Lecture 11.

4.2. **Malliavin operators I:** D **&** δ**.** In this section, we will introduce several important operators in Malliavin calculus, which we have already studied in Section 2 with finitely many Gaussian random variables.

• **Malliavin derivative.** Recall that for a random variable $F = f(G_1, ..., G_d)$ for some $f \in C^\infty_{\text{poly}}(\mathbb{R}^d)$ and $(G_1, ..., G_d) \in \mathcal{N}_d(0, \mathbf{I})$, we have

$$\nabla f(G_1, ..., G_d) = \sum_{j=1}^{d} \partial_j f(G_1, ..., G_d)\mathfrak{e}_j \tag{4.9}$$

with $\{\mathfrak{e}_j : j = 1, ..., d\}$ denoting the canonical basis of $\mathbb{R}^d$.[27] In particular, we can realize the standard Gaussian vector in the isonormal framework as in Example 3. Putting $G_j = W(e_j)$ with $\{e_j\}_{j\geq 1}$ orthonormal basis of $\mathfrak{H}$, we can rewrite (4.9) as

$$DF = Df(W(e_1), ..., W(e_d)) = \sum_{j=1}^{d} \partial_j f(W(e_1), ..., W(e_d))e_j \in \mathfrak{H}.$$

This gives us the Malliavin derivative of $F = f(W(e_1), ..., W(e_d))$, which is a $\mathfrak{H}$-valued random variable (indeed, $DF \in L^2(\Omega; \mathfrak{H})$).

Now let us consider the case where $F' = g(W(h_1), ..., W(h_m))$ for some $g \in C^\infty_{\text{poly}}(\mathbb{R}^m)$ and $h_1, ..., h_m \in \mathfrak{H}$. Then, $\mathbf{X} = (W(h_1), ..., W(h_m))$ is a m-dimensional centered Gaussian vector with covariance matrix $C = QQ$ for some symmetric positive-definite matrix $Q \in \mathbb{R}^{m\times m}$. Assume $\mathbf{X} = Q\mathbf{G}$ with $\mathbf{G} = (W(e_1), ..., W(e_m))$ a standard Gaussian vector. Then, we have

$$\begin{aligned} DF' = Dg(Q\mathbf{G}) &= \sum_{j=1}^{m} \bigg(\sum_{i=1}^{m} \partial_i g(Q\mathbf{G})Q_{ij} \bigg) e_j \\ &= \sum_{i=1}^{m} \partial_i g\big(W(h_1), ..., W(h_m)\big) \sum_{j=1}^{m} Q_{ij}e_j \\ &= \sum_{i=1}^{m} \partial_i g\big(W(h_1), ..., W(h_m)\big) h_i, \end{aligned}$$

where the last equality follows from $W(h_j) = (Q\mathbf{G})_j = \sum_{j=1}^{m} Q_{ij}W(e_j) = W\big(\sum_{j=1}^{m} Q_{ij}e_j\big)$. This discussion suggests the following definition.

Definition 4.6. (i) *Suppose $F \in \mathcal{S}_{\text{poly}}$ can be expressed as $F = f(W(h_1), ..., W(h_m))$ with $f \in C^\infty_{\text{poly}}(\mathbb{R}^m)$, $h_1, ..., h_m \in \mathfrak{H}$, and $m \in \mathbb{N}_{\geq 1}$. The Malliavin derivative DF of F is defined by*

$$DF = \sum_{i=1}^{m} \partial_i f\big(W(h_1), ..., W(h_m)\big)h_i. \tag{4.10}$$

In particular, $DW(h) = h$.

(ii) *Let $\mathbb{D}^{1,2}$ be the closure of $\mathcal{S}_{\text{poly}}$ with respect to the following inner product*

$$\langle F, G\rangle_{\mathbb{D}^{1,2}} := \mathbb{E}[FG] + \langle DF, DG\rangle_{L^2(\Omega;\mathfrak{H})}.$$

In particular, $\mathbb{D}^{1,2}$ is a Hilbert space.

