2. **Gaussian analysis on** $\mathbb{R}^d$

In Section 1, we studied some basic Gaussian analysis that is involved with a single Gaussian random variable.

Let us briefly review Section 1 while introducing a few new things. Throughout this sections, we write

$$\gamma_d(dx) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}|x|^2} dx$$

for the standard Gaussian measure on $\mathbb{R}^d$.

2.1. **Refresh on week 1.** In Section 1, we introduced D, δ with a common domain

$$\mathbb{D}^{1,2}(\mathbb{R}) := \{f \in L^2(\gamma_1) : f \text{ is absolutely continuous and } f' \in L^2(\gamma_1)\}$$

The Gaussian Poincaré inequality (Proposition 1.14) ensures that $f \in L^2(\gamma_1)$ whenever $f' \in L^2(\gamma_1)$. We defined Hermite polynomials $\{H_p : p \in \mathbb{N}\}$, which are the orthogonal polynomials for γ_1. More precisely, for any $f \in L^2(\gamma_1)$, we can find $\{c_q : q \geq 0\}$ such that

$$f = \sum_{q \in \mathbb{N}} c_q H_q \quad \text{and} \quad \sum_{q \geq 0} c_q^2 q! < \infty.$$

With $Df = f'$ and $(\delta g)(x) = xg(x) - g'(x)$, we observe that

$$\mathcal{L}f := -\delta D f = f'' - xf'(x),$$

which gives us the generator of the real Ornstein-Uhlenbeck process described by the following stochastic differential equation (SDE)

$$\begin{cases} dX_t = -X_t dt + \sqrt{2} dB_t \\ X_0 = x \in \mathbb{R} \end{cases}$$

with B a standard real Brownian motion. One can easily solve the above SDE:

$$X_t = xe^{-t} + \sqrt{2} \int_0^t e^{-(t-s)} dB_s.$$

It is easy to verify that $X_t \sim \mathcal{N}(xe^{-t}, 1 - e^{-2t})$, which weakly converges (or converges in law) to the standard Gaussian measure γ_1 as $t \to +\infty$, regardless of the initial value $x \in \mathbb{R}$. The process $(X_t : t \in \mathbb{R}_+)$ is a Markov process and admits a semigroup (called Ornstein-Uhlenbeck semigroup)

$$(P_t f)(x) = \mathbb{E}[f(X_t) | X_0 = x] = \mathbb{E}[f(xe^{-t} + \sqrt{1 - e^{-2t}} Z)]$$

with $Z \sim \mathcal{N}(0, 1)$, and the second representation is known as Mehler's formula.

Exercise 2.1. (i) Show $P_{t+s} = P_t P_s$ for any $t, s \in \mathbb{R}_+$, $P_t H_q = e^{-qt} H_q$, $\mathcal{L} H_q = -q H_q$, and $DP_t = e^{-t} P_t D$.
(ii) Show that $\mathcal{L}f = \lim_{t \to 0} \frac{P_t f - f}{t}$ for every $f \in \text{dom}(\mathcal{L})$ after you figure out $\text{dom}(\mathcal{L})$.
(iii) Show that for every $f, g \in \mathbb{D}^{1,2}(\mathbb{R})$, with $Z \sim \mathcal{N}(0, 1)$, we have

$$\text{Cov}(f(Z), g(Z)) = \int_0^\infty \langle Df(Z), DP_t g(Z) \rangle_{L^2(\gamma_1)} dt.$$

`Hint for (iii): Hermite expansion.`

2.2. **Week 1 extension to $\mathbb{R}^d$.** Now let us extend what we have learned to $\mathbb{R}^d$. We collect a few facts on Gaussian vectors in Appendix B, and students could refer to it when the need arises.

