## 1. Gaussian integration by parts on $\mathbb{R}$

In this section, we begin by introducing the one-dimensional version of the Malliavin calculus, or the stochastic analysis for random variables of the form $g(Z)$, with $Z \sim \mathcal{N}(0,1)$ a standard normal random variable. Let us give a brief review on the real Gaussian random variables. We say $Z \sim \mathcal{N}(0,1)$ is a standard normal if $Z$ is a continuous random variable with the probability density function (PDF)

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \ z \in \mathbb{R}. \tag{1.1}$$

Throughout this course, we will use $\gamma$ to denote the standard Gaussian measure on $\mathbb{R}$ unless specified otherwise:

$$\gamma(dz) = \phi(z)dz \quad \text{with } \phi \text{ as in (1.1).}$$

For example, $L^2(\gamma)$ is the space of measurable functions from $\mathbb{R}$ to $\mathbb{R}$ that are square-integrable with respect to the standard Gaussian measure $\gamma$. Similarly, we write for $p \in [1,\infty)$,[1]

$$f \in L^p(\gamma) \Leftrightarrow \int_{\mathbb{R}} |f(z)|^p \gamma(dz) < \infty.$$

Let us first do some easy Gaussian computations. Using the above PDF (1.1), we can compute the fourth moment of $Z \sim \mathcal{N}(0,1)$ as follows:

$$\begin{aligned}
\mathbb{E}[Z^4] &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} z^4 dz = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} z^3 d(z^2/2) \\
&= -\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} z^3 de^{-\frac{1}{2}z^2} = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz^3 \quad \text{(by integration by parts)} \\
&= 3\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} z^2 dz = 3\mathbb{E}[Z^2] = 3.
\end{aligned} \tag{1.2}$$

**Exercise 1.1.** *Verify the following moment formulae for $Z \sim \mathcal{N}(0,1)$:*

$$\mathbb{E}[Z^m] = \begin{cases} 0 & \text{if } m \in 2\mathbb{N}+1 \\ (2n)!! = \frac{(2n)!}{2^n n!} & \text{if } m = 2n \in 2\mathbb{N}. \end{cases} \tag{1.3}$$

Students are invited to think about the following warm-up questions.[2]

**Question 1.2.** *Let $Z$ be a standard normal and $g: \mathbb{R} \to \mathbb{R}$ be a measurable function. Is the following equivalence true?*

*"$g(Z)$ is normal distributed iff $g$ is affine"*

Question 1.2 appears to be straightforward, at least for the "if" direction. For the "only if" direction, it is somehow wrong in general. See the following example:

$$g(z) = z\mathbf{1}_{\{|z|\geq 1\}} - z\mathbf{1}_{\{|z|<1\}}.$$

What if we assume additionally that the function $g$ is bijective and increasing?

**Question 1.3.** *Can you provide a direct proof of the following inequality?*

$$\|P_d\|_{L^p(\gamma)} \leq C_{p,d}\|P_d\|_{L^2(\gamma)} \tag{1.4}$$

*for any polynomial $P_d$ with degree at most $d$ and for any $p \in [2,\infty)$, where $C_{p,d}$ is a constant that only depends on $p$ and $d$.*

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[1] $f \in L^\infty(\gamma)$ iff $f$ is bounded almost everywhere with respect to the Lebesgue measure.

[2] In this lecture notes, the Exercises are left for students to solve, while the Questions will be addressed in classes.

There are various reasons for the question

"*why the Gaussian objects are so important?*".

One of them is that the Gaussian distribution often arises as the limit of various important probabilistic/statistical objects. For example, the **law of large number** (LLN) asserts that (in the language of a undergraduate statistic course)

**(LLN)** for a random sample from a population distribution with mean zero and variance one, the sample mean statistic tends to zero[3] as the sample size tends to infinity.

If we assume further that the population distribution has finite second moment (or equivalently, finite variance), the classical central limit theorem asserts that after a proper normalization, the sample mean statistic admits *Gaussian fluctuation* as the sample size tends to infinity. Here, we see that the Gaussian distribution arises as the *universal* limiting distribution regardless of the population distribution. Results on asymptotic normality are always welcomed, e.g., in statistics, since one can use them for various statistical inferences (e.g., confidence interval estimation, hypothesis testing).

