Lecture 10.

4. Redo the analysis in an isonormal framework

Previously, we have developed basic Gaussian analysis with finitely many Gaussian random variables. From now on, we will develop relevant stochastic analysis on infinitely many Gaussian random variables or a Gaussian process (e.g., Brownian motion, white noises). We will work on the isonormal framework.

Definition 4.1 (Isonormal Gaussian process)**.** Let $\mathfrak{H}$ be a real separable Hilbert space[23]. We say $W = \{W(h)\}_{h\in\mathfrak{H}}$ is an isonormal Gaussian process indexed by $\mathfrak{H}$ if W is a centered Gaussian family with $\mathbb{E}[W(f)W(g)] = \langle f, g\rangle_{\mathcal{H}}$, $\forall f, g \in \mathfrak{H}$.

Remark 4.2. The mapping $W : \mathfrak{H} \to L^2(\Omega, \mathbb{P})$ is a bounded linear map: for any $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathfrak{H}$, we have

$$W(\alpha f + \beta g) = \alpha W(f) + \beta W(g) \quad \text{in } L^2(\mathbb{P}) \text{ and a.s..}$$

Indeed, it is easy to verify by using the covariance structure of W that

$$\|W(\alpha f + \beta g) - \alpha W(f) - \beta W(g)\|^2_{L^2(\mathbb{P})} = 0,$$

which concludes the desired linearity. The operator norm of W is clearly one, since $\|W(h)\|_2 = \|h\|_{\mathfrak{H}}$ for any $h \in \mathfrak{H}$.

Proposition 4.3 (Existence of isonormal Gaussian process)**.** Let $\{e_i\}_{i\geq 1}$ be an orthonormal basis of $\mathfrak{H}$. Let $\{G_i\}_{i\geq 1}$ be i.i.d. standard normal random variables. Define

$$W(h) = \sum_{i=1}^{\infty} \langle h, e_i\rangle_{\mathcal{H}} G_i,$$

where the above series converges in $L^2(\mathbb{P})$. Then, W is an isonormal Gaussian process indexed by $\mathfrak{H}$.

Proof. From the orthonormality of $\{e_i\}$, we have

$$\sum_{i=1}^{\infty} \langle h, e_i\rangle^2_{\mathfrak{H}} = \|h\|^2_{\mathfrak{H}}.$$

In particular, we can see $W(h)$ as a $L^2(\Omega)$-limit of $\sum_{i=1}^{N} \langle h, e_i\rangle_{\mathcal{H}} G_i$ as $N \to +\infty$. Therefore, for any $h, g \in \mathfrak{H}$, we have

$$\begin{aligned}
\mathbb{E}[W(h)W(g)] &= \lim_{N\to+\infty} \mathbb{E}\Big(\sum_{i,j\leq N} \langle h, e_i\rangle_{\mathfrak{H}} G_i \langle g, e_j\rangle_{\mathfrak{H}} G_j \Big) \\
&= \lim_{N\to+\infty} \sum_{i,j\leq N} \langle h, e_i\rangle_{\mathfrak{H}} \langle g, e_j\rangle_{\mathfrak{H}} \mathbb{E}[G_i G_j] \\
&= \lim_{N\to+\infty} \sum_{i\leq N} \langle h, e_i\rangle_{\mathfrak{H}} \langle g, e_i\rangle_{\mathfrak{H}} = \langle h, g\rangle_{\mathfrak{H}}.
\end{aligned}$$

Hence, the covariance structure holds and the proof is completed. □

In the following, we present a few examples of isonormal processes.

[23]By the basic Hilbert space theory, $\mathfrak{H}$ is isometric to the sequence space $\ell^2(\mathbb{N})$, which is the set of square-summable sequences of real numbers $a = \{a_j\}_{j\in\mathbb{N}}$ with $\sum_{i\in\mathbb{N}} a_i^2 < \infty$.

Example 3. (i) [Standard Gaussian vector on $\mathbb{R}^d$] Let $\mathbf{G} = (G_1, \dots, G_d) \sim \mathcal{N}_d(0, \mathbf{I})$ be a standard d-dimensional Gaussian vector, and take $\mathfrak{H} = \mathbb{R}^d$ equipped with the standard Euclidean inner product

$$\langle h, k\rangle_{\mathbb{R}^d} = \sum_{i=1}^d h_i k_i.$$

With $\{e_j\}_{j=1,\dots,d}$ the canonical basis of $\mathbb{R}^d$, we can realize $\mathbf{G}$ as an isonormal process indexed by $\mathfrak{H} = \mathbb{R}^d$:

$$G_j := W(e_j)$$

for $j = 1, ..., d$.

