for $h \in C_c^\infty(\mathbb{R})$. If we replace the Lebesgue measure by the standard Gaussian measure γ, we have

$$\int_{\mathbb{R}} h'(x)\gamma(dx) = \int_{\mathbb{R}} h'(x)\phi(x)dx = -\int_{\mathbb{R}} h(x)\phi'(x)dx = \int_{\mathbb{R}} xh(x)\phi(x)dx,$$

which is a particular case of the usual integration by parts. For this reason, we often call (1.13) a Gaussian integration by parts formula.

Let $\mathrm{dom}(\delta)$ be the set of absolutely continuous functions g with $g' \in L^2(\gamma)$. Let $\mathbb{D}$ be the set of absolutely continuous functions f with $f' \in L^2(\gamma)$. Note that $\mathrm{dom}(\delta)$ is the domain for the divergence operator and $\mathbb{D}$ is the domain for the derivative operator on $L^2(\gamma)$, and in our current simple setting, $\mathrm{dom}(\delta)$ coincides with $\mathbb{D}$. In a more general setting, these two domains are totally different. In Proposition 1.7, we can only see that $\delta(f) \in L^1(\gamma)$ when $f' \in L^2(\gamma)$. In fact, $f' \in L^2(\gamma)$ ensures $\delta(f) \in L^2(\gamma)$; see Remark 1.15.

$\propto\propto\propto\propto\propto\propto$ **End of Lecture 1 (Sept. 2, 2025)** $\infty\infty\infty\infty\infty\infty\infty$

Let us first introduce Hermite polynomials and then return for more discussions on the divergence operator δ and its domain $\mathrm{dom}(\delta)$.

1.2. **Hermite polynomials.**

Definition 1.9. *Put $H_0(x) = 1$, and $H_{p+1} = \delta(H_p)$ for $p \in \mathbb{N}$. H_p is called the Hermite polynomial of order p. The first few ones are given by*

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \ldots$$

It is not difficult to verify by induction that

$$\frac{d}{dx} H_p(x) = pH_{p-1}(x) \quad \text{and} \quad H_{p+1}(x) = xH_p(x) - pH_{p-1}(x) \tag{1.15}$$

for any integer $p \geq 1$. In a compact manner, we can also define Hermite polynomials using Rodrigues' formula:

$$H_p(x) = (-1)^p \frac{1}{\phi(x)} \phi^{(p)}(x) \tag{1.16}$$

with ϕ as in (1.1). We leave the verification as an exercise to students. Another way to define the Hermite polynomials is via the Gaussian integral formula $H_p(x) = \mathbb{E}[(x+iZ)^p]$ with $Z \sim \mathcal{N}(0,1)$; see the expository article [NO25] for various applications.

Exercise 1.10. (i) *Verify the relations* (1.15) *and* (1.16).

(ii) *Show that any monomial can be expressed as a finite linear combination of Hermite polynomials.*

The family of Hermite polynomials play a crucial role in the Gaussian analysis. They are the orthogonal polynomials with respect to the standard Gaussian measure on $\mathbb{R}$.

Proposition 1.11. *The set of polynomials* $\{H_p/\sqrt{p!} : p \in \mathbb{N}\}$ *form an orthonormal basis for* $L^2(\gamma)$. *As a consequence, the following results hold true.*
(i) *With* $Z \sim \mathcal{N}(0,1)$, *we have*

$$\mathbb{E}[H_p(Z)H_q(Z)] = p!\mathbf{1}_{\{p=q\}}. \tag{1.17}$$

(ii) *For any* $f \in L^2(\gamma)$, *we have the following* **Hermite expansion**:

$$f = \sum_{q\geq 0} c_q H_q \quad \textit{with } c_q = \frac{1}{q!}\int_{\mathbb{R}} H_q(z)f(z)\gamma(dz),$$

where the above series converge in $L^2(\gamma)$ *and* $\sum_{q\geq 0} c_q^2 q! < \infty$.