[27] $\mathfrak{e}_j \in \mathbb{R}^d$ is a unit vector with only one nonzero entry (i.e., $+1$) in the j-th coordinate.

The following question indicates that the definition (4.10) of Malliavin derivative DF is independent of the choices of representations.

Question 4.7. (i) *Given two representations of* $F = f(W(h_1), ..., W(h_m)) = g(W(f_1), ..., W(f_n))$, *prove that the following expressions are identical:*

$$\sum_{i=1}^{m} \partial_i f\big(W(h_1), ..., W(h_m)\big) h_i = \sum_{j=1}^{n} \partial_j g\big(W(f_1), ..., W(f_n)\big) f_j.$$

(ii) [Directional derivative] *Given* $F = f(W(h_1), ..., W(h_m))$ *and* $h \in \mathfrak{H}$. *Verify that*

$$\langle DF, h\rangle_{\mathfrak{H}} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big[- F + f\big(W(h_1) + \varepsilon\langle h_1, h\rangle_{\mathfrak{H}}, ..., W(h_m) + \varepsilon\langle h_m, h\rangle_{\mathfrak{H}}\big)\Big].$$

The quantity $\langle DF, h\rangle_{\mathfrak{H}}$ *is called the directional derivative of* F *in the direction* h, *which measures the sensitivity of the random variable* F *to the "small" amount of perturbations from* W *to* $W + \varepsilon h$.

Let us now recall a few basic properties of the Malliavin derivative operator.

Proposition 4.8. (i) *The derivative operator* D *is closable: if* $F_n \in \mathbb{D}^{1,2}$ *with* $DF_n \to V$ *in* $L^2(\Omega; \mathfrak{H})$ *for some* V *and* $F_n \to 0$ *in* $L^2(\mathbb{P})$ *as* $n \to +\infty$, *then* $V = 0$.
(ii) *Let* $\mathbb{C}_k^W$ *denote the* k*-Wiener chaos as in Theorem 4.4. Then,* $\mathbb{C}_k^W \subset \mathbb{D}^{1,2}$ *and*

$$\|DJ_kF\|^2_{L^2(\Omega;\mathfrak{H})} = k\|J_kF\|_2^2 \tag{4.11}$$

for any $F \in L^2(\Omega, \sigma\{W\}, \mathbb{P})$.
(iii) *Given* $G \in L^2(\Omega, \sigma\{W\}, \mathbb{P})$, *we have* $G \in \mathbb{D}^{1,2}$ *if and only if* $\sum_{k\geq 1} k\|J_kG\|_2^2 < \infty$. *In each case, we have* $\mathbb{E}[\|DG\|^2_{\mathfrak{H}}] = \sum_{k\geq 1} k\|J_kG\|_2^2$. *Here,* J_k *denotes the projection operator onto* $\mathbb{C}_k^W$.
(iv) [Gaussian Poincaré inequality] *For any* $F \in \mathbb{D}^{1,2}$, *we have*

$$\mathrm{Var}(F) \leq \mathbb{E}[\|DF\|^2_{\mathfrak{H}}],$$

where the equality holds only when $F \in \mathbb{C}_0^W \oplus \mathbb{C}_1^W$.

Proof. Given any $G \in \mathcal{S}_c, h \in \mathfrak{H}$, we can write

$$\begin{aligned}
\mathbb{E}[\langle V, h\rangle_{\mathfrak{H}} G] &= \lim_{n\to+\infty} \mathbb{E}[\langle DF_n, h\rangle_{\mathfrak{H}} G] \\
&= \lim_{n\to+\infty} \mathbb{E}[\langle D(F_nG) - F_nDG, h\rangle_{\mathfrak{H}}] \\
&= \lim_{n\to+\infty} \mathbb{E}[\langle F_nGW(h)] + \lim_{n\to+\infty} \mathbb{E}[F_n\langle DG, h\rangle_{\mathfrak{H}}] \\
&= 0,
\end{aligned} \tag{4.12}$$