(ii) [General Gaussian vector on $\mathbb{R}^d$] Let $\mathbf{Z} \sim \mathcal{N}_d(0, C)$ be a centered Gaussian vector on $\mathbb{R}^d$ with covariance matrix C. With the decomposition $C = Q^2$ for some symmetric positive-definite matrix Q, we can write $\mathbf{Z} = Q\mathbf{G}$. Using (i) and with

$$h_i := \sum_{j=1}^d Q_{ij} e_j \in \mathbb{R}^d,$$

we have $Z_i = W(h_i)$ for every $i \in \{1, ..., d\}$.

(iii) [Continuous Gaussian process] Let $(G_t)_{t\geq 0}$ be a real-valued centered Gaussian process that is continuous in time almost surely. Define the following L^2-Gaussian Hilbert space

$$\mathcal{H}_G := L^2(\mathbb{P})\text{-closure of } \left\{ \sum_{i=1}^m \alpha_i G_{t_i} : \alpha_i \in \mathbb{Q}, t_i \in \mathbb{Q}_+, m \in \mathbb{N} \right\}.$$

By continuity, it is clear that G_t is contained in $\mathcal{H}_G$ for any $t \in \mathbb{R}_+$. By construction, $\mathcal{H}_G$ is a real separable Hilbert space, and thus isometric to some real separable Hilbert space $\mathfrak{H}$ (this is an existence argument, without further information on G, we can not really say more on $\mathfrak{H}$). Let W be the isometry mapping from $\mathfrak{H}$ to $\mathcal{H}_G$:

$$h \in \mathfrak{H} \mapsto W(h) \in \mathcal{H}_G \subset L^2(\Omega, \sigma\{G\}, \mathbb{P}),$$

which is the (abstract) isonormal Gaussian process generated by the Gaussian process G. Moreover, there exist a family $\{e_t\}_{t\geq 0} \subset \mathfrak{H}$ such that $G_t = W(e_t)$, $t \geq 0$. It is clear that

$$\mathbb{E}[W(e_s)W(e_t)] = \langle e_s, e_t\rangle_{\mathfrak{H}}.$$

Once we know more information on the covariance structure of G, we can say more on the associated inner product $\langle \bullet, \bullet\rangle_{\mathfrak{H}}$, with $\mathfrak{H}$ known as the **reproducing kernel Hilbert space** of G.

(iv) [Brownian motion] Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard real Brownian motion on $[0, \infty)$. It is a real-valued, centered continuous Gaussian process with

$$\mathbb{E}[B_t B_s] = t \wedge s, \ \forall t, s \in \mathbb{R}_+.$$

For $f \in L^2(\mathbb{R}_+)$ (i.e., $\int_0^\infty f(s)^2 ds < \infty$), we can define the **Wiener integral**

$$W(f) := \int_0^\infty f(s) dB_s,$$

which is a centered Gaussian random variable with variance $\|f\|^2_{L^2(\mathbb{R}_+)}$. It is easy to see that $W = \{W(f)\}_{f\in L^2(\mathbb{R}_+)}$ is an isonormal process indexed by $\mathfrak{H} = L^2(\mathbb{R}_+)$. For any $t \in \mathbb{R}_+$, we have

$$B_t = W(\mathbf{1}_{[0,t]}) = \int_0^\infty \mathbf{1}_{[0,t]}(s) dB_s.$$

That is, the reproducing kernel Hilbert space for the Brownian motion is given by $\mathfrak{H} = L^2(\mathbb{R}_+)$.

