Now let us pick m large enough such that (2.21) holds and let $K(x,y)$ denote the LHS and $\widetilde{K}(x,y)$ denote the RHS. Then $K(x,y)$ is the covariance kernel of $\Phi_n = \varphi_1(x)+...+\varphi_n(x)$ and $\widetilde{K}$ is the covariance kernel of $\sqrt{\varepsilon}Z+\widetilde{\Phi}_m = \sqrt{\varepsilon}Z+\widetilde{\varphi}_1(x)+...+\widetilde{\varphi}_n(x)$, with $Z\sim\mathcal{N}_1(0,1)$ independent of $\widetilde{\Phi}_m$. Therefore, it follows from Proposition 2.13-(ii) (applied to the measure $g(x)\nu(dx)$ with compact support A) that

$$
\begin{aligned}
\mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} g(x)M_n(dx)\Big)\Big] &= \mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} e^{\Phi_n(x)-\frac{1}{2}\mathrm{Var}\Phi_n(x)}g(x)\nu(dx)\Big)\Big]\\
&\le \mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} e^{\widetilde{\Phi}_m(x)-\frac{1}{2}\mathrm{Var}\widetilde{\Phi}_m(x)+\sqrt{\varepsilon}Z-\frac{1}{2}\varepsilon}g(x)\nu(dx)\Big)\Big]\\
&= \mathbb{E}\Big[\exp\Big(-\lambda e^{\sqrt{\varepsilon}Z-\frac{1}{2}\varepsilon}\int_{\mathcal{D}} g(x)\widetilde{M}_m(dx)\Big)\Big] \qquad (2.22)
\end{aligned}
$$

for any $m\ge m_{n,\varepsilon}$. Then, passing $m\to+\infty$ and then $\varepsilon\to 0$, $n\to+\infty$ yields

$$
\mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} g(x)M_\infty(dx)\Big)\Big] \le \mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} g(x)\widetilde{M}_\infty(dx)\Big)\Big] \qquad (2.23)
$$

and by the same argument and switching the role of C and $\widetilde{C}$, we can get the reverse inequality, implying that the above inequality is indeed an equality, which is (2.20). $\square$

$\propto\propto\propto\propto\propto\propto$ **End of Lecture 4 (Sept. 11, 2025)** $\infty\infty\infty\infty\infty\infty\infty$

Lecture 5.

Proof of Proposition 2.13. Let us first prove (ii) by assuming (i).

Since ν is a Radon measure on $\mathcal{D}$, we can construct a sequence of measures ν_n of the form

$$
\nu_n = \sum_{j=1}^n p_j\delta_{x_j} \quad \text{with } p_j\in\mathbb{R}_+ \text{ and } x_j\in\mathcal{D}
$$

such that ν_n weakly converges to ν meaning that for any bounded continuous function $h:\mathcal{D}\to\mathbb{R}$,

$$
\int_{\mathcal{D}} h(x)\nu_n(dx) = \sum_{j=1}^n p_j h(x_j) \xrightarrow{n\to+\infty} \int_{\mathcal{D}} h(x)\nu(dx).
$$

Note that part (i) implies that

$$
\mathbb{E}\Big[F\Big(\int_A e^{\Phi(x)-\frac{1}{2}\mathrm{Var}\Phi(x)}\nu_n(dx)\Big)\Big] \le \mathbb{E}\Big[F\Big(\int_A e^{\widetilde{\Phi}(x)-\frac{1}{2}\mathrm{Var}\widetilde{\Phi}(x)}\nu_n(dx)\Big)\Big]. \qquad (2.24)
$$

Note that the two exponentials (as functions in x) in (2.24) are bounded continuous on the compact set A, and thus we have the almost sure convergence of the two integrals over A to the corresponding integrals with respect to ν. In other words, passing $n\to\infty$ in (2.24) yields (2.19).[13] This proves (ii).

Now let us prove (i) via the smart-path method. The basic idea of comparing the terms in (2.18) is to interpolate them via some "smart" path and then show certain monotonicity along

[13] Actually, a bit more work is needed. For example, we should also check the uniform integrability of $\{F(\int_A e^{\Phi(x)-\frac{1}{2}\mathrm{Var}\Phi(x)}\nu_n(dx)\}_n$, which is ensured by the finiteness of $\sup_{x\in A}\mathbb{E}[e^{p\Phi(x)}]$ and $\nu(A)$.

such a path. Without losing any generality, we will assume that the two vectors $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ are independent and we define

$$G_j^t = tX_j + \sqrt{1-t^2}Y_j \quad \text{and} \quad v_j^t = \mathrm{Var}G_j^t = t^2\mathrm{Var}X_j + (1-t^2)\mathrm{Var}Y_j.$$

Define

$$\Psi(t) = \mathbb{E}\Big[F\Big(\sum_{i=1}^n p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big)\Big]$$

so that LHS of (2.18) is $\Psi(1)$ while RHS of (2.18) is $\Psi(0)$. Therefore,

$$\Psi(1) - \Psi(0) = \int_0^1 \frac{d}{dt}\Psi(t)dt \le 0$$

when $\Psi'(t) = \frac{d}{dt}\Psi(t) \le 0$ for all $t \in (0,1)$.