where the second equality follows from the usual product rule $D(FG) = FDG + GDF$ for smooth random variables $F, G \in \mathcal{S}_{\text{poly}}$,[28], the third equality follows from the Gaussian integration by parts formula in Proposition 2.11, and the last equality is due to the L^2-convergence of $F_n \to 0$ with $GW(h), DG, h\rangle_{\mathfrak{H}} \in L^2(\Omega)$. Recall from Corollary 4.5 that $\mathcal{S}_c(\mathfrak{H}) = \{Gh : G \in \mathcal{S}_c, h \in \mathfrak{H}\}$ is dense in $L^2(\Omega; \mathfrak{H})$. This implies that $V = 0$. That is, part (i) is proved.

Now let us prove (ii). Suppose $F \in \mathbb{C}_k^W$ has the following form

$$F = \sum_{\mathbf{q}\in\mathbb{N}^\infty} \mathbf{1}_{\{|\mathbf{q}|=\sum_{j\geq 1} q_j = k\}} c_{\mathbf{q}} \prod_{j=1}^{\infty} H_{q_j}(W(e_j))$$

[28] In view of Question (4.7), we can express F and G as two functions of a common Gaussian vector, so that the desired product rule is simply the usual product rule in calculus.

with $\{e_j\}_{j\geq 1}$ orthonormal basis of $\mathfrak{H}$. As in the proof of Theorem 4.4, we define

$$F_m = \sum_{q_1,\dots,q_m\geq 0} \mathbf{1}_{\{\sum_{j=1}^m q_j=k\}} c_{q_1,\dots,q_m} \prod_{j=1}^m H_{q_j}(W(e_j)),$$

which converges to F in $L^2(\Omega)$. Then, F_m is indeed a polynomial in independent Gaussians and belongs to $\mathcal{S}_{\text{poly}}$ with

$$DF_m = \sum_{q_1,\dots,q_m\geq 0} \mathbf{1}_{\{\sum_{j=1}^m q_j=k\}} c_{q_1,\dots,q_m} \sum_{i=1}^m q_i H_{q_i-1}(W(e_i))\Big(\prod_{j\neq i} H_{q_j}(W(e_j))\Big) e_i.$$

It follows from the results in Section 2 that

$$\mathbb{E}\big[\|DF_m\|_{\mathfrak{H}}^2\big] = k\mathbb{E}[F_m^2] \quad \text{and} \quad \mathbb{E}\big[\|D(F_m-F_n)\|_{\mathfrak{H}}^2\big] = k\mathbb{E}[(F_m-F_n)^2], \tag{4.13}$$

where we also used $\mathbb{E}[F_mF_n] = \mathbb{E}[F_m^2]$ and $\mathbb{E}[\langle DF_m, DF_n\rangle_{\mathfrak{H}}] = \mathbb{E}[\|DF_m\|_{\mathfrak{H}}^2]$ for $m \leq n$ to obtain the second equality. In particular, $\{DF_m\}_m$ is Cauchy in $L^2(\Omega;\mathfrak{H})$ with limit

$$DF = \sum_{\mathbf{q}\in\mathbb{N}^\infty} \mathbf{1}_{\{|\mathbf{q}|=k\}} c_{\mathbf{q}} \sum_{i=1}^\infty q_i H_{q_i-1}(W(e_i))\Big(\prod_{j\neq i} H_{q_j}(W(e_j))\Big) e_i.$$

It is clear that $DF \in L^2(\Omega;\mathfrak{H})$ and passing $m\to+\infty$ in (4.13) yields (4.11).