(v) [white noise as derivative of Brownian motion] If we take the formal derivative of Brownian motion $\xi(t) := \frac{d}{dt}B_t$, we can write

$$W(f) = \int_0^\infty f(s)\xi(s)ds$$

and the (generalized) correlation kernel of ξ is given by

$$\mathbb{E}[\xi(t)\xi(s)] = \delta_0(t-s),$$

where δ_0 is the Dirac-delta function at zero.[24] Indeed, via a *formal* calculation, we have

$$\begin{aligned}\mathbb{E}[W(f)W(g)] &= \int_0^\infty \int_0^\infty g(t)f(s)\mathbb{E}[\xi(t)\xi(s)]dsdt \\ &= \int_0^\infty \int_0^\infty g(t)f(s)\delta_0(t-s)dsdt = \langle f, g\rangle_{L^2(\mathbb{R}_+)},\end{aligned} \tag{4.1}$$

and confirms the heuristic that the white noise on $\mathbb{R}_+$ is the formal derivative of the Brownian motion. Moreover, the generalized function δ_0 is the Fourier transform of constant power spectral density, and thus bears the name "white" noise:

$$\delta_0(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}dx.$$

(vi) [Space-time white noise] The space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$ generalizes the idea in (v) to generalized random fields indexed by both time and space. Formally, a **space-time white noise** ξ is a centered Gaussian family $\xi = \{\xi(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ with correlation structure

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta_0(t-s)\delta_0(x-y),$$

where the above two Dirac delta functions are defined on $\mathbb{R}_+$ and $\mathbb{R}^d$ respectively. Heuristically speaking, ξ is "very *rough/irregular*" and should be interpreted as a random generalized function, and thus it is only defined by paring with a nice test function: given any $f \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, the pairing

$$\langle f, \xi\rangle := \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} f(t,x)\xi(t,x)dtdx$$

is expected to a centered Gaussian random variable. Performing the same formal calculation as in (4.1), we have for any $f, g \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$\begin{aligned}\mathbb{E}[\langle f,\xi\rangle\langle g,\xi\rangle] &= \mathbb{E}\left[\left(\int f(t,x)\xi(t,x)dtdx\right)\left(\int g(s,y)\xi(s,y)dsdy\right)\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^d}\int_{\mathbb{R}_+\times\mathbb{R}^d} f(t,x)g(s,y)\xi(t,x)\xi(s,y)dtdxdsdy\right] \\ &= \int_{\mathbb{R}_+\times\mathbb{R}^d}\int_{\mathbb{R}_+\times\mathbb{R}^d} f(t,x)g(s,y)\underbrace{\mathbb{E}[\xi(t,x)\xi(s,y)]}_{\delta_0(t-s)\delta_0(x-y)}dtdxdsdy \\ &= \ldots\ldots \\ &= \langle f, g\rangle_{L^2(\mathbb{R}_+\times\mathbb{R}^d)}.\end{aligned}$$

That is, we have defined a centered Gaussian family $\{W(f) : f \in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^d)\}$, whose existence is guaranteed by Kolmogorov's theorem. Then, by a standard density argument, we can extend the definition to the whole space $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$. In this way, we have realized the space-time white noise $\xi = \{\xi(f) = \langle f, \xi\rangle : f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)\}$ as an isonormal Gaussian process indexed by

[24] δ_0 is a generalized function defined by testing it with test functions in $C^\infty_{\mathrm{Poly}}(\mathbb{R})$: $\langle \delta_0, \varphi\rangle = \varphi(0)$ for any $\varphi \in C^\infty_{\mathrm{Poly}}(\mathbb{R})$.

the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}_+ \times \mathbb{R}^d)$. The space-time white noise often appears as the natural driving noise in stochastic partial differential equations (SPDEs). For example, the following stochastic heat equation with multiplicative white noise

$$\underbrace{\partial_t u(t,x) - \frac{1}{2}\Delta u(t,x)}_{\text{heat equation}} = \underbrace{u(t,x)\xi(t,x)}_{\text{random potential}}$$

is the well-known **parabolic Anderson model** (PAM). We will study the well-posedness of this PAM in a later section.

4.1. **Chaos expansion.**

Theorem 4.4 (Wiener–Itô chaos decomposition). *Let $W = \{W(h)\}_{h\in\mathfrak{H}}$ be an isonormal Gaussian process. Then, the space $L^2(\Omega, \sigma(W), \mathbb{P})$ admits an orthogonal decomposition:*

$$L^2(\Omega, \sigma(W), \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathbb{C}_n^W, \tag{4.2}$$

where $\mathbb{C}_0^W$ is the set of constant random variables, $\mathbb{C}_k^W$ is called the k-th Wiener chaos and defined as

$$\mathbb{C}_k^W = L^2(\mathbb{P})\text{-closure of span}\left\{\prod_{j=1}^{\infty} H_{q_j}(W(e_j)) : \sum_{j=1}^{\infty} q_j = k\right\} \tag{4.3}$$

with H_q the q-th Hermite polynomial and $\{e_j\}_{j\geq 1}$ an orthonormal basis of $\mathfrak{H}$.[25] *Moreover, we can further represent $\mathbb{C}_k^W$ as*

$$\mathbb{C}_k^W = L^2(\mathbb{P})\text{-closure of span}\big\{H_k(W(h)) : \|h\|_{\mathfrak{H}} = 1\big\}. \tag{4.4}$$