Proof. First we prove that

$$\text{the linear span of the family } \{H_p/\sqrt{p!} : p \in \mathbb{N}\} \text{ is dense in } L^2(\gamma). \qquad (\#)$$

In view of Exercise 1.10, it is enough to show that the linear span of the family $\{x^p : p \in \mathbb{N}\}$ is dense in $L^2(\gamma)$. Suppose $f \in L^2(\gamma)$ satisfies

$$\int_{\mathbb{R}} f(z)z^k\gamma(dz) = 0, \ \forall k \in \mathbb{N}. \tag{1.18}$$

For any $t \in \mathbb{R}$, we have $\int_{\mathbb{R}} e^{2|tz|}\gamma(dz) < \infty$ and thus

$$\int_{\mathbb{R}} |f(z)| \sum_{k=0}^{\infty} \frac{|tz|^k}{k!}\gamma(dz) = \int_{\mathbb{R}} |f(z)|e^{|tz|}\gamma(dz) < \infty$$

by Cauchy-Schwarz inequality. This ensures that we can write

$$\int_{\mathbb{R}} f(z) \sum_{k=0}^{\infty} \frac{(itz)^k}{k!}\gamma(dz) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} f(z)\frac{(itz)^k}{k!}\gamma(dz),$$

which is zero due to the condition (1.18). This implies in particular that $\int_{\mathbb{R}} f(z)e^{itz}\phi(z)dz = 0$ for any $t \in \mathbb{R}$. With $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, we get $f = f^+ - f^-$ and

$$\int_{\mathbb{R}} e^{itz}f^+(z)\phi(z)dz = \int_{\mathbb{R}} e^{itz}f^-(z)\phi(z)dz. \tag{1.19}$$

If one of f^+ and f^- is a zero function (a.e.), then the other one has to be zero almost everywhere. Now assume otherwise so that (taking $t = 0$ in (1.19))

$$C := \int_{\mathbb{R}} f^+(z)\phi(z)dz = \int_{\mathbb{R}} f^-(z)\phi(z)dz \in (0, \infty),$$

In this case, we can deduce from (1.19) that $\frac{1}{C}f^+(z)\phi(z)$ and $\frac{1}{C}f^-(z)\phi(z)$ are two probability density functions with the same Fourier transform (i.e., characteristic function), and thus they must coincide with each other almost everywhere. That is, we have $f^+ = f^-$, or equivalently, $f = f^+ - f^- = 0$ almost everywhere. Now we can conclude that for a function $f \in L^2(\gamma)$ satisfying (1.18), $f = 0$ a.e., which shows the desired conclusion (#).

Next, we show the orthogonality relation (1.17). Suppose $p \leq q$ and recall the relation (1.15). We can deduce from the duality relation (Proposition 1.7) that

$$\begin{aligned}\mathbb{E}[H_p(Z)H_q(Z)] &= \mathbb{E}[H_p(Z)(\delta H_{q-1})(Z)] \\ &= \mathbb{E}[pH_{p-1}(Z)H_{q-1}(Z)] \\ &= p!\mathbb{E}[H_{q-p}(Z)] = p!\mathbf{1}_{\{p=q\}}.\end{aligned}$$

Hence, we just proved that $\{H_p/\sqrt{p!} : p \in \mathbb{N}\}$ form an orthonormal basis for $L^2(\gamma)$, and the Hermite expansion in part (ii) follows immediately. $\square$

Proposition 1.12 (Product formula for Hermite polynomials)**.** *Let p, q be positive integers. Then,*

$$H_p(x)H_q(x) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} H_{p+q-2r}(x).$$

Proof. Note that $H_p(x)$ is a polynomial of degree p, then $H_p(x)H_q(x)$ is a polynomial of degree $p+q$. Thanks to Exercise 1.10, we can express $H_p(x)H_q(x)$ as a finite linear combination of Hermite polynomials:

$$H_p(x)H_q(x) = \sum_{r=0}^{p+q} c_r H_r(x)$$

with the coefficients c_r's to be determined. Since $\mathbb{E}[H_k(Z)] = 0$ with $Z \sim \mathcal{N}(0,1)$ for any $k \geq 1$ and $\frac{d^k}{dx^k}H_k(x) = k!$ (see (1.15)), we have

$$c_r r! = \mathbb{E}\Big(\frac{d^r}{dx^r}\big[H_p(x)H_q(x)\big]\big|_{x=Z}\Big). \tag{1.20}$$

