Lecture 4.

• **Ornstein-Uhlenbeck semigroups** $(P_t)_{t\in\mathbb{R}_+}$.

$$P_t = e^{tL} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathcal{L}^k$$

Let $f \in L^2(\gamma_d)$, we deduce from Proposition 2.7 that

$$P_t J_q f = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathcal{L}^k J_q f = \sum_{k=0}^{\infty} \frac{t^k}{k!}(-q)^k J_q f = e^{-qt} J_q f.$$

Then, we have

$$P_t f = \sum_{q=0}^{\infty} e^{-qt} J_q f, \tag{2.11}$$

which is well defined for any $f \in L^2(\gamma_d)$. It is also evident that

$$\|P_t f\|^2_{L^2(\gamma_d)} = \sum_{q=0}^{\infty} e^{-2qt}\|J_q f\|^2_{L^2(\gamma_d)} \le \|f\|^2_{L^2(\gamma_d)},$$

which is known as the contraction property for the OU semigroup; see also Proposition 2.9.

Exercise 2.8. (i) *Show that for $f \in L^2(\gamma_d)$, $f \in \mathrm{dom}\mathcal{L}$ if and only if*

$$\lim_{t\downarrow 0} \frac{P_t f - f}{t}$$

exists in $L^2(\gamma_d)$ and it is equal to $\mathcal{L}f$.

(ii) *Verify that semigroup properties: $P_0 f = f$ and $P_{t+s} f = P_t P_s f$ for any $f \in L^2(\gamma_d)$.*

(iii) *Show that $DP_t f = e^{-t}P_t Df$ for any $f \in \mathbb{D}^{1,2}(\mathbb{R}^d)$ and $\partial_t P_t = \mathcal{L}P_t$ on $L^2(\gamma_d)$. Here, we should understand $P_t Df$ as a vector $(P_t\partial_1 f, ..., P_t\partial_d f)$.*

Proposition 2.9. (i) [Mehler's formula] For any $f \in L^2(\gamma_d)$, we have

$$(P_t f)(x) = \mathbb{E}\big[f(e^{-t}x + \sqrt{1-e^{-2t}}\mathbf{G})\big].$$

(ii) [contraction property] For any $f \in L^p(\gamma_d)$ with $p \in [2,\infty]$, we have

$$\|P_t f\|_{L^p(\gamma_d)} \le \|f\|_{L^p(\gamma_d)}.$$

Proof. To show Mehler's formula, it suffices to prove it for $f(x) = \prod_{j=1}^d H_{q_j}(x_j)$. and then the general result follows by linearity. In this case, $P_t f = e^{-(q_1+...+q_d)t} f$ and

$$\begin{aligned}\mathbb{E}\prod_{j=1}^d H_{q_j}(e^{-t}x_j + \sqrt{1-e^{-2t}}G_j) &= \prod_{j=1}^d \mathbb{E}H_{q_j}(e^{-t}x_j + \sqrt{1-e^{-2t}}G_j)\\ &= \prod_{j=1}^d e^{-q_j t}H_{q_j}(x),\end{aligned}$$

which follows from the binomial formula (see Question 1.13). Therefore, part (i) is proved.

Now let $f \in L^p(\gamma_d)$ for $p \in [2,\infty)$.[9] Then, the result in (i) ensures that we can write

$$(P_t f)(x) = \mathbb{E}\big[f(e^{-t}x + \sqrt{1-e^{-2t}}\mathbf{G})\big].$$

[9]The case $p = \infty$ is trivial by noting that $\|f\|_{L^\infty(\gamma_d)}$ coincides with the usual L^∞-norm of f.