Definition 2.2. *Let $C^\infty_{\mathrm{Poly}}(\mathbb{R}^d)$ be the set of smooth functions $f:\mathbb{R}^d\to\mathbb{R}$ such that f and all its partial derivatives have at most polynomial growth at infinity.*

Proposition 2.3 (chaos expansion)**.** The Hilbert space $L^2(\gamma_d)$ can be decomposed into mutually orthogonal closed subspaces:

$$L^2(\gamma_d)=\bigoplus_{p\in\mathbb{N}}\mathbb{C}_p,$$

where $\mathbb{C}_p$ is the $L^2(\gamma_d)$-closure of the span of[7]

$$\Big\{(x_1,\dots,x_d)\in\mathbb{R}^d\mapsto\prod_{j=1}^d H_{q_j}(x_j):\sum_{j=1}^d q_j=p\Big\}.$$

That is, the following statements hold true.

(i) For any $p_i,q_i\in\mathbb{N}$, we have

$$\mathbb{E}\bigg(\prod_{i=1}^d H_{p_i}(G_i)H_{q_i}(G_i)\bigg)=\mathbf{1}_{\{p_i=q_i,\forall i\}}\prod_{j=1}^d q_j!. \tag{2.1}$$

with $\mathbf{G}\sim\mathcal{N}_d(0,\mathbf{I})$.

(ii) For any $f\in L^2(\gamma_d)$, there exist a family of coefficients $c_{q_1,\dots,q_d}$'s such that

$$f=\sum_{k=0}^\infty\underbrace{\sum_{\substack{q_1,\dots,q_d\in\mathbb{N}\\ q_1+\dots+q_d=k}}c_{q_1,\dots,q_d}\prod_{j=1}^d H_{q_j}(x_j)}_{\in\mathbb{C}_k}\quad\text{and}\quad\sum_{q_1,\dots,q_d\in\mathbb{N}}c^2_{q_1,\dots,q_d}\prod_{j=1}^d q_j!<\infty. \tag{2.2}$$

Proof. First note that (2.1) follows from the independence of different Gaussian entries and the orthogonality between different Hermite polynomials. Now let us prove (2.2) via an induction argument on $d\ge1$. The 1D case is established in Proposition 1.11. Suppose that (2.2) holds true for any $d\le n$ with $n\in\mathbb{N}_0$, and we will show it for $d=n+1$.

Let $f\in L^2(\gamma_{n+1})$, i.e.,

$$\int_{\mathbb{R}^n}|f(x_1,\dots,x_n,y)|^2\gamma_d(dx)\gamma_d(dy)<\infty. \tag{2.3}$$

By Fubini's theorem, for γ_1-almost every $y\in\mathbb{R}$, the mapping

$$(x_1,\dots,x_n)\in\mathbb{R}^n\mapsto f(x_1,\dots,x_n,y)\in\mathbb{R}$$

is square-integrable with respect to γ_d. Then, the induction hypothesis ensures that one can find a family of (y-dependent!) coefficients $c_{q_1,\dots,q_d}(y)$ such that

$$f(x_1,\dots,x_n,y)=\sum_{q_1,\dots,q_n\in\mathbb{N}}c_{q_1,\dots,q_n}(y)\prod_{j=1}^n H_{q_j}(x_j)\quad\text{in }L^2(\mathbb{R}^n,\gamma_n) \tag{2.4}$$

and the equation (2.3) is equivalent to

$$\sum_{q_1,\dots,q_n\in\mathbb{N}}\bigg(\prod_{j=1}^n q_j!\bigg)\int_{\mathbb{R}}c_{q_1,\dots,q_n}(y)^2\gamma_1(dy)<\infty.$$

[7] $\mathbb{C}_0\simeq\mathbb{R}$ is the set of constant random variables.

Then, applying induction hypothesis again (for $d = 1$), we can expand the function $y \in \mathbb{R} \mapsto c_{q_1,\dots,q_n}(y)$ as

$$c_{q_1,\dots,q_n}(y) = \sum_{q_{n+1}\geq 0} c_{q_1,\dots,q_n,q_{n+1}} H_{q_{n+1}}(y).$$

Plugging it into (2.4) yields

$$f(x_1,\dots,x_n,x_{n+1}) = \sum_{q_1,\dots,q_{n+1}\in\mathbb{N}} c_{q_1,\dots,q_{n+1}} \prod_{j=1}^{n+1} H_{q_j}(x_j) \quad \text{in } L^2(\gamma_{n+1}).$$