Proof. The k-th chaos $\mathbb{C}_k^W$ is a closed subspace of $L^2(\Omega, \sigma\{W\}, \mathbb{P})$, then we define J_k to be the projection operator onto $\mathbb{C}_k^W$. Then, it suffices to show that for all $F \in L^2(\Omega, \sigma(W), \mathbb{P})$, we can write

$$F = \sum_{k\geq 0} J_k F.$$

Now let us fix an orthonormal basis $\{e_j\}_{j\geq 1}$ of $\mathfrak{H}$ and set $G_j = W(e_j)$. Then $\{G_j\}_{j\geq 1}$ are i.i.d. standard normal random variables and we can view F as an element in $L^2(\Omega, \sigma\{G_j : j \geq 1\}, \mathbb{P})$. For any $m \geq 1$, we define

$$\mathcal{F}_m = \sigma\{G_j : j \leq m\},\ m \geq 1,$$

which is a filtration on $(\Omega, \sigma\{W\}, \mathbb{P})$ with $\mathcal{F}_\infty = \sigma\{W\}$. For each $m \geq 1$, we define

$$F_m = \mathbb{E}[F|\mathcal{F}_m].$$

Therefore, we deduce from the martingale convergence theorem (Proposition B.6) that

$$F_m \xrightarrow[m\to+\infty]{\text{a.s. and in } L^2(\mathbb{P})} F.$$

Since F_m is measurable with respect to $\sigma\{G_1, \dots, G_m\}$, there exists a measurable function $\varphi_m : \mathbb{R}^m \to \mathbb{R}$ such that

$$F_m = \varphi_m(G_1, \dots, G_m) \in L^2(\mathbb{P})$$

[25] It is routine to check that the choice of the orthonormal basis is immaterial.

as a consequence of *Doob-Dynkin's lemma*. Next, using the chaos expansion for finitely many Gaussian random variables (Proposition 2.3), we can write

$$F_m = \sum_{k\geq 0} J_k F_m$$

with J_k the projection operator as in (2.8). Note that this argument also yields the orthogonality between different chaoses: suppose $U \in \mathbb{C}_p^W$ and $V \in \mathbb{C}_q^W$ with $p \neq q$, we define

$$U_m = \mathbb{E}[U|\mathcal{F}_m] \quad \text{and} \quad V_m = \mathbb{E}[V|\mathcal{F}_m],$$

which live in the p-th and q-th chaoses (respectively) generated by $G_1, ..., G_m$. Then, the orthogonality between U_m and V_m can be preserved in passing $m \to +\infty$. That is, $\mathbb{C}_p^W$ and $\mathbb{C}_q^W$ are mutually orthogonal for $p \neq q$. Finally, to conclude the orthogonal decomposition in (4.2), it suffices to pass $m \to \infty$ to obtain

$$F = \sum_{k\geq 0} J_k F \quad \text{in } L^2(\mathbb{P})$$

with $J_k F$ being the $L^2(\mathbb{P})$-limit of $J_k F_m$ as $m \to \infty$.

It is clear that J_k is a bounded linear operator on $L^2(\mathbb{P})$. Then,

$$\|J_k F_m - J_k F_n\|_{L^2(\mathbb{P})} \leq \|F_m - F_n\|_{L^2(\mathbb{P})},$$

that is, $\{J_k F_m\}_{m\geq 1}$ is Cauchy for each fixed k. Since $\mathbb{C}_k^W$ is a closed subspace of $L^2(\Omega, \sigma\{W\}, \mathbb{P})$, $J_k F_m$ converges in $L^2(\mathbb{P})$ to some $H_k \in \mathbb{C}_k^W$. We claim that

(i) the seires $\sum_{k\geq 0} H_k$ converge in $L^2(\mathbb{P})$,
(ii) as $m \to \infty$, F_m converges in $L^2(\mathbb{P})$ to $\sum_{k\geq 0} H_k$.