Let $\mathbf{G}'$ be an independent copy of $\mathbf{G}$, we can write $\|P_t f\|^p_{L^p(\gamma_d)}$ as

$$\begin{aligned}\|P_t f\|^p_{L^p(\gamma_d)} &= \mathbb{E}\Big[\big|\mathbb{E}\big[f(e^{-t}\mathbf{G}' + \sqrt{1-e^{-2t}}\mathbf{G})|\mathbf{G}'\big]\big|^p\Big]\\ &\le \mathbb{E}\Big[\mathbb{E}\big[|f(e^{-t}\mathbf{G}' + \sqrt{1-e^{-2t}}\mathbf{G})|^p|\mathbf{G}'\big]\Big]\\ &= \mathbb{E}\Big[|f(e^{-t}\mathbf{G}' + \sqrt{1-e^{-2t}}\mathbf{G})|^p\Big],\end{aligned}$$

where the last two steps follows from Jensen's inequality and tower property for conditional expectation. Note that $e^{-t}\mathbf{G}' + \sqrt{1-e^{-2t}}\mathbf{G}$ has the same law as $\mathbf{G}$, so that we just proved $\|P_t f\|^p_{L^p(\gamma_d)} \le \|f\|^p_{L^p(\gamma_d)}$. □

Exercise 2.10. *Prove that the Mehler formula also holds for $f \in L^1(\gamma_d)$.*

Proposition 2.11 (Gaussian integration by parts)**.** (i) Let $Y \sim \mathcal{N}(0,\sigma^2)$ and $\mathbf{X} \sim \mathcal{N}_d(0,C)$ be jointly Gaussian. Then, for any $h \in C^\infty_{\text{Poly}}(\mathbb{R}^d)$, it holds that

$$\mathbb{E}[Yh(\mathbf{X})] = \sum_{j=1}^d \mathbb{E}[YX_j]\mathbb{E}[\partial_j h(\mathbf{X})].$$

(ii) For $f \in C^2_{\text{Poly}}(\mathbb{R}^d)$ and $\mathbf{X} \sim \mathcal{N}_d(0,C)$, we have

$$\mathbb{E}\big[\mathbf{X}\cdot\nabla f(\mathbf{X})\big] = \sum_{i,j=1}^d C_{ij}\mathbb{E}[\partial_{ij} f(\mathbf{X})] = \mathbb{E}\big[\langle C, \text{Hess} f(\mathbf{X})\rangle_{\text{HS}}\big]. \tag{2.12}$$

Proof. Let h be given as in the statement, and by scaling, we can assume $Y \sim \mathcal{N}(0,1)$. Rewriting

$$X_j = \mathbb{E}[YX_j]Y + (X_j - \mathbb{E}[YX_j]Y) =: \rho_j Y + W_j$$

as a sum of two independent Gaussians, we are able to transform our proof to the 1D case. Applying the integration by parts for Y yields

$$\begin{aligned}\mathbb{E}[Yh(\mathbf{X})] &= \mathbb{E}\big[Yh(\rho_1 Y + W_1, ..., \rho_d Y + W_d)\big]\\ &= \mathbb{E}\sum_{j=1}^d \rho_j\partial_j h(\rho_1 Y + W_1, ..., \rho_d Y + W_d)\big]\\ &= \sum_{j=1}^d \mathbb{E}[YX_j]\mathbb{E}[\partial_j h(\mathbf{X})].\end{aligned}$$

This proves (i). To prove (ii), it suffices to apply (i) to obtain

$$\mathbb{E}\big[X_i\partial_i f(\mathbf{X})\big] = \sum_{j=1}^d \mathbb{E}[X_iX_j]\mathbb{E}\big[\partial_j\partial_i f(\mathbf{X})\big]$$

and conclude by linearity. □

2.3. **Application I: A gentle touch on GMC.** For an in-depth introduction to the theory of Gaussian multiplicative chaos (GMC), please refer to the survey [RV14] by Rhodes and Vargas.

Let $\mathcal{D} \subset \mathbb{R}^d$ be bounded and open and ν be a finite Radon measure on $\mathcal{D}$. A standard Gaussian multiplicative chaos (GMC) is a random measure on D formally defined as

$$M(A) = \int_A e^{\varphi(x) - \frac{1}{2}\text{Var}\varphi(x)}\nu(dx), \tag{2.13}$$

where $\{\varphi(x) : x \in \mathcal{D}\}$ is a centered Gaussian field with covariance kernel

$$C(x,y) = \mathbb{E}[\varphi(x)\varphi(y)].$$

The random measure in (2.13) is well defined when φ is a regular field meaning $C(x,y)$ is finite for every $x, y \in \mathcal{D}$, which excludes the case