Next, we prove (iii). Given $G \in L^2(\Omega, \sigma\{W\}, \mathbb{P})$, we define

$$G_N = \sum_{k\leq N} J_k G.$$

Then, by orthogonality relation and part (ii), we get

$$\|DG_N\|_{L^2(\Omega;\mathfrak{H})}^2 = \sum_{k\leq N} k\|J_kG\|_2^2.$$

If $G \in \mathbb{D}^{1,2}$, then due to (i) with $G_N \xrightarrow[L^2(\Omega)]{N\to+\infty} G$, we have $DG_N \to DG$ in $L^2(\Omega;\mathfrak{H})$. Therefore,

$$\sum_{k\geq 1} k\|J_kG\|_2^2 = \lim_{N\to+\infty} \|DG_N\|_{L^2(\Omega;\mathfrak{H})}^2 = \|DG\|_{L^2(\Omega;\mathfrak{H})}^2 < \infty.$$

Suppose $\sum_{k\geq 1} k\|J_kG\|_2^2 < \infty$. Then, $\{DG_N\}_N$ is Cauchy in $L^2(\Omega;\mathfrak{H})$, which together with $G_N \xrightarrow[L^2(\Omega)]{N\to+\infty} G$, implies that $G\in\mathbb{D}^{1,2}$ with DG the limit of DG_N in $L^2(\Omega;\mathfrak{H})$.

Finally, it is easy to see that part (iv) is an immediate consequence of part (iii). □

Remark 4.9. For $F \in \mathcal{S}_{\text{poly}}$, the definition of DF is independent of its representation; see Question 4.7-(i). The space $\mathbb{D}^{1,2}$ is defined by closing $\mathcal{S}_{\text{poly}}$ under the $\mathbb{D}^{1,2}$-norm, and the closability of D ensures that the definition of DF does not depend on the particular choice of approximating sequence $F_n \in \mathcal{S}_{\text{poly}}$. Part (iii) of Proposition 4.8 gives an equivalent definition of the space $\mathbb{D}^{1,2}$ using the projection operators.

In (4.12), we used the product rule for smooth random variables. It is not difficult to generalize it for more general random variables; see also [NP12, Exercise 2.3.10].

Proposition 4.10 (Product rule). *Suppose $F \in \mathbb{D}^{1,2}$ and $G \in \mathcal{S}_b$. Then, $FG \in \mathbb{D}^{1,2}$ and*

$$D(FG) = FDG + GDF. \tag{4.14}$$

Proof. Due to $F \in \mathbb{D}^{1,2}$ and $G \in \mathcal{S}_b$, we have the RHS of (4.14) belongs to $L^2(\Omega; \mathfrak{H})$ and can be represented as a limit of

$$F_n DG + GDF_n$$

with $F_n \in \mathcal{S}_{\text{poly}}$. For each n, by the usual product rule in calculus, we have $F_n DG + GDF_n = D(F_n G)$. Since $F_n G \in \mathcal{S}_{\text{poly}}$ converges to FG in $L^2(\Omega)$, we can conclude that its limit FG belongs to $\mathbb{D}^{1,2}$ and the equality (4.14) holds. $\square$

• **Higher-order Malliavin derivatives.** For $k \in \mathbb{N}_{\geq 2}$, we can define the k-th Malliavin derivative D^k in a similar way.

(i) For $F \in \mathcal{S}_{\text{poly}}$ given by $F = f(W(h_1), ..., W(h_m))$, we define

$$D^k F = \sum_{i_1, \dots, i_k = 1}^{m} \partial_{i_1 \dots i_k} f(W(h_1), ..., W(h_m)) h_{i_1} \otimes \cdots \otimes h_{i_k},$$

which is clearly a random element in $\mathfrak{H}^{\odot k}$; here $\mathfrak{H}^{\odot k}$ denotes the k-th symmetric tensor product of $\mathfrak{H}$ (see Appendix (A) for more details).