By the general Leibniz rule, we have for $r \leq p+q$,

$$\begin{aligned}\frac{d^r}{dx^r}\big[H_p(x)H_q(x)\big] &= \sum_{k=0}^{r}\binom{r}{k}\big(\frac{d^k}{dx^k}H_p(x)\big)\big(\frac{d^{r-k}}{dx^{r-k}}H_q(x)\big)\\ &= \sum_{k=0}^{r}\binom{r}{k}\frac{p!}{(p-k)!}H_{p-k}(x)\frac{q!}{(q+k-r)!}H_{q+k-r}(x)\mathbf{1}_{\left\{\substack{k\leq p\\ r-k\leq q}\right\}}.\end{aligned} \tag{1.21}$$

Using the orthogonality relation in Proposition 1.11-(i), we have

$$\mathbb{E}\big[H_{p-k}(Z)H_{q+k-r}(Z)\big] = (p-k)!\mathbf{1}_{\{k=\frac{p-q+r}{2}\}}.$$

Therefore, it follows from (1.20) and (1.21) that

$$c_r r! = \sum_{k=0}^{r}\binom{r}{k}\frac{p!}{(p-k)!}\frac{q!}{(q+k-r)!}\mathbf{1}_{\left\{\substack{k\leq p\\ r-k\leq q}\right\}}(p-k)!\mathbf{1}_{\{k=\frac{p-q+r}{2}\}}.$$

Then, taking the above restrictions on k, r into account with $k = \frac{p-q+r}{2}$, we have

$$c_r = \frac{p!q!}{(\frac{p-q+r}{2})!(\frac{q+r-p}{2})!(\frac{p+q-r}{2})!}\mathbf{1}_{\left\{\substack{|p-q|\leq r\leq p+q\\ p+q+r \text{ is even}}\right\}}.$$

Then, making the change of variables $r' = (p+q-r)/2$ (thus $r = p+q-2r'$), we get

$$c_{p+q-2r'} = \frac{p!q!}{(p-r')!(q-r')!(r')!}\mathbf{1}_{\{0\leq r'\leq p\wedge q\}}.$$

Hence, this concludes our proof. $\square$

Question 1.13. *Use the same proof strategy to show the following "binomial formula" for Hermite polynomials:*

$$H_n(ax+by) = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}H_k(x)H_{n-k}(y), \tag{1.22}$$

where $a^2+b^2 = 1$.

Proposition 1.14 (Gaussian Poincaré inequality revisited)**.** *Suppose $f \in \mathbb{D}$ and $Z \sim \mathcal{N}(0,1)$. Then,*

$$\mathrm{Var}(f(Z)) \leq \mathbb{E}[|f'(Z)|^2],$$

where the equality holds if and only if $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Proof. In Proposition 1.7, we gave a proof of the Gaussian Poincaré inequality using the fundamental theorem of calculus. Now let us present a different proof of the Gaussian Poincaré inequality using the Hermite expansion.

Suppose $f \in L^2(\gamma)$ admits the following expansion $f = \sum_{q\in\mathbb{N}} c_q H_q$ with $\sum_{q\in\mathbb{N}} c_q^2 q! < \infty$. Then,

$$\mathrm{Var}(f(Z)) = \sum_{q\in\mathbb{N}_0} c_q^2 q!$$

and with $f' = \sum_{q\in\mathbb{N}_0} c_q q H_{q-1}$

$$\mathbb{E}[|f'(Z)|^2] = \sum_{q\in\mathbb{N}_0} c_q^2 q^2 (q-1)! = \sum_{q\in\mathbb{N}_0} c_q^2 q! q.$$

Thus,

$$\mathrm{Var}(f(Z)) \leq \mathbb{E}[|f'(Z)|^2]$$

with equality only when $c_q = 0$ for all $q \geq 2$. That is, the equality holds only when f is affine. $\square$

