude that

$$\|q_i-\mu\| \leq \sqrt{N(\sigma_1^d+\sigma_3^d+M_{\sigma_1}+M_{\sigma_3})}$$

Finally, using (28), we have

$$\begin{aligned}\|q_i-\mu^d\| &= \|q_i-\mu+\mu-\mu^d\| \leq \|q_i-\mu\|+\|\mu-\mu^d\| \\ &\leq \sqrt{N(\sigma_1^d+\sigma_3^d+M_{\sigma_1}+M_{\sigma_3})}+M_\mu\end{aligned}$$

which concludes the proof.

Remark 1. It is easy to see that, in the assumed non-degenerate case as defined in Section 3.1, the individual velocities $\dot{q}_i$, $i=1,\ldots,N$ are bounded if the velocity $\dot{g}$ on the group manifold is bounded. Future work will focus on adding terms in the individual controls so that degenerate situations cannot occur.

Moreover, from (21) and (23) it is easy to see that if $\sigma_1^d>0$, $\sigma_3^d>0$ then $\sigma_1(t)>0$, $\sigma_3(t)>0$, $\forall t\geq 0$, which means that if the robots were not coincident at time 0, they will never become coincident if the proposed control is applied.

In the stabilization to a point case, the boundness and globally asymptotic convergence to the desired values of the group variables $g=[\mu^T,\ \sigma_1,\ \sigma_2,\ \sigma_3]^T$ are guaranteed by (20), (21), (22), (23). Proposition 2 proves the boundness of the internal dynamics. We still need to study the equilibria and regions of convergence for each robot. We have the following Proposition:

Proposition 3. *The closed loop system (13), (20), (21), (22), (23),(6), (9), (10), (11) is in equilibrium at each point on the set* $\mu=\mu^d$, $\sigma_1=\sigma_1^d$, $\sigma_2=\sigma_2^d$, $\sigma_3=\sigma_3^d$. *All solutions of the closed loop system globally asymptotically converge to* $\mu=\mu^d$, $\sigma_1=\sigma_1^d$, $\sigma_2=\sigma_2^d$, $\sigma_3=\sigma_3^d$ *when* $t\to\infty$

Proof. For the first part, from (13) and the definitions of the group variables, it is easy to see that the group is in equilibrium ($\dot{g}=0$) if and only if each agent is in equilibrium ($\dot{q}_i=0$, $i=1,\ldots,N$). Therefore, the equilibria of the closed loop system are sets described by $\mu=\mu^d$, $\sigma_1=\sigma_1^d$, $\sigma_2=\sigma_2^d$, $\sigma_3=\sigma_3^d$.

For the second part, consider the following Lyapunov function defined on Q:

$$V(q)=\frac{1}{2}\|\mu^d-\mu\|^2+\frac{1}{2}\|\sigma_1^d-\sigma_1\|^2+\frac{1}{2}\|\sigma_2^d-\sigma_2\|^2+\frac{1}{2}\|\sigma_3^d-\sigma_3\|^2 \tag{31}$$

and consider the derivative of V along the vector field on Q:

$$\dot{V}(q) = -K_\mu\|\mu^d - \mu\|^2 - k_{\sigma_1}\|\sigma_1^d - \sigma_1\|^2 - k_{\sigma_2}\|\sigma_2^d - \sigma_2\|^2 - k_{\sigma_3}\|\sigma_3^d - \sigma_3\|^2 \tag{32}$$

Therefore, $\dot{V}(q) \leq 0$, $\forall q \in \mathbb{R}^{2N}$ and $\dot{V} = 0$ if and only if $\mu = \mu^d$, $\sigma_1 = \sigma_1^d$, $\sigma_2 = \sigma_2^d$, and $\sigma_3 = \sigma_3^d$, which is also an invariant set for the closed loop system. According to the Global Invariant Set Theorem (LaSalle), to prove the proposition we only have to prove that $V(q) \to \infty$ as $\|q\| \to \infty$. We prove this by contradiction. Suppose $\|q\| \to \infty$ and there exists some $L > 0$ so that $V(q) < L$. This implies

$$\|\mu - \mu^d\| \leq \sqrt{2L},\ \|\sigma_1 - \sigma_1^d\| \leq \sqrt{2L},$$
$$\|\sigma_2 - \sigma_2^d\| \leq \sqrt{2L},\ \|\sigma_3 - \sigma_3^d\| \leq \sqrt{2L}$$

By an argument similar to the one used in the proof of Proposition 2, we can conclude that

$$\|q_i - \mu^d\| \leq \sqrt{N(\sigma_1^d + \sigma_3^d + 2\sqrt{2L})} + \sqrt{2L}$$

which means that all q_i are bounded. But $\|q\| \to \infty$ implies that, for at least one i, $i = 1, \ldots, N$, $\|q_i\| \to \infty$. Therefore, we reached a contradiction and the theorem is proved.