By Lebesgue's differentiation theorem, we have

$$\begin{aligned}
\Psi'(t) &= \mathbb{E}\Big[F'\Big(\sum_{i=1}^n p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big)\sum_{j=1}^n p_j e^{G_j^t - \frac{1}{2}v_j^t}\frac{d}{dt}\big[G_j^t - \frac{1}{2}v_j^t\big]\Big] \\
&= \sum_{j=1}^n p_j \mathbb{E}\Big[F'\Big(\sum_{i=1}^n p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big)e^{G_j^t - \frac{1}{2}v_j^t}\Big[X_j - \frac{t}{\sqrt{1-t^2}}Y_j - t(\mathrm{Var}X_j - \mathrm{Var}Y_j)\Big]\Big].
\end{aligned}$$

Applying the Gaussian integration by parts (Proposition 2.11-(i)), we have

$$\begin{aligned}
&\mathbb{E}\Big[\overbrace{F'\Big(\sum_{i=1}^n p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big)e^{G_j^t - \frac{1}{2}v_j^t}}^{=h(G_1^t, ..., G_t^n)} X_j\Big] \\
&= \sum_{i=1}^n \mathbb{E}[X_j G_i^t]\mathbb{E}\Big[\frac{\partial}{\partial G_i^t}h(G_1^t, ..., G_t^n)\Big] \\
&= \sum_{i=1}^n t\mathbb{E}[X_i X_j]\mathbb{E}\Big[F''\Big(\sum_{i=1}^n p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big)e^{G_j^t - \frac{1}{2}v_j^t}p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big] \\
&\quad + t\mathrm{Var}(X_j)\mathbb{E}\Big[F'\Big(\sum_{i=1}^n p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big)e^{G_j^t - \frac{1}{2}v_j^t}\Big]
\end{aligned} \tag{2.25}$$

and similarly,

$$\begin{aligned}
&\frac{t}{\sqrt{1-t^2}}\mathbb{E}\Big[F'\Big(\sum_{i=1}^n p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big)e^{G_j^t - \frac{1}{2}v_j^t}Y_j\Big] \\
&= \sum_{i=1}^n t\mathbb{E}[Y_i Y_j]\mathbb{E}\Big[F''\Big(\sum_{i=1}^n p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big)e^{G_j^t - \frac{1}{2}v_j^t}p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big] \\
&\quad + t\mathrm{Var}(Y_j)\mathbb{E}\Big[F'\Big(\sum_{i=1}^n p_i e^{G_i^t - \frac{1}{2}v_i^t}\Big)e^{G_j^t - \frac{1}{2}v_j^t}\Big].
\end{aligned} \tag{2.26}$$

Therefore, it follows from (2.25), (2.26), and $F'' \ge 0$ that $\Psi'(t) \le 0$. Hence, the proof is completed. $\square$

• **Gaussian Poincaré inequality** for $\mathbf{X} \sim \mathcal{N}_d(0, C)$. The Gaussian Poincaré inequality in Proposition 2.5 that for $f \in \mathbb{D}^{1,2}(\mathbb{R}^d)$, the variance of $f(\mathbf{G})$ with $\mathbf{G} \sim \mathcal{N}_d(0, \mathbf{I})$, is bounded by the corresponding second moment of $|Df(\mathbf{G})|$. When we replace $\mathbf{G}$ by $\mathbf{X} \sim \mathcal{N}_d(0, C)$, the same inequality holds up to a constant.

Proposition 2.14. *Let $f \in \mathbb{D}^{1,2}(\mathbb{R}^d)$ (defined with respect to γ_d) and $\mathbf{X} \sim \mathcal{N}_d(0, C)$. Then,*

$$\mathrm{Var}(f(\mathbf{X})) \le \lambda_* \mathbb{E}\big[\|(Df)(\mathbf{X})\|^2_{\mathbb{R}^d}\big],$$

where λ_* denotes the largest eigenvalue of the covariance matrix C.