Remark 1.15. Let us return to the description of the domains $\mathbb{D} = \mathrm{dom}(\delta)$ and prove the anticipated result

$$\delta(f) \in L^2(\gamma) \text{ for } f \in \mathrm{dom}(\delta).$$

For $f \in L^2(\gamma)$ with the Hermite expansion

$$f = \sum_{q\geq 0} c_q H_q, \tag{1.23}$$

we have $\sum_{q\geq 0} c_q^2 q! < \infty$. The set $\mathbb{D}$ is then the set of functions f with the Hermite expansion as in (1.23) such that

$$\sum_{q\geq 0} c_q^2 q! q < \infty. \tag{1.24}$$

Indeed, $f' = \sum_{q\geq 1} c_q H_q' = \sum_{q\geq 1} c_q q H_{q-1} \in L^2(\gamma)$ implies

$$\sum_{q\geq 1} c_q^2 q^2 (q-1)! < \infty,$$

which is exactly the condition (1.24). For f as in (1.23) with (1.24), we deduce from the recurrence relation 1.15 that

$$xf(x) - f'(x) = \sum_{q=0}^{\infty} c_q [xH_q(x) - qH_{q-1}(x)] = \sum_{q=0}^{\infty} c_q H_{q+1}(x),$$

which is square-integrable with respect to the standard Gaussian measure, that is, $\delta(f) \in L^2(\gamma)$.

It is not difficult to show that for $f \in \mathrm{dom}(\delta)$ as in (1.23) with (1.24), we have

$$\delta(f) = xf(x) - f'(x) = \sum_{q\geq 1} c_q \delta(H_q) = \sum_{q=0}^{\infty} c_q H_{q+1} \in L^2(\gamma).$$

Question 1.16. (i) *Show that the derivative operator (i.e., $Df = f'$) is* **closable** *meaning that if $f_n \in \mathbb{D}$ with $Df_n \to g$ in $L^2(\gamma)$ for some g and $f_n \to 0$ in $L^2(\gamma)$ as $n \to +\infty$, then $g = 0$.*

(ii) *Show that δ (as the adjoint operator of D) is* **closed** *meaning that for $g_n \in \mathrm{dom}(\delta)$ with $\delta(g_n)$ convergent to h in $L^2(\gamma)$, one can find $g \in \mathrm{dom}(\delta)$ such that $h = \delta(g)$ and $g_n \to g$ in $L^2(\gamma)$.*

Exercise 1.17. *Let $\mathcal{A}$ be the set of functions $g \in L^2(\gamma)$ such that there is a constant $C = C_g$ such that*

$$\left| \int_{\mathbb{R}} f'(z) g(z) \gamma(dz) \right| \leq C \|f\|_{L^2(\gamma)}$$

for any $f \in \mathbb{D}$. Prove that $\mathcal{A} = \mathrm{dom}(\delta)$.

Exercise 1.18. *Verify the duality relation in Proposition 1.7 by using the Hermite expansions.*

1.3. **Application I: two series representations of π (optional reading).**

$$\pi = \sum_{k=0}^{\infty} \frac{1}{2^{2k-1}(2k+1)} \binom{2k}{k} \qquad \text{Ramanujan's 1912 identity} \tag{1.25}$$

$$\pi = \sum_{k=0}^{\infty} \frac{3}{2^{4k}(2k+1)} \binom{2k}{k} \qquad \text{Newton's 1676 identity.} \tag{1.26}$$

We will prove the above two formulae using Hermite expansions.

• **Preparation.** Let $h_u(x) = \mathbf{1}_{\{x>u\}}$. We claim that

$$h_u = \mathbb{P}(Z > u) + \sum_{q=1}^{\infty} a_q H_q \quad \text{with } a_q = \frac{1}{q!} \phi(u) H_{q-1}(u). \tag{1.27}$$

Proof of claim (1.27). It is trivial that $h \in L^2(\gamma)$ and thus admits a Hermite expansion

$$h = \sum_{q \geq 0} a_q H_q,$$

where $a_0 = \mathbb{P}(Z > u)$ and for $q \geq 1$,

$$\begin{aligned} a_q : &= \frac{1}{q!} \int_{\mathbb{R}} H_q(z) h_u(z) \gamma(dz) = \frac{1}{q!} \int_u^{\infty} H_q(z) \phi(z) dz \\ &= \frac{1}{q!} \int_u^{\infty} (-1)^q \phi^{(q)}(z) dz \quad \text{using (1.16)} \\ &= \frac{1}{q!} (-1)^{q+1} \phi^{(q-1)}(u) = \frac{1}{q!} \phi(u) H_{q-1}(u) \quad \text{using (1.16) again.} \end{aligned}$$