4 Mean and Variance Control for Fully Actuated Planar Robots

This section is a particular case of the previous Section 3 when the group variable g as defined by (8) is restricted to the 3 - dimensional $\bar{g} = [\mu^T,\ \sigma]^T$ where μ is the mean given by (6) and the variance σ is defined by

$$\sigma = \sigma_1 + \sigma_3 = \frac{1}{N}\sum_{i=1}^{N}(q_i - \mu)^T(q_i - \mu) \tag{33}$$

Then it is easy to see that *each agent $q_i(t)$ is inside a circle centered at $\mu(t)$ and with radius $\sqrt{N\sigma(t)}$*. The proof is obvious by noting that from (33), for each $i = 1, \ldots, N$, we have

$$\|q_i - \mu\|^2 \leq \sum_{j=1}^{N}\|q_j - \mu\|^2 = N\sigma$$

It is straightforward to check that the decoupling, boundness of internal dynamics, and stability results proved above remain valid. The individual control laws assume a much simpler form in this case:

$$\dot{q}_i = u_i = \dot{\mu} + \frac{q_i - \mu}{2\sigma}\dot{\sigma},\ i = 1, \ldots, N \tag{34}$$

Therefore, one can design trajectories on the 3 - dimensional manifold $[\mu^T \ \sigma]^T$, *i.e.,* generate a moving circle with varying radius on the plane, and have the guarantee that, if each agent is applied the corresponding control law (34), stays inside the moving circle. Obvious applications would be clustering, obstacle avoidance, tunnel passing, etc.

For the simplified controllers (34), the following interesting result holds:

Proposition 4. *The decoupling, minimum energy controllers (34) preserve the shape and orientation of the structure formed by the position vectors q_i in the given inertial frame.*

Proof. Let $l_{ij} = \|q_i - q_j\|$, $i \neq j$. Using (34) and $l_{ij}^2 = (q_i - q_j)^T(q_i - q_j)$, it is easy to see by integration that

$$l_{ij}(t) = l_{ij}(0)\sqrt{\frac{\sigma(t)}{\sigma(0)}},\ \forall t > 0$$

and therefore

$$\frac{l_{ij}(t)}{l_{kl}(t)} = \frac{l_{ij}(0)}{l_{kl}(0)},\ \forall i, j, k, l = 1, \ldots, N,\ \forall t > 0$$

from which we conclude that the geometric shape is preserved and the scale factor is proportional to $\sqrt{\sigma(t)}$. Also, straightforward calculations show that

$$\frac{d}{dt}\left(\frac{q_i - q_j}{\|q_i - q_j\|}\right) = 0,\ i \neq j$$

proving that the orientation of the structure is also preserved during the motion.

Remark 2. The group abstraction is, in this case, reduced to the position of the centroid and the scale factor of a geometric figure of given shape and orientation determined by the initial positions of the robots.

4.1 Example

Consider the task of controlling $N = 10$ planar fully-actuated robots with arbitrary initial positions so that they pass through a tunnel of given geometry in 2 seconds. See Figure 2. The initial positions are

$$q_1(0) = \begin{bmatrix} 5 \\ 5 \end{bmatrix},\ q_2(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix},\ q_3(0) = \begin{bmatrix} 4 \\ 3 \end{bmatrix},\ q_4(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix},\ q_5(0) = \begin{bmatrix} 9 \\ 6 \end{bmatrix},$$

$$q_6(0) = \begin{bmatrix} 7 \\ 8 \end{bmatrix},\ q_7(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix},\ q_8(0) = \begin{bmatrix} 2 \\ 7 \end{bmatrix},\ q_9(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix},\ q_{10}(0) = \begin{bmatrix} 6 \\ 9 \end{bmatrix}.$$

The corresponding initial group variable is given by

$$\mu(0) = [5.1,\ 5.7]^T, \sigma(0) = 7.3$$

By the simplified abstraction presented above, the problem can be reduced to controlling the 3 - dimensional group variable $\bar{g} = [\mu^T,\ \sigma]^T$ where μ and σ are given by (6) and (33), respectively. From the geometry of the tunnel, it can be seen that an enclosing circle of radius 2 is enough to pass the tunnel. The corresponding variance σ for radius 2 is 0.4 ($\sqrt{10 \cdot 0.4} = 2$).