As a result, we can conclude our proof by induction. $\square$

Remark 2.4. The space $C^\infty_{\text{Poly}}(\mathbb{R}^d)$ is dense in $L^2(\gamma_d)$. Indeed, for any $f \in L^2(\gamma_d)$ as in (2.2), we have

$$f_N = \sum_{k=0}^{N} \sum_{\substack{q_1,\dots,q_d\in\mathbb{N}\\ q_1+\dots+q_d=k}} c_{q_1,\dots,q_d} \prod_{j=1}^{d} H_{q_j}(x_j) \in C^\infty_{\text{Poly}}(\mathbb{R}^d) \tag{2.5}$$

converges in $L^2(\gamma_d)$ to f.

• **Derivative operator** $D = \nabla$. When $f \in C^\infty_{\text{Poly}}(\mathbb{R}^d)$,

$$Df = \nabla f = (\partial_1 f, \dots, \partial_d f) : \mathbb{R}^d \to \mathbb{R}^d$$

is well defined. For f_N as in (2.5), we have

$$\partial_i f_N(x) = \sum_{k=1}^{N} \sum_{\substack{q_1,\dots,q_d\in\mathbb{N}\\ q_1+\dots+q_d=k}} c_{q_1,\dots,q_d} q_i H_{q_i-1}(x_i) \prod_{j=1}^{d} \mathbf{1}_{\{j\neq i\}} H_{q_j}(x_j)$$

with the convention $H_{-1}(x_i) = 0$. This implies that

$$\|\partial_i f_N\|^2_{L^2(\gamma_d)} = \sum_{k=1}^{N} \sum_{\substack{q_1,\dots,q_d\in\mathbb{N}\\ q_1+\dots+q_d=k}} c^2_{q_1,\dots,q_d} q_i \prod_{j=1}^{d} q_j!$$

and thus

$$\int_{\mathbb{R}^d} \|\nabla f_N(x)\|^2_{\mathbb{R}^d} \gamma_d(dx) = \sum_{i=1}^{d} \|\partial_i f_N\|^2_{L^2(\gamma_d)} = \sum_{k=1}^{N} k \sum_{\substack{q_1,\dots,q_d\in\mathbb{N}\\ q_1+\dots+q_d=k}} c^2_{q_1,\dots,q_d} \prod_{j=1}^{d} q_j!.$$

The **domain** of D, denoted by $\mathbb{D}^{1,2}(\mathbb{R}^d)$, should be the set of $f \in L^2(\gamma_d)$ absolutely continuous with $|\nabla f| \in L^2(\gamma_d)$. Then the above discussion suggests an equivalent definition, that is, for f as in (2.2), it belongs to $\mathbb{D}^{1,2}(\mathbb{R}^d)$ if and only if

$$\sum_{k=1}^{\infty} k \sum_{\substack{q_1,\dots,q_d\in\mathbb{N}\\ q_1+\dots+q_d=k}} c^2_{q_1,\dots,q_d} \prod_{j=1}^{d} q_j! < \infty. \tag{2.6}$$

• **Orthogonal projections**. Let J_k denote the orthogonal projection operator onto the k-th chaos $\mathbb{C}_k$. Then, for f as in (2.2), we can write $f = \sum_{k=0}^\infty J_k f$, with

$$J_k f = \sum_{\substack{q_1,\dots,q_d\in\mathbb{N}\\ q_1+\dots+q_d=k}} c_{q_1,\dots,q_d} \prod_{j=1}^{d} H_{q_j}(x_j). \tag{2.7}$$

Then, we have $f \in \mathbb{D}^{1,2}(\mathbb{R}^d)$ if and only if $\sum_{k\geq 1} k\|J_k f\|^2_{L^2(\gamma_d)} < \infty$.

GPI2 **Proposition 2.5** (Gaussian Poincaré inequality)**.** *Let $f \in \mathbb{D}^{1,2}(\mathbb{R}^d)$ and $\mathbf{G} \sim \mathcal{N}_d(0, \mathbf{I})$. Then,*

$$\mathrm{Var}(f(\mathbf{G})) \leq \mathbb{E}[|\nabla f(\mathbf{G})|^2]$$

with equality only when $f(\mathbf{G}) = a_0 + \sum_{i=1}^d a_i G_i$ is a linear combination of the components, i.e., only when $f(\mathbf{G})$ is Gaussian.