This verifies the claim (1.27). □

Let us first present the formula for $H_q(0)$:

$$H_{2p-1}(0) = 0 \quad \text{and} \quad H_{2p}(0) = \frac{(-1)^p (2p)!}{2^p p!} \quad \text{for any } p \in \mathbb{N}_0. \tag{1.28}$$

This can be easily verified by induction (Exercise!). For general $x \in \mathbb{R}$, we have for $q \in \mathbb{N}_0$,

$$H_q(x) = \sum_{k=0}^{\lfloor q/2 \rfloor} \frac{q! (-1)^k}{k! (q-2k)! 2^k} x^{q-2k}, \tag{1.29}$$

which can be derived by using (1.28) and (1.15) (Exercise!).

• *Proof of* (1.25). From the above discussions, we can first write, with $h_0(x) = \mathbf{1}_{\{x>0\}}$, that

$$\begin{aligned}
h_0 &= \frac{1}{2} + \sum_{q=1}^{\infty} \frac{1}{q!\sqrt{2\pi}} H_{q-1}(0) H_q \\
&= \frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!\sqrt{2\pi}} \frac{(-1)^k (2k)!}{2^k k!} H_{2k+1} \\
&= \frac{1}{2} + \sum_{k=0}^{\infty} d_{2k+1} H_{2k+1},
\end{aligned}$$

where

$$d_{2k+1} = \frac{(-1)^k}{2^k k! (2k+1)\sqrt{2\pi}}. \tag{1.30}$$

Then, using the orthogonality relation of Hermite polynomials, we have

$$\mathrm{Var}(h_0(Z)) = \sum_{k=0}^{\infty} d_{2k+1}^2 (2k+1)! = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{2k+1}\pi(2k+1)}.$$

It is easy to get $\mathrm{Var}(h_0(Z)) = 1/4$, and hence we get (1.25). □

• *Proof of* (1.26). Recall Φ is the CDF of a standard normal $Z \sim \mathcal{N}(0,1)$. Then, by the probability integral transform $\Phi(Z)$ is uniformly distributed over $(0,1)$, so that $\mathrm{Var}(\Phi(Z)) = \frac{1}{12}$. On the other hand, we can write by using the above expansion of h_0 and the binomial formula (1.22),

$$\begin{aligned}
\Phi(x) &= \mathbb{E}\big[h_0(\tfrac{Z}{\sqrt{2}} + \tfrac{x}{\sqrt{2}})\big] \\
&= \frac{1}{2} + \mathbb{E}\bigg(\sum_{k=0}^{\infty} d_{2k+1} H_{2k+1}(\tfrac{Z}{\sqrt{2}} + \tfrac{x}{\sqrt{2}})\bigg) \\
&= \frac{1}{2} + \mathbb{E}\bigg(\sum_{k=0}^{\infty} d_{2k+1} \sum_{r=0}^{2k+1} \binom{2k+1}{r} 2^{-k-\frac{1}{2}} H_r(Z) H_{2k+1-r}(x)\bigg).
\end{aligned}$$

Since $\mathbb{E}[H_r(Z)] = 0$ for $r \geq 1$ and $H_0 \equiv 1$, we can simplify the above expression to

$$\Phi = \frac{1}{2} + \sum_{k=0}^{\infty} d_{2k+1} 2^{-k-\frac{1}{2}} H_{2k+1}.$$

Therefore, we get

$$\begin{aligned}
\frac{1}{12} = \mathrm{Var}(\Phi(Z)) &= \sum_{k=0}^{\infty} d_{2k+1}^2 2^{-2k-1} (2k+1)! \\
&= \sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{4k+2}\pi(2k+1)},
\end{aligned}$$

from which we can deduce (1.26). □

1.4. **Application II : computation of Greek letters (optional reading).** In the end of this section, we present a financial-math application of the Gaussian integration by parts formula. Let us first give a bit the financial context, which is by no means comprehensive, while we refer interested readers to the books [Sh04, Hull22] for the preliminaries in finance and relevant mathematics.