**Theorem 1.4** (Classical central limit theorem)**.** *Let $\{X_i : i \in \mathbb{N}\}$ be i.i.d. random variables with mean zero and variance one. Then,*

$$V_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

*converges in law/distribution to a standard normal distribution $\mathcal{N}(0,1)$ as $n \to \infty$. That is, for any $t \in \mathbb{R}$,*

$$\mathbb{P}(V_n \le t) \xrightarrow{n\to\infty} \Phi(t) := \int_{-\infty}^{t} \phi(z)dz \tag{1.5}$$

*with $\phi$ as in (1.1).*

The above central limit theorem (CLT) is of qualitative nature. The convergence in (1.5) is the point-wise convergence of the cumulative distribution function (CDF) of $V_n$ to the standard normal CDF $\Phi$. Under stronger assumptions (e.g., $\mathbb{E}[|X_1|^3] < \infty$), the famous **Berry-Esseen bound** in Theorem 1.5 gives a more quantitative result by specifying the rate at which this convergence takes place. The rate of convergence is measured by the Kolmogorov distance

$$d_{\mathrm{Kol}}(X,Y) := \sup_{t\in\mathbb{R}} \big|\mathbb{P}(X \le t) - \mathbb{P}(Y \le t)\big|. \tag{1.6}$$

There are other frequently used distributional metrics that measure the proximity of distributions (see Section 3).

**Theorem 1.5** (Berry-Esseen bound)**.** *Let the assumptions in Theorem 1.4 hold and assume additionally that $\mathbb{E}[|X_1|^3] < \infty$. Then,*

$$d_{\mathrm{Kol}}(V_n, Z) \le \frac{C\mathbb{E}[|X_1|^3]}{\sqrt{n}},$$

*where $Z \sim \mathcal{N}(0,1)$ and the absolute constant $C$ does not depend on the population distribution nor the sample size $n$.*[4] *We postpone the proof to Section 3.*

Let us now move to the core of this section.

[3]This convergence occurs almost surely and in $L^1(\Omega)$.

[4]References on the absolute constant **XXX**; Without assuming the finite absolute third moment, we can derive $d_{\mathrm{Kol}}(V_n, Z) \to 0$ as $n \to +\infty$.

### 1.1. Gaussian integration by parts formula.

**Lemma 1.6.** *Let $Z \sim \mathcal{N}(0,1)$ and $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function such that $f' \in L^1(\gamma)$. Then, $\mathbb{E}[|Zf(Z)|] < \infty$ and*

$$\mathbb{E}[f'(Z)] = \mathbb{E}[Zf(Z)] \tag{1.7}$$

*or equivalently,*

$$\int_{\mathbb{R}} f'(z)\gamma(dz) = \int_{\mathbb{R}} zf(z)\gamma(dz). \tag{1.8}$$

*Proof.* Let $f, Z$ be given as above. To show the finiteness of $\mathbb{E}[|Zf(Z)|]$, it suffices to prove

$$\mathbb{E}[|Z[f(Z) - f(0)]|] < \infty. \tag{1.9}$$

Indeed, using the fundamental theorem of calculus (Theorem A.1), we can write

$$\begin{aligned}
\mathbb{E}[|Z[f(Z) - f(0)]|] &= \int_{-\infty}^{\infty} |z[f(z) - f(0)]|\phi(z)dz \\
&= \int_{-\infty}^{0} \left| z\int_{z}^{0} f'(t)dt \right| \phi(z)dz + \int_{0}^{\infty} \left| z\int_{0}^{z} f'(t)dt \right| \phi(z)dz \\
&\le \int_{-\infty}^{0} |z| \int_{z}^{0} |f'(t)|dt\phi(z)dz + \int_{0}^{\infty} |z| \int_{0}^{z} |f'(t)|dt\phi(z)dz \\
&= \int_{0}^{\infty} |f'(t)|\Big(\int_{t}^{\infty} z\phi(z)dz\Big)dt - \int_{-\infty}^{0} |f'(t)|\Big(\int_{-\infty}^{t} z\phi(z)dz\Big)dt.
\end{aligned} \tag{1.10}$$