$$C(x,y) = \max\{\log|x-y|^{-1}, 0\} + g(x,y) \tag{2.14}$$

with g a bounded continuous function on $\mathcal{D} \times \mathcal{D}$. The corresponding Gaussian field (with covariance kernel in (2.14)) can only be interpreted in a generalized sense, and φ can not be defined pointwise. Hence, the exponential $e^{\varphi(x)}$ does not make any sense. In 1985, J.-P. Kahane introduced GMC in his paper [Kah85] to make Mandelbrot's initial model of energy dissipation in intermittent turbulence, the so-called Kolmogorov-Obukhov model [Man72].

Specifically, Kahane started from the covariance kernel C of σ-positive type, meaning

$$C(x,y) = \sum_{k=1}^{\infty} C_k(x,y), \tag{2.15}$$

where $\{C_k\}_k$ is a sequence of $[0,\infty)$-valued, continuous, positive-definite kernels:

$$\int_{\mathcal{D}^2} f(x)f(y)C_k(x,y)\nu(dx)\nu(dy) < \infty$$

for any 'test function' $f : \mathcal{D} \to \mathbb{R}$. Suppose we have a sequence of centered independent continuous Gaussian fields $\{\varphi_k\}_k$ with covariance kernels $\{C_k\}_k$. Then,

$$\Phi_n = \sum_{k=1}^{n} \varphi_k$$

is a centered Gaussian field with covariance kernel $\sum_{k=1}^{n} C_k$. Kahane proved that the random measure M_n on $\mathcal{D}$, with

$$A \in \mathcal{B}(\mathcal{D}) \mapsto M_n(A) := \int_A e^{\Phi_n(x) - \frac{1}{2}\mathrm{Var}\Phi_n(x)}\nu(dx),$$

converges almost surely in the space of Radon measure (equipped with the topology of weak convergence) towards a random measure M_∞. This random measure M_∞, formally written as

$$M_\infty(A) = \int_A e^{\Phi_\infty(x) - \frac{1}{2}\mathrm{Var}\Phi_\infty(x)}\nu(dx) \quad \text{or} \quad \int_A e^{\varphi(x) - \text{``}\infty\text{''}}\nu(dx)$$

is called Gaussian multiplicative chaos with kernel C acting on ν. Indeed, from the Doob's convergence theorem for positive martingales[10], we can deduce that for any Borel set $A \subset \mathcal{D}$, as $n \to +\infty$, $M_n(A)$ converges almost surely to some random variable $M_\infty(A)$ with finite expectation. Note that the construction of GMC seems to depend on the decomposition (2.15) of C. Kahane proved in his 1985 paper that this is not the case.

Theorem 2.12 (Uniqueness in law). *Let $\{\widetilde{\varphi}_k\}$ be another independent sequence of centered continuous Gaussian field with nonnegative covariance kernels $\{\widetilde{C}_k\}$ such that*

$$C(x,y) = \sum_{k=1}^{\infty} C_k(x,y) = \sum_{k=1}^{\infty} \widetilde{C}_k(x,y) \tag{2.16}$$

[10]You can find this classic result in most books on Ito calculus.

for any $(x,y) \in \mathcal{D}\times\mathcal{D}$. *Let* $\widetilde{M}_\infty$ *be the GMC constructed in the same way as* M_∞. *Then,* $\widetilde{M}_\infty = M_\infty$ *in law, meaning that for any* $f:\mathcal{D}\to\mathbb{R}$ *bounded continuous,*

$$\int_{\mathcal{D}} g(x)\widetilde{M}_\infty(dx) = \int_{\mathcal{D}} g(x)M_\infty(dx) \quad \text{in law.} \tag{2.17}$$

We will prove Theorem 2.12 by using the following convexity inequality (or Gaussian comparison principle), which is a consequence of the Gaussian integration by parts formula (Proposition 2.11).