(ii) Suppose D^{k-1} and its domain $\mathbb{D}^{k-1,2}$ have been defined. We define $\mathbb{D}^{k,2}$ to be the closure of $\mathcal{S}_{\text{poly}}$ under the following inner product

$$\begin{aligned} \langle F, G \rangle_{\mathbb{D}^{k,2}} &= \langle F, G \rangle_{\mathbb{D}^{k-1,2}} + \langle D^k F, D^k G \rangle_{L^2(\Omega; \mathfrak{H}^{\otimes k})} \\ &= \sum_{i=0}^{k} \mathbb{E}\big[\langle D^i F, D^i G \rangle_{\mathfrak{H}^{\otimes i}}\big] \quad \text{with } D^0 F = F. \end{aligned}$$

(iii) As in Proposition 4.8, one can show that D^k is closable and the following equivalence holds:

$$F \in \mathbb{D}^{k,2} \text{ if and only if } \sum_{p=1}^{\infty} p^k \|J_p F\|_2^2 < \infty. \tag{4.15}$$

We leave this as an exercise to interested students.

• **Divergence operator.** The divergence operator δ is the adjoint operator of D, which is characterized by the **duality relation**

$$\mathbb{E}[\langle DF, V \rangle_{\mathfrak{H}}] = \mathbb{E}[F \delta(V)]. \tag{4.16}$$

Let $\text{dom}(\delta)$ be the set of random vectors $V \in L^2(\Omega; \mathfrak{H})$ such that there is some finite constant $C = C_V$ such that $|\mathbb{E}[\langle DF, V \rangle_{\mathfrak{H}}]| \leq C \|F\|_2$ for any $F \in \mathcal{S}_{\text{poly}}$. In particular, the functional $F \mapsto \Psi(F) := \mathbb{E}[\langle DF, V \rangle_{\mathfrak{H}}]$ is bounded linear on $\mathcal{S}_{\text{poly}}$. Then, due to density of $\mathcal{S}_{\text{poly}}$, one can extend Ψ to a bounded linear function on the Hilbert space $L^2(\Omega, \sigma\{W\}, \mathbb{P})$, and thus Riesz's representation theorem indicates that there exists a unique element (denoted by $\delta(V)$) in $L^2(\Omega, \sigma\{W\}, \mathbb{P})$ such that (4.16) holds true. One can proceed in the same way to define the iterated divergence operator δ^k as the adjoint operator of D^k. We summarize the above discussion in the following.

Lemma 4.11. *For any $F \in \mathbb{D}^{k,2}$ and $V \in \text{dom}(\delta^k)$, we have*

$$\mathbb{E}[\langle D^k F, V \rangle_{\mathfrak{H}^{\otimes k}}] = \mathbb{E}[F \delta^k(V)].$$

Question 4.12. *Given a unit vector $h \in \mathfrak{H}$. Show that $\delta^k(h^{\otimes k}) = H_k(W(h))$.*

We conclude this section with a few properties of the divergence operator and its domain $\text{dom}(\delta)$.

Proposition 4.13. (i) *Recall* $\mathcal{S}_c(\mathfrak{H})$ *from Corollary 4.5 and define similarly* $\mathcal{S}_0(\mathfrak{H}) = \operatorname{span}\{Gh : h \in \mathfrak{H}, G \in \mathcal{S}_0\}$. *Then, both* $\mathcal{S}_0(\mathfrak{H})$ *and* $\mathcal{S}_c(\mathfrak{H})$ *are contained in* $\operatorname{dom}(\delta)$. *In particular,* $\operatorname{dom}(\delta)$ *is dense in* $L^2(\Omega; \mathfrak{H})$.

(ii) *Assume* $G \in \mathbb{D}^{1,2}$ *and* $V \in \operatorname{dom}(\delta)$ *satisfy*

$$\mathbb{E}\big[G^2\|V\|_{\mathfrak{H}}^2\big] + \mathbb{E}\big[G^2\delta(V)^2\big] + \mathbb{E}\big[\langle DG, V\rangle_{\mathfrak{H}}^2\big] < \infty. \tag{4.17}$$