• **Basics of financial derivatives.** A *financial derivative* is a financial instrument whose value is derived from the value of an underlying asset, index, or rate. In this subsection, the underlying assets will be restricted to stocks whose prices follow the law of the classical geometric Brownian motion. Concretely, let S_t denote the stock price at time t, and it can be modeled by

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}, \tag{1.31}$$

where $S_0 > 0$ is the initial stock price, $r > 0$ is the risk-free interest rate, $\sigma > 0$ is the volatility, and $(W_t)_{t\geq 0}$ denotes the real-valued, standard Brownian motion.[5] Suppose a financial derivative's underlying asset is a stock with the above price (1.31) and its payoff at the terminal time $T > 0$ is described by $\psi(S_T)$, that is, the payoff of the financial derivative at time T is assumed to only depend on the terminal stock price. According to the principle of risk-neutral pricing (see, e.g., [Sh04, Chapter 5]), the initial price of such a financial derivative is given by

$$H = e^{-rT}\mathbb{E}[\psi(S_T)],$$

where e^{-rT} is the discounting factor of a future money value to the present value (continuous compounding, in the language of finance). In this simple example, it is not difficult to see that the initial value H depends on various parameters like r, σ, and the initial stock price S_0. Here is a natural and important question:

"how sensitive H is when the parameters r, σ, S_0 change?"

In financial markets, traders and portfolio managers use **Greek letters** to measure the sensitivity of prices of financial derivatives to various market parameters. These sensitivities help manage risk efficiently and construct effective hedging strategies.

The primary Greek letters are:

(i) Delta (Δ): Sensitivity of the option price to the underlying asset price;
(ii) Gamma (Γ): Rate of change of Delta with respect to the underlying price;
(iii) Vega ($\mathcal{V}$): Sensitivity to changes in volatility;
(iv) Rho (ρ): Sensitivity to interest rates.

They are defined by

$$\Delta := \frac{\partial}{\partial S_0}H, \quad \Gamma := \frac{\partial}{\partial S_0}\Delta = \frac{\partial^2}{\partial^2 S_0}H, \quad \mathcal{V} := \frac{\partial}{\partial \sigma}H, \quad \text{and} \quad \rho := \frac{\partial}{\partial r}H. \tag{1.32}$$

In what follows, we only present the formula for the Delta and leave the proof of other Greek letters for students.

Proposition 1.19 (formula of Delta). *Consider the above framework and assume that $\psi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous (see Footnote 7). Then, we have*

$$\Delta = \frac{e^{-rT}}{\sigma S_0 T}\,\mathbb{E}\left[\psi(S_T)W_T\right]. \tag{1.33}$$

[5]Here is a quick definition of Brownian motion. $(W_t)_{t\geq 0}$ is a family of centered Gaussian random variables satisfying the following three conditions: (i) any finite linear combinations of W_t's is Gaussian (ii) $t \in \mathbb{R}_+ \mapsto W_t$ is a.s. continuous (iii) $\mathbb{E}[W_t W_s] = t \wedge s$ for any $s, t \in \mathbb{R}_+$. In particular, we have $W_t \sim \mathcal{N}(0,t)$ for any $t \in \mathbb{R}_+$. This is the only property we need for this section.

Proof. Recall from Footnote 5 that $W_T \sim \mathcal{N}(0,T)$, so we can write $W_T = \sqrt{T}\cdot Z$ with $Z \sim \mathcal{N}(0,1)$. Then, using Lemma 1.6, we get for any absolutely continuous function f with $\mathbb{E}[|f'(\sqrt{T}Z)|] < \infty$,

$$\mathbb{E}[W_T f(W_T)] = \mathbb{E}[\sqrt{T}Z \cdot f(\sqrt{T}Z)] = T\cdot\mathbb{E}[f'(\sqrt{T}Z)] = T\cdot\mathbb{E}[f'(W_T)],$$

that is, we have

$$\mathbb{E}[W_T f(W_T)] = T\mathbb{E}[f'(W_T)]. \tag{1.34}$$

Since

$$H = e^{-rT}\mathbb{E}[\psi(S_T)] \quad \text{with } S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T},$$