Note that for $a \in \mathbb{R}_+$, one can easily verify that

$$\int_{a}^{\infty} z\phi(z)dz = -\int_{-\infty}^{-a} z\phi(z)dz = \phi(a). \tag{1.11}$$

It follows immediately that

$$\mathbb{E}[|Z[f(Z) - f(0)]|] \le \int_{-\infty}^{\infty} |f'(t)|\phi(t)dt < \infty.$$

That is, we just proved (1.9) and thus proved the finiteness of $\mathbb{E}[|Zf(Z)|]$. The proof of (1.8) can be done in the same way. Here is a cheap way. Removing the absolute signs in (1.10), we can get similar expressions with all equal signs, and particularly we have,

$$\mathbb{E}[Z[f(Z) - f(0)]] = \int_{0}^{\infty} f'(t)\Big(\int_{t}^{\infty} z\phi(z)dz\Big)dt - \int_{-\infty}^{0} f'(t)\Big(\int_{-\infty}^{t} z\phi(z)dz\Big)dt.$$

Note that the LHS coincides with $\mathbb{E}[Zf(Z)]$, and the RHS is exactly $\mathbb{E}[f'(Z)]$ in view of (1.11). This concludes our proof. □

• **Divergence operator $\delta$.** For $f : \mathbb{R} \to \mathbb{R}$ absolutely continuous, we put

$$(\delta f)(x) = xf(x) - f'(x),$$

which is well defined for almost every $x \in \mathbb{R}$. From Lemma 1.6, we know that

$$\delta f \in L^1(\gamma) \text{ with } \mathbb{E}[\delta(f)(Z)] = 0, \text{ provided } f' \in L^1(\gamma). \tag{1.12}$$

Next, we present a generalization of Lemma 1.6.

**Proposition 1.7.** (i) [Gaussian Poincaré inequality] *Suppose $f$ is absolutely continuous with $f'$ in $L^2(\gamma)$. Then, $f \in L^2(\gamma)$ with*

$$\mathrm{Var}(f(Z)) \le \mathbb{E}[|f'(Z)|^2].$$

*The equality holds true only when $f$ is affine, i.e., $f(z) = az + b$ for some $a, b \in \mathbb{R}$.*

(ii) [Duality relation] *For $f, g$ absolutely continuous with $f', g'$ in $L^2(\gamma)$, we have*

$$\int_{\mathbb{R}} f'(z)g(z)\gamma(dz) = \int_{\mathbb{R}} f(z)\delta(g)(z)\gamma(dz). \tag{1.13}$$

*Proof.* Let us first prove part (i) using the fundamental theorem of calculus as in the proof of Lemma 1.6. Similarly, we write

$$\begin{aligned}
&\int_{\mathbb{R}} |f(z) - f(0)|^2 \phi(z)dz = \int_0^\infty \left|\int_0^z f'(t)dt\right|^2 \phi(z)dz + \int_{-\infty}^0 \left|\int_z^0 f'(t)dt\right|^2 \phi(z)dz \\
&\le \int_0^\infty |z| \int_0^z |f'(t)|^2 dt\phi(z)dz + \int_{-\infty}^0 |z| \int_z^0 |f'(t)|^2 dt\phi(z)dz \\
&= \int_0^\infty dt|f'(t)|^2 \left(\int_t^\infty z\phi(z)dz\right) - \int_{-\infty}^0 dt|f'(t)|^2 \left(\int_{-\infty}^t z\phi(z)dz\right) \\
&= \int_{\mathbb{R}} dt|f'(t)|^2 \phi(t) < \infty,
\end{aligned} \tag{1.14}$$

where the last equality is due to (1.11). This proves $f \in L^2(\gamma)$. Moreover,

$$\mathrm{Var}(f(Z)) \le \mathbb{E}[|f(Z) - f(0)|^2] \le \int_{\mathbb{R}} dt|f'(t)|^2\phi(t) = \mathbb{E}[|f'(Z)|^2].$$

Note that the inequality in (1.14) becomes an equality only when

$$\left|\int_0^z f'(t)dt\right|^2 = z\int_0^z |f'(t)|^2 dt \quad \text{and} \quad \left|\int_y^0 f'(t)dt\right|^2 = |y|\int_y^0 |f'(t)|^2 dt$$

for any $y < 0 < z$. This occurs only when $f'$ is a constant (*why?*). Therefore, the proof of part (i) is completed.