First we write

$$\sum_{k=0}^{\infty} \mathbb{E}[H_k^2] = \sum_{k=0}^{\infty} \lim_{m\to\infty} \mathbb{E}[(J_k F_m)^2] \overset{\text{Fatou}}{\leq} \liminf_{m\to\infty} \sum_{k=0}^{\infty} \mathbb{E}[(J_k F_m)^2] = \liminf_{m\to\infty} \mathbb{E}[F_m^2] = \mathbb{E}[F^2] < \infty,$$

which together with the orthogonality verifies the claim (i). To see the claim (ii), we can write

$$\left\| F_m - \sum_{k=0}^{\infty} H_k \right\|_{L^2(\mathbb{P})}^2 = \sum_{k=0}^{\infty} \underbrace{\|J_k F_m - H_k\|_{L^2(\mathbb{P})}^2}_{=: a_{k,m}}. \tag{4.5}$$

This is of the form $\sum_{k=0}^{\infty} a_{k,m}$, where

- $0 \leq a_{k,m} \xrightarrow{m\to\infty} 0$ for each $k \geq 0$;
- $a_{k,m} \leq b_{k,m}$ with $b_{k,m} = 2\|J_k F_m\|_2^2 + 2\|H_k\|_{L^2(\mathbb{P})}^2$;
- $\sum_{k\geq 0} b_{k,m} = 2\mathbb{E}[F_m^2] + 2\sum_{k=0}^{\infty} \mathbb{E}[H_k^2] \xrightarrow{m\to\infty} 2\mathbb{E}[F^2] + 2\sum_{k=0}^{\infty} \mathbb{E}[H_k^2] < \infty$.

By the generalized dominated convergence theorem (Proposition A.3), we see that the expression in (4.5) converges to zero as $m \to +\infty$. Hence, the proof of (4.2) is completed.

Next, we prove (4.4). Denote the RHS of (4.4) by $\mathcal{C}_k$. Then, we have

$$\bigoplus_{k\geq 0} \mathcal{C}_k \subset \bigoplus_{k\geq 0} \mathbb{C}_k^W = L^2(\Omega, \sigma\{W\}, \mathbb{P}).$$

Due to the orthogonality, it suffices to show that $\bigcup_{k\geq 0} \mathcal{C}_k$ is dense in $L^2(\Omega, \sigma\{W\}, \mathbb{P})$. That is, if $V \in L^2(\Omega, \sigma\{W\}, \mathbb{P})$ satisfies

$$\mathbb{E}\big[V H_k(W(h))\big] = 0 \tag{4.6}$$

for any unit vector $h \in \mathfrak{H}$ (i.e., $\|h\|_{\mathfrak{H}} = 1$) and for any $k \geq 0$, we must show that $V = 0$ almost surely.

Assume (4.6) holds for any unit vector $h \in \mathfrak{H}$ and for any $k \geq 0$. Since every monomial is a linear combination of Hermite polynomials, we have

$$\mathbb{E}\big[VW(h)^k\big] = 0,$$

and thus

$$\mathbb{E}\left[V\sum_{k=0}^{\infty}\frac{(it)^k W(h)^k}{k!}\right] = 0, \quad \text{i.e.,} \quad \mathbb{E}\big[Ve^{itW(h)}\big] = 0.$$

Therefore, with the orthonormal basis $\{e_j\}_{j\geq 1}$ of $\mathfrak{H}$, we can get

$$\mathbb{E}\Big[Ve^{i\sum_{j=1}^m \lambda_j W(e_j)}\Big] = 0$$

for any $\lambda_j \in \mathbb{R}$ and $m \geq 1$. Taking conditional expectation with respect to $\mathcal{F}_m$, we get

$$\mathbb{E}[V_m e^{i\sum_{j=1}^m \lambda_j W(e_j)}\Big] = 0$$

with $V_m = \mathbb{E}[V|\mathcal{F}_m] = \varphi_m(W(e_1), ..., W(e_m))$ for some $\varphi_m \in L^2(\gamma_m)$. By the same argument as in the proof of Proposition 1.11, we can conclude that $\varphi_m = 0$ almost everywhere, i.e., $V_m = 0$ almost surely. Then, another application of martingale convergence theorem yields $V = 0$ almost surely.

Hence, the proof of Theorem 4.4 is completed. □

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