Proof. First, we write $\mathbf{X} = Q\mathbf{G}$ with $\mathbf{G} \sim \mathcal{N}_d(0, \mathbf{I})$ and $Q = \sqrt{C}$.[14] Then, Proposition 2.5 implies that

$$\begin{aligned}
\mathrm{Var}(f(\mathbf{X})) &= \mathrm{Var}(f(Q\mathbf{G})) \le \mathbb{E}\langle Df(Q\mathbf{G}), Df(Q\mathbf{G})\rangle_{\mathbb{R}^d} \\
&= \mathbb{E}\langle Q \cdot (Df)(Q\mathbf{G}), Q \cdot (Df)(Q\mathbf{G})\rangle_{\mathbb{R}^d} = \mathbb{E}\langle (Df)(Q\mathbf{G}), C \cdot (Df)(Q\mathbf{G})\rangle_{\mathbb{R}^d} \qquad (2.27) \\
&\le \lambda_* \mathbb{E}\big[\|(Df)(\mathbf{X})\|^2_{\mathbb{R}^d}\big],
\end{aligned}$$

where we used the chain rule in (2.27) and the fact that $\langle Qx, y\rangle_{\mathbb{R}^d} = \langle x, Qy\rangle_{\mathbb{R}^d}$ due to the symmetry of Q and $QQ = C$; in the last step, we used the fact that $\langle Cx, x\rangle \le \lambda_* \|x\|^2_{\mathbb{R}^d}$. □

2.4. **Applications II: Maximum of Gaussians.**

In this section, we will look at the maximum of a centered Gaussian vector.

Proposition 2.15. *Let $\mathbf{X} \sim \mathcal{N}_d(0, C)$ and define $M_d = \max\{X_1, ..., X_d\}$. Then,*

$$\mathbb{E}[M_d] \le \sqrt{2C_* \log d} \quad \text{and} \quad \mathrm{Var}(M_d) \le C_*$$

with $C_* = \max\{C_{ii} : i = 1, ..., d\}$. In particular, when $\mathbf{X}$ is standard normal, we have $C_* = 1$.

Proof. Put $f(x) = \max\{x_1, ..., x_d\}$. Then, it is easy to verify that $|f(x) - f(y)| \le \max_i\{|x_i - y_i|\}$.[15] In particular, f is absolutely continuous (even Lipschitz continuous) with

$$\partial_i f(x) = \mathbf{1}_{\{x_i = f(x)\}} \qquad (2.28)$$

for almost every $x \in \mathbb{R}^d$. Therefore, it follows from the computations in (2.27) that

$$\begin{aligned}
\mathrm{Var}(f(\mathbf{X})) &\le \mathbb{E}\langle (Df)(\mathbf{X}), C \cdot (Df)(\mathbf{X})\rangle_{\mathbb{R}^d} \\
&= \mathbb{E}\sum_{i=1}^d (\partial_i f)(\mathbf{X}) \sum_{j=1}^d C_{ij} (\partial_j f)(\mathbf{X}) = \mathbb{E}\sum_{i=1}^d C_{ii} |(\partial_i f)(\mathbf{X})|^2 \quad \text{by (2.28)} \\
&\le C_* \mathbb{E}\sum_{i=1}^d \mathbf{1}_{\{X_i = M_d\}} = C_*.
\end{aligned}$$

Next, we prove the upper bound for the expectation. Note that for any $\beta > 0$,

$$\max\{x_1, ..., x_d\} \le \frac{1}{\beta} \log \sum_{i=1}^d e^{\beta x_i},$$

[14] This means that Q is a symmetric and positive-definite matrix such that $QQ = C$.

[15] Suppose $x_i = f(x)$ and $f(y) = y_j \ge x_i$, then $|f(x) - f(y)| = y_j - x_i \le y_j - x_j = |x_j - x_j| \le \max_i\{|x_i - y_i|\}$.

from which, we deduce

$$
\begin{aligned}
\mathbb{E}[M_d] &\leq \frac{1}{\beta}\mathbb{E}\log\sum_{i=1}^{d} e^{\beta X_i} \\
&\leq \frac{1}{\beta}\log\sum_{i=1}^{d}\mathbb{E}e^{\beta X_i} \quad \text{by Jensen's inequality} \\
&= \frac{1}{\beta}\log\sum_{i=1}^{d} e^{\frac{1}{2}\beta^2 C_{ii}} \leq \frac{1}{\beta}\log\left(d\cdot e^{\frac{1}{2}\beta^2 C_*}\right) \\
&= \frac{\log d}{\beta} + \frac{1}{2}\beta C_*.
\end{aligned}
$$

An easy optimization (with $\beta = \sqrt{\frac{2}{C_*}\log d}$) leads to $\mathbb{E}[M_d] \leq \sqrt{2C_*\log d}$. □

A natural question after Proposition 2.15 is the sharpness of the bounds. In fact, the order of the expectation is optimal, but the order of the variance is not., as illustrated by the following question.