First, we generate controls so that, in 1 second, the robots gather inside a circle of radius 2 centered at (10,0) in front of the tunnel. This can be simply done by designing constant vector fields on the group manifold as

$$\dot{\sigma} = 0.4 - \sigma(0) = -6.9,\ \dot{\mu} = [10,\ 0]^T - \mu(0) = [4.9,\ -5.7]^T.$$

After 1 second, we need to control the robots so that they pass through the tunnel. This can be done by keeping the radius of the moving circle constant and move the center along the x - axis (the axis of the tunnel) for 1sec until the point (15,0) is reached. The correponding vector fields on the group manifold are constant and given by

$$\dot{\sigma} = 0,\ \dot{\mu} = [15,\ 0]^T - \mu(1) = [5,\ 0]^T$$

Eight snapshots from the generated motion of the team are shown for illustration in Figure 2 together with the enclosing circle.

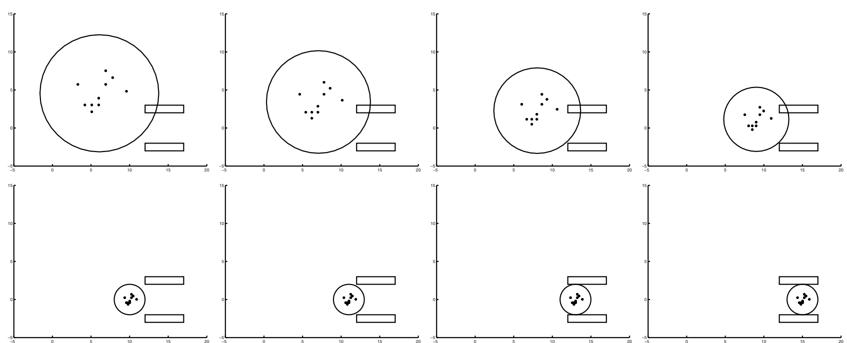

Fig. 2. 10 planar robots moving through a tunnel. The control was designed on the 3 - dimensional $[\mu^T, \sigma]$. The individual control laws are given by (34).

5 Conclusion

In this paper we propose a control method for a large number of robots based on an abstraction of the team to a small dimensional group manifold whose dimension does not scale with the number of robots. The task to be accomplished by the team suggests a natural feedback control system on the group manifold. From the multititude of individual control laws which determine the desired behavior of the group, we select those that only use partial state feedback so that the inter - robot information exchange in the overall control architecture is kept as small as possible. In this paper, we only consider fully-actuated planar robots abstracted to mean and covariance of the positions with respect to some reference frame. We prove that in this case the controllers that minimize the overall energy of the group are also *decoupling* controllers, *i.e.,* the control law of each robot only depends on the state of the robot and the small dimensional state of the group manifold. Illustrative examples of trajectory tracking on the group manifold and globally asymptotic stabilization to a point on the group manifols are included.

References

1. Belta C., Kumar V. (2002) Trajectory design for formations of robots by kinetic energy shaping. In: IEEE International Conference on Robotics and Automation, Washington, DC, 2002
2. Desai J., Ostrowski J.P., Kumar V. (1998) Controlling formations of multiple mobile robots. In: Proc. IEEE Int. Conf. Robot. Automat., Leuven, Belgium, May, 1998. 2864–2869
3. Eren T., Belhumeur P.N., Morse A.S. (2002) Closing ranks in vehicle formations based rigidity. In: IEEE Conference on Decision and Control, Las Vegas, NV, December, 2002
4. Isidori A. (1995) Nonlinear Control Systems, 3rd edn. Springer-Verlag, London
5. Leonard N.E., Fiorelli E. (2001) Virtual leaders, artificial potentials, and coordinated control of groups. In: 40th IEEE Conference on Decision and Control, Orlando, FL, December, 2001. 2968 – 2973
6. Ogren P., Fiorelli E., Leonard N.E. (2002) Formations with a mission: stable coordination of vehicle group maneuvers. In: Proc. Symp. on Mathematical Theory of Networks and Systems, Notre Dame, IN, August, 2002
7. Olfati-Saber R., Murray R.M. (2002) Distributed cooperative control of multiple vehicle formations using structural potential functions. In: IFAC World Congress, Barcelona, Spain, July, 2002
8. Tabuada P., Pappas G.J., Lima P. (2001) Feasible formations of multi-agent systems. In: American Control Conference, Arlington, VA, June, 2001
9. Tanner H., Pappas G.J., Kumar V. (2002) Input-to-state stability on formation graphs. In: Proceedings of the IEEE International Conference on Decision and Control, Las Vegas, NV, December, 2002