Proof. First, we write $f = \sum_{k=0}^\infty J_k f$, then

$$\mathrm{Var}(f(\mathbf{G})) = \sum_{k=1}^\infty \|J_k f\|^2_{L^2(\gamma_d)} \leq \sum_{k=1}^\infty k\|J_k f\|^2_{L^2(\gamma_d)} = \mathbb{E}[|\nabla f(\mathbf{G})|^2].$$

It is clear that the above inequality becomes an equality only when $J_k f = 0$ for any $k \geq 2$. That is, $f(x) = a_0 + a \cdot x$ for some $a_0 \in \mathbb{R}$ and $a \in \mathbb{R}^d$. □

• **Divergence operator δ.** Let us now introduce the adjoint operator of D. Let $\mathrm{dom}(\delta)$ be the set of functions $g = (g_1, ..., g_d) : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\|g\|^2_{L^2(\gamma_d)} = \sum_{j=1}^d \|g_j\|^2_{L^2(\gamma_d)} < \infty$$

and for any $f \in C^\infty_{\text{Poly}}(\mathbb{R}^d)$,

$$\big|\mathbb{E}\langle \nabla f(\mathbf{G}), g(\mathbf{G})\rangle_{\mathbb{R}^d}\big| \leq C_g \|f\|_{L^2(\gamma_d)},$$

where $\mathbf{G} \sim \mathcal{N}_d(0, \mathbf{I})$ and $C_g \in (0, \infty)$ is a constant that depends only on g.

Note that $f \in C^\infty_{\text{Poly}} \mapsto \mathbb{E}\langle \nabla f(\mathbf{G}), g(\mathbf{G})\rangle_{\mathbb{R}^d}$ defines bounded linear functional on C^∞_{Poly}, and thus extends to $L^2(\gamma_d)$ by density. Therefore, by Riesz's representation theorem (see Rudin's book), there exists a unique element in $L^2(\gamma_d)$, denoted by $\delta(g)$, such that

$$\langle \nabla f, g\rangle_{L^2(\gamma_d)} = \mathbb{E}\langle \nabla f(\mathbf{G}), g(\mathbf{G})\rangle_{\mathbb{R}^d} = \langle f, \delta(g)\rangle_{L^2(\gamma_d)} \tag{2.8}$$ L3_dual

for every $f \in \mathbb{D}^{1,2}(\mathbb{R}^d)$.

Remark 2.6. For $g = (g_1, ..., g_d) : \mathbb{R}^d \to \mathbb{R}^d$ with $g_j \in C^\infty_{\text{Poly}}(\mathbb{R}^d)$, we have the following explicit expression

$$\delta(g)(x) = x \cdot g - \mathrm{div} g \tag{2.9}$$ L3_dl

where $\mathrm{div} g = \sum_{j=1}^d \partial_j g_j$. Indeed, the LHS of (2.8) is equal to

$$\sum_{j=1}^d \mathbb{E}[g_j(\mathbf{G})\partial_j f(\mathbf{G})] = \sum_{j=1}^d \mathbb{E}\Big[\big(G_j g_j(\mathbf{G}) - \partial_j g_j(\mathbf{G})\big) f(\mathbf{G})\Big]$$

where the last equality follows from the 1D duality relation (1.13). This implies (2.9).

• **Ornstein-Uhlenbeck generator $\mathcal{L}$.** For $f \in C^\infty_{\text{Poly}}(\mathbb{R}^d)$,

$$\mathcal{L} f = \Delta f - x \cdot \nabla f.$$

Note that $\mathcal{L} = -\delta D f$ for $f \in C^\infty_{\text{Poly}}(\mathbb{R}^d)$. This suggests that the domain of $\mathcal{L}$ is

$$\mathrm{dom}(\mathcal{L}) = \big\{ f \in \mathbb{D}^{1,2}(\mathbb{R}^d) : Df \in \mathrm{dom}(\delta) \big\}.$$

L3_prop_L **Proposition 2.7.** (i) *For $f \in \mathbb{C}_k$, we have $\mathcal{L} f = -kf$.*
(ii) *$f \in \mathrm{dom}(\mathcal{L})$ if and only if*

$$\sum_{k=1}^{\infty} k^2 \|J_k f\|^2_{L^2(\gamma_d)} < \infty,$$

where J_k denotes the projection operator onto $\mathbb{C}_k$ as in (2.7).