Then, $GV \in \operatorname{dom}(\delta)$ *with*

$$\delta(GV) = G\delta(V) - \langle DG, V\rangle_{\mathfrak{H}}.$$

Proof. Suppose $F, G \in \mathcal{S}_{\text{poly}}$ and $h \in \mathfrak{H}$. Then, we can proceed as in (4.12)

$$\begin{aligned}\mathbb{E}\big[\langle DF, h\rangle_{\mathfrak{H}} G\big] &= \mathbb{E}\big[\langle D(FG), h\rangle_{\mathfrak{H}}\big] - \mathbb{E}\big[F\langle DG, h\rangle_{\mathfrak{H}}\big] \\ &= \mathbb{E}[FGW(h)] - \mathbb{E}\big[F\langle DG, h\rangle_{\mathfrak{H}}\big]\end{aligned} \tag{4.18}$$

using the product rule. Therefore, we deduce from Cauchy-Schwarz that

$$\big|\mathbb{E}\big[\langle DF, h\rangle_{\mathfrak{H}} G\big]\big| \leq \|F\|_2\big(\|GW(h)\|_2 + \|DG\|_{L^2(\Omega;\mathfrak{H})}\|h\|_{\mathfrak{H}}\big). \tag{4.19}$$

Due to $\mathcal{S}_c \cup \mathcal{S}_0 \subset \mathbb{D}^{1,2}$, we can approximate any element in $\mathcal{S}_c \cup \mathcal{S}_0$ by a sequence of random variables in $\mathcal{S}_{\text{poly}}$ with respect to the norm $\|\bullet\|_{\mathbb{D}^{1,2}}$. That is, via a limiting argument (if necessary), we can obtain the bound (4.19) for any $G \in \mathcal{S}_c \cup \mathcal{S}_0$. This implies that $\mathcal{S}_0(\mathfrak{H}) \cup \mathcal{S}_c(\mathfrak{H}) \subset \operatorname{dom}(\delta)$ with

$$\delta(Gh) = GW(h) - \langle DG, h\rangle_{\mathfrak{H}} \tag{4.20}$$

for any $G \in \mathcal{S}_{\text{poly}} \cup \mathcal{S}_0$ in view of (4.18). Thus, the proof of part (i) is concluded by invoking Corollary 4.5.

Next, we prove part (ii), which generalizes (4.20). Let $F \in \mathcal{S}_b$, then we can deduce from the product rule (Proposition 4.10) and duality relation that

$$\begin{aligned}\mathbb{E}\big[\langle DF, GV\rangle_{\mathfrak{H}}\big] &= \mathbb{E}\big[\langle D(FG) - FDG, V\rangle_{\mathfrak{H}}\big] \\ &= \mathbb{E}\big[\langle D(FG), V\rangle_{\mathfrak{H}}\big] - \mathbb{E}\big[F\langle DG, V\rangle_{\mathfrak{H}}\big] \\ &= \mathbb{E}\big[FG\delta(V)\big] - \mathbb{E}\big[F\langle DG, V\rangle_{\mathfrak{H}}\big] \qquad \text{due to (4.17)}\end{aligned}$$

and thus, we get $GV \operatorname{dom}(\delta)$ and $\delta(GV) = G\delta(V) - \langle DG, V\rangle_{\mathfrak{H}}$. Note that we used the density of $\mathcal{S}_b$ in $L^2(\Omega, \sigma\{W\}, \mathbb{P})$ and $FG \in \mathbb{D}^{1,2}$ (a consequence of Proposition 4.10).

Hence, the proof of Proposition 4.13 is completed. □

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Lecture 12.

4.3. **(Generalized) chain rule.** XXX

• **Application: (local) Bouleau-Hirsch's criterion.**

Example 4 (Nonzero V with $\delta(V)=0$)**.** Let $\{e_1, e_2\}$ be an orthonormal system in $\mathfrak{H}$ and define

$$f = e_1 \otimes e_2 - e_2 \otimes e_1 \in \mathfrak{H}^{\otimes 2}.$$

Then, $V := I_1(f) = W(e_1)\, e_2 \; - \; W(e_2)\, e_1$ is nonzero almost surely and

$$\delta(V) = I_2\big(\widetilde{f}\big) = 0.$$

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