We can write $S_T = h(W_T)$ as a function of W_T with $h(x) = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma x}$. That is, we have

$$H = e^{-rT}\mathbb{E}[\psi\circ h(W_T)].$$

Note that the function $f = \psi\circ h$ is absolutely continuous with

$$f'(x) = \psi'(h(x))h'(x) = \sigma\psi'(h(x))h(x) \quad \text{and thus } f'(W_T) = \sigma S_T\psi'(S_T). \tag{1.35}$$

see, e.g., (A.1). Since ψ' is uniformly bounded (see, e.g., Footnote 7) and $h(W_T) = S_T$ has finite expectation, we see that the function f satisfy the assumptions for applying (1.34). Doing so leads to

$$\begin{aligned}
\Delta = \frac{\partial}{\partial S_0}H &= e^{-rT}\frac{\partial}{\partial S_0}\mathbb{E}\left[\psi\big(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T}\big)\right] \\
&= e^{-rT}\,\mathbb{E}\left[\psi'(S_T)\cdot\frac{S_T\sigma}{S_0\sigma}\right] &(1.36)\\
&= \frac{e^{-rT}}{\sigma S_0}\,\mathbb{E}\left[f'(W_T)\right] &(1.37)\\
&= \frac{e^{-rT}}{\sigma S_0 T}\,\mathbb{E}\left[f(W_T)W_T\right] \quad \text{by (1.34)}.
\end{aligned}$$

Note that the step (1.36) follows from Lebesgue's differentiation theorem with ψ' uniformly bounded and $S_T \in L^1(\Omega)$, and the equality (1.37) is due to (1.35). Finally, the formula (1.33) is proved by noticing that $f(W_T) = \psi(S_T)$. □

Remark 1.20. (i) The above derivation of the formula for Δ is quite elementary and only utilizes the Gaussian integration by parts formula in Lemma 1.6. However, the above proof strategy fails if the payoff does not just depend on the terminal stock price S_T, for example, when the payoff depends on $(S_t)_{t\in[0,T]}$, i.e., the whole stock price dynamics on the duration of the financial derivative. See Chapter 6 of [Nua06] for more discussions and in particular, the interested readers may want to compare our proof with the argument on [Nua06, page 332] that uses the full power of the Malliavin calculus. See also Chapter 2 of the book [MT06] by Malliavin and Thalmaier.

(ii) Consider the case where $\psi(x) = \max\{x-K, 0\}$ for some given $K > 0$. This corresponds to the payoff of a European call option with strike price K and terminal time T. The European call option gives its holder a right (not an obligation) to purchase the underlying asset (stock in our example) at the price K at time T. It is of fundamental importance in finance, and its price is given by the famous Black-Scholes-Merton formula:

$$H_{\text{call}} = S_0\Phi\big(d_1\big) - Ke^{-rT}\Phi\big(d_2\big)$$

with

$$d_1 = \frac{\log\big(S_0/K\big) + \big(r + \frac{\sigma^2}{2}\big)\,T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

By directly differentiating the explicit formula of H_{call} with respect to S_0, we can get[6]

$$\Delta_{\text{call}} = \Phi(d_1).$$

Alternatively, one can first use (1.33) to write

$$\Delta_{\text{call}} = \frac{e^{-rT}}{\sigma S_0 T}\mathbb{E}\big[\max\{S_T - K, 0\}W_T\big]$$

and standard (although tedious) Gaussian computations will lead to the same formula $\Delta_{\text{call}} = \Phi(d_1)$. The formula (1.33) is advantageous when we do not have an explicit price formula.

Exercise 1.21. *Under the same assumptions as in Proposition 1.19, prove the following formulae:*

$$\Gamma = \frac{e^{-rT}}{\sigma^2 T S_0^2}\,\mathbb{E}\left[\left(\frac{W_T^2}{T} - 1\right)\psi(S_T)\right]; \qquad \mathcal{V} = \mathbb{E}\left[\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\psi(S_T)\right];$$

$$\rho = \frac{e^{-rT}}{\sigma}\mathbb{E}\big[\psi(S_T)W_T\big] - TH.$$

$\propto\propto\propto\propto\propto\propto$ **End of Lecture 2 (Sept. 4, 2025)** $\infty\infty\infty\infty\infty\infty\infty\infty$

[6]Note that the expressions d_1, d_2 are involved with S_0 such that $S_0\Phi'(d_1) = Ke^{-rT}\Phi'(d_2)$.