Question 2.16. Let $\{g_i\}_{i\geq 1}$ be i.i.d. standard normal and define $M_n = \max\{g_1, ..., g_n\}$. Prove

$$\mathbb{P}\Big(\frac{M_n - a_n}{b_n} \leq x\Big) \xrightarrow{n\to+\infty} \exp(-e^{-x})$$

for every $x \in \mathbb{R}_+$, where

$$a_n = \frac{1}{b_n} = \sqrt{2\log n}.$$

The above limit is the cumulative distribution function of the so-called Gumbel distribution. You may use the tail-ratio

$$\frac{1-\Phi(u+\frac{x}{u})}{1-\Phi(u)} \xrightarrow{u\to+\infty} e^{-x},$$

where Φ stands for the standard normal CDF.

Remark 2.17. (i) The above question tells us that the variance of M_n is of order $\frac{1}{\log n}$ and the expectation is of order $\sqrt{\log n}$. But the application of the Gaussian Poincaré inequality only allows us to bound the variance by a constant (i.e., 1).

(ii) The famous L^1/L^2 variance inequality of M. Talagrand asserts that for $\mathbf{G} \sim \mathcal{N}_d(0, \mathbf{I})$ and $f \in \mathbb{D}^{1,2}(\mathbb{R}^d)$,

$$\mathrm{Var}(f(\mathbf{G})) \leq C\sum_{i=1}^{d}\frac{\|\partial_i f\|^2_{L^2(\gamma_d)}}{1+\log\left(\|\partial_i f\|_{L^2(\gamma_d)}/\|\partial_i f\|_{L^1(\gamma_d)}\right)}. \tag{2.29}$$

This inequality will allow us to obtain the optimal variance bound $\mathrm{Var}(M_n) \leq \frac{C}{\log n}$ in the setting of Question 2.16 (*Exercise!*). We will prove (2.29) later via an application of the hypercontractivity property of the Ornstein-Uhlenbeck semigroups.

2.5. **Applications III: Gaussian random polymer.** For a generic path of a 1D random walk starting from (0,0), we can represent it as

$$(0,0) \to (1, a_1) \to (2, a_2) \to \cdots \to (n, a_n) \to \cdots,$$

where $a_i \in \mathbb{Z}$ and $|a_i - a_{i+1}| = 1$ with $a_0 = 0$. Let $\mathcal{P}_n$ denote the set of such paths with length n, then the cardinality of $\mathcal{P}_n$ is 2^n.

• **Gaussian environment.** Let $\{g_v : v \in \mathbb{Z}^2\}$ be i.i.d. real standard normal random variables modeling the random medium. A path $p \in \mathcal{P}_n$ immersed in this medium posses the "**energy**"

$$H_n(p) = -\sum_{v \in p} g_v,$$

where "$v \in p$" means that the vertex v belongs to the path p. The quantity H_n is also known as the **Hamiltonian**. The so-called **ground-state energy** is defined by

$$E_n = \min_{p \in \mathcal{P}_n} H_n(p),$$

that is, it is the minimum energy over all possible paths of length n.

The following result indicates that the variance of the ground-state energy is at most of linear order.

Proposition 2.18. *It holds that* $\mathrm{Var}(E_n) \le n + 1$.

Proof. It is clear that E_n can be viewed as an absolutely continuous function of the Gaussian environment $\{g_v : v \in \mathbb{Z}^2\}$ (in fact only finitely many of g_v's). Let

$$p_* = \arg\min_{p \in \mathcal{P}_n} H_n(p).$$

By definition, p_* depends on the randomness inherited from the medium (and it is almost surely unique (*why?*)). It is also evident that if we perturbate a bit the Gaussian weight g_v assigned to the vertex v, E_n will vary if and only if v belongs to the optimal path p_*. As a result, we have

$$\frac{\partial}{\partial g_v} E_n = -\mathbf{1}_{\{v \in p_*\}}$$

and thus by Gaussian Poincaré inequality, we have

$$\begin{aligned}\mathrm{Var}(E_n) &\le \mathbb{E} \sum_{v \in \mathbb{Z}^2} |\frac{\partial}{\partial g_v} E_n|^2 \\ &= \mathbb{E} \sum_{v \in \mathbb{Z}^2} \mathbf{1}_{\{v \in p_*\}} = n + 1,\end{aligned}$$

since there are $n + 1$ vertices in any path of length n. This concludes our proof. □

The above bound is not optimal and S. Chatterjee proved in 2008 that $\mathrm{Var}(E_n) \le C \frac{n}{\log n}$, where the constant C does not depend on n; we refer interested students to [Cha14] for more details.

∝∝∝∝∝∝ **End of Lecture 5 (Sept. 16, 2025)** ∞∞∞∞∞∞∞∞