Proof. It suffices to consider $f(x_1, ..., x_d) = \prod_{i=1}^d H_{q_i}(x_i)$ with $q_1 + ... + q_d = k$. Then,[8]

$$\begin{aligned}
\mathcal{L} f &= \sum_{j=1}^{d} (\partial_j^2 - x_j \partial_j) f = \sum_{j=1}^{d} (\partial_j - x_j)\partial_j f \\
&= \sum_{j=1}^{d} q_j \underbrace{(\partial_j - x_j) H_{q_j - 1}(x_j)}_{= -H_{q_j}(x_j)} \prod_{i=1}^{d} \mathbf{1}_{\{i \neq j\}} H_{q_i}(x_i) \\
&= \left(\sum_{j=1}^{d} q_j \right) \prod_{i=1}^{d} H_{q_i}(x_i) = -kf,
\end{aligned}$$

where we noticed that $\partial_j g(x_j) - x_j g(x_j) = -\delta(g)(x_j)$ for $g : \mathbb{R} \to \mathbb{R}$. This proves (i).

Suppose $f \in \mathrm{dom}(\mathcal{L})$, i.e., $f \in \mathbb{D}^{1,2}(\mathbb{R}^d)$ and $Df \in \mathrm{dom}(\delta)$. In particular, we have $\mathcal{L} f = -\delta(Df) \in L^2(\gamma_d)$. Therefore, it follows from part (i) that

$$\|\mathcal{L} f\|^2_{L^2(\gamma_d)} = \|\mathcal{L} \sum_{k=0}^{\infty} J_k f\|^2_{L^2(\gamma_d)} = \|\sum_{k=0}^{\infty} \mathcal{L} J_k f\|^2_{L^2(\gamma_d)} = \sum_{k=1}^{\infty} k^2 \|J_k f\|^2_{L^2(\gamma_d)} < \infty.$$

Now let us assume that $\sum_{k=1}^{\infty} k^2 \|J_k f\|^2_{L^2(\gamma_d)} < \infty$ and we need to prove $f \in \mathrm{dom}(\mathcal{L})$. It is clear that $f \in \mathbb{D}^{1,2}(\mathbb{R}^d)$ and

$$\|Df_N - Df\|^2_{L^2(\gamma_d)} \xrightarrow{N \to +\infty} 0,$$

where $f_N = \sum_{k \le N} J_k f$. It remains to show that $Df \in \mathrm{dom}(\delta)$. Given $h \in C^{\infty}_{\mathrm{Poly}}(\mathbb{R}^d)$, we can write

$$\begin{aligned}
\langle Dh, Df \rangle_{L^2(\gamma_d)} &= \lim_{N \to +\infty} \langle Dh, Df_N \rangle_{L^2(\gamma_d)} \\
&= \lim_{N \to +\infty} \langle h, -\mathcal{L} f_N \rangle_{L^2(\gamma_d)} \quad \text{by duality,}
\end{aligned}$$

which is bounded by

$$\|h\|_{L^2(\gamma_d)} \sup_N \|\mathcal{L} f_N\|_{L^2(\gamma_d)} = \|h\|_{L^2(\gamma_d)} \sqrt{\sum_{k=1}^{\infty} k^2 \|J_k f\|^2_{L^2(\gamma_d)}}.$$

This shows $Df \in \mathrm{dom}(\delta)$. Hence, $f \in \mathrm{dom}(\mathcal{L})$ and our proof is completed. □

$\propto\propto\propto\propto\propto\propto$ **End of Lecture 3 (Sept. 9, 2025)** $\infty\infty\infty\infty\infty\infty\infty\infty$

[8] $x_j f(x)$ is viewed as the multiplication operator.