Proposition 2.13 (Kahane's convexity inequality)**.** (i) *Let* $(X_1,...,X_n)$ *and* $(Y_1,...,Y_n)$ *be two centered Gaussian vectors such that*

$$\mathbb{E}[X_iX_j] \le \mathbb{E}[Y_iY_j]\,, \quad \forall i,j\in\{1,...,n\}.$$

Then, for any $(p_1,...,p_n)\in\mathbb{R}_+^n$ *and for all convex functions* $F:\mathbb{R}_+\to\mathbb{R}$ *twice differentiable with at most polynomial growth at infinity,*[11] *we have*

$$\mathbb{E}\Big[F\Big(\sum_{i=1}^n p_i e^{X_i-\frac12\mathbb{E}[X_i^2]}\Big)\Big] \le \mathbb{E}\Big[F\Big(\sum_{i=1}^n p_i e^{Y_i-\frac12\mathbb{E}[Y_i^2]}\Big)\Big]. \tag{2.18}$$

(ii) *Let* $\mathcal{D}$ *be a bounded open subset of* $\mathbb{R}^d$ *and* ν *be a finite Radon measure on* $\mathcal{D}$.[12] *Let* $\{\Phi(x)\}_{x\in\mathcal{D}}$, $\{\widetilde{\Phi}(x)\}_{x\in\mathcal{D}}$ *be real centered continuous Gaussian fields on* $\mathcal{D}$ *with covariance kernel* $K,\widetilde{K}$ *respectively such that*

$$K(x,y)\le\widetilde{K}(x,y), \forall x,y\in\mathcal{D}.$$

Let F *be as in* (i). *Then, we have for any compact* $A\subset\mathcal{D}$

$$\mathbb{E}\Big[F\Big(\int_A e^{\Phi(x)-\frac12\mathrm{Var}\Phi(x)}\nu(dx)\Big)\Big] \le \mathbb{E}\Big[F\Big(\int_A e^{\widetilde{\Phi}(x)-\frac12\mathrm{Var}\widetilde{\Phi}(x)}\nu(dx)\Big)\Big]. \tag{2.19}$$

We will postpone the proof of Proposition 2.13 to the end of this section.

Proof of Theorem 2.12. To show (2.17), it suffices to consider the case where g is nonnegative and continuous with *compact* support (say, $\mathrm{supp}(g)=A$). In this case, the two random variables in (2.17) are nonnegative. Then, to show the equality in law, it suffices to show that they have the same Laplace transform. That is, for any $\lambda\in\mathbb{R}_+$,

$$\mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} g(x)\widetilde{M}_\infty(dx)\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} g(x)M_\infty(dx)\Big)\Big]. \tag{2.20}$$

Note that from (2.16), we can **claim** that given a compact set $A\subset\mathcal{D}$, for any $\varepsilon>0$ and for any $n\in\mathbb{N}$, there exists some large $m_{n,\varepsilon}\in\mathbb{N}$ such that for any $m\ge m_{n,\varepsilon}$,

$$\sum_{k=1}^n C_k(x,y) < \varepsilon + \sum_{k=1}^m \widetilde{C}_k(x,y) \tag{2.21}$$

for any $x,y\in A$. Indeed, letting F_m be the set of $(x,y)\in A\times A$ such that (2.21) fails, then F_m is closed due to the continuity of C_k and $\widetilde{C}_k$, and thus F_m is compact. It is also clear that $F_{m+1}\subset F_m$ for each m and $\bigcap_{m\ge1}F_m =$ due to (2.16). Therefore, we deduce from Heine-Borel theorem that F_m must be empty for all large m, that is, the desired claim is verified.

[11]The polynomial growth at infinity ensures that the expectations in (2.18) are finite.

[12]In particular, ν is inner regular meaning that for any open $U\subset\mathcal{D}$, $\nu(U)=\sup\{\nu(K):K\subset U$ and K is compact$\}$. This ensures that the measure ν can be approximated by weighted sum of Dirac masses of the form $\sum_{j=1}^n p_{j,n}\delta_{x_{j,n}}$, with $p_j\in\mathbb{R}_+$ and $x_{j,n}\in\mathcal{D}$.