Now let us prove part (ii). Let $g$ be any absolutely continuous function with $g' \in L^2(\gamma)$, then $g \in L^2(\gamma)$ and $fg$ is absolutely continuous with

$$(fg)' = fg' + f'g \in L^1(\gamma).$$

See Theorem A.2. Therefore, Lemma 1.6 (see also (1.12)) indicates that $\delta(fg) \in L^1(\gamma)$ with

$$\begin{aligned}
0 = \int_{\mathbb{R}} \delta(fg)d\gamma &= \int_{\mathbb{R}} \big[zf(z)g(z) - f(z)g'(z) - f'(z)g(z)\big]\gamma(dz) \\
&= \int_{\mathbb{R}} f(z)\delta(g)(z)\gamma(dz) - \int_{\mathbb{R}} f'(z)g(z)\gamma(dz),
\end{aligned}$$

which is exactly the equality (1.13). Hence, the proof is completed. □

**Remark 1.8.** Consider $f, g \in C_c^\infty(\mathbb{R})$, the integration by parts formula $\int_{\mathbb{R}} f'gdx = -\int_{\mathbb{R}} fg'dx$ in undergraduate calculus essentially follows from the fact that

$$\int_{\mathbb{R}} h'(x)dx = 0$$

for $h \in C_c^\infty(\mathbb{R})$. If we replace the Lebesgue measure by the standard Gaussian measure $\gamma$, we have

$$\int_{\mathbb{R}} h'(x)\gamma(dx) = \int_{\mathbb{R}} h'(x)\phi(x)dx = -\int_{\mathbb{R}} h(x)\phi'(x)dx = \int_{\mathbb{R}} xh(x)\phi(x)dx,$$

which is a particular case of the usual integration by parts. For this reason, we often call (1.13) a Gaussian integration by parts formula.

Let $\mathrm{dom}(\delta)$ be the set of absolutely continuous functions $g$ with $g' \in L^2(\gamma)$. Let $\mathbb{D}$ be the set of absolutely continuous functions $f$ with $f' \in L^2(\gamma)$. Note that $\mathrm{dom}(\delta)$ is the domain for the divergence operator and $\mathbb{D}$ is the domain for the derivative operator on $L^2(\gamma)$, and in our current simple setting, $\mathrm{dom}(\delta)$ coincides with $\mathbb{D}$. In a more general setting, these two domains are totally different. In Proposition 1.7, we can only see that $\delta(f) \in L^1(\gamma)$ when $f' \in L^2(\gamma)$. In fact, $f' \in L^2(\gamma)$ ensures $\delta(f) \in L^2(\gamma)$; see Remark 1.15.

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Let us first introduce Hermite polynomials and then return for more discussions on the divergence operator $\delta$ and its domain $\mathrm{dom}(\delta)$.

### 1.2. **Hermite polynomials.**

**Definition 1.9.** *Put $H_0(x) = 1$, and $H_{p+1} = \delta(H_p)$ for $p \in \mathbb{N}$. $H_p$ is called the Hermite polynomial of order $p$. The first few ones are given by*

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \ldots$$

It is not difficult to verify by induction that

$$\frac{d}{dx}H_p(x) = pH_{p-1}(x) \quad \text{and} \quad H_{p+1}(x) = xH_p(x) - pH_{p-1}(x) \tag{1.15}$$

for any integer $p \geq 1$. In a compact manner, we can also define Hermite polynomials using Rodrigues' formula:

$$H_p(x) = (-1)^p \frac{1}{\phi(x)} \phi^{(p)}(x) \tag{1.16}$$

with $\phi$ as in (1.1). We leave the verification as an exercise to students. Another way to define the Hermite polynomials is via the Gaussian integral formula $H_p(x) = \mathbb{E}[(x+iZ)^p]$ with $Z \sim \mathcal{N}(0,1)$; see the expository article [NO25] for various applications.

**Exercise 1.10.** (i) *Verify the relations* (1.15) *and* (1.16).

(ii) *Show that any monomial can be expressed as a finite linear combination of Hermite polynomials.*

The family of Hermite polynomials play a crucial role in the Gaussian analysis. They are the orthogonal polynomials with respect to the standard Gaussian measure on $\mathbb{R}$.