Now let us pick m large enough such that (2.21) holds and let $K(x,y)$ denote the LHS and $\widetilde{K}(x,y)$ denote the RHS. Then $K(x,y)$ is the covariance kernel of $\Phi_n = \varphi_1(x)+\ldots+\varphi_n(x)$ and $\widetilde{K}$ is the covariance kernel of $\sqrt{\varepsilon}Z+\widetilde{\Phi}_m = \sqrt{\varepsilon}Z+\widetilde{\varphi}_1(x)+\ldots+\widetilde{\varphi}_n(x)$, with $Z \sim \mathcal{N}_1(0,1)$ independent of $\widetilde{\Phi}_m$. Therefore, it follows from Proposition 2.13-(ii) (applied to the measure $g(x)\nu(dx)$ with compact support A) that

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} g(x)M_n(dx)\Big)\Big] &= \mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} e^{\Phi_n(x)-\frac{1}{2}\mathrm{Var}\Phi_n(x)}g(x)\nu(dx)\Big)\Big]\\ &\le \mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} e^{\widetilde{\Phi}_m(x)-\frac{1}{2}\mathrm{Var}\widetilde{\Phi}_m(x)+\sqrt{\varepsilon}Z-\frac{1}{2}\varepsilon}g(x)\nu(dx)\Big)\Big]\\ &= \mathbb{E}\Big[\exp\Big(-\lambda e^{\sqrt{\varepsilon}Z-\frac{1}{2}\varepsilon}\int_{\mathcal{D}} g(x)\widetilde{M}_m(dx)\Big)\Big] \qquad (2.22)\end{aligned}$$

for any $m \ge m_{n,\varepsilon}$. Then, passing $m \to +\infty$ and then $\varepsilon \to 0$, $n \to +\infty$ yields

$$\mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} g(x)M_\infty(dx)\Big)\Big] \le \mathbb{E}\Big[\exp\Big(-\lambda\int_{\mathcal{D}} g(x)\widetilde{M}_\infty(dx)\Big)\Big] \qquad (2.23)$$

and by symmetry, the above inequality is indeed an equality, which is (2.20). $\square$

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Lecture 5.

Proof of Proposition 2.13. Let us first prove (ii) by assuming (i).

Since ν is a Radon measure on $\mathcal{D}$, we can construct a sequence of measures ν_n of the form

$$\nu_n = \sum_{j=1}^{n} p_j \delta_{x_j} \quad \text{with } p_j \in \mathbb{R}_+ \text{ and } x_j \in \mathcal{D}$$

such that ν_n weakly converges to ν meaning that for any bounded continuous function $h : \mathcal{D} \to \mathbb{R}$,

$$\int_{\mathcal{D}} h(x)\nu_n(dx) = \sum_{j=1}^{n} p_j h(x_j) \xrightarrow{n\to+\infty} \int_{\mathcal{D}} h(x)\nu(dx).$$

Note that part (i) implies that

$$\mathbb{E}\Big[F\Big(\int_A e^{\Phi(x)-\frac{1}{2}\mathrm{Var}\Phi(x)}\nu_n(dx)\Big)\Big] \le \mathbb{E}\Big[F\Big(\int_A e^{\widetilde{\Phi}(x)-\frac{1}{2}\mathrm{Var}\widetilde{\Phi}(x)}\nu_n(dx)\Big)\Big]. \qquad (2.24)$$

Note that the two exponentials (as functions in x) in (2.24) are bounded continuous on the compact set A, and thus we have the almost sure convergence of the two integrals over A to the corresponding integrals with respect to ν. In other words, passing $n \to \infty$ in (2.24) yields (2.19).[13] This proves (ii).

Now let us prove (i) via the smart-path method. The basic idea of comparing the terms in (2.18) is to interpolate them via some "smart" path and then show certain monotonicity along

[13] Actually, a bit more work is needed. For example, we should also check the uniform integrability of $\{F(\int_A e^{\Phi(x)-\frac{1}{2}\mathrm{Var}\Phi(x)}\nu_n(dx)\}_n$, which is ensured by the finiteness of $\sup_{x\in A}\mathbb{E}[e^{p\Phi(x)}]$ and $\